Fourieranalys MVE030 och Fourier Metoder MVE290 2025.juni.10
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
5 Timmar.
Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & Chalmers godkänt miniräknare.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

Uppgifter

1. (**Fourierserier gör det rätt — närmevärdet blir helt rätt!**) Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum H. Om $f \in H$, och

$$\sum_{n\in\mathbb{N}} c_n \phi_n \in H,$$

bevisa att

$$\|f - \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n\| \leq \|f - \sum_{n\in\mathbb{N}} c_n \phi\|.$$

(English: **Fourier series pass the test, they can approximate the best!**) Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space H. If $f \in H$, and

$$\sum_{n\in\mathbb{N}} c_n \phi_n \in H,$$

then

$$\|f - \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n\| \leq \|f - \sum_{n\in\mathbb{N}} c_n \phi\|.$$

(10p)

2. (**Besselfunktionerna är genererade av en funktion som är exponentierad**). Bevisa att Besselfunktionerna J_n uppfyller

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}, \quad x \in \mathbb{R}, \quad z \in \mathbb{C} \setminus \{0\}.$$

(English: **The Bessel functions are generated by a function that's exponentiated**) Prove that for all $x \in \mathbb{R}$ and for all $z \in \mathbb{C}$

with $z \neq 0$, the Bessel functions, J_n, satisfy

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}.$$

3. Beräkna, beroende på a: (Compute depending on a):

$$\sum_{n=2}^{\infty} \frac{a^2}{4n^2 - 4n + 1}.$$

(10p)

4. Lös problemet: (solve):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = (x\cos(t))^3, & t > 0,\ 0 < x < 2, \\ u(0,t) = 0, & t > 0, \\ u(2,t) = 0, & t > 0, \\ u(x,0) = 5x, & 0 \leq x \leq 2. \end{cases}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.) (10 p)

5. Lös problemet (solve):

$$u_t(r,\theta,t) = \Delta u(r,\theta,t) + \sin(r),\ 0 < t,\ 0 < r < 1,\ -\pi < \theta < \pi,$$
$$u(1,\theta,t) = 0,$$
$$u(r,\theta,0) = r^3.$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.)

(10p)

6. Lös problemet (solve):

$$u_t - u_{xx} = 0,\ x,t > 0,$$
$$u(0,t) = 0,$$
$$u(x,0) = \frac{1}{1+x^2}.$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.) (10p)

7. Låt f vara en kontinuerlig udda funktion på $[-1, 1]$. Funktionen $p(x)$ är ett polynom av grad mindre än eller lika med 8 som minimerar

$$\|f(x) - p(x)\| = \sqrt{\int_{-1}^{1} (f(x) - p(x))^2 \mathrm{d}x}.$$

Visa att $p(x)$ också är en udda funktion.
(English) Let f be a continuous odd function on $[-1, 1]$. The function $p(x)$ is a polynomial of degree at most 8 that minimizes

$$\|f(x) - p(x)\| = \sqrt{\int_{-1}^{1} (f(x) - p(x))^2 \mathrm{d}x}.$$

Show that p is also an odd function.

8. Lös den partiella differentialekvationen (solve):

$$\begin{aligned} &u_t + u_x + u = 0, \; x > 0, \quad t > 0, \\ &u(0, t) = \cos(t), \\ &u(x, 0) = 0. \end{aligned}$$

(10p)

1 Fun facts!

1.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

1.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with u the unknown function and p and g specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called μ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the Mμthod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital C. The solution is:

$$u(t) = \frac{\int_0^t (\mu(s)g(s)ds) + C}{\mu(t)}.$$

1.3 How to solve second order linear ODEs

Theorem 1 (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function u that depends on one variable, with constants b and c given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:

1. ***If** $b^2 \neq 4ac$, then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$, then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

Theorem 2 (Particular solution to linear second order ODEs)**.** *Assume that y_1 und y_2 are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

Then, if y_p is a particular solution of the inhomogeneous ODE, so that

$$L(y_p) = g,$$

then all solutions to $L(y) = g$ can be expressed as

$$c_1 y_1 + c_2 y_2 + y_p,$$

for y_1 and y_2 as above, for coefficients c_1 and c_2. One way to find a particular solution to the ODE

$$L(y) = g(t)$$

is to calculate

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

The Wronskian of y_1 and y_2, denoted by $W(y_1, y_2)$ above, is defined to be

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t).$$

1.4 Euler differential equations

An Euler differential equation is an equation of the form

$$x^2u''(x) + axu'(x) + bu(x) = 0,$$

for some constants a and b.

1. If the quadratic equation
$$r^2 + (a-1)r + b = 0$$
has two distinct solutions, r_1 and r_2 then $\{x^{r_1}, x^{r_2}\}$ is a basis of solutions.

2. If the quadratic equation has one solution, call it r, then $\{x^r, x^r \log(x)\}$ is a basis of solutions.

OBSERVE that the solutions to the quadratic equation are in $\mathbb{C}$ so they can be complex numbers! The distinction is whether we have two different complex numbers that solve the equation (case 1) or whether only one complex number solves the equation (so it's a double root).

1.5 Orthogonal bases in certain cases

1. The functions
$$\{e^{inx}\}, \quad n \in \mathbb{Z}$$
are an orthogonal base for L^2 on the interval $[-\pi, \pi]$.

2. The functions $\{\sin(nx)\}_{n\geq 1}$ together with $\{\cos(nx)\}_{n\geq 0}$ are an orthogonal base for L^2 on the interval $[-\pi, \pi]$.

3. The functions $\{\sin(nx)\}_{n\geq 1}$ are an orthogonal base for L^2 on the interval $[0, \pi]$.

4. The functions $\{\cos(nx)\}_{n\geq 0}$ are an orthogonal base for L^2 on the interval $[0, \pi]$.

5. The functions $\{e^{in\pi x/L}\}_{n\in\mathbb{Z}}$ are an orthogonal base for $L^2(-L, L)$.

6. The functions $\{\sin(n\pi x/L)\}_{n\geq 1}$ are an orthogonal base for L^2 on the interval $[0, L]$.

7. The functions $\{\cos(n\pi x/L)\}_{n\geq 0}$ re an orthogonal base for L^2 on the interval $[0, L]$.

8. The functions $\{e^{in\pi(x-a)/L}\}_{n\in\mathbb{Z}}$ are an orthogonal base for L^2 on the interval $[a-L, a+L]$.

1.6 The Plancharel theorem and Fourier Inversion

Theorem 3 (Plancharel)**.** *Assume that f and g are in $L^2(\mathbb{R})$. Then we have the relationship between the scalar product of f and g and that of their Fourier transforms*

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\langle \hat{f}, \hat{g} \rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Theorem 4 (FIT (Fourier inversion))**.** *Assume that $f \in L^2(\mathbb{R})$. Then there is a unique element of $L^2(\mathbb{R})$ that is defined to be the Fourier transform of f and expressed as*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

Moreover,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

1.7 The Laplace transform

Proposition 1. *Assume that f and all derivatives of f up to the k^{th} are Laplace transformable. Then*

$$\widetilde{(f^{(k)})}(z) = z^k \tilde{f}(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}.$$

Here $\widetilde{\text{something}}$ denotes the Laplace transform of 'something.'

1.8 Dirichlet, Neumann, Periodic, and Robin boundary conditions in SLp's

1. The Dirichlet boundary condition requires a function to be equal to zero at the boundary.

2. The Neumann boundary condition requires the normal derivative of a function to be equal to zero at the boundary. So at a boundary point $x = a$ it looks like $f'(a) = 0$.

3. The Robin boundary condition requires the normal derivative to be equal to a scalar multiple of the function at the boundary. So if the boundary is at a point like $x = a$ then it would require $f'(a) = cf(a)$ for some constant c.

4. The periodic boundary condition requires a function to be the same at the endpoints of an interval, and also the derivative of the function to be the same at the endpoints of the interval. So if the interval for the problem is $[a, b]$ this would require $f(a) = f(b)$, and usually we also require $f'(a) = f'(b)$.

1.9 Bessel facts

Definition 1 (The Bessel function J of order ν)**.** The Bessel function J of order ν is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Bessel function satisfies the Bessel equation:

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0.$$

A second linearly independent solution to the Bessel equation is the Weber Bessel function $Y_\nu(x)$. This function tends to infinity as $x \to 0$. The modified Bessel equation is satisfied by I_ν and K_ν

$$x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x).$$

For real values of ν the function $I_\nu(x) \neq 0$ for all $x > 0$. One way to see this is that

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

So for $x > 0$ it's a sum of positive terms. The function $K_\nu(x)$ tends to ∞ as $x \to 0$. The Γ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{1}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of ν, it is an entire function of $x \in \mathbb{C}$.

Theorem 5 (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the m^{th} positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n} r/L)\}_{m \geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0, L)$. *Recall that this is the weighted* $\mathcal{L}^2$ *space on the interval* $(0, L)$ *with respect to the weight function* r, *so the scalar product*

$$\langle f, g\rangle = \int_0^L f(r)\overline{g(r)} r dr.$$

More generally, for any $\nu \in \mathbb{R}$, *and for any* $L > 0$, *the functions*

$$\{J_\nu(\pi_m r/L)\}_{m\geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0, L)$, *where above* π_m *denotes the* m^{th} *zero of the Bessel function* J_ν. *They have norms equal to*

$$\int_0^L |J_\nu(\pi_m r/L)|^2 r dr = \frac{L^2}{2} \left(J_{\nu+1}(\pi_m)\right)^2 .$$

Corollary 1 (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n} r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

are an orthogonal basis for $\mathcal{L}^2$ *on the disk of radius* L.

Theorem 6 (Bessel functions as bases in some other cases)**.** *Assume that* $L > 0$. *Let the weight function* $w(x) = x$. *Fix* $\nu \in \mathbb{R}$. *Then* J'_ν *has infinitely many positive zeros. Let*

$$\{\pi'_k\}_{k\geq 1}$$

be the positive zeros of J'_ν. *Then we define*

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu > 0, \quad k \geq 1.$$

In case $\nu = 0$, *define further* $\psi_0(x) = 1$. *(If* $\nu \neq 0$, *then this case is omitted.) Then* $\{\psi_k\}_{k\geq 1}$ *for* $\nu \neq 0$ *is an orthogonal basis for* $\mathcal{L}^2_w(0, L)$. *For* $\nu = 0$, $\{\psi_k\}_{k\geq 0}$ *is an orthogonal basis for* $\mathcal{L}^2_w(0, L)$. *Moreover the norm*

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \geq 1, \quad ||\psi_0||_w^2 = \frac{L^{2\nu+2}}{2\nu+2}.$$

Next, fix a constant $c > 0$. *Then there are infinitely many positive solutions of*

$$\mu J'_\nu(\mu) + cJ_\nu(\mu) = 0,$$

that can be enumerated as $\{\mu_k\}_{k\geq 1}$. *Then*

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}^2_w(0, L)$.

1.10 Orthogonal polynomials

Definition 2. The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right). \tag{2}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1,1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

The first few Legendre polynomials are $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, and $P_3 = \frac{1}{2}(5x^3-3x)$.

Definition 3. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}^2_2(\mathbb{R})$ with respect to the weight function e^{-x^2}. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2}dx = 2^n n! \int_{\mathbb{R}} e^{-x^2}dx = 2^n n!\sqrt{\pi}.$$

Definition 4. The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x}).$$

The Laguerre polynomials $\{L_n^\alpha\}_{n\geq 0}$ are an orthogonal basis for $\mathcal{L}^2_\alpha$ on $(0,\infty)$ with the weight function $\alpha(x) = x^\alpha e^{-x}$. Their norms squared,

$$||L_n^\alpha||^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

1.11 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

1.	$f(x)=x$	$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}.$
2.	$f(x)=\lvert x\rvert$	$\frac{\pi}{2}+\sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$
3.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ x, & 0<x<\pi.\end{cases}$	$\frac{\pi}{4}+\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\frac{(-1)^{n+1}}{2in}+\frac{(-1)^n-1}{2\pi n^2}\right]e^{inx}$
4.	$f(x)=\sin^2(x)$	$\frac{1}{2}-\frac{1}{4}\left(e^{2ix}+e^{-2ix}\right)$
5.	$f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{2}{i\pi}\sum_{n\geq 1}\frac{e^{(2n-1)ix}-e^{-(2n-1)ix}}{2n-1}$
6.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{1}{2}+\sum_{n\geq 1}\frac{e^{i(2n-1)x}-e^{-i(2n-1)x}}{i\pi(2n-1)}$
7.	$f(x)=\lvert\sin(x)\rvert$	$\frac{2}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{e^{2inx}+e^{-2inx}}{4n^2-1}$
8.	$f(x)=\lvert\cos(x)\rvert$	$\frac{2}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{(-1)^n\left[e^{inx}+e^{-inx}\right]}{4n^2-1}$
9.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin(x), & 0<x<\pi.\end{cases}$	$\frac{1}{\pi}-\frac{1}{\pi}\sum_{n\geq 1}\frac{e^{2inx}+e^{-2inx}}{4n^2-1}+\frac{1}{4i}\left(e^{ix}-e^{-ix}\right)$
10.	$f(x)=x^2$	$\frac{\pi^2}{3}+2\sum_{n\geq 1}\frac{(-1)^n(e^{inx}+e^{-inx})}{n^2}$
11.	$f(x)=x(\pi-\lvert x\rvert)$	$\frac{4}{i\pi}\sum_{n\geq 1}\frac{e^{i(2n-1)x}-e^{-i(2n-1)x}}{(2n-1)^3}$
12.	$f(x)=e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{b-in}e^{inx}$
13.	$f(x)=\sinh x$	$\frac{\sinh(\pi)}{i\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}n}{n^2+1}\left[e^{inx}-e^{-inx}\right]$

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x\notin(-\pi,\pi)$.

1.	$f(x) = x$	$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$
2.	$f(x) = \lvert x\rvert$	$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2}$
3.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$	$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2} + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin(nx)$
4.	$f(x) = \sin^2(x)$	$\frac{1}{2} - \frac{1}{2}\cos(2x)$
5.	$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{4}{\pi}\sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$
6.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{1}{2} + \frac{2}{\pi}\sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$
7.	$f(x) = \lvert\sin(x)\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1}$
8.	$f(x) = \lvert\cos(x)\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1} \frac{(-1)^n\cos(2nx)}{4n^2-1}$
9.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$	$\frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1} + \frac{1}{2}\sin(x)$
10.	$f(x) = x^2$	$\frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n\cos(nx)}{n^2}$
11.	$f(x) = x(\pi - \lvert x\rvert)$	$\frac{8}{\pi}\sum_{n\geq 1} \frac{\sin((2n-1)x)}{(2n-1)^3}$
12.	$f(x) = e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\left(\frac{1}{b} + \sum_{n\geq 1} \frac{(-1)^n}{b^2+n^2}\left[2b\cos(nx) - 2n\sin(nx)\right]\right)$
13.	$f(x) = \sinh x$	$\frac{2\sinh(\pi)}{\pi}\sum_{n\geq 1} \frac{(-1)^{n+1}n}{n^2+1}\sin(nx)$

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1, \cos(nx), \sin(nx)\}_{n\geq 1}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a > 0$ and $c \in \mathbb{R}$.

1.	$\Theta(t)f(t)$	$\widetilde{f}(z)$
2.	$\Theta(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
3.	$e^{ct}\Theta(t)f(t)$	$\widetilde{f}(z-c)$
4.	$\Theta(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
5.	$\Theta(t)f'(t)$	$z\widetilde{f}(z)-f(0)$
6.	$\Theta(t)f^{(k)}(t)$	$z^k\widetilde{f}(z)-\sum_0^{k-1} z^{k-1-j}f^{(j)}(0)$
7.	$\Theta(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
8.	$\Theta(t)tf(t)$	$-\widetilde{f}'(z)$
9.	$\Theta(t)t^{-1}f(t)$	$\int_z^\infty \widetilde{f}(w)dw$
10.	$\Theta f * \Theta g(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
11.	$\Theta(t)t^\nu e^{ct}$	$\Gamma(\nu+1)(z-c)^{-\nu-1}$
12.	$\Theta(t)(t+a)^{-1}$	$e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$
13.	$\Theta(t)\sin(ct)$	$\frac{c}{z^2+c^2}$
14.	$\Theta(t)\cos(ct)$	$\frac{z}{z^2+c^2}$

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$. $\Theta(t)$ is the heaviside function, that is zero when $t < 0$ and one when $t > 0$.

15.	$\Theta(t)\sinh(ct)$	$\frac{c}{z^2-c^2}$
16.	$\Theta(t)\cosh(ct)$	$\frac{z}{z^2-c^2}$
17.	$\Theta(t)\sin(\sqrt{at})$	$\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$
18.	$\Theta(t)t^{-1}\sin(\sqrt{at})$	$\pi\,\mathrm{erf}(\sqrt{a/(4z)})$
19.	$\Theta(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$
20.	$\Theta(t)\,\mathrm{erf}(at)$	$z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$
21.	$\Theta(t)\,\mathrm{erf}(\sqrt{t})$	$(z\sqrt{z+1})^{-1}$
22.	$\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$	$((z-1)\sqrt{z})^{-1}$
23.	$\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$	$z^{-1}e^{-a\sqrt{z}}$
24.	$\Theta(t)t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$
25.	$\Theta(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
26.	$\Theta(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
27.	$\Theta(t)t^{\nu}J_{\nu}(t)$	$2^{\nu}\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$
28.	$\Theta(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.

Fourieranalys MVE030 och Fourier Metoder MVE290 2025.juni.10
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
5 Timmar.
Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & Chalmers godkänt miniräknare.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

Uppgifter

1. (**Fourierserier gör det rätt — närmevärdet blir helt rätt!**) Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum H. Om $f \in H$, och

$$\sum_{n\in\mathbb{N}} c_n\phi_n \in H,$$

bevisa att

$$\|f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n\| \leq \|f - \sum_{n\in\mathbb{N}} c_n\phi.$$

(English:) (**Fourier series pass the test, they can approximate the best!**) Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space H. If $f \in H$, and

$$\sum_{n\in\mathbb{N}} c_n\phi_n \in H,$$

then

$$\|f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n\| \leq \|f - \sum_{n\in\mathbb{N}} c_n\phi.$$

(10p)

Solution

It is convenient (but optional) to introduce some terminology. Let

$$\hat{f}_n = \langle f, \phi_n\rangle, \quad g = \sum_{n\in\mathbb{N}} \hat{f}_n\phi_n, \quad \psi = \sum_{n\in\mathbb{N}} c_n\phi_n$$

$\rightarrow$ (3p)

$$
\begin{aligned}
(1p)\|f-g+g-\psi\|^2 &= \langle f-g+g-\psi, f-g+g-\psi\rangle \\
(1p) &= \|f-g\|^2 + \|g-\psi\|^2 + \langle f-g, g-\psi\rangle + \langle g-\psi, f-g\rangle \\
(1p) &= \|f-g\|^2 + \|g-\psi\|^2 + 2\mathrm{Re}\langle f-g, g-\psi\rangle
\end{aligned}
$$

[Note: One can arrive at the last conclusion here directly since this computation was done often in the course]

We consider first the last term:

$$
\begin{aligned}
&(1p)\langle f-g, g-\psi\rangle = \langle f,g\rangle - \langle f,\psi\rangle - \langle g,g\rangle + \langle g,\psi\rangle \\
&(1p) = \sum_{n\in\mathbb{N}} \bar{\hat{f}}_n \langle f,\phi_n\rangle - \sum_{n\in\mathbb{N}} \bar{c_n}\langle f,\phi_n\rangle - \sum_{n\in\mathbb{N}} \hat{f}_n\langle \phi_n, \sum_{m\in\mathbb{N}} \hat{f_m}\phi_m\rangle + \sum_{n\in\mathbb{N}} \hat{f}_n\langle\phi_n, \sum_{m\in\mathbb{N}} c_m\phi_m\rangle \\
&(2p) = \sum_{n\in\mathbb{N}} |\hat{f}_n|^2 - \sum_{n\in\mathbb{N}} \bar{c_n}\hat{f}_n - \sum_{n\in\mathbb{N}} |\hat{f}_n|^2 + \sum_{n\in\mathbb{N}} \hat{f}_n\bar{c_n} = 0,
\end{aligned}
$$

where we used the fact that $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal set.

Thereby, we have shown that

$$(1p)\|f-\psi\|^2 = \|f-g\|^2 + \|g-\psi\|^2 \geq \|f-g\|^2.$$

Equality holds if and only if

$$(1p)\|g-\psi\|^2 = 0 \iff g = \psi$$

(1p) By their definitions and the fact that the ϕ_n are orthogonal, this holds if and only if for all $_{n\in\mathbb{N}}$,

$$\hat{f}_n = c_n$$

2. (Bevisa - The Bessel functions are generated by a function that's exponentiated) Prove that for all $x \in \mathbb{R}$ and for all $z \in \mathbb{C}$ with $z \neq 0$, the Bessel functions, J_n, satisfy

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z-\frac{1}{z})}.$$

Solution

(1p) Write out the Taylor series expansion for the exponential functions:

$$e^{\frac{xz}{2}} = \sum_{j=0}^{\infty} \frac{(\frac{xz}{2})^j}{j!},$$
$$e^{-\frac{x}{2z}} = \sum_{k=0}^{\infty} \frac{(-\frac{x}{2z})^k}{k!}.$$

(1p) These converge absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$. So, since $z \neq 0$, we can multiply these series and rearrange the terms as we like. Thus, we write

$$(2p) \quad e^{\frac{xz}{2}} e^{-\frac{x}{2z}} = \sum_{j=0}^{\infty} \frac{(\frac{xz}{2})^j}{j!} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2z})^k}{k!} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}.$$

(2p) Next, keep k as it is and let $n = j - k$, so that $j = n + k$ and $j + k = n + 2k$. Then n will range from $-\infty$ to ∞. In particular, $j! = (n+k)!$, but this is problematic when $n + k < 0$ because the sum only contains $j!$ for non-negative j. However, we can remedy this by introducing the Gamma function:

$$j! = \Gamma(j+1), \;\; k! = \Gamma(k+1).$$

We have $\frac{1}{\Gamma(m)} = 0$ for $m \in \mathbb{Z}, \;\; m \leq 0$. Hence, we can write

$$(1p) \quad e^{\frac{xz}{2}} e^{-\frac{x}{2z}} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n+k+1)!k!}.$$

(1p) Note that terms with $n + k + 1 \leq 0$ correspond to $(n + k)!$ with $n + k < 0$, so they should not be there, but indeed $\frac{1}{\Gamma(n+k+1)}$ causes those terms to vanish! Finally, recall that

$$(1p) \quad J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{n+2k}}{k!\Gamma(n+k+1)}.$$

Thus, it follows that

$$(1p) \quad e^{\frac{x}{2}(z-\frac{1}{z})} = e^{\frac{xz}{2}} e^{-\frac{x}{2z}} = \sum_{n=-\infty}^{\infty} J_n(x) z^n.$$

(10 p)

3. Beräkna, berorende på a: (Compute depending on a):

$$\sum_{n=2}^{\infty} \frac{a^2}{4n^2 - 4n + 1}$$

(10p)

Solution

→ (4p) We first need to find a series in the table that could give us a solution. First, we see that we do not find a series in the table that has $4n^2 - 4n - 1$ in the denominator. But we note that $4n^2 - 4n - 1 = (2n-1)^2$, and find two possible Fourier series to help us! We pick here

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi \end{cases} = \frac{1}{2} + \frac{2}{\pi} \sum_{n \geq 1} \frac{\sin((2n-1)x)}{2n-1}$$

Note carefully that the sum starts at 1 here, not at 2. We will deal with this later.

→ (2p) We apply Parseval's equality to obtain

$$\|f\|^2 = \left\| \frac{1}{2} + \frac{2}{\pi} \sum_{n \geq 1} \frac{\sin((2n-1)x)}{2n-1} \right\|^2 = \frac{2\pi}{4} + \frac{4}{\pi^2} \sum_{n \geq 1} \frac{\|\sin((2n-1)x)\|^2}{(2n-1)^2}$$
$$= \frac{\pi}{2} + \frac{4}{\pi^2} \sum_{n \geq 1} \frac{\pi}{(2n-1)^2} = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n \geq 1} \frac{1}{(2n-1)^2}$$

→ (1p) On the other hand, we have

$$\|f\|^2 = \int_{-\pi}^{\pi} f(x)\overline{f(x)}dx = \int_0^{\pi} 1dx = \pi$$

→ (1p) We can now solve for (almost) our series: We get

$$\frac{\pi}{2} + \frac{4}{\pi} \sum_{n \geq 1} \frac{1}{(2n-1)^2} = \pi$$

and, therefore,

$$\sum_{n \geq 1} \frac{1}{(2n-1)^2} = \frac{\pi}{2}\frac{\pi}{4} = \frac{\pi^2}{8}$$

→ (2p) Finally,

$$\sum_{n=2}^{\infty} \frac{a^2}{4n^2 - 4n + 1} = a^2 \sum_{n \geq 1} \frac{1}{(2n-1)^2} - a^2 = \frac{a^2 \pi^2}{8} - a^2.$$

4. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = (x\cos(t))^3, & t > 0,\ 0 < x < 2, \\ u(0,t) = 0, & t > 0, \\ u(2,t) = 0, & t > 0, \\ u(x,0) = 5x, & 0 \leq x \leq 2. \end{cases}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.) (10 p)

Solution

This is an inhomogeneous heat equation on a bounded interval, where the inhomogeneity depends on time.

(a) (1p) SLPs are keys to solving inhomogeneous pde's. Even if you do nothing else, this rhyme is worth a point. If you don't do this rhyme, you still get a point if you set up the SLP to solve

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(2) = 0$$

(or written equivalently).

(b) (2p) Solve this SLP. You should obtain (see the vibrating string example in Chapter 1 of the textbook for the derivation of these solutions)

$$X_n(x) = \sin\left(\frac{n\pi x}{2}\right),\ \lambda_n = \frac{n^2\pi^2}{4},\ n \geq 1.$$

(c) (1p) Set up the solution you seek to be a series

$$u(x,t) = \sum_{n \geq 1} T_n(t) X_n(x),$$

where we will solve for the T_n functions using the inhomogeneous pde together with the initial condition.

(d) (2p) Expand the inhomogeneity in terms of the X_n base. This is possible because $\{X_n\}_{n=1}^{\infty}$ is an orthogonal base for $L^2(0,2)$ by the spectral theorem for SLPs:

$$(x\cos(t))^3 = \sum_{n\geq 1} \cos^3(t) \frac{\langle x^3, X_n\rangle}{||X_n||^2} X_n(x) = \sum_{n\geq 1} c_n \cos^3(t) X_n(x)$$

with

$$c_n = \frac{\langle x^3, X_n\rangle}{||X_n||^2},\ \langle x^3, X_n\rangle = \int_0^2 x^3 \overline{X_n(x)} dx,\ ||X_n||^2 = \int_0^2 |X_n(x)|^2 dx.$$

It is okay if you leave these integrals like this (without calculating them) as long as you have correctly defined the scalar product and the norm squared. However, if you do decide to compute them, you will get

$$c_n = \frac{16(-1)^{n+1}(\pi^2 n^2 - 6)}{\pi^3 n^3}.$$

(e) (1p) Plug u into the heat equation (and use that $X_n'' = -\lambda_n X_n$) to obtain

$$u_t - u_{xx} = \sum_{n\geq 1} (T_n'(t) + \frac{n^2\pi^2}{4} T_n(t)) X_n(x) = \sum_{n\geq 1} c_n \cos^3(t) X_n(x).$$

(f) (1p) Identify coefficients to obtain the equation for T_n:

$$T_n'(t) + \frac{n^2\pi^2}{4} T_n(t) = c_n \cos^3(t).$$

(g) (1p) Set up the correct initial condition:

$$u(x,0) = \sum_{n\geq 1} X_n(x) T_n(0) = 5x = \sum_{n\geq 1} C_n X_n(x)$$

with

$$T_n(0) = C_n = \frac{\langle 5x, X_n\rangle}{||X_n||^2}.$$

Again, it is okay if you don't compute this scalar product and norm squared. In fact, if you have correctly defined the scalar product and norm squared before, you do not even need to write these out as integrals again. However, if you do decide to compute them, you will get

$$C_n = \frac{20(-1)^{n+1}}{\pi n}.$$

(h) (1p) Solve the ODE for $T_n(t)$. The method of integrating factor will give you

$$T_n(t) = e^{-\frac{n^2\pi^2}{4}t}\left(\int_0^t e^{\frac{n^2\pi^2}{4}s} c_n \cos^3(s)ds + C_n\right).$$

5. Lös problemet (solve):

$$u_t(r,\theta,t) = \Delta u(r,\theta,t) + \sin(r),\ 0 < t,\ 0 < r < 1,\ -\pi < \theta < \pi,$$
$$u(1,\theta,t) = 0,$$
$$u(r,\theta,0) = r^3.$$

Solution and point distribution

(1p) The solution is independent of θ, which simplifies things.

(2p) SLp's are the keys to solving inhomogeneous pde's. Let's pretend that the pde is homogeneous and solve the part for r since it has the nice Dirichlet condition at $r = 1$. The Laplacian in radial coordinates is

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

We need to find the associated Bessel equation for R, and solve it. We need to find solutions of the form

$$R(r)T(t).$$

If the pde were homogeneous, we would get the equation

$$RT' = R''T + r^{-1}R'T$$

and then

$$\frac{R''}{R} + r^{-1}\frac{R'}{R} = \frac{T}{T'} = \lambda.$$

We will solve for R and then remember to stop because our equation for the T function will need to incorporate that inhomogeneity. For R the equation is

$$R'' + r^{-1}R' - \lambda R = 0,$$

or

$$r^2R'' + rR' - \lambda r^2 R = 0.$$

If $\lambda = 0$, then this becomes an Euler equation with solutions 1 and $\log(r)$. The function $\log(r)$ is excluded because it blows up at $r = 0$. The other function 1 does not satisfy the Dirichlet boundary condition at $r = 1$. If $\lambda > 0$, then this is a modified Bessel equation of order 0 with solutions K_0 and I_0. The K_0 Bessel function also blows up at $r = 0$, while the I_0 Bessel function does not have any positive real zeros, so it won't satisfy the Dirichlet boundary condition at $r = 5$. The only viable case is therefore that $\lambda < 0$, and the solutions are J_0 and Y_0. The Y_0 Bessel function blows up at $r = 0$ so we say goodbye to it. We are then left with the Bessel function J_0 with argument $\sqrt{-\lambda}r$. Considering the boundary condition $u(1, \theta, t) = 0$ and we get $\sqrt{-\lambda} = \pi_k$, with π_k one of the zeros of the Bessel function of degree 0, and

$$-\pi_k^2 = \lambda.$$

Note that $(R_k)_k$ is a base for $L_r^2([0, 1])$.

(1p) Realizing how we can write the solution in terms of R_k and T_k, as

$$u(r, t) = \sum_k T_k(t)R_k(r).$$

(2p) Use the Bessel equation to transform it. The differential equation says that we need

$$\sum_k T_k'(t)R_k(r) = \sin(r)+\sum_k T_k(t)(R_k''(r)+r^{-1}R_k'(r)) = \sin(r)-\sum_k \pi_k^2 T_k(t)R_k(r).$$

Here we used that R_k satisfies the differential equation we had above

$$r^2R_k'' + rR_k' = \lambda r^2R_k \iff R_k'' + r^{-1}R_k' = \lambda R_k.$$

This implies that

$$\sin(r) = \sum_k R_k(T_k' + \pi_k^2 T_k/25).$$

(2p) Writing $\sin(r)$ in terms of the basis R_k and coefficients.

$$\sin(r) = \sum_k \frac{\langle \sin(r), R_k\rangle}{\langle R_k, R_k\rangle} R_k.$$

Now

$$\langle R_k, R_k\rangle = \int_0^5 |J_0(\pi_k r)|^2 r dr,$$

and

$$\langle \sin(r), R_k \rangle = \int_0^5 J_0(\pi_k r) r dr.$$

For simplicity let's call

$$c_k = \frac{\langle \sin(r), R_k \rangle}{\langle R_k, R_k \rangle}$$

(2p) Finding all the T_k. We need

$$\sum_k c_k R_k = \sum_k R_k (T_k' + \pi_k^2 T_k) \implies c_k = T_k' + \pi_k^2 T_k.$$

So we find using the method of integrating factor that

$$T_k(t) = e^{-\pi_k^2 t} \left[\int_0^t c_k e^{\pi_k^2 s} ds + T_k(0) \right].$$

For the initial condition we need

$$f(r) = r^3 = \sum_k R_k(r) T_k(0),$$

so

$$T_k(0) = \frac{\langle f, R_k \rangle}{\langle R_k, R_k \rangle},$$

with the scalar product and norm defined analogously to the above.

6. Lös problemet

$$\begin{aligned} &u_t - u_{xx} = 0, \ x, t > 0, \\ &u(0, t) = 0, \\ &u(x, 0) = \frac{1}{1 + x^2}. \end{aligned}$$

Solution and point distribution

(1p) Realize the functions u and f need to be extended oddly because there is a Dirichlet condition $u(0, t) = 0$.

(1p) We do a Fourier transform and get

$$\hat{u}_t(\xi, t) = -\xi^2 \hat{u}(\xi, t).$$

(2p) We solve this and get

$$\hat{u}(\xi, t) = c(\xi)e^{-\xi^2 t}.$$

(4p) We realize

$$c(\xi) = \hat{u}(\xi, 0) = \int_0^\infty \frac{1}{1+x^2} e^{-ix\xi} dx - \int_{-\infty}^0 \frac{1}{1+x^2} e^{-ix\xi} dx$$
$$= \int_0^\infty \frac{1}{1+x^2} e^{-ix\xi} dx - \int_0^\infty \frac{1}{1+x^2} e^{ix\xi} dx = -2i \int_0^\infty \frac{\sin(x\xi)}{1+x^2} dx$$

(2p) So we need to have

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi, t) e^{i\xi x} dx = \frac{-i}{\pi} \int_{\mathbb{R}} \int_0^\infty \frac{\sin(x\xi) e^{ix\xi - \xi^2 t}}{1+x^2} dx d\xi.$$

It's fine to leave this integral as it is.

7. Låt f vara en kontinuerlig udda funktion på $[-1, 1]$. Funktionen $p(x)$ är ett polynom av grad mindre än eller lika med 8 som minimerar

$$\|f(x) - p(x)\| = \sqrt{\int_{-1}^1 (f(x) - p(x))^2 \mathrm{d}x}.$$

Visa att $p(x)$ också är en udda funktion.

Solution and points

(5p) Writing out the best approximation of f in terms of the Legendre polynomials correctly. One point is lost for each error in doing this (but you won't get negative points!).

(1p) If you say that the n^{th} degree Legendre polynomial is odd if n is odd and even if n is even this is worth 1 point.

(2p) Explain or refer to Beta to justify the preceding fact.

(1p) Explain why the terms

$$\frac{\langle f, P_n \rangle}{||P_n||^2}$$

vanish when n is even, and thus only the terms when n is odd remain. (1p) Explain that all that remains is a sum of odd functions which is therefore odd.

8. Lös den partiella differentialekvationen

$$\begin{aligned}
&u_t + u_x + u = 0, \ x > 0, \quad t > 0, \\
&u(0,t) = \cos(t), \\
&u(x,0) = 0.
\end{aligned}$$

Solution and points

(2p) Writing that you do the Laplace transform in the t variable. (1p for LT and 1p for correct variable).

(2p) Actually doing the LT of the pde correctly. You should get

$$z\widetilde{u}(x,z) + \widetilde{u}_x(x,z) + \widetilde{u}(x,z) = 0.$$

(2p) Solving this ode in x to get

$$\widetilde{u}(x,z) = c(z)e^{-(z+1)x} = c(z)e^{-xz}e^{-x}.$$

(2p) Using the boundary condition to say that

$$c(z) = \widetilde{\cos}(z).$$

(2p) Correctly unravelling the Laplace transform to get

$$u(x,t) = e^{-x}\Theta(t-x)\cos(t-x).$$

1 Fun facts!

1.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

1.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with u the unknown function and p and g specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called μ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the Mμthod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital C. The solution is:

$$u(t) = \frac{\int_0^t (\mu(s)g(s)ds) + C}{\mu(t)}.$$

1.3 How to solve second order linear ODEs

Theorem 1 (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function u that depends on one variable, with constants b and c given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:

1. ***If** $b^2 \neq 4ac$, then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$, then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

Theorem 2 (Particular solution to linear second order ODEs)**.** *Assume that y_1 und y_2 are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

Then, if y_p is a particular solution of the inhomogeneous ODE, so that

$$L(y_p) = g,$$

then all solutions to $L(y) = g$ can be expressed as

$$c_1 y_1 + c_2 y_2 + y_p,$$

for y_1 and y_2 as above, for coefficients c_1 and c_2. One way to find a particular solution to the ODE

$$L(y) = g(t)$$

is to calculate

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

The Wronskian of y_1 and y_2, denoted by $W(y_1, y_2)$ above, is defined to be

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t).$$

1.4 Euler differential equations

An Euler differential equation is an equation of the form

$$x^2u''(x) + axu'(x) + bu(x) = 0,$$

for some constants a and b.

1. If the quadratic equation
$$r^2 + (a-1)r + b = 0$$
has two distinct solutions, r_1 and r_2 then $\{x^{r_1}, x^{r_2}\}$ is a basis of solutions.

2. If the quadratic equation has one solution, call it r, then $\{x^r, x^r \log(x)\}$ is a basis of solutions.

OBSERVE that the solutions to the quadratic equation are in $\mathbb{C}$ so they can be complex numbers! The distinction is whether we have two different complex numbers that solve the equation (case 1) or whether only one complex number solves the equation (so it's a double root).

1.5 Orthogonal bases in certain cases

1. The functions
$$\{e^{inx}\}, \quad n \in \mathbb{Z}$$
are an orthogonal base for L^2 on the interval $[-\pi, \pi]$.

2. The functions $\{\sin(nx)\}_{n\geq 1}$ together with $\{\cos(nx)\}_{n\geq 0}$ are an orthogonal base for L^2 on the interval $[-\pi, \pi]$.

3. The functions $\{\sin(nx)\}_{n\geq 1}$ are an orthogonal base for L^2 on the interval $[0, \pi]$.

4. The functions $\{\cos(nx)\}_{n\geq 0}$ are an orthogonal base for L^2 on the interval $[0, \pi]$.

5. The functions $\{e^{in\pi x/L}\}_{n\in\mathbb{Z}}$ are an orthogonal base for $L^2(-L, L)$.

6. The functions $\{\sin(n\pi x/L)\}_{n\geq 1}$ are an orthogonal base for L^2 on the interval $[0, L]$.

7. The functions $\{\cos(n\pi x/L)\}_{n\geq 0}$ re an orthogonal base for L^2 on the interval $[0, L]$.

8. The functions $\{e^{in\pi(x-a)/L}\}_{n\in\mathbb{Z}}$ are an orthogonal base for L^2 on the interval $[a-L, a+L]$.

1.6 The Plancharel theorem and Fourier Inversion

Theorem 3 (Plancharel)**.** *Assume that f and g are in $L^2(\mathbb{R})$. Then we have the relationship between the scalar product of f and g and that of their Fourier transforms*

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Theorem 4 (FIT (Fourier inversion))**.** *Assume that $f \in L^2(\mathbb{R})$. Then there is a unique element of $L^2(\mathbb{R})$ that is defined to be the Fourier transform of f and expressed as*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

Moreover,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

1.7 The Laplace transform

Proposition 1. *Assume that f and all derivatives of f up to the k^{th} are Laplace transformable. Then*

$$\widetilde{(f^{(k)})}(z) = z^k \widetilde{f}(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}.$$

Here $\widetilde{\text{something}}$ denotes the Laplace transform of 'something.'

1.8 Dirichlet, Neumann, Periodic, and Robin boundary conditions in SLp's

1. The Dirichlet boundary condition requires a function to be equal to zero at the boundary.

2. The Neumann boundary condition requires the normal derivative of a function to be equal to zero at the boundary. So at a boundary point $x = a$ it looks like $f'(a) = 0$.

3. The Robin boundary condition requires the normal derivative to be equal to a scalar multiple of the function at the boundary. So if the boundary is at a point like $x = a$ then it would require $f'(a) = cf(a)$ for some constant c.

4. The periodic boundary condition requires a function to be the same at the endpoints of an interval, and also the derivative of the function to be the same at the endpoints of the interval. So if the interval for the problem is $[a, b]$ this would require $f(a) = f(b)$, and usually we also require $f'(a) = f'(b)$.

1.9 Bessel facts

Definition 1 (The Bessel function J of order ν)**.** The Bessel function J of order ν is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Bessel function satisfies the Bessel equation:

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0.$$

A second linearly independent solution to the Bessel equation is the Weber Bessel function $Y_\nu(x)$. This function tends to infinity as $x \to 0$. The modified Bessel equation is satisfied by I_ν and K_ν

$$x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x).$$

For real values of ν the function $I_\nu(x) \neq 0$ for all $x > 0$. One way to see this is that

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

So for $x > 0$ it's a sum of positive terms. The function $K_\nu(x)$ tends to ∞ as $x \to 0$. The Γ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{1}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of ν, it is an entire function of $x \in \mathbb{C}$.

Theorem 5 (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the m^{th} positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n} r/L)\}_{m \geq 1}$$

are an orthogonal base for $\mathcal{L}_r^2(0,L)$*. Recall that this is the weighted* $\mathcal{L}^2$ *space on the interval* $(0,L)$ *with respect to the weight function* r*, so the scalar product*

$$\langle f, g\rangle = \int_0^L f(r)\overline{g(r)} r dr.$$

More generally, for any $\nu \in \mathbb{R}$*, and for any* $L > 0$*, the functions*

$$\{J_\nu(\pi_m r/L)\}_{m\geq 1}$$

are an orthogonal base for $\mathcal{L}_r^2(0,L)$*, where above* π_m *denotes the* m^{th} *zero of the Bessel function* J_ν*. They have norms equal to*

$$\int_0^L |J_\nu(\pi_m r/L)|^2 r dr = \frac{L^2}{2}\left(J_{\nu+1}(\pi_m)\right)^2.$$

Corollary 1 (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n} r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

are an orthogonal basis for $\mathcal{L}^2$ *on the disk of radius* L*.*

Theorem 6 (Bessel functions as bases in some other cases)**.** *Assume that* $L > 0$*. Let the weight function* $w(x) = x$*. Fix* $\nu \in \mathbb{R}$*. Then* J'_ν *has infinitely many positive zeros. Let*

$$\{\pi'_k\}_{k\geq 1}$$

be the positive zeros of J'_ν*. Then we define*

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu > 0, \quad k \geq 1.$$

In case $\nu = 0$*, define further* $\psi_0(x) = 1$*. (If* $\nu \neq 0$*, then this case is omitted.) Then* $\{\psi_k\}_{k\geq 1}$ *for* $\nu \neq 0$ *is an orthogonal basis for* $\mathcal{L}_w^2(0,L)$*. For* $\nu = 0$*,* $\{\psi_k\}_{k\geq 0}$ *is an orthogonal basis for* $\mathcal{L}_w^2(0,L)$*. Moreover the norm*

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \geq 1, \quad ||\psi_0||_w^2 = \frac{L^{2\nu+2}}{2\nu+2}.$$

Next, fix a constant $c > 0$*. Then there are infinitely many positive solutions of*

$$\mu J'_\nu(\mu) + cJ_\nu(\mu) = 0,$$

that can be enumerated as $\{\mu_k\}_{k\geq 1}$*. Then*

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}_w^2(0,L)$*.*

1.10 Orthogonal polynomials

Definition 2. The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right). \tag{2}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1,1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

The first few Legendre polynomials are $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, and $P_3 = \frac{1}{2}(5x^3-3x)$.

Definition 3. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}^2_2(\mathbb{R})$ with respect to the weight function e^{-x^2}. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2}dx = 2^n n! \int_{\mathbb{R}} e^{-x^2}dx = 2^n n!\sqrt{\pi}.$$

Definition 4. The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x}).$$

The Laguerre polynomials $\{L_n^\alpha\}_{n\geq 0}$ are an orthogonal basis for $\mathcal{L}^2_\alpha$ on $(0,\infty)$ with the weight function $\alpha(x) = x^\alpha e^{-x}$. Their norms squared,

$$||L_n^\alpha||^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

1.11 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

1.	$f(x) = x$	$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}$.
2.	$f(x) = \|x\|$	$\frac{\pi}{2} + \sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$
3.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$	$\frac{\pi}{4} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\frac{(-1)^{n+1}}{2in} + \frac{(-1)^n - 1}{2\pi n^2}\right] e^{inx}$
4.	$f(x) = \sin^2(x)$	$\frac{1}{2} - \frac{1}{4}\left(e^{2ix} + e^{-2ix}\right)$
5.	$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{2}{i\pi} \sum_{n\geq 1} \frac{e^{(2n-1)ix} - e^{-(2n-1)ix}}{2n-1}$
6.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{1}{2} + \sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{i\pi(2n-1)}$
7.	$f(x) = \|\sin(x)\|$	$\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2 - 1}$
8.	$f(x) = \|\cos(x)\|$	$\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{(-1)^n\left[e^{inx} + e^{-inx}\right]}{4n^2 - 1}$
9.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$	$\frac{1}{\pi} - \frac{1}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2 - 1} + \frac{1}{4i}\left(e^{ix} - e^{-ix}\right)$
10.	$f(x) = x^2$	$\frac{\pi^2}{3} + 2\sum_{n\geq 1} \frac{(-1)^n (e^{inx} + e^{-inx})}{n^2}$
11.	$f(x) = x(\pi - \|x\|)$	$\frac{4}{i\pi} \sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{(2n-1)^3}$
12.	$f(x) = e^{bx}$	$\frac{\sinh(b\pi)}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b - in} e^{inx}$
13.	$f(x) = \sinh x$	$\frac{\sinh(\pi)}{i\pi} \sum_{n\geq 1} \frac{(-1)^{n+1} n}{n^2 + 1}\left[e^{inx} - e^{-inx}\right]$

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi, \pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi, \pi)$. On the rest of the real line, outside of the interval $(-\pi, \pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi, \pi)$.

1.	$f(x)=x$	$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx).$
2.	$f(x)=\|x\|$	$\frac{\pi}{2}-\frac{4}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2}$
3.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ x, & 0<x<\pi.\end{cases}$	$\frac{\pi}{4}-\frac{2}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2}+\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sin(nx)$
4.	$f(x)=\sin^2(x)$	$\frac{1}{2}-\frac{1}{2}\cos(2x)$
5.	$f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{4}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
6.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{1}{2}+\frac{2}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
7.	$f(x)=\|\sin(x)\|$	$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1}$
8.	$f(x)=\|\cos(x)\|$	$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n\geq 1}\frac{(-1)^n\cos(2nx)}{4n^2-1}$
9.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin(x), & 0<x<\pi.\end{cases}$	$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$
10.	$f(x)=x^2$	$\frac{\pi^2}{3}+4\sum_{n\geq 1}\frac{(-1)^n\cos(nx)}{n^2}$
11.	$f(x)=x(\pi-\|x\|)$	$\frac{8}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{(2n-1)^3}$
12.	$f(x)=e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\left(\frac{1}{b}+\sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2}\left[2b\cos(nx)-2n\sin(nx)\right]\right)$
13.	$f(x)=\sinh x$	$\frac{2\sinh(\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}n}{n^2+1}\sin(nx)$

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1,\cos(nx),\sin(nx)\}_{n\geq 1}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x\notin(-\pi,\pi)$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a > 0$ and $c \in \mathbb{R}$.

1.	$\Theta(t)f(t)$	$\widetilde{f}(z)$
2.	$\Theta(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
3.	$e^{ct}\Theta(t)f(t)$	$\widetilde{f}(z-c)$
4.	$\Theta(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
5.	$\Theta(t)f'(t)$	$z\widetilde{f}(z)-f(0)$
6.	$\Theta(t)f^{(k)}(t)$	$z^k\widetilde{f}(z)-\sum_0^{k-1} z^{k-1-j}f^{(j)}(0)$
7.	$\Theta(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
8.	$\Theta(t)tf(t)$	$-\widetilde{f}'(z)$
9.	$\Theta(t)t^{-1}f(t)$	$\int_z^\infty \widetilde{f}(w)dw$
10.	$\Theta f * \Theta g(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
11.	$\Theta(t)t^\nu e^{ct}$	$\Gamma(\nu+1)(z-c)^{-\nu-1}$
12.	$\Theta(t)(t+a)^{-1}$	$e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$
13.	$\Theta(t)\sin(ct)$	$\frac{c}{z^2+c^2}$
14.	$\Theta(t)\cos(ct)$	$\frac{z}{z^2+c^2}$

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$. $\Theta(t)$ is the heaviside function, that is zero when $t < 0$ and one when $t > 0$.

15.	$\Theta(t)\sinh(ct)$	$\frac{c}{z^2-c^2}$
16.	$\Theta(t)\cosh(ct)$	$\frac{z}{z^2-c^2}$
17.	$\Theta(t)\sin(\sqrt{at})$	$\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$
18.	$\Theta(t)t^{-1}\sin(\sqrt{at})$	$\pi\,\mathrm{erf}(\sqrt{a/(4z)})$
19.	$\Theta(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$
20.	$\Theta(t)\,\mathrm{erf}(at)$	$z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$
21.	$\Theta(t)\,\mathrm{erf}(\sqrt{t})$	$(z\sqrt{z+1})^{-1}$
22.	$\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$	$((z-1)\sqrt{z})^{-1}$
23.	$\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$	$z^{-1}e^{-a\sqrt{z}}$
24.	$\Theta(t)t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$
25.	$\Theta(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
26.	$\Theta(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
27.	$\Theta(t)t^{\nu}J_{\nu}(t)$	$2^{\nu}\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$
28.	$\Theta(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.