**Fourieranalys MVE030 och Fourier Metoder MVE290 2025.mars.21**
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
5 Timmar.
Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & Chalmers godkänt miniräknare.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

# 1 Uppgifter

1. Bevisa DOG-satsen: Antar att $\{\phi_n\}_{n\geq 0}$ är ortonormala i ett Hilbertrum $H$. Bevisa att följande är äkvilenta:

   (a) Om $f \in H$, sedan gäller $f = \sum_{n\geq 0}\langle f, \phi_n\rangle \phi_n$.
   (b) $\|f\|^2 = \sum_{n\geq 0}|\langle f, \phi_n\rangle|^2$.
   (c) Om $v \in H$ och $\langle v, \phi_n\rangle = 0 \ \forall \ n \geq 0$, sedan $v = 0$.

   (English:) Assume that $\{\phi_n\}_{n\geq 0}$ are orthonormal in a Hilbert space $H$. Prove that the following are equivalent:

   (a) If $f \in H$, then it holds that $f = \sum_{n\geq 0}\langle f, \phi_n\rangle \phi_n$.
   (b) $\|f\|^2 = \sum_{n\geq 0}|\langle f, \phi_n\rangle|^2$.
   (c) If $v \in H$ and $\langle v, \phi_n\rangle = 0$ for all $n \geq 0$, then $v = 0$.

   (10p)

2. Bevisa att hermitpolynomen $\{H_n\}_{n=0}^{\infty}$ är ortogonala på $\mathbb{R}$ med avseende på viktfunktionen $w(x) = e^{-x^2}$. Kom ihåg det här

   $$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

   (English): Prove that the Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Recall here that

   $$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

   (10 p)

3. Beräkna (Compute):
$$\sum_{n=0}^{\infty} \frac{e}{4n^2 - 4n + 1}.$$
(10p)

4. Lös problemet: (Solve the following problem):
$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = x\sin^2(t), & t > 0,\ 0 < x < 1, \\ u(0,t) = 0, & t > 0, \\ u(1,t) = 0, & t > 0, \\ u(x,0) = x^2, & 0 \le x \le 1. \end{cases}$$
(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.) (10 p)

5. Lös problemet (solve):
$$\begin{aligned} &u_t(r,\theta,t) = \Delta u(r,\theta,t) + e^t,\ 0 < t,\ 0 < r < 5,\ -\pi < \theta < \pi, \\ &u(5,\theta,t) = 0, \\ &u(r,\theta,0) = re^{-r^2} + 5r. \end{aligned}$$
(10p)

6. Lös problemet
$$\begin{aligned} &u_t - u_{xx} = 0,\ x,t > 0, \\ &u_x(0,t) = 0, \\ &u(x,0) = e^{-x^2}. \end{aligned}$$
(10p)

7. Låt $f$ vara en kontinuerlig jämn funktion på $[-1,1]$. Funktionen $p(x)$ är ett polynom av grad mindre än eller lika med 9 som minimerar
$$\|f(x) - p(x)\| = \sqrt{\int_{-1}^{1} (f(x) - p(x))^2 \mathrm{d}x}.$$

Visa att $p(x)$ också är en jämn funktion.
**Ledtråd:** Funktionen $r(x) = \frac{1}{2}(p(x)+p(-x))$ är också ett polynom av grad mindre än eller lika med 9 som uppfyller att $r(x) = r(-x)$. Vad kan du säga om $\|f(x) - r(x)\|$? Triangelolikheten $\|h_1(x) + h_2(x)\| \leq \|h_1(x)\| + \|h_2(x)\|$ kan visa sig vara användbar.

8. Lös den partiella differentialekvationen

$$\begin{aligned} &u_t + u_x + u = 0,\ x > 0,\ t > 0, \\ &u(0,t) = \sin(t), \\ &u(x,0) = 0. \end{aligned}$$

(10p)

# 2 Fun facts!

## 2.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

## 2.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with $u$ the unknown function and $p$ and $g$ specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called $\mu$ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the M$\mu$thod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital $C$. The solution is:

$$u(t) = \frac{\int_0^t(\mu(s)g(s)ds) + C}{\mu(t)}.$$

### 2.3 How to solve second order linear ODEs

**Theorem 1** (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function $u$ that depends on one variable, with constants $b$ and $c$ given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

*A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:*

1. ***If** $b^2 \neq 4ac$**,** then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$**,** then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

**Theorem 2** (Particular solution to linear second order ODEs)**.** *Assume that $y_1$ und $y_2$ are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

*Then, if $y_p$ is a particular solution of the inhomogeneous ODE, so that*

$$L(y_p) = g,$$

*then all solutions to $L(y) = g$ can be expressed as*

$$c_1 y_1 + c_2 y_2 + y_p,$$

*for $y_1$ and $y_2$ as above, for coefficients $c_1$ and $c_2$. One way to find a particular solution to the ODE*

$$L(y) = g(t)$$

*is to calculate*

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

*The Wronskian of $y_1$ and $y_2$, denoted by $W(y_1, y_2)$ above, is defined to be*

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t).$$

### 2.4 Euler differential equations

An Euler differential equation is an equation of the form

$$x^2 u''(x) + axu'(x) + bu(x) = 0,$$

for some constants $a$ and $b$.

1. If the quadratic equation
$$r^2 + (a-1)r + b = 0$$
has two distinct solutions, $r_1$ and $r_2$ then $\{x^{r_1}, x^{r_2}\}$ is a basis of solutions.

2. If the quadratic equation has one solution, call it $r$, then $\{x^r, x^r \log(x)\}$ is a basis of solutions.

OBSERVE that the solutions to the quadratic equation are in $\mathbb{C}$ so they can be complex numbers! The distinction is whether we have two different complex numbers that solve the equation (case 1) or whether only one complex number solves the equation (so it's a double root).

### 2.5 Orthogonal bases in certain cases

1. The functions
$$\{e^{inx}\}, \quad n \in \mathbb{Z}$$
are an orthogonal base for $L^2$ on the interval $[-\pi, \pi]$.

2. The functions $\{\sin(nx)\}_{n\geq 1}$ together with $\{\cos(nx)\}_{n\geq 0}$ are an orthogonal base for $L^2$ on the interval $[-\pi, \pi]$.

3. The functions $\{\sin(nx)\}_{n\geq 1}$ are an orthogonal base for $L^2$ on the interval $[0, \pi]$.

4. The functions $\{\cos(nx)\}_{n\geq 0}$ are an orthogonal base for $L^2$ on the interval $[0, \pi]$.

5. The functions $\{e^{in\pi x/L}\}_{n\in\mathbb{Z}}$ are an orthogonal base for $L^2(-L, L)$.

6. The functions $\{\sin(n\pi x/L)\}_{n\geq 1}$ are an orthogonal base for $L^2$ on the interval $[0, L]$.

7. The functions $\{\cos(n\pi x/L)\}_{n\geq 0}$ re an orthogonal base for $L^2$ on the interval $[0, L]$.

8. The functions $\{e^{in\pi(x-a)/L}\}_{n\in\mathbb{Z}}$ are an orthogonal base for $L^2$ on the interval $[a-L, a+L]$.

### 2.6 The Plancharel theorem and Fourier Inversion

**Theorem 3** (Plancharel)**.** *Assume that $f$ and $g$ are in $L^2(\mathbb{R})$. Then we have the relationship between the scalar product of $f$ and $g$ and that of their Fourier transforms*

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

**Theorem 4** (FIT (Fourier inversion))**.** *Assume that $f \in L^2(\mathbb{R})$. Then there is a unique element of $L^2(\mathbb{R})$ that is defined to be the Fourier transform of $f$ and expressed as*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

*Moreover,*

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

### 2.7 The Laplace transform

**Proposition 1.** *Assume that $f$ and all derivatives of $f$ up to the $k^{th}$ are Laplace transformable. Then*

$$\widetilde{(f^{(k)})}(z) = z^k\widetilde{f}(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}.$$

*Here $\widetilde{\text{something}}$ denotes the Laplace transform of 'something.'*

### 2.8 Dirichlet, Neumann, Periodic, and Robin boundary conditions in SLp's

1. The Dirichlet boundary condition requires a function to be equal to zero at the boundary.

2. The Neumann boundary condition requires the normal derivative of a function to be equal to zero at the boundary. So at a boundary point $x = a$ it looks like $f'(a) = 0$.

3. The Robin boundary condition requires the normal derivative to be equal to a scalar multiple of the function at the boundary. So if the boundary is at a point like $x = a$ then it would require $f'(a) = cf(a)$ for some constant $c$.

4. The periodic boundary condition requires a function to be the same at the endpoints of an interval, and also the derivative of the function to be the same at the endpoints of the interval. So if the interval for the problem is $[a,b]$ this would require $f(a) = f(b)$, and usually we also require $f'(a) = f'(b)$.

### 2.9 Bessel facts

**Definition 1** (The Bessel function $J$ of order $\nu$)**.** The Bessel function $J$ of order $\nu$ is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Bessel function satisfies the Bessel equation:

$$x^2 f''(x) + xf'(x) + (x^2 - \nu^2)f(x) = 0.$$

A second linearly independent solution to the Bessel equation is the Weber Bessel function $Y_\nu(x)$. This function tends to infinity as $x \to 0$. The modified Bessel equation is satisfied by $I_\nu$ and $K_\nu$

$$x^2 f''(x) + xf'(x) - (x^2 + \nu^2)f(x).$$

For real values of $\nu$ the function $I_\nu(x) \neq 0$ for all $x > 0$. One way to see this is that

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

So for $x > 0$ it's a sum of positive terms. The function $K_\nu(x)$ tends to $\infty$ as $x \to 0$. The $\Gamma$ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{1}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of $\nu$, it is an entire function of $x \in \mathbb{C}$.

**Theorem 5** (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the $m^{th}$ positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n} r/L)\}_{m \geq 1}$$

*are an orthogonal base for $\mathcal{L}^2_r(0,L)$. Recall that this is the weighted $\mathcal{L}^2$ space on the interval $(0,L)$ with respect to the weight function $r$, so the scalar product*

$$\langle f, g\rangle = \int_0^L f(r)\overline{g(r)} r dr.$$

*More generally, for any $\nu \in \mathbb{R}$, and for any $L > 0$, the functions*

$$\{J_\nu(\pi_m r/L)\}_{m\geq 1}$$

*are an orthogonal base for $\mathcal{L}^2_r(0,L)$, where above $\pi_m$ denotes the $m^{th}$ zero of the Bessel function $J_\nu$. They have norms equal to*

$$\int_0^L |J_\nu(\pi_m r/L)|^2 r dr = \frac{L^2}{2}\left(J_{\nu+1}(\pi_m)\right)^2.$$

**Corollary 1** (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n} r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

*are an orthogonal basis for $\mathcal{L}^2$ on the disk of radius $L$.*

**Theorem 6** (Bessel functions as bases in some other cases)**.** *Assume that $L > 0$. Let the weight function $w(x) = x$. Fix $\nu \in \mathbb{R}$. Then $J'_\nu$ has infinitely many positive zeros. Let*

$$\{\pi'_k\}_{k\geq 1}$$

*be the positive zeros of $J'_\nu$. Then we define*

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu > 0, \quad k \geq 1.$$

*In case $\nu = 0$, define further $\psi_0(x) = 1$. (If $\nu \neq 0$, then this case is omitted.) Then $\{\psi_k\}_{k\geq 1}$ for $\nu \neq 0$ is an orthogonal basis for $\mathcal{L}^2_w(0,L)$. For $\nu = 0$, $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w(0,L)$. Moreover the norm*

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \geq 1, \quad ||\psi_0||^2_w = \frac{L^{2\nu+2}}{2\nu+2}.$$

*Next, fix a constant $c > 0$. Then there are infinitely many positive solutions of*

$$\mu J'_\nu(\mu) + cJ_\nu(\mu) = 0,$$

*that can be enumerated as $\{\mu_k\}_{k\geq 1}$. Then*

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

*is an orthogonal basis for $\mathcal{L}^2_w(0,L)$.*

## 2.10 Orthogonal polynomials

**Definition 2.** The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( (x^2 - 1)^n \right). \tag{2}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1, 1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

The first few Legendre polynomials are $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2 - 1)$, and $P_3 = \frac{1}{2}(5x^3 - 3x)$.

**Definition 3.** The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}^2_2(\mathbb{R})$ with respect to the weight function $e^{-x^2}$. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

**Definition 4.** The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x}).$$

The Laguerre polynomials $\{L_n^\alpha\}_{n \geq 0}$ are an orthogonal basis for $\mathcal{L}^2_\alpha$ on $(0, \infty)$ with the weight function $\alpha(x) = x^\alpha e^{-x}$. Their norms squared,

$$||L_n^\alpha||^2 = \frac{\Gamma(n + \alpha + 1)}{n!}.$$

## 2.11 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

| | | |
|---|---|---|
| 1. | $f(x) = x$ | $\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}$. |
| 2. | $f(x) = \lvert x\rvert$ | $\frac{\pi}{2} + \sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$ |
| 3. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$ | $\frac{\pi}{4} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\frac{(-1)^{n+1}}{2in} + \frac{(-1)^n - 1}{2\pi n^2}\right] e^{inx}$ |
| 4. | $f(x) = \sin^2(x)$ | $\frac{1}{2} - \frac{1}{4}\left(e^{2ix} + e^{-2ix}\right)$ |
| 5. | $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{2}{i\pi}\sum_{n\geq 1} \frac{e^{(2n-1)ix} - e^{-(2n-1)ix}}{2n-1}$ |
| 6. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{1}{2} + \sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{i\pi(2n-1)}$ |
| 7. | $f(x) = \lvert\sin(x)\rvert$ | $\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1}$ |
| 8. | $f(x) = \lvert\cos(x)\rvert$ | $\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{(-1)^n\left[e^{inx} + e^{-inx}\right]}{4n^2-1}$ |
| 9. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$ | $\frac{1}{\pi} - \frac{1}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1} + \frac{1}{4i}\left(e^{ix} - e^{-ix}\right)$ |
| 10. | $f(x) = x^2$ | $\frac{\pi^2}{3} + 2\sum_{n\geq 1} \frac{(-1)^n (e^{inx} + e^{-inx})}{n^2}$ |
| 11. | $f(x) = x(\pi - \lvert x\rvert)$ | $\frac{4}{i\pi}\sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{(2n-1)^3}$ |
| 12. | $f(x) = e^{bx}$ | $\frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b-in} e^{inx}$ |
| 13. | $f(x) = \sinh x$ | $\frac{\sinh(\pi)}{i\pi}\sum_{n\geq 1} \frac{(-1)^{n+1} n}{n^2+1}\left[e^{inx} - e^{-inx}\right]$ |

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all $2\pi$ periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

| | | |
|---|---|---|
| 1. | $f(x) = x$ | $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$ |
| 2. | $f(x) = \|x\|$ | $\frac{\pi}{2} - \frac{4}{\pi} \sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2}$ |
| 3. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$ | $\frac{\pi}{4} - \frac{2}{\pi} \sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2} + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin(nx)$ |
| 4. | $f(x) = \sin^2(x)$ | $\frac{1}{2} - \frac{1}{2}\cos(2x)$ |
| 5. | $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{4}{\pi} \sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$ |
| 6. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{1}{2} + \frac{2}{\pi} \sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$ |
| 7. | $f(x) = \|\sin(x)\|$ | $\frac{2}{\pi} - \frac{4}{\pi} \sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1}$ |
| 8. | $f(x) = \|\cos(x)\|$ | $\frac{2}{\pi} - \frac{4}{\pi} \sum_{n\geq 1} \frac{(-1)^n \cos(2nx)}{4n^2-1}$ |
| 9. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$ | $\frac{1}{\pi} - \frac{2}{\pi} \sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1} + \frac{1}{2}\sin(x)$ |
| 10. | $f(x) = x^2$ | $\frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n \cos(nx)}{n^2}$ |
| 11. | $f(x) = x(\pi - \|x\|)$ | $\frac{8}{\pi} \sum_{n\geq 1} \frac{\sin((2n-1)x)}{(2n-1)^3}$ |
| 12. | $f(x) = e^{bx}$ | $\frac{\sinh(b\pi)}{\pi} \left( \frac{1}{b} + \sum_{n\geq 1} \frac{(-1)^n}{b^2+n^2} \left[2b\cos(nx) - 2n\sin(nx)\right] \right)$ |
| 13. | $f(x) = \sinh x$ | $\frac{2\sinh(\pi)}{\pi} \sum_{n\geq 1} \frac{(-1)^{n+1} n}{n^2+1} \sin(nx)$ |

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1, \cos(nx), \sin(nx)\}_{n\geq 1}$. The series on the right are all $2\pi$ periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

| | |
|---|---|
| $f(x)$ | $\hat{f}(\xi)$ |
| $f(x-c)$ | $e^{-ic\xi}\hat{f}(\xi)$ |
| $e^{ixc}f(x)$ | $\hat{f}(\xi-c)$ |
| $f(ax)$ | $a^{-1}\hat{f}(a^{-1}\xi)$ |
| $f'(x)$ | $i\xi\hat{f}(\xi)$ |
| $xf(x)$ | $i(\hat{f})'(\xi)$ |
| $(f*g)(x)$ | $\hat{f}(\xi)\hat{g}(\xi)$ |
| $f(x)g(x)$ | $(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$ |
| $e^{-ax^2/2}$ | $\sqrt{2\pi/a}e^{-\xi^2/(2a)}$ |
| $(x^2+a^2)^{-1}$ | $(\pi/a)e^{-a\|\xi\|}$ |
| $e^{-a\|x\|}$ | $2a(\xi^2+a^2)^{-1}$ |
| $\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$ | $2\xi^{-1}\sin(a\xi)$ |
| $x^{-1}\sin(ax)$ | $\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$ |

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a > 0$ and $c \in \mathbb{R}$.

| | | |
|---|---|---|
| 1. | $\Theta(t)f(t)$ | $\widetilde{f}(z)$ |
| 2. | $\Theta(t-a)f(t-a)$ | $e^{-az}\widetilde{f}(z)$ |
| 3. | $e^{ct}\Theta(t)f(t)$ | $\widetilde{f}(z-c)$ |
| 4. | $\Theta(t)f(at)$ | $a^{-1}\widetilde{f}(a^{-1}z)$ |
| 5. | $\Theta(t)f'(t)$ | $z\widetilde{f}(z)-f(0)$ |
| 6. | $\Theta(t)f^{(k)}(t)$ | $z^k\widetilde{f}(z)-\sum_0^{k-1} z^{k-1-j}f^{(j)}(0)$ |
| 7. | $\Theta(t)\int_0^t f(s)ds$ | $z^{-1}\widetilde{f}(z)$ |
| 8. | $\Theta(t)tf(t)$ | $-\widetilde{f}'(z)$ |
| 9. | $\Theta(t)t^{-1}f(t)$ | $\int_z^\infty \widetilde{f}(w)dw$ |
| 10. | $\Theta f * \Theta g(t)$ | $\widetilde{f}(z)\widetilde{g}(z)$ |
| 11. | $\Theta(t)t^\nu e^{ct}$ | $\Gamma(\nu+1)(z-c)^{-\nu-1}$ |
| 12. | $\Theta(t)(t+a)^{-1}$ | $e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$ |
| 13. | $\Theta(t)\sin(ct)$ | $\frac{c}{z^2+c^2}$ |
| 14. | $\Theta(t)\cos(ct)$ | $\frac{z}{z^2+c^2}$ |

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a>0$ is constant and $c\in\mathbb{C}$. $\Theta(t)$ is the heaviside function, that is zero when $t<0$ and one when $t>0$.

| | | |
|---|---|---|
| 15. | $\Theta(t)\sinh(ct)$ | $\frac{c}{z^2-c^2}$ |
| 16. | $\Theta(t)\cosh(ct)$ | $\frac{z}{z^2-c^2}$ |
| 17. | $\Theta(t)\sin(\sqrt{at})$ | $\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$ |
| 18. | $\Theta(t)t^{-1}\sin(\sqrt{at})$ | $\pi\,\mathrm{erf}(\sqrt{a/(4z)})$ |
| 19. | $\Theta(t)e^{-a^2t^2}$ | $(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$ |
| 20. | $\Theta(t)\,\mathrm{erf}(at)$ | $z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$ |
| 21. | $\Theta(t)\,\mathrm{erf}(\sqrt{t})$ | $(z\sqrt{z+1})^{-1}$ |
| 22. | $\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$ | $((z-1)\sqrt{z})^{-1}$ |
| 23. | $\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$ | $z^{-1}e^{-a\sqrt{z}}$ |
| 24. | $\Theta(t)t^{-1/2}e^{-\sqrt{at}}$ | $\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$ |
| 25. | $\Theta(t)t^{-1/2}e^{-a^2/(4t)}$ | $\sqrt{\pi/z}e^{-a\sqrt{z}}$ |
| 26. | $\Theta(t)t^{-3/2}e^{-a^2/(4t)}$ | $2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$ |
| 27. | $\Theta(t)t^{\nu}J_{\nu}(t)$ | $2^{\nu}\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$ |
| 28. | $\Theta(t)J_0(\sqrt{t})$ | $z^{-1}e^{-1/(4z)}$ |

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.

## 1. Theory problem

Prove the DOG: Assume that $\{\phi_n\}_{n\geq 0}$ are orthonormal in a Hilbert space $H$. Prove that the following are equivalent:

(1) If $f \in H$, then it holds that $f = \sum_{n\geq 0}\langle f, \phi_n\rangle\phi_n$
(2) $\|f\|^2 = \sum_{n\geq 0}|\langle f, \phi_n\rangle|^2$.
(3) If $v \in H$ and $\langle v, \phi_n\rangle = 0$ for all $n \geq 0$, then $v = 0$.

1.1. **Solution.** 1. (3p) Assume the first statement. (1p) Then, since the $\{\phi_n\}_{n\geq 0}$ are orthogonal, so are the $\{\langle f, \phi_n\rangle\phi_n\}_{n\geq 0}$, because we are just rescaling them.
(2p) So, by the infinite-dimensional Pythagorean theorem:

$$\|f\|^2 = \left\|\sum_{n\geq 0}\langle f, \phi_n\rangle\phi_n\right\|^2 = \sum_{n\geq 0}\|\langle f, \phi_n\rangle\phi_n\|^2 = \sum_{n\geq 0}|\langle f, \phi_n\rangle|^2\,\|\phi_n\|^2 = \sum_{n\geq 0}|\langle f, \phi_n\rangle|^2 .$$

The last equality comes from the fact that $\{\phi_n\}_{n\geq 0}$ are ortho*normal*, which implies that $\|\phi_n\| = 1$.
2. (3p) Now, we assume the second statement and show the third.
(1p) We consider some $v \in H$ that satisfies $\langle v, \phi_n\rangle = 0$ for all $n \geq 0$.
(1p) Therefore, we obtain

$$\|v\|^2 = \sum_{n\geq 0}|\langle v, \phi_n\rangle|^2 = \sum_{n\geq 0} 0 = 0$$

(1p) Since the norm in a Hilbert space can only be zero for the zero element, we know that this implies $v = 0$.
3. (4p) Finally, we assume the third statement and show the first, closing the circle. Consider any $f \in H$.
(1p) By Bessel's projection inequality, we know that

$$g = \sum_{n\geq 0}\langle f, \phi_n\rangle\phi_n$$

is also in $H$.
(1p) All that is left is to show that $g = f$. In order to use (3), we rephrase this to $f - g = 0$ and show that $\langle f - g, \phi_n\rangle = 0$ for all $n \geq 0$, which then gives us that $f - g = 0$.
(2p) But by linearity of the scalar product,

$$\langle f - g, \phi_m\rangle = \langle f, \phi_m\rangle - \langle g, \phi_m\rangle = 0,$$

because

$$\langle g, \phi_m\rangle = \langle\sum_{n\geq 0}\langle f, \phi_n\rangle\phi_n, \phi_m\rangle = \sum_{n\geq 0}\langle f, \phi_n\rangle\langle\phi_n, \phi_m\rangle = \langle f, \phi_m\rangle$$

Here, we use the linearity and continuity of the scalar product and that $\{\phi_n\}_{n\geq 0}$ are orthogonal, i.e., $\langle\phi_n, \phi_m\rangle = 0$ if $m \neq n$.
Therefore, $f = g$ and (1) is proven.

Note that you can reorder this proof in different ways, so do not worry if you have a different order.

## 2. Theory problem

(Bevisa - The Hermite polynomials are so tight, the angles between them are always right!) Prove that the Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Recall here that

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

(10 p)

**Solution.** We want to show that the weighted inner product of $H_n$ and $H_m$ is zero if $n \neq m$. Hence, we may assume without loss of generality that $n > m$ (if $n < m$, simply rename $n$ as $m$ and $m$ as $n$). Since the Hermite polynomials $H_n$ begin with $n = 0$, this means that $m \geq 0$ and $n \geq 1$. The above argument gives one point (1p). Next, we insert the definition of $H_n$ into the weighted inner product:

$$(2p) \quad \langle H_n, H_m \rangle_{e^{-x^2}} = \int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = \int_{\mathbb{R}} (-1)^n e^{x^2} \left( \frac{d^n}{dx^n} e^{-x^2} \right) H_m(x) e^{-x^2} dx$$
$$= (-1)^n \int_{\mathbb{R}} \left( \frac{d^n}{dx^n} e^{-x^2} \right) H_m(x) dx.$$

Since we know that $n \geq 1$, we apply integration by parts to obtain

$$(2p) \quad \langle H_n, H_m \rangle_{e^{-x^2}} = \left[ (-1)^n \left( \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H_m(x) \right]_{-\infty}^{\infty} + (-1)^{n+1} \int_{\mathbb{R}} \left( \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H'_m(x) dx.$$

Next, we use that any number of derivatives of $e^{-x^2}$ is of the form

$$(1p) \quad \frac{d^n}{dx^n} e^{-x^2} = p_n(x) e^{-x^2},$$

where $p_n(x)$ is a polynomial. This can be proven by induction, but it is not required here. Instead, we will simply use this fact to note that the boundary terms in the integration by parts vanish:

$$(1p) \quad \left[ (-1)^n \left( \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H_m(x) \right]_{-\infty}^{\infty} = \left[ (-1)^n p_{n-1}(x) e^{-x^2} H_m(x) \right]_{-\infty}^{\infty}$$
$$= \lim_{x \to \infty} (-1)^n p_{n-1}(x) e^{-x^2} H_m(x) - \lim_{x \to -\infty} (-1)^n p_{n-1}(x) e^{-x^2} H_m(x) = 0,$$

due to the fact that $(-1)^n p_{n-1}(x) H_m(x)$ is a polynomial, and $e^{-x^2} \to 0$ as $x \to \pm\infty$ faster than any polynomial. Thus, we obtain

$$(1p) \quad \langle H_n, H_m \rangle_{e^{-x^2}} = (-1)^{n+1} \int_{\mathbb{R}} \left( \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H'_m(x) dx.$$

We can repeat the same argument until we run out of derivatives. We've got $n$ derivatives, so we repeat this argument $n$ times, arriving at

$$(1p) \quad \langle H_n, H_m \rangle_{e^{-x^2}} = (-1)^{n+n} \int_{\mathbb{R}} e^{-x^2} \left( \frac{d^n}{dx^n} H_m(x) \right) dx.$$

Finally, recall that $H_m$ is a polynomial of degree $m$. If you differentiate a polynomial of degree $m$ more than $m$ times, you get zero. Since $n > m$, it follows that

$$(1p) \quad \frac{d^n}{dx^n} H_m(x) = 0.$$

Hence,

$$(1p) \quad \langle H_n, H_m \rangle_{e^{-x^2}} = (-1)^{n+n} \int_{\mathbb{R}} e^{-x^2} \left( \frac{d^n}{dx^n} H_m(x) \right) dx = \int_{\mathbb{R}} e^{-x^2} 0 dx = 0.$$

## 3. Computing series using Fourier expansions

Compute

$$\sum_{n=0}^{\infty} \frac{e}{4n^2 - 4n + 1}$$

3.1. **Solution.** → (4p) We first need to find a series in the table that could give us a solution. First, we see that we do not find a series in the table that has $4n^2 - 4n - 1$ in the denominator. But we note that $4n^2 - 4n - 1 = (2n - 1)^2$, and find two possible Fourier series to help us! We pick here

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi \end{cases} = \frac{1}{2} + \frac{2}{\pi} \sum_{n \geq 1} \frac{\sin((2n-1)x)}{2n-1}$$

Note carefully that the sum starts at 1 here, not at 0. We will deal with this later.
→ (2p) We apply Parseval's equality to obtain

$$\|f\|^2 = \left\| \frac{1}{2} + \frac{2}{\pi} \sum_{n \geq 1} \frac{\sin((2n-1)x)}{2n-1} \right\|^2 = \frac{2\pi}{4} + \frac{4}{\pi^2} \sum_{n \geq 1} \frac{\|\sin((2n-1)x)\|^2}{(2n-1)^2}$$
$$= \frac{\pi}{2} + \frac{4}{\pi^2} \sum_{n \geq 1} \frac{\pi}{(2n-1)^2} = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n \geq 1} \frac{1}{(2n-1)^2}$$

→ (1p) On the other hand, we have

$$\|f\|^2 = \int_{-\pi}^{\pi} f(x)\overline{f(x)} dx = \int_0^{\pi} 1 dx = \pi$$

→ (1p) We can now solve for (almost) our series: We get

$$\frac{\pi}{2} + \frac{4}{\pi} \sum_{n \geq 1} \frac{1}{(2n-1)^2} = \pi$$

and, therefore,

$$\sum_{n\geq 1} \frac{1}{(2n-1)^2} = \frac{\pi}{2}\frac{\pi}{4} = \frac{\pi^2}{8}$$

→ (2p) Finally,

$$\sum_{n=0}^{\infty} \frac{e}{4n^2-4n+1} = e + e\sum_{n\geq 1} \frac{1}{(2n-1)^2} = e + \frac{e\pi^2}{8}.$$

## 4. PDE on bounded interval

Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = x\sin^2(t), & t > 0,\ 0 < x < 1, \\ u(0,t) = 0, & t > 0, \\ u(1,t) = 0, & t > 0, \\ u(x,0) = x^2, & 0 \leq x \leq 1. \end{cases}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.) (10 p)

4.1. **Solution.** This is an inhomogeneous heat equation on a bounded interval, where the inhomogeneity depends on time.
(a) (1p) SLPs are keys to solving inhomogeneous pde's. Even if you do nothing else, this rhyme is worth a point. If you don't do this rhyme, you still get a point if you set up the SLP to solve

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(1) = 0$$

(or written equivalently).
(b) (2p) Solve this SLP. You should obtain (see the vibrating string example in Chapter 1 of the textbook for the derivation of these solutions)

$$X_n(x) = \sin(n\pi x),\ \lambda_n = n^2\pi^2,\ n \geq 1.$$

One point for the correct function and eigenvalues and one point for the correct range on $n$.
(c) (1p) Set up the solution you seek to be a series

$$u(x,t) = \sum_{n\geq 1} T_n(t)X_n(x),$$

where we will solve for the $T_n$ functions using the inhomogeneous pde together with the initial condition.
(d) (2p) Expand the inhomogeneity in terms of the $X_n$ base. This is possible because $\{X_n\}_{n=1}^{\infty}$ is an orthogonal base for $L^2(0,1)$ by the spectral theorem for SLPs:

$$x\sin^2(t) = \sum_{n\geq 1} \sin^2(t)\frac{\langle x, X_n\rangle}{||X_n||^2}X_n(x) = \sum_{n\geq 1} c_n \sin^2(t)X_n(x)$$

with

$$c_n = \frac{\langle x, X_n \rangle}{||X_n||^2}, \ \langle x, X_n \rangle = \int_0^1 x\overline{X_n(x)}dx, \ ||X_n||^2 = \int_0^1 |X_n(x)|^2 dx.$$

It is okay if you leave these integrals like this (without calculating them) as long as you have correctly defined the scalar product and the norm squared. Each of these correctly defined is worth one point. However, if you do decide to compute them, you will get

$$c_n = \frac{2(-1)^{n+1}}{\pi n}.$$

(e) (1p) Plug $u$ into the heat equation (and use that $X_n'' = -\lambda_n X_n$) to obtain

$$u_t - u_{xx} = \sum_{n \geq 1} (T_n'(t) + n^2\pi^2 T_n(t)) X_n(x) = \sum_{n \geq 1} c_n \sin^2(t) X_n(x).$$

(f) (1p) Identify coefficients to obtain the equation for $T_n$:

$$T_n'(t) + n^2\pi^2 T_n(t) = c_n \sin^2(t).$$

(g) (1p) Set up the correct initial condition:

$$u(x,0) = \sum_{n \geq 1} X_n(x) T_n(0) = x^2 = \sum_{n \geq 1} C_n X_n(x)$$

with

$$T_n(0) = C_n = \frac{\langle x^2, X_n \rangle}{||X_n||^2}.$$

Again, it is okay if you don't compute this scalar product and norm squared. In fact, if you have correctly defined the scalar product and norm squared before, you do not even need to write these out as integrals again. However, if you do decide to compute them, you will get

$$C_n = \frac{2}{n\pi}\left((-1)^{n+1} + \frac{2(-1)^n - 2}{n^2\pi^2}\right).$$

(h) (1p) Solve the ODE for $T_n(t)$. The method of integrating factor will give you

$$T_n(t) = e^{-n^2\pi^2 t}\left(\int_0^t e^{n^2\pi^2 s} c_n \sin^2(s) ds + C_n\right).$$

## 5. Bessel 10 points

Lös problemet

$$\begin{aligned}
&u_t(r,\theta,t) = \Delta u(r,\theta,t) + e^t, \ 0 < t, \ 0 < r < 5, \ -\pi < \theta < \pi, \\
&u(5,\theta,t) = 0, \\
&u(r,\theta,0) = re^{-r^2} + 5r.
\end{aligned}$$

5.1. **Solution and point distribution. OBS!** There was a typo on the exam (fixed it here but missed it on the exam) with the boundary condition given at $u(1, \theta, t) = 0$. SO - we accept if you solved it with the BC at $r = 1$ OR if you solved with BC at $r = 5$. Below is the solution with BC at $r = 5$. (Basically a bunch of 5s in denominators of things and then some 5s squared (25) will disappear if you used the $r = 1$ boundary condition). Both are okay!
(1p) The solution is independent of $\theta$, which simplifies things.
(2p) SLp's are the keys to solving inhomogeneous pde's. Let's pretend that the pde is homogeneous and solve the part for $r$ since it has the nice Dirichlet condition at $r = 5$. The Laplacian in radial coordinates is

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

We need to find the associated Bessel equation for $R$, and solve it. We need to find solutions of the form

$$R(r)T(t).$$

If the pde were homogeneous, we would get the equation

$$RT' = R''T + r^{-1}R'T$$

and then

$$\frac{R''}{R} + r^{-1}\frac{R'}{R} = \frac{T}{T'} = \lambda.$$

We will solve for $R$ and then remember to stop because our equation for the $T$ function will need to incorporate that inhomogeneity. For $R$ the equation is

$$R'' + r^{-1}R' - \lambda R = 0,$$

or

$$r^2R'' + rR' - \lambda r^2 R = 0.$$

If $\lambda = 0$, then this becomes an Euler equation with solutions 1 and $\log(r)$. The function $\log(r)$ is excluded because it blows up at $r = 0$. The other function 1 does not satisfy the Dirichlet boundary condition at $r = 5$. If $\lambda > 0$, then this is a modified Bessel equation of order 0 with solutions $K_0$ and $I_0$. The $K_0$ Bessel function also blows up at $r = 0$, while the $I_0$ Bessel function does not have any positive real zeros, so it won't satisfy the Dirichlet boundary condition at $r = 5$. The only viable case is therefore that $\lambda < 0$, and the solutions are $J_0$ and $Y_0$. The $Y_0$ Bessel function blows up at $r = 0$ so we say goodbye to it. We are then left with the Bessel function $J_0$ with argument $\sqrt{-\lambda}r$. Considering the boundary condition $u(5, \theta, t) = 0$ and we get $\sqrt{-\lambda} = \frac{\pi_k}{5}$, with $\pi_k$ one of the zeros of the Bessel function of degree 0, and

$$-\frac{\pi_k^2}{25} = \lambda.$$

Note that $(R_k)_k$ is a base for $L_r^2([0, 5])$.
(1p) Realizing how we can write the solution in terms of $R_k$ and $T_k$, as

$$u(r, t) = \sum_k T_k(t)R_k(r).$$

(2p) Use the Bessel equation to transform it. The differential equation says that we need

$$\sum_k T_k'(t)R_k(r) = e^t + \sum_k T_k(t)(R_k''(r) + r^{-1}R_k'(r)) = e^t + \sum_k -\frac{\pi_k^2}{25}T_k(t)R_k(r).$$

Here we used that $R_k$ satisfies the differential equation we had above

$$r^2R_k'' + rR_k' = \lambda r^2 R_k \iff R_k'' + r^{-1}R_k' = \lambda R_k.$$

This implies that

$$e^t = \sum_k R_k(T_k' + \pi_k^2 T_k/25).$$

(2p) Writing $e^t$ in terms of the basis $R_k$ and coefficients.

$$e^t = e^t 1 = e^t \sum_k \frac{\langle 1, R_k\rangle}{\langle R_k, R_k\rangle} R_k.$$

Now

$$\langle R_k, R_k\rangle = \int_0^5 |J_0(\pi_k r/5)|^2 r dr,$$

and

$$\langle 1, R_k\rangle = \int_0^5 J_0(\pi_k r/5) r dr.$$

For simplicity let's call

$$c_k = \frac{\langle 1, R_k\rangle}{\langle R_k, R_k\rangle}$$

(2p) Finding all the $T_k$. We need

$$\sum_k c_k e^t R_k = \sum_k R_k(T_k' + \pi_k^2 T_k/25) \implies c_k e^t = T_k' + \pi_k^2 T_k/25.$$

So we find using the method of integrating factor that

$$T_k(t) = e^{-\pi_k^2 t/25}\left[\int_0^t e^{\pi_k^2 s/25 + s} ds + T_k(0)\right].$$

For the initial condition we need

$$f(r) = re^{-r^2} + 5r = \sum_k R_k(r)T_k(0),$$

so

$$T_k(0) = \frac{\langle f, R_k\rangle}{\langle R_k, R_k\rangle},$$

with the scalar product and norm defined analogously to the above.

## 6. Fourier inversion 10 points

Lös problemet

$$\begin{aligned} &u_t - u_{xx} = 0, \ x, t > 0, \\ &u_x(0,t) = 0, \\ &u(x,0) = e^{-x^2}. \end{aligned}$$

**Solution and point distribution.** (1p) Realize the functions $u$ and $f$ need to be extended evenly because there is a von Neumann condition $u_x(0,t) = 0$.
(1p) We do a Fourier transform and get

$$\hat{u}_t(\xi,t) = -\xi^2 \hat{u}(\xi,t).$$

(2p) We solve this and get

$$\hat{u}(\xi,t) = c(\xi)e^{-\xi^2 t}.$$

(4p) We realize

$$c(\xi) = \hat{u}(\xi,0) = \int_{\mathbb{R}} e^{-x^2} e^{-ix\xi} dx = \sqrt{\pi} e^{-\frac{\xi^2}{4}}.$$

(2p) So we need to have

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi,t) e^{i\xi x} dx = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-(t+\frac{1}{4})\xi^2} e^{ix\xi} d\xi = \frac{1}{2\sqrt{\pi}} \sqrt{\frac{2\pi}{2t+\frac{1}{2}}} e^{-\frac{x^2}{4t+1}} = \frac{1}{\sqrt{4t+1}} e^{-\frac{x^2}{4t+1}}.$$

An alternative method involves observing that $u(x,0)$ is already an even function of $x$, and the Fourier transform preserves even functions (we can see by the direct calculation above that the Fourier transform of this function is an even function of the Fourier transform variable $\xi$). So, using the tables we find the anti-Fourier-transform of $e^{-\xi^2 t}$ and then obtain $u$ as the convolution

$$u(x,t) = \int_{\mathbb{R}} e^{-(x-y)^2} e^{-y^2/(4t)} \frac{1}{\sqrt{4\pi t}} dy.$$

Or we could write it as

$$u(x,t) = \int_{\mathbb{R}} e^{-y^2} e^{-(x-y)^2/(4t)} \frac{1}{\sqrt{4\pi t}} dy,$$

because the convolution is commutative. It's fine to leave this integral as it is.

## 7. Best approximation problem

There are several ways of solving this problem. Here we present one way, which is the solution that is hinted at. The idea is based on expanding the norm of $r(x)$.

(1) Realize that the norm $\|f(x) - r(x)\|$ can be expanded (3 points)
(2) Arrive at the correct expression by applying the triangle inequality $\frac{1}{2}\|f(x)-p(x)\| + \frac{1}{2}\|f(x) - p(-x)\|$ (3 points)
(3) Rewrite $\|f(x) - p(-x)\|$ as $\|f(x) - p(x)\|$ (1 point)

(4) Conclude using the best approximation theorem that due to the uniqueness of $p(x)$, $p(x) = r(x)$.

Here is another way.

(1) Let $P_n$ be the Legendre polynomial of degree $n$. Then we know that the best approximation is

$$p(x) = \sum_{n=0}^{9} \frac{\langle f, P_n \rangle}{||P_n||^2} P_n(x).$$

(2) We know (or we prove it) that $P_n$ is an even function if $n$ is even, and it's an odd function if $n$ is odd.

To prove this, observe that (up to some silly constant factor) $P_n$ is obtained by differentiating $(x^2-1)^n$ precisely $n$ times. In the binomial expansion of $(x^2-1)^n$ all the terms are like $x^2$ raised to some power (starting from 0 and going up to $n$, with various coefficients). Well these are all $x$ raised to an even power with whatever bla coefficient. Thus if we differentiate any such term an even number of times, each term gets its power of $x$ lowered by that even number of times we differentiated it and therewith remains an even power (or if it gets differentiated enough times it turns into zero). This shows that $P_n$ is an even function if $n$ is even because it is a sum of even powers of $x$ times their whatever coefficients.

On the other hand, if we differentiate $(x^2 - 1)^n$ an odd number of times, each term gets its (even) power of $x$ lowered by an odd amount and therewith either vanishes (if we differentiate it enough times) or it becomes an odd power. So in the end our $P_n$ for $n$ odd is a sum of $x$ raised to odd powers times whatever coefficient.

(3) Since

$$\langle f, P_n \rangle = \int_{-1}^{1} f(x)\overline{P_n(x)}dx,$$

and $f$ is even, whenever $n$ is odd, the integrand is odd, and the integral is zero. Thus the only terms in $p(x)$ are the terms with the coefficient multiplied by $P_n$ for $n$ even, and as we have just explained, each $P_n$ with $n$ even is an even function. Consequently this whole polynomial is even.

## 8. Laplace problem

OBS - there was a typo! The problem had $x \in \mathbb{R}$ but it should have been $x > 0$.

(1) Realize that you should use Laplace transform in the $t$ variable (say it!) (1 point)

(2) Find the transformed equation

$$sU(x,s) - u(x,0) + U_x(x,s) + U(x,s) = 0.$$

This gives (2 points)

(3) Insert the initial condition to obtain the ODE $U_x = -(s+1)U$. (1 point)

(4) Find the solution $U(x,s) = C(s)e^{(-1-s)x}$. (1 point)

(5) Correctly identify $C(s) = 1/(s^2+1)$ using the boundary condition (2 points) that is $U(0,t) = 1/(s^2+1)$ (1 point).

(6) Find the inverse transform of $\frac{e^{-x}}{s^2+1}e^{-sx}$ to be the solution

$$u(x,t) = e^{-x}\sin(t-x)\Theta(t-x)$$

where $\Theta(t-x)$ is the Heaviside step function. (3 points)