**Fourieranalys MVE030, Fourier Metoder MVE290 2024.augusti**
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
5 Timmar.
Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & Chalmers godkänt miniräknare.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

# 1 Uppgifter

1. (Bevisa tF-C@T) Låt $f$ vara en $2\pi$ periodisk funktion. Antar att $f$ är styckvis deriverbar. Låt

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Bevisa att gäller:

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right), \quad \forall x \in \mathbb{R}.$$

(10p)

**Solution and points:** Fix a point $x \in \mathbb{R}$. This first step is more getting into a frame of mind. Think of $x$ as fixed. Then the numbers $f(x_-)$ and $f(x_+)$ are just the left and right limits of $f$ at $x$, so these are also fixed. Our goal is to prove that:

$$\lim_{N\to\infty} \sum_{-N}^{N} c_n e^{inx} = \frac{1}{2}\left(f(x_-) + f(x_+)\right).$$

This is completely equivalent to proving

$$\lim_{N\to\infty} \sum_{-N}^{N} c_n e^{inx} - \frac{1}{2}\left(f(x_-) + f(x_+)\right) = 0.$$

Let us call

$$\star = \sum_{-N}^{N} c_n e^{inx} - \frac{1}{2}\left(f(x_-) + f(x_+)\right).$$

The main idea is to try to make the two things look like each other, that is we want to make $\sum c_n e^{inx}$ look like the average of the left and right limits of $f$. To get $\sum c_n e^{inx}$ looking more like $f$, we write out what each term in this sum really is:

$$c_n e^{inx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} e^{inx} dy.$$

We must use a different variable for integration because $x$ is a fixed point.

**2 points for correct tF series with correct definition of coefficients.**

We want to get some $x$ inside of $f$ to be able to relate to $\frac{f(x_+)+f(x_-)}{2}$. Make a change of variables to do this:

$$y = t + x, \quad dy = dt \implies c_n e^{inx} = \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt.$$

**1 point for this step.**

Slide the integral back to being from $-\pi$ to $\pi$ because the integrand is $2\pi$ periodic, so the integral over any interval of length $2\pi$ is the same. Then we have

$$c_n e^{inx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) e^{-int} dt.$$

**1 point for this step.**

Investigate the sum, because

$$\star = \sum_{-N}^{N} c_n e^{inx} - \frac{1}{2} \left( f(x_-) + f(x_+) \right) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) e^{-int} dt - \frac{1}{2} \left( f(x_-) + f(x_+) \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) \sum_{-N}^{N} e^{int} dt - \frac{1}{2} \left( f(x_-) + f(x_+) \right).$$

So, let us see what we can say about

$$\sum_{-N}^{N} e^{int}.$$

Use the fact that

$$e^{int} + e^{-int} = 2\cos(nt) \implies \sum_{-N}^{N} e^{int} = 1 + 2 \sum_{1}^{N} \cos(nt)$$

to compute

$$\frac{1}{2} = \frac{1}{2\pi}\int_{-\pi}^{0}\sum_{-N}^{N} e^{int}dt, \quad \frac{1}{2} = \frac{1}{2\pi}\int_{0}^{\pi}\sum_{-N}^{N} e^{int}dt.$$

**1 point for this calculation.**

Use this to express

$$\frac{1}{2}f(x_-) = \frac{1}{2\pi}\int_{-\pi}^{0} f(x_-)\sum_{-N}^{N} e^{int}dt, \quad \frac{1}{2}f(x_+) = \frac{1}{2\pi}\int_{0}^{\pi} f(x_+)\sum_{-N}^{N} e^{int}dt,$$

so that we can equivalently write

$$\star = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+x)\sum_{-N}^{N} e^{int}dt - \frac{1}{2\pi}\int_{-\pi}^{0} f(x_-)\sum_{-N}^{N} e^{int}dt - \frac{1}{2\pi}\int_{0}^{\pi} f(x_+)\sum_{-N}^{N} e^{int}dt$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(f(t+x)-f(x_-))\sum_{-N}^{N} e^{int}dt + \int_{0}^{\pi}(f(t+x)-f(x_+))\sum_{-N}^{N} e^{int}dt\right).$$

Return to that sum to see if we can simplify it somehow: $\sum_{-N}^{N} e^{int}$. This is like a geometric series, that can be explicitly computed

$$\sum_{-N}^{N} e^{int} = e^{-iNt}\sum_{0}^{2N} e^{int} = e^{-iNt}\frac{e^{i(2N+1)t}-1}{e^{it}-1} = \frac{e^{i(N+1)t}-e^{-iNt}}{e^{it}-1},$$

**1 point for this calculation.** having used the fact about geometric series that if $z \neq 1$ then

$$\sum_{0}^{M} z^m = \frac{z^{M+1}-1}{z-1}.$$

So,

$$\star = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(f(t+x)-f(x_-))\frac{e^{i(N+1)t}-e^{-iNt}}{e^{it}-1}dt + \int_{0}^{\pi}(f(t+x)-f(x_+))\frac{e^{i(N+1)t}-e^{-iNt}}{e^{it}-1}dt\right).$$

Collect everything except the $e^{i(N+1)t}$ and $e^{-iNt}$ and use it to define a new function

$$g(t) := \begin{cases} \frac{f(t+x)-f(x_-)}{e^{it}-1} & -\pi < t < 0 \\ \frac{f(t+x)-f(x_+)}{e^{it}-1} & 0 < t < \pi \end{cases}$$

**1 point for defining this funky function.**

Check to see that this does not do anything terrible at $t = 0$ by evaluating its left and right limits. Note that at all other points in $[-\pi, \pi]$ $g$ inherits the properties of $f$ because the denominator is non-zero. We use l'hopital's rule to compute

$$\lim_{t\to 0^-} g(t) = \frac{f'(x_-)}{i}, \quad \lim_{t\to 0^+} g(t) = \frac{f'(x_+)}{i}.$$

These are just the left and right limits of $f'$ at $x$ which both exist since $f$ is piecewise $\mathcal{C}^1$. So $g$ is also piecewise $\mathcal{C}^1$ and therefore bounded and therefore also in $\mathcal{L}^2(-\pi, \pi)$. **1 point for justifying the boundedness of the funky function.**

Recognize

$$\star = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)(e^{i(N+1)t} - e^{-iNt})dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{i(N+1)t}dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-iNt}dt$$

are Fourier coefficients of $g$, specifically

$$\star = \hat{g}_{-N-1} - \hat{g}_N.$$

Since $g$ is in $\mathcal{L}^2$, we have the equality for the ONB $\{e^{inx}/\sqrt{2\pi}\}$

$$||g||^2 = \sum_{\mathbb{Z}} \left|\langle g, \frac{e^{inx}}{\sqrt{2\pi}}\rangle\right|^2 = \frac{1}{2\pi}\sum_{\mathbb{Z}} |\langle g, e^{inx}\rangle|^2.$$

The norm on the left is finite, so this series converges, and so the individual terms tend to zero as $|n| \to \infty$. Consequently

$$\hat{g}_{-N-1} = \frac{\langle g, e^{-i(N+1)x}\rangle}{2\pi} \to 0, \quad \hat{g}_N = \frac{\langle g, e^{iNx}\rangle}{2\pi} \to 0$$

as $N \to \infty$, thereby guaranteeing that $\star \to 0$. The last step is to observe that if $f$ is continuous at $x$, then the left and right limits are both equal to $f(x)$ so their average is also equal to $f(x)$. **1 point for recognizing tF coefficients of the funky function and 1 point for justifying why they shall tend to zero.**

2. (Bevisa - when a function is derived its tF coefficients get multiplied!) Låt $f$ vara en $2\pi$ periodisk funktion och antar att $f \in \mathcal{C}^2(\mathbb{R})$. Bevisa att de trigonometriska Fourierkoefficienterna $C_n$ av $f$ i den ortogonal

bas $\{e^{inx}\}_{n\in\mathbb{Z}}$ på Hilbertrummet $\mathcal{L}^2(-\pi,\pi)$ och de Fourierkoefficienterna $c_n$ av $f'$ ($f'$ är derivatan av $f$) uppfyller:

$$c_n = inC_n.$$

**Solution:** We use the definitions of the Fourier series and coefficients of $f$ and $f'$ respectively with respect to the given orthogonal base in the given Hilbert space.

$$(2p) \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

and

$$(2p) \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx.$$

You could use either of these expressions and integrate by parts. So, just the idea to integrate by parts is worth 2 points, because it is the key idea. Next, actually integrating by parts correctly will be worth another 2 points. If you take the expression for $c_n$ and integrate by parts, you get

$$(2p) \quad c_n = \frac{1}{2\pi}\left( f(x)e^{-inx}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-in)e^{-inx}\right).$$

Now you get two points (2p) by observing that the first term vanishes due to the $2\pi$ periodicity of both $f$ and $e^{-inx}$, so that we end up with

$$c_n = inC_n.$$

If you instead take the expression for $C_n$ and integrate by parts there, it's slightly tricky because you need to separate the cases $n=0$ and $n \neq 0$. For $n=0$, it is worth one point to show that

$$c_0 = 0$$

by the $2\pi$ periodicity of $f$. For $n \neq 0$, the idea to integrate by points is worth 1 point, and getting the following expression right for $n \neq 0$ is worth 2 points:

$$C_n = \frac{1}{2\pi}\left( f(x)\frac{e^{-inx}}{-in}\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}dx\right).$$

It is worth two points (2p) for observing that the first term vanishes due to the $2\pi$ periodicity of both $f$ and $e^{-inx}$. So, we get

$$c_0 = 0,\ C_n = \frac{c_n}{in},\ n \in \mathbb{Z} \setminus \{0\}.$$

Thus we get $c_n = inC_n$ holds for all $n \in \mathbb{Z}$.

3. Lös problemet: (Solve the following problem):

$$v_t(r,\theta,t) = \Delta v(r,\theta,t) + \cos(t), \quad 0 < r < 5,\ t > 0,\ -\pi < \theta < \pi,$$
$$v(5,\theta,t) = \sin(t) + 2,$$
$$v(r,\theta,0) = (r-5)\cos(\theta) + 2.$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated).

(10 p)

**Rättningsmall**

(1p) Vi har en inhomogen PDE med ett inhomogent randvillkor, där både inhomogeniteter beror på $t$. Därför använder vi metoden "Linear coalitions handle time-dependent boundary conditions". Vi letar efter en funktion $S(r,\theta,t)$ som är lika med $\sin(t)+2$ där $r=5$, och är so enkel som mögligt. (Obs: $v(5,\theta,t) = \sin(t)+2$ ar det enda randvillkoret.) Den enklaste funktionen vi kan hitta är

$$S(r,\theta,t) = \sin(t) + 2.$$

Nu kan vi skapa ett problem vi gillar bättre, genom att definiera $u(r,\theta,t) = v(r,\theta,t) - S(r,\theta,t)$:

$$\Delta u(r,\theta,t) = \Delta v(r,\theta,t) - \Delta S(r,\theta,t) = \Delta v(r,\theta,t),$$

$$u_t(r,\theta,t) = v_t(r,\theta,t) - S_t(r,\theta,t) = v_t(r,\theta,t) - \cos(t)$$

Problemet blir:

$$u_t(r,\theta,t) = \Delta u(r,\theta,t), \quad 0 < r < 5,\ t > 0,\ -\pi < \theta < \pi,$$
$$u(5,\theta,t) = 0,$$
$$u(r,\theta,0) = (r-5)\cos(\theta).$$

Lyckligtvis har vi nu inga inhomogeniteter kvar och vi kan fortsätta som vanligt.

(1p) Variabelseparera $u_t = \Delta u$ i polära koordinater $(r, \theta)$ och tidsvariabeln $t$.

$$\frac{T'}{T} = \frac{R'' + r^{-1}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

Låt $\lambda = T'/T$ och skriv

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} = \lambda r^2$$

eller, ekvivalent,

$$\frac{r^2R'' + rR'}{R} - \lambda r^2 = -\frac{\Theta''}{\Theta}.$$

(1p) Eftersom höger- och vänsterled beror på $\theta$ respektive $r$ är detta också en separerad ekvation och både sidor måste vara konstant.

(1p) Vi argumenterar för att $\Theta$ ska vara $2\pi$-periodisk. Det betyder att $\Theta'' =$ konstant gång $\Theta$ och $\Theta(-\pi) = \Theta(\pi)$ och $\Theta'(-\pi) = \Theta'(\pi)$. Vi löser den här ekvation (se boken kapitel 1 exempel med Rings of Saturn).

(1p) Vi skriver ner vår slutsats:

$$\Theta_n(\theta) = e^{in\theta}, \quad \Theta_n'' = -n^2\Theta_n, \quad n \in \mathbb{Z}.$$

(3p) Vi går igenom fallen $\lambda > 0$, $\lambda = 0$ och $\lambda < 0$. Om $\lambda > 0$ vi får den modifierade Besselekvationen. Lösningarna $K$ går mot oändligheten när $r \to 0$, och $I$ saknar nollställen för $r > 0$. Alltså kan inte $\lambda > 0$ ge några fysikaliska lösningar som samtidigt uppfyller randvillkoret vid $r = 5$. Säg att $\lambda = 0$ i stället. Då gäller att $r^2R''(r) + rR'(r) - n^2R(r) = 0$, Eulers ekvation. Lösningarna är $ar^n + br^{-n}$ för konstanter $a, b$ om $n > 0$, eller $R(r) = a + b\log(r)$ ifall $n = 0$. Alltså kan inte heller $\lambda = 0$ ge några fysikaliska lösningar som samtidigt uppfyller randvillkoret vid $r = 5$. Antag $\lambda < 0$. Låt $\lambda = -\nu^2$. Vi erhåller Besselekvationen $r^2R''(r) + rR'(r) + (\nu^2r^2 - n^2)R(r) = 0$, som har lösningarna $J_n$ eller $Y_n$. Lösningarna $Y_n$ går mot $\infty$ när $r \to 0$, så vi utesluter dem. Randvillkoret $u(5, \theta, t) = 0$ ger $R(5) = 0$ och alltså $J_n(5\nu) = 0$. Det betyder att det finns lösningar för alla $\lambda$ sådana att

$$\lambda = -\left(\frac{\pi_{n,k}}{5}\right)^2,$$

där $\pi_{n,k}$ är nollställe nummer $k$ till $J_n$. Så

$$R_{n,k}(r) = J_n(\pi_{n,k}r/5).$$

(1p) Vi löser för $T$ funktionen:

$$T' = \lambda T, \quad \lambda = -\nu^2, \quad \nu = \frac{\pi_{n,k}}{5},$$

altså

$$T_{n,k}(t) = c_{n,k} e^{-\lambda_{n,k} t}, \quad \lambda_{n,k} = \frac{(\pi_{n,k})^2}{25}.$$

(1p) Superposition!

$$u(r, \theta, t) = \sum_{k>0, n\in\mathbb{Z}} T_{n,k}(t)\Theta_n(\theta) R_{n,k}(r)$$

För att uppfylla begynnelsevillkoret ska vi lösa

$$u(r, \theta, 0) = \sum_{k>0, n\in\mathbb{Z}} T_{n,k}(0)\Theta_n(\theta) R_{n,k}(r) = (r-5)\cos(\theta).$$

Så vi har

$$T_{n,k}(0) = c_{n,k} = \frac{\int_{-\pi}^{\pi} \int_0^5 (r-5)\cos(\theta)\overline{R_{n,k}(r)\Theta_n(\theta)} r dr d\theta}{\int_{-\pi}^{\pi} \int_0^5 |R_{n,k}(r)\Theta_n(\theta)|^2 r dr d\theta}.$$

Till slut har vi:

$$v(r, \theta, 0) = \sum_{k>0, n\in\mathbb{Z}} T_{n,k}(0)\Theta_n(\theta) R_{n,k}(r) + \sin(t) + 2.$$

4. Sök en begränsad lösning till problemet (find a bounded solution to this problem)

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \ t > 0,$$
$$u(x, 0) = e^{-|x|}.$$

### Rättningsmall

(3p) Det här är värmeledningsekvationen för $x \in \mathbb{R}$ med begynnelsevillkor $u(x, 0) = f(x)$ där $f \in L^2(\mathbb{R})$. Det osar Fouriertransform! Vi transformerar ekvationen:

$$\hat{u}_t = -\xi^2 \hat{u},$$

där

$$\hat{u}(\xi, t) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x, t)\, dx.$$

(1p Fouriertransform, 1p x variabel, 1p får rätt Fouriertransform av $u_{xx}$.)

(2p) Löser vi denna nya ekvation får vi, med någon godtycklig funktion av $\xi$, som vi kallar $a$,

$$\hat{u}(\xi, t) = a(\xi)e^{-\xi^2 t}.$$

(1p) Begynnelsevillkoret kan Fouriertransformeras eftersom

$$\|f\|^2 = \int_{-\infty}^{\infty} e^{-2|x|}\mathrm{d}x = 1$$

Alltså

$$a(\xi) = \hat{f}(\xi) \implies \hat{u}(\xi, t) = \hat{f}(\xi)e^{-\xi^2 t},$$

där $\hat{f}(\xi) = 2/(\xi^2 + 1)$.

(4p) Om man bara skriver

$$u(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)e^{-\xi^2 t} e^{ix\xi} d\xi$$

får man 2p. Om man hittar funktionen som har Fouriertransform $e^{-\xi^2 t}$ och skriva som en faltning

$$u(x, t) = \int_{\mathbb{R}} f(x-y)e^{-y^2/(4t)}(4\pi t)^{-1/2}dy = \int_{\mathbb{R}} f(y)e^{-(x-y)^2/(4t)}(4\pi t)^{-1/2}dy$$

får man 4p.

5. Expand, by hand, the function $f$ given as

$$f = \begin{cases} 1 & |x| \leq a, \\ 0 & \text{otherwise}, \end{cases}$$

with parameter $a \in (0, \pi)$, as a trigonometric Fourier series and use it to compute the series

$$\sum_{n=1}^{\infty} \frac{\sin(an)}{n}$$

**Solution**

A trigonometric Fourier series can be written either with respect to the basis $\{e^{inx}\}_{n\in\mathbb{Z}}$ or $1 \cup \{\sin(nx)\}_{n\geq 1} \cup \{\cos(nx)\}_{n\geq 1}$. Here, we choose to use the latter basis consisting of sine and cosine functions.

→ We compute the coefficients according to the definition of a Fourier expansion with the given basis:

→ (1p) $a_0$, the coefficient of the basis function 1.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-a}^{a} 1\,\mathrm{d}x = \frac{a}{\pi}$$

→ (2p) $a_n$, the coefficients of the basis functions $\sin(nx)$.

$$a_n = \frac{\langle f(x), \sin(nx)\rangle}{\|\sin(nx)\|^2}$$

The numerator evaluates to

$$\begin{aligned}\langle f(x), \sin(nx)\rangle &= \int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \int_{-a}^{a} \sin(nx)\,\mathrm{d}x \\ &= \left[-\frac{1}{n}\cos(nx)\right]_{x=-a}^{a} = -\frac{1}{n}(\cos(an) - \cos(-an)) = 0.\end{aligned}$$

All we need to know about the denominator is then that it is not equal to zero. The norm of a basis function can never be zero, so we're in the clear here, and we can say without further computation that $a_n = 0\ \forall n \geq 1$.

→ (3p) $b_n$, the coefficients of the basis functions $\cos(nx)$.

$$b_n = \frac{\langle f(x), \cos(nx)\rangle}{\|\cos(nx)\|^2}$$

Here, we will see that the numerator does not become 0, so we need to evaluate both numerator and denominator:

$$\begin{aligned}\langle f(x), \cos(nx)\rangle &= \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \int_{-a}^{a} \cos(nx)\,\mathrm{d}x \\ &= \left[\frac{1}{n}\sin(nx)\right]_{x=-a}^{a} = \frac{1}{n}(\sin(an) - \sin(-an)) = \frac{2}{n}\sin(an) \\ \|\cos(nx)\|^2 &= \int_{-\pi}^{\pi} \cos^2(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1}{2} + \frac{\cos(2nx)}{2}\,\mathrm{d}x = \pi + \left[\frac{\sin(2nx)}{4n}\right]_{x=-\pi}^{\pi} = \pi.\end{aligned}$$

To compute the integral in the denominator, we used the double-angle formula

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Taking both together, we get

$$b_n = \frac{2\sin(an)}{n\pi}$$

→ (1p) So, our full series becomes:

$$\frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(an)\cos(nx)$$

→ (3p) Using the tF-C@T, we get for all $x$ in $(-\pi, \pi)$ where $f$ is continuous:

$$f(x) = \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(an)\cos(nx)$$

$f$ is continuous at $x = 0$ for all $a \in (0, \pi)$, so we can plug in $x = 0$:

$$1 = \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(an)\cos(0) = \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(an)$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(an) = 1 - \frac{a}{\pi}$$
$$\iff \sum_{n=1}^{\infty} \frac{\sin(an)}{n} = \frac{\pi}{2} - \frac{a}{2} = \frac{a-\pi}{2}$$

(10p)

6. Solve the following initial boundary value problem for the heat equation:

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 \le x \le l, t > 0 \\ u_x(0,t) &= 21 \\ u(l,t) &= 0 \\ u(x,0) &= g(x). \end{aligned}$$

**Solution**

→ We are dealing with an inhomogeneous problem here. The inhomogeneities are independent of time, so we can use the method of steady states.

→ (2p) First, we find a steady-state solution, i.e., we set the time derivative $u_t$ to 0. We obtain the problem:

$$\begin{aligned} -s''(x) &= 0, \quad 0 \leq x \leq l \\ s'(0) &= 21 \\ s(l) &= 0. \end{aligned}$$

We can find $s(x)$ by integrating twice:

$$\begin{aligned} s''(x) &= 0 \\ s'(x) &= c \\ s(x) &= cx + d \end{aligned}$$

Plugging in the boundary conditions, we get:

$$\begin{aligned} s'(0) = c = 21 &\Rightarrow c = 21 \\ s(l) = 21l + d = 0 &\Rightarrow d = -21l \end{aligned}$$

Thus, our steady state is

$$s(x) = 21x - 21l$$

→ (1p) Defining $v(x,t) = u(x,t) - s(x)$, we obtain the homogeneous problem

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 \leq x \leq l, t > 0 \\ v_x(0,t) &= 0 \\ v(l,t) &= 0 \\ v(x,0) &= \tilde{g}(x) = g(x) - s(x). \end{aligned}$$

→ (1p) We solve this using our favorite method ♡. We start by separating our variables: $v(x,t) = X(x)T(t)$, and obtain:

$$T'X - TX'' = 0$$

and

$$\frac{T'}{T} = \frac{X''}{X} = \lambda$$

→ (2p) *The boundary conditions will help us see what the value of* $\lambda$ *needs to be!*

$$X'' = \lambda X$$

By Theorem 1.0.1, this gives us

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

With the boundary conditions, we obtain

$$X'(0) = 0 \Rightarrow A\sqrt{\lambda}e^{\sqrt{\lambda}0} - B\sqrt{\lambda}e^{-\sqrt{\lambda}0} \Rightarrow A = B$$

$$X(l) = 0 \Rightarrow A(e^{\sqrt{\lambda}l} + e^{-\sqrt{\lambda}l}) = 0 \Rightarrow A = 0 \text{ (the boring case) } \vee e^{2\sqrt{\lambda}l} = -1$$

The non-boring case is equivalent to:

$$2\sqrt{\lambda}l = (2n-1)\pi i \iff \sqrt{\lambda} = \frac{(2n-1)\pi i}{2l}, \quad n \in \mathbb{Z}$$

This leaves us with (up to constants):

$$X_n(x) = \frac{1}{2}\left(e^{\frac{(2n-1)\pi i}{2l}x} + e^{-\frac{(2n-1)\pi i}{2l}x}\right) = \cosh\left(\frac{(2n-1)\pi i}{2l}x\right) = \cos\left(\frac{(2n-1)\pi}{2l}x\right)$$

and the eigenvalues

$$\lambda_n = \left(\frac{(2n-1)\pi i}{2l}\right)^2 = -\frac{(2n-1)^2\pi^2}{4l^2}$$

→ (1p) We can now solve for $T$:

$$T' = \lambda T \Rightarrow T(t) = ce^{\lambda t}$$

Thus,

$$T_n(t) = c_n e^{-\frac{(2n-1)^2\pi^2}{4l^2}t}$$

→ We can now do a superposition:

$$v(x,t) = \sum_{n\geq 1} c_n e^{-\frac{(2n-1)^2\pi^2}{4l^2}t} \cos\left(\frac{(2n-1)\pi}{2l}x\right)$$

where we took all terms for $2n-1 < 0$ together with the corresponding term $-(2n-1)$, since all functions are even in $(2n-1)$.

→ (2p) *The initial condition will help us find, coefficients that depend on time*

$$v(x,t) = \sum_{n\geq 1} c_n \cos\left(\frac{(2n-1)\pi}{2l}x\right) \stackrel{!}{=} \tilde{g}(x)$$

This can be solved by a Fourier expansion:

$$c_n = \frac{\int_0^l \tilde{g}(x)\cos\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x}{\int_0^l \cos^2\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x} = \frac{l}{2}\int_0^l \tilde{g}(x)\cos\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x$$

By a double-angle formula,

$$\int_0^l \cos^2\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x = \int_0^l \frac{1}{2} + \frac{\cos\left(\frac{2(2n-1)\pi}{2l}x\right)}{2}\,\mathrm{d}x = \frac{l}{2}$$

→ (1p) Now, we just take it all together, and note that $u(x,t) = v(x,t) + s(x)$:

$$u(x,t) = \sum_{n\geq 1} \frac{l}{2}\int_0^l \tilde{g}(x)\cos\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x \cdot e^{-\frac{(2n-1)^2\pi^2}{4l^2}t}\cos\left(\frac{(2n-1)\pi}{2l}x\right)$$
$$+ 21x - 21l$$

7. Lös problemet:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t) \quad t,x > 0, \\ &u(x,0) = 0, \\ &u(0,t) = f(t), t > 0. \end{aligned}$$

Du kan anta att $f(t)$ är en snäll funktion och att alla transformer vi sett i kursen kan appliceras på den.

### 1.1 Solution

Problemet har ett randvillkor i $x = 0$ som beror på $t$, dvs, dags för Laplace-transform i $t$. Laplacetransform i $t$ ger:

$$zU(x,z) - U_{xx}(x,z).$$

Vi löser den här ODE:en och får

$$U(x,z) = a(z)e^{-x\sqrt{z}} + b(z)e^{x\sqrt{z}}.$$

Eftersom $U(x,z)$ måste vara Laplacetransformerbar såär nödvändigtvis $b(z) = 0$. Lösningen till ODE:en är alltså $U(x,z) = F(z)e^{-x\sqrt{z}}$. Om Laplacetransformen av en funktion är produkt så är funktionen en faltning. Enligt tabellen är den inversa Laplacetransformen av $e^{-x\sqrt{z}}$ funktionen

$$\frac{x}{2\sqrt{\pi}t^{3/2}}H(t)e^{-x^2/(4t)},$$

så $u(x,t)$ ges av

$$u(x,t) = \int_0^t f(s)\frac{x}{2\sqrt{\pi}(t-s)^{3/2}}e^{-x^2/(4(t-s))}\mathrm{d}s$$

(2p) Välj Laplacetransform i rätt variabel.

(2p) Laplacetransforma ekvationen rätt.

(2p) Lös den resulterande ODE:en.

(2p) Korrekt argumentera för att $b(z) = 0$.

(2p) Invertera Laplacetransformen rätt, dvs, skriv ned faltningsintegralen.

8. Låt $C[-1,1]$ vara rummet av alla kontinuerliga reelvärda funktioner på intervallet $[-1,1]$. Låt $G = \{g(x) \in C[-1,1]; g(x) = g(-x)\}$ vara mängden av alla jämna funktioner på $[-1,1]$. Hitta den bästa $L^2$-approximationen av $f(x) = x^3 + x^2 + e^{-x}$ i $G$, det vill säga, hitta funktionen $g(x) \in G$ som minimerar

$$\|g - f\|^2 = \int_{-1}^1 (g - x^3 - x^2 - e^{-x})^2 \mathrm{d}x.$$

(Tips: Använd Pytagoras sats, att $e^{-x} = \sinh x + \cosh x$ samt att om $h(x)$ är en udda funktion, dvs, $h(x) \in H = \{h(x) \in C[-1,1]; h(-x) = -h(x)\}$ så gäller att $\int g(x)h(x)\mathrm{d}x = 0$).

**Lösning.**

Notera att $f(x) = x^3 + x^2 + \sinh(x) + \cosh(x)$. Funktionen $x^3 + \sinh(x)$ är udda, och $x^2 + \cosh(x)$ är jämn. Ta nu ett $g \in G$ och använd Pythagoras sats:

$$\|g - f\|^2 = \left\| \big(g - (x^2 + \cosh(x))\big) + \big(x^3 + \sinh(x)\big) \right\|^2 .$$

Eftersom $\big(g - (x^2 + \cosh(x))\big) \in G$ och $\big(x^3 + \sinh(x)\big) \in H$ så är de ortogonala, och Pytagoras sats gäller och så

$$\|g - f\|^2 = \left\| \big(g - (x^2 + \cosh(x))\big) \right\|^2 + \left\| \big(x^3 + \sinh(x)\big) \right\|^2 .$$

För att minimiera uttrycket ovan så måste $\left\| \big(g - (x^2 + \cosh(x))\big) \right\|^2 = 0$, dvs, $g(x) = x^2 + \cosh(x)$.

(1p) Dela upp $e^{-x}$ i $\cosh(x)$ och $\sinh(x)$.

(2p) Inse att ena delen av $f(x)$ är jämn och att den andra är udda.

(2p) Korrekt använda Pytagoras sats.

(5p) Inse att för att minimera uttrycket så måste $g(x) = x^2 + \cosh(x)$.

Lycka till! May the FourierForce be with you! ♡ Julie, Carl-Joar, Björn, Erik, & Kolya

# 2 Fun facts!

## 2.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

## 2.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with $u$ the unknown function and $p$ and $g$ specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called $\mu$ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the M$\mu$thod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital $C$. The solution is:

$$u(t) = \frac{\int_0^t (\mu(s)g(s)ds) + C}{\mu(t)}.$$

## 2.3 How to solve second order linear ODEs

**Theorem 1** (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function $u$ that depends on one variable, with constants $b$ and $c$ given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

*A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:*

1. ***If** $b^2 \neq 4ac$**,** then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$**,** then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

**Theorem 2** (Particular solution to linear second order ODEs)**.** *Assume that $y_1$ und $y_2$ are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

*Then, if $y_p$ is a particular solution of the inhomogeneous ODE, so that*

$$L(y_p) = g,$$

*then all solutions to $L(y) = g$ can be expressed as*

$$c_1 y_1 + c_2 y_2 + y_p,$$

*for $y_1$ and $y_2$ as above, for coefficients $c_1$ and $c_2$. One way to find a particular solution to the ODE*

$$L(y) = g(t)$$

*is to calculate*

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

*The Wronskian of $y_1$ and $y_2$, denoted by $W(y_1, y_2)$ above, is defined to be*

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t).$$

### 2.4 Bessel facts

**Definition 1** (The Bessel function $J$ of order $\nu$)**.** The Bessel function $J$ of order $\nu$ is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Bessel function satisfies the Bessel equation:

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0.$$

The modified Bessel equation is satisfied by $I_\nu$ and $K_\nu$

$$x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x).$$

For real values of $\nu$ the function $I_\nu(x) \neq 0$ for all $x > 0$. The function $K_\nu(x)$ tends to $\infty$ as $x \to 0$. The $\Gamma$ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{1}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of $\nu$, it is an entire function of $x \in \mathbb{C}$.

**Theorem 3** (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the $m^{th}$ positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n} r/L)\}_{m \geq 1}$$

*are an orthogonal base for $\mathcal{L}^2_r(0, L)$. Recall that this is the weighted $\mathcal{L}^2$ space on the interval $(0, L)$ with respect to the weight function $r$, so the scalar product*

$$\langle f, g \rangle = \int_0^L f(r) \overline{g(r)} r dr.$$

*More generally, for any $\nu \in \mathbb{R}$, and for any $L > 0$, the functions*

$$\{J_\nu(\pi_m r/L)\}_{m \geq 1}$$

*are an orthogonal base for $\mathcal{L}^2_r(0, L)$, where above $\pi_m$ denotes the $m^{th}$ zero of the Bessel function $J_\nu$. They have norms equal to*

$$\int_0^L |J_\nu(\pi_m r/L)|^2 r dr = \frac{L^2}{2} \left(J_{\nu+1}(\pi_m)\right)^2.$$

**Corollary 1** (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n}r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

*are an orthogonal basis for $\mathcal{L}^2$ on the disk of radius $L$.*

**Theorem 4** (Bessel functions as bases in some other cases)**.** *Assume that $L > 0$. Let the weight function $w(x) = x$. Fix $\nu \in \mathbb{R}$. Then $J'_\nu$ has infinitely many positive zeros. Let*

$$\{\pi'_k\}_{k\geq 1}$$

*be the positive zeros of $J'_\nu$. Then we define*

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu > 0, \quad k \geq 1.$$

*In case $\nu = 0$, define further $\psi_0(x) = 1$. (If $\nu \neq 0$, then this case is omitted.) Then $\{\psi_k\}_{k\geq 1}$ for $\nu \neq 0$ is an orthogonal basis for $\mathcal{L}^2_w(0, L)$. For $\nu = 0$, $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w(0, L)$. Moreover the norm*

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \geq 1, \quad ||\psi_0||_w^2 = \frac{L^{2\nu+2}}{2\nu+2}.$$

*Next, fix a constant $c > 0$. Then there are infinitely many positive solutions of*

$$\mu J'_\nu(\mu) + cJ_\nu(\mu) = 0,$$

*that can be enumerated as $\{\mu_k\}_{k\geq 1}$. Then*

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

*is an orthogonal basis for $\mathcal{L}^2_w(0, L)$.*

## 2.5 Orthogonal polynomials

**Definition 2.** The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2 - 1)^n\right). \tag{2}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1, 1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

The first few Legendre polynomials are $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2 - 1)$, and $P_3 = \frac{1}{2}(5x^3 - 3x)$.

**Definition 3.** The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}_2^2(\mathbb{R})$ with respect to the weight function $e^{-x^2}$. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

**Definition 4.** The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x}).$$

The Laguerre polynomials $\{L_n^\alpha\}_{n\geq 0}$ are an orthogonal basis for $\mathcal{L}_\alpha^2$ on $(0, \infty)$ with the weight function $\alpha(x) = x^\alpha e^{-x}$. Their norms squared,

$$||L_n^\alpha||^2 = \frac{\Gamma(n + \alpha + 1)}{n!}.$$

## 2.6 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

| | | |
|---|---|---|
| 1. | $f(x) = x$ | $\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}$. |
| 2. | $f(x) = \|x\|$ | $\frac{\pi}{2} + \sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$ |
| 3. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$ | $\frac{\pi}{4} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\frac{(-1)^{n+1}}{2in} + \frac{(-1)^n - 1}{2\pi n^2}\right] e^{inx}$ |
| 4. | $f(x) = \sin^2(x)$ | $\frac{1}{2} - \frac{1}{4}\left(e^{2ix} + e^{-2ix}\right)$ |
| 5. | $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{2}{i\pi} \sum_{n\geq 1} \frac{e^{(2n-1)ix} - e^{-(2n-1)ix}}{2n-1}$ |
| 6. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{1}{2} + \sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{i\pi(2n-1)}$ |
| 7. | $f(x) = \|\sin(x)\|$ | $\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1}$ |
| 8. | $f(x) = \|\cos(x)\|$ | $\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{(-1)^n\left[e^{inx} + e^{-inx}\right]}{4n^2-1}$ |
| 9. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$ | $\frac{1}{\pi} - \frac{1}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1} + \frac{1}{4i}\left(e^{ix} - e^{-ix}\right)$ |
| 10. | $f(x) = x^2$ | $\frac{\pi^2}{3} + 2\sum_{n\geq 1} \frac{(-1)^n(e^{inx} + e^{-inx})}{n^2}$ |
| 11. | $f(x) = x(\pi - \|x\|)$ | $\frac{4}{i\pi}\sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{(2n-1)^3}$ |
| 12. | $f(x) = e^{bx}$ | $\frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b-in} e^{inx}$ |
| 13. | $f(x) = \sinh x$ | $\frac{\sinh(\pi)}{i\pi}\sum_{n\geq 1} \frac{(-1)^{n+1} n}{n^2+1}\left[e^{inx} - e^{-inx}\right]$ |

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all $2\pi$ periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

| | | |
|---|---|---|
| 1. | $f(x) = x$ | $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$. |
| 2. | $f(x) = \|x\|$ | $\frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2}$ |
| 3. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$ | $\frac{\pi}{4} - \frac{2}{\pi}\sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2} + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin(nx)$ |
| 4. | $f(x) = \sin^2(x)$ | $\frac{1}{2} - \frac{1}{2}\cos(2x)$ |
| 5. | $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{4}{\pi}\sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$ |
| 6. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$ | $\frac{1}{2} + \frac{2}{\pi}\sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$ |
| 7. | $f(x) = \|\sin(x)\|$ | $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1}$ |
| 8. | $f(x) = \|\cos(x)\|$ | $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1} \frac{(-1)^n\cos(2nx)}{4n^2-1}$ |
| 9. | $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$ | $\frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1} + \frac{1}{2}\sin(x)$ |
| 10. | $f(x) = x^2$ | $\frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n\cos(nx)}{n^2}$ |
| 11. | $f(x) = x(\pi - \|x\|)$ | $\frac{8}{\pi}\sum_{n\geq 1} \frac{\sin((2n-1)x)}{(2n-1)^3}$ |
| 12. | $f(x) = e^{bx}$ | $\frac{\sinh(b\pi)}{\pi}\left(\frac{1}{b} + \sum_{n\geq 1} \frac{(-1)^n}{b^2+n^2}\left[2b\cos(nx) - 2n\sin(nx)\right]\right)$ |
| 13. | $f(x) = \sinh x$ | $\frac{2\sinh(\pi)}{\pi}\sum_{n\geq 1} \frac{(-1)^{n+1}n}{n^2+1}\sin(nx)$ |

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1, \cos(nx), \sin(nx)\}_{n\geq 1}$. The series on the right are all $2\pi$ periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

| | |
|---|---|
| $f(x)$ | $\hat{f}(\xi)$ |
| $f(x-c)$ | $e^{-ic\xi}\hat{f}(\xi)$ |
| $e^{ixc}f(x)$ | $\hat{f}(\xi-c)$ |
| $f(ax)$ | $a^{-1}\hat{f}(a^{-1}\xi)$ |
| $f'(x)$ | $i\xi\hat{f}(\xi)$ |
| $xf(x)$ | $i(\hat{f})'(\xi)$ |
| $(f*g)(x)$ | $\hat{f}(\xi)\hat{g}(\xi)$ |
| $f(x)g(x)$ | $(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$ |
| $e^{-ax^2/2}$ | $\sqrt{2\pi/a}e^{-\xi^2/(2a)}$ |
| $(x^2+a^2)^{-1}$ | $(\pi/a)e^{-a\|\xi\|}$ |
| $e^{-a\|x\|}$ | $2a(\xi^2+a^2)^{-1}$ |
| $\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$ | $2\xi^{-1}\sin(a\xi)$ |
| $x^{-1}\sin(ax)$ | $\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$ |

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a > 0$ and $c \in \mathbb{R}$.

| | | |
|---|---|---|
| 1. | $\Theta(t)f(t)$ | $\widetilde{f}(z)$ |
| 2. | $\Theta(t-a)f(t-a)$ | $e^{-az}\widetilde{f}(z)$ |
| 3. | $e^{ct}\Theta(t)f(t)$ | $\widetilde{f}(z-c)$ |
| 4. | $\Theta(t)f(at)$ | $a^{-1}\widetilde{f}(a^{-1}z)$ |
| 5. | $\Theta(t)f'(t)$ | $z\widetilde{f}(z)-f(0)$ |
| 6. | $\Theta(t)f^{(k)}(t)$ | $z^k\widetilde{f}(z)-\sum_0^{k-1} z^{k-1-j}f^{(j)}(0)$ |
| 7. | $\Theta(t)\int_0^t f(s)ds$ | $z^{-1}\widetilde{f}(z)$ |
| 8. | $\Theta(t)tf(t)$ | $-\widetilde{f}'(z)$ |
| 9. | $\Theta(t)t^{-1}f(t)$ | $\int_z^\infty \widetilde{f}(w)dw$ |
| 10. | $\Theta f * \Theta g(t)$ | $\widetilde{f}(z)\widetilde{g}(z)$ |
| 11. | $\Theta(t)t^\nu e^{ct}$ | $\Gamma(\nu+1)(z-c)^{-\nu-1}$ |
| 12. | $\Theta(t)(t+a)^{-1}$ | $e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$ |
| 13. | $\Theta(t)\sin(ct)$ | $\frac{c}{z^2+c^2}$ |
| 14. | $\Theta(t)\cos(ct)$ | $\frac{z}{z^2+c^2}$ |

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.

| | | |
|---|---|---|
| 15. | $\Theta(t)\sinh(ct)$ | $\frac{c}{z^2-c^2}$ |
| 16. | $\Theta(t)\cosh(ct)$ | $\frac{z}{z^2-c^2}$ |
| 17. | $\Theta(t)\sin(\sqrt{at})$ | $\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$ |
| 18. | $\Theta(t)t^{-1}\sin(\sqrt{at})$ | $\pi\,\mathrm{erf}(\sqrt{a/(4z)})$ |
| 19. | $\Theta(t)e^{-a^2t^2}$ | $(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$ |
| 20. | $\Theta(t)\,\mathrm{erf}(at)$ | $z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$ |
| 21. | $\Theta(t)\,\mathrm{erf}(\sqrt{t})$ | $(z\sqrt{z+1})^{-1}$ |
| 22. | $\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$ | $((z-1)\sqrt{z})^{-1}$ |
| 23. | $\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$ | $z^{-1}e^{-a\sqrt{z}}$ |
| 24. | $\Theta(t)t^{-1/2}e^{-\sqrt{at}}$ | $\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$ |
| 25. | $\Theta(t)t^{-1/2}e^{-a^2/(4t)}$ | $\sqrt{\pi/z}e^{-a\sqrt{z}}$ |
| 26. | $\Theta(t)t^{-3/2}e^{-a^2/(4t)}$ | $2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$ |
| 27. | $\Theta(t)t^\nu J_\nu(t)$ | $2^\nu\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$ |
| 28. | $\Theta(t)J_0(\sqrt{t})$ | $z^{-1}e^{-1/(4z)}$ |

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.