Fourieranalys MVE030 och Fourier Metoder MVE290 2024.mars.15
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
5 Timmar.
Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & Chalmers godkänt miniräknare.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

1 Uppgifter

1. (CAT) Låt $g \in L^1(\mathbb{R})$. Antar att f är begränsad och styckvis kontinuerlig. Låt

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}, \quad \epsilon > 0.$$

Bevisa att för varenda punkt $x \in \mathbb{R}$ gäller

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = f(x_+) \int_{-\infty}^{0} g(y)dy + f(x_-) \int_{0}^{\infty} g(y)dy.$$

Här betyder $*$ faltning eller "convolution" på engelska.
English: assume that $g \in L^1(\mathbb{R})$. Define g_ϵ as above. Assume that f is piecewise continuous and bounded. Then prove that for each point $x \in \mathbb{R}$

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = f(x_+) \int_{-\infty}^{0} g(y)dy + f(x_-) \int_{0}^{\infty} g(y)dy.$$

Note that $f * g$ is the convolution. This is one of the two cases in the CAT theorem which assumed that *either* f is bounded *or* g has compact support. So you just need to do the proof for the f is bounded case!

(10p)

2. Vad säger Fourier-Inverse-Formel (FIT)?
English: what's the Fourier inverse theorem (FIT) say? (10 p)

3. Lös problemet: (Solve the following problem):

$$\begin{aligned} u_t - u_{xx} &= e^x, 0 \le x \le l, t > 0, \\ u_x(0,t) &= 0, \\ u(l,t) &= 21, \\ u(x,0) &= f(x). \end{aligned}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated). (10 p)

4. Lös problemet: (Solve the following problem):

$$\begin{aligned} &u_t(r,\theta,t) = \Delta u(r,\theta,t), \quad 0 < r < 3,\ t > 0,\ -\pi < \theta < \pi, \\ &u(3,\theta,t) = 0, \\ &u(r,\theta,0) = r\cos(\theta). \end{aligned}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated).

(10 p)

5. Beräkna: (Compute):

$$\sum_{n=1}^{\infty} \frac{1}{4n^4 - n^2}$$

Hint: Hitta a och b så att

$$\frac{1}{4n^4 - n^2} = \frac{a}{n^2} + \frac{b}{4n^2 - 1}$$

och sedan andvänd tabellen av Fourierserier. English: compute the sum and the hint is that to do so find a and b as above and then find something helpful from the tables of tF-series in here!

(10p)

6. Sök en begränsad lösning till problemet (find a bounded solution to this problem)

$$\begin{aligned} &u_t = u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = \frac{x}{x^2 + 2}. \end{aligned}$$

(If horrible Dark Souls integrals appear you don't need to calculate them just explain why they appear and make sure they're correct.)

(10 p)

7. Lös problemet: (Solve the following problem):

$$
\begin{aligned}
&u(x,t) + \int_0^t (t-s)u_{xx}(x,s)ds = 2,\ \ x,t>0\\
&u(x,0) = 0,\\
&u(0,t) = \sin(t).
\end{aligned}
$$

(10 p)

8. Bestäm det *maximala* värdet av

$$\int_{-1}^{1} x^3 f(x)\mathrm{d}x$$

bland alla kontinuerliga begränsade reella funktioner på $[-1,1]$ som uppfyller

$$\int_{-1}^{1} |f(x)|^2 dx = 1, \quad \int_{-1}^{1} x f(x)\mathrm{d}x = 0.$$

(Tips: expandera f med Legendre-polynom basen, titta på P_1 och P_3, och använd Hilbertrum fakta om ortogonala baser.).

(English): Determine the *maximum* value of

$$\int_{-1}^{1} x^3 f(x)dx$$

among all continuous bounded real-valued functions in $[-1,1]$ that satisfy

$$\int_{-1}^{1} |f(x)|^2 dx = 1, \quad \int_{-1}^{1} x f(x)dx = 0.$$

(Hint: expand f using the Legendre polynomial base, look at P_1 and P_3, and use Hilbert space orthogonal base facts.)

(10p)

Lycka till! May the FourierForce be with you! ♡ Julie, Carl-Joar, Björn, Erik, & Kolya

2 Fun facts!

2.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

2.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with u the unknown function and p and g specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called μ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the Mμthod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital C. The solution is:

$$u(t) = \frac{\int_0^t(\mu(s)g(s)ds) + C}{\mu(t)}.$$

2.3 How to solve second order linear ODEs

Theorem 1 (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function u that depends on one variable, with constants b and c given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:

1. ***If** $b^2 \neq 4ac$**,** then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$**,** then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

Theorem 2 (Particular solution to linear second order ODEs)**.** *Assume that y_1 und y_2 are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

Then, if y_p is a particular solution of the inhomogeneous ODE, so that

$$L(y_p) = g,$$

then all solutions to $L(y) = g$ can be expressed as

$$c_1 y_1 + c_2 y_2 + y_p,$$

for y_1 and y_2 as above, for coefficients c_1 and c_2. One way to find a particular solution to the ODE

$$L(y) = g(t)$$

is to calculate

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

The Wronskian of y_1 and y_2, denoted by $W(y_1, y_2)$ above, is defined to be

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t).$$

2.4 Bessel facts

Definition 1 (The Bessel function J of order ν)**.** The Bessel function J of order ν is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Bessel function satisfies the Bessel equation:

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0.$$

The modified Bessel equation is satisfied by I_ν and K_ν

$$x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x).$$

For real values of ν the function $I_\nu(x) \neq 0$ for all $x > 0$. The function $K_\nu(x)$ tends to ∞ as $x \to 0$. The Γ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{1}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of ν, it is an entire function of $x \in \mathbb{C}$.

Theorem 3 (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the m^{th} positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n} r/L)\}_{m \geq 1}$$

are an orthogonal base for $\mathcal{L}_r^2(0, L)$. Recall that this is the weighted $\mathcal{L}^2$ space on the interval $(0, L)$ with respect to the weight function r, so the scalar product

$$\langle f, g \rangle = \int_0^L f(r)\overline{g(r)} r dr.$$

More generally, for any $\nu \in \mathbb{R}$, and for any $L > 0$, the functions

$$\{J_\nu(\pi_m r/L)\}_{m \geq 1}$$

are an orthogonal base for $\mathcal{L}_r^2(0, L)$, where above π_m denotes the m^{th} zero of the Bessel function J_ν. They have norms equal to

$$\int_0^L |J_\nu(\pi_m r/L)|^2 r dr = \frac{L^2}{2} \left(J_{\nu+1}(\pi_m)\right)^2.$$

Corollary 1 (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n}r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

are an orthogonal basis for $\mathcal{L}^2$ on the disk of radius L.

Theorem 4 (Bessel functions as bases in some other cases)**.** *Assume that $L > 0$. Let the weight function $w(x) = x$. Fix $\nu \in \mathbb{R}$. Then J_ν' has infinitely many positive zeros. Let*

$$\{\pi_k'\}_{k\geq 1}$$

be the positive zeros of J_ν'. Then we define

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu > 0, \quad k \geq 1.$$

In case $\nu = 0$, define further $\psi_0(x) = 1$. (If $\nu \neq 0$, then this case is omitted.) Then $\{\psi_k\}_{k\geq 1}$ for $\nu \neq 0$ is an orthogonal basis for $\mathcal{L}_w^2(0, L)$. For $\nu = 0$, $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}_w^2(0, L)$. Moreover the norm

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \geq 1, \quad ||\psi_0||_w^2 = \frac{L^{2\nu+2}}{2\nu+2}.$$

Next, fix a constant $c > 0$. Then there are infinitely many positive solutions of

$$\mu J_\nu'(\mu) + c J_\nu(\mu) = 0,$$

that can be enumerated as $\{\mu_k\}_{k\geq 1}$. Then

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}_w^2(0, L)$.

2.5 Orthogonal polynomials

Definition 2. The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2-1)^n\right). \tag{2}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1, 1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

The first few Legendre polynomials are $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2 - 1)$, and $P_3 = \frac{1}{2}(5x^3 - 3x)$.

Definition 3. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}_2^2(\mathbb{R})$ with respect to the weight function e^{-x^2}. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

Definition 4. The Laguerre polynomials,

$$L_n^{\alpha}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x}).$$

The Laguerre polynomials $\{L_n^{\alpha}\}_{n\geq 0}$ are an orthogonal basis for $\mathcal{L}_{\alpha}^2$ on $(0, \infty)$ with the weight function $\alpha(x) = x^{\alpha} e^{-x}$. Their norms squared,

$$||L_n^{\alpha}||^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

2.6 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

1.	$f(x) = x$	$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}.$
2.	$f(x) = \|x\|$	$\frac{\pi}{2} + \sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$
3.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$	$\frac{\pi}{4} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\frac{(-1)^{n+1}}{2in} + \frac{(-1)^n-1}{2\pi n^2}\right] e^{inx}$
4.	$f(x) = \sin^2(x)$	$\frac{1}{2} - \frac{1}{4}\left(e^{2ix} + e^{-2ix}\right)$
5.	$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{2}{i\pi} \sum_{n\geq 1} \frac{e^{(2n-1)ix} - e^{-(2n-1)ix}}{2n-1}$
6.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{1}{2} + \sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{i\pi(2n-1)}$
7.	$f(x) = \|\sin(x)\|$	$\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1}$
8.	$f(x) = \|\cos(x)\|$	$\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{(-1)^n\left[e^{inx} + e^{-inx}\right]}{4n^2-1}$
9.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$	$\frac{1}{\pi} - \frac{1}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1} + \frac{1}{4i}\left(e^{ix} - e^{-ix}\right)$
10.	$f(x) = x^2$	$\frac{\pi^2}{3} + 2\sum_{n\geq 1} \frac{(-1)^n(e^{inx} + e^{-inx})}{n^2}$
11.	$f(x) = x(\pi - \|x\|)$	$\frac{4}{i\pi}\sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{(2n-1)^3}$
12.	$f(x) = e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b-in} e^{inx}$
13.	$f(x) = \sinh x$	$\frac{\sinh(\pi)}{i\pi}\sum_{n\geq 1} \frac{(-1)^{n+1}n}{n^2+1}\left[e^{inx} - e^{-inx}\right]$

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi, \pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi, \pi)$. On the rest of the real line, outside of the interval $(-\pi, \pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi, \pi)$.

1.	$f(x) = x$	$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$
2.	$f(x) = \|x\|$	$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2}$
3.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$	$\frac{\pi}{4} - \frac{2}{\pi} \sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2} + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin(nx)$
4.	$f(x) = \sin^2(x)$	$\frac{1}{2} - \frac{1}{2}\cos(2x)$
5.	$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{4}{\pi} \sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$
6.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{1}{2} + \frac{2}{\pi} \sum_{n\geq 1} \frac{\sin((2n-1)x)}{2n-1}$
7.	$f(x) = \|\sin(x)\|$	$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1}$
8.	$f(x) = \|\cos(x)\|$	$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n\geq 1} \frac{(-1)^n \cos(2nx)}{4n^2-1}$
9.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$	$\frac{1}{\pi} - \frac{2}{\pi} \sum_{n\geq 1} \frac{\cos(2nx)}{4n^2-1} + \frac{1}{2}\sin(x)$
10.	$f(x) = x^2$	$\frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n \cos(nx)}{n^2}$
11.	$f(x) = x(\pi - \|x\|)$	$\frac{8}{\pi} \sum_{n\geq 1} \frac{\sin((2n-1)x)}{(2n-1)^3}$
12.	$f(x) = e^{bx}$	$\frac{\sinh(b\pi)}{\pi} \left(\frac{1}{b} + \sum_{n\geq 1} \frac{(-1)^n}{b^2+n^2} \left[2b\cos(nx) - 2n\sin(nx)\right] \right)$
13.	$f(x) = \sinh x$	$\frac{2\sinh(\pi)}{\pi} \sum_{n\geq 1} \frac{(-1)^{n+1} n}{n^2+1} \sin(nx)$

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1, \cos(nx), \sin(nx)\}_{n\geq 1}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a > 0$ and $c \in \mathbb{R}$.

1.	$\Theta(t)f(t)$	$\widetilde{f}(z)$
2.	$\Theta(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
3.	$e^{ct}\Theta(t)f(t)$	$\widetilde{f}(z-c)$
4.	$\Theta(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
5.	$\Theta(t)f'(t)$	$z\widetilde{f}(z)-f(0)$
6.	$\Theta(t)f^{(k)}(t)$	$z^k\widetilde{f}(z)-\sum_0^{k-1} z^{k-1-j}f^{(j)}(0)$
7.	$\Theta(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
8.	$\Theta(t)tf(t)$	$-\widetilde{f}'(z)$
9.	$\Theta(t)t^{-1}f(t)$	$\int_z^\infty \widetilde{f}(w)dw$
10.	$\Theta f * \Theta g(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
11.	$\Theta(t)t^\nu e^{ct}$	$\Gamma(\nu+1)(z-c)^{-\nu-1}$
12.	$\Theta(t)(t+a)^{-1}$	$e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$
13.	$\Theta(t)\sin(ct)$	$\frac{c}{z^2+c^2}$
14.	$\Theta(t)\cos(ct)$	$\frac{z}{z^2+c^2}$

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a>0$ is constant and $c\in\mathbb{C}$.

15.	$\Theta(t)\sinh(ct)$	$\frac{c}{z^2-c^2}$
16.	$\Theta(t)\cosh(ct)$	$\frac{z}{z^2-c^2}$
17.	$\Theta(t)\sin(\sqrt{at})$	$\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$
18.	$\Theta(t)t^{-1}\sin(\sqrt{at})$	$\pi\,\mathrm{erf}(\sqrt{a/(4z)})$
19.	$\Theta(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$
20.	$\Theta(t)\,\mathrm{erf}(at)$	$z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$
21.	$\Theta(t)\,\mathrm{erf}(\sqrt{t})$	$(z\sqrt{z+1})^{-1}$
22.	$\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$	$((z-1)\sqrt{z})^{-1}$
23.	$\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$	$z^{-1}e^{-a\sqrt{z}}$
24.	$\Theta(t)t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$
25.	$\Theta(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
26.	$\Theta(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
27.	$\Theta(t)t^{\nu}J_{\nu}(t)$	$2^{\nu}\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$
28.	$\Theta(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.

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1 Uppgifter

1. Låt $g \in L^1(\mathbb{R})$. Antar att f är kontinuerlig och begränsad. Låt

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}, \quad \epsilon > 0.$$

Observera att då f är kontinuerlig gäller $f(x_+) = f(x_-)$ $\forall x \in \mathbb{R}$. Sedan bevisa att för varenda punkt $x \in \mathbb{R}$ gäller

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = f(x) \int_{-\infty}^{\infty} g(y) dy.$$

Här betyder $*$ faltning eller "convolution" på engelska. Det här är en förenklad versionen av den CAT.
English: assume that $g \in L^1(\mathbb{R})$. Assume that f is continuous and bounded. Define g_ϵ as above. Note that the left and right limits $f(x_+) = f(x_-)$ $\forall x \in \mathbb{R}$ since f is continuous. Then prove that for each point $x \in \mathbb{R}$

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = f(x) \int_{-\infty}^{\infty} g(y) dy.$$

Note that $f * g$ is the convolution, and that this is one of the cases subsumed by the more general CAT, thus a simplified version of that theorem!

Solution

1p. Fix the point x. Show that it is enough to prove that

$$\lim_{\epsilon \to 0} \int_{-\infty}^{0} f(x-y) g_\epsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy = 0$$

and also

$$\lim_{\epsilon\to 0}\int_0^\infty f(x-y)g_\epsilon(y)dy - \int_0^\infty f(x-)g(y)dy = 0.$$

The argument is same for both, so choose one. I choose the first one.

2p. Do a substitution in the second integral, setting $z = \epsilon y$, so $y = z/\epsilon$, and $dz/\epsilon = dy$. This shows that:

$$\int_{-\infty}^0 \left(f(x-y)g_\epsilon(y) - f(x+)g(y)\right)dy = \int_{-\infty}^0 g_\epsilon(y)\left(f(x-y) - f(x+)\right)dy.$$

1p. To estimate

$$\int_{-\infty}^0 g_\epsilon(y)\left(f(x-y) - f(x+)\right)dy,$$

split the integral into the part out near $-\infty$, $\int_{-\infty}^{y_0}$ added to the part close to zero $\int_{y_0}^0$. CAT's face and CAT's tail.

3p. Estimate the integral close to zero first so that you can figure out what y_0 needs to be. In CAT language, pet the face first. To do this, use the fact that the integral is over negative values of y, so $x - y > x$, together with the definition of $f(x+)$ as the right-hand-limit. In this way make $|f(x-y) - f(x+)|$ super small by choosing $y_0 < 0$ small. Then you can pull out a factor of "super small" and estimate

$$(\text{super small})\int_{y_0}^0 |g_\epsilon(y)|dy \le (\text{super small})\int_{-\infty}^0 |g_\epsilon(y)|dy \le (\text{super small})||g||_{L^1}.$$

This is fine because the L^1 norm of g is finite.

3p. Next estimate the CAT's tail

$$\int_{-\infty}^{y_0} g_\epsilon(y)\left(f(x-y) - f(x+)\right)dy.$$

Since we assume f is bounded, note that $|f(x-y) - f(x+)| \le 2(\text{the number that bounds } f)$. So you pull this out. Change variables to make the integral go from $-\infty$ to $-\delta/\epsilon$. Use the fact that the tail of a convergent integral can be made small to make this small.

2. Vad är Fourier-Inverse-Formel (FIT)? (10 p)

If f is in $\mathcal{L}^2(\mathbb{R})$ then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi.$$

You probably either get this or not, but if I have to break down points I'd say 5 points for $f \in \mathcal{L}^2(\mathbb{R})$ and 5 points for the equation. Minus one point for each mistake in the equation, if you made any.

3. Lös problemet: (Solve the following problem):

$$\begin{aligned} u_t - u_{xx} &= e^x, 0 \leq x \leq l, t > 0 \\ u_x(0,t) &= 0 \\ u(l,t) &= 21 \\ u(x,0) &= f(x). \end{aligned}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated). (10 p)

If you are curious about the physics behind this problem here it is, but this is just for curiosity not part of the task! A rod of length l is hit with high-energy particles on the left end ($x = 0$). The amount of energy (i.e., heat) these particles transfer to the rod depends on how far they have already traveled through the rod.Surprisingly, up to a certain point, more energy is transferred, the further the particles have moved through the rod. This phenomenon has applications, for example, in radiation shielding and is described by the so-called Bragg curve, which can take many different shapes depending on the type of particle and its speed. In our case, we model the Bragg curve as an exponential function. The left end of the rod is insulated from its surroundings, while the right end is kept at room temperature 21°C.

Solution

→ We are dealing with an inhomogeneous problem here. The inhomogeneities are independent of time, so we can use the method of steady states.

$\rightarrow$ (2p) First, we find a steady-state solution, i.e., we set the time derivative u_t to 0. We obtain the problem:

$$\begin{aligned} -s''(x) &= e^x, \quad 0 \leq x \leq l \\ s'(0) &= 0 \\ s(l) &= 21. \end{aligned}$$

We can find $s(x)$ by integrating twice:

$$\begin{aligned} s''(x) &= -e^x \\ s'(x) &= -e^x + c \\ s(x) &= -e^x + cx + d \end{aligned}$$

Plugging in the boundary conditions, we get:

$$\begin{aligned} s'(0) = -1 + c = 0 &\Rightarrow c = 1 \\ s(l) = -e^l + l + d = 21 &\Rightarrow d = 20 + e^l \end{aligned}$$

Thus, our steady state is

$$s(x) = -e^x + x + 20 + e^l$$

$\rightarrow$ (1p) Defining $v(x,t) = u(x,t) - s(x)$, we obtain the homogeneous problem

$$\begin{aligned} v_t - v_{xx} &= 0, 0 \leq x \leq l, t > 0 \\ v_x(0,t) &= 0 \\ v(l,t) &= 0 \\ v(x,0) &= f(x). \end{aligned}$$

$\rightarrow$ (1p) We solve this using our favorite method $\heartsuit$. We start by separating our variables: $v(x,t) = X(x)T(t)$, and obtain:

$$T'X - TX'' = 0$$

and

$$\frac{T'}{T} = \frac{X''}{X} = \lambda$$

→ (2p) *The boundary conditions will help us see what the value of λ needs to be!*

$$X'' = \lambda X$$

By Theorem 1.0.1, this gives us

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

With the boundary conditions, we obtain

$$X'(0) = 0 \Rightarrow A\sqrt{\lambda}e^{\sqrt{\lambda}0} - B\sqrt{\lambda}e^{-\sqrt{\lambda}0} \Rightarrow A = B$$

$$X(l) = 0 \Rightarrow A(e^{\sqrt{\lambda}l} + e^{-\sqrt{\lambda}l}) = 0 \Rightarrow A = 0 \text{ (the boring case) } \vee e^{2\sqrt{\lambda}l} = -1$$

The non-boring case is equivalent to:

$$2\sqrt{\lambda}l = (2n-1)\pi i \iff \sqrt{\lambda} = \frac{(2n-1)\pi i}{2l}, \quad n \in \mathbb{Z}$$

This leaves us with (up to constants):

$$X_n(x) = \frac{1}{2}\left(e^{\frac{(2n-1)\pi i}{2l}x} + e^{-\frac{(2n-1)\pi i}{2l}x}\right) = \cosh\left(\frac{(2n-1)\pi i}{2l}x\right) = \cos\left(\frac{(2n-1)\pi}{2l}x\right)$$

and the eigenvalues

$$\lambda_n = \left(\frac{(2n-1)\pi i}{2l}\right)^2 = -\frac{(2n-1)^2\pi^2}{4l^2}$$

→ (1p) We can now solve for T:

$$T' = \lambda T \Rightarrow T(t) = ce^{\lambda t}$$

Thus,

$$T_n(t) = c_n e^{-\frac{(2n-1)^2\pi^2}{4l^2}t}$$

→ We can now do a superposition:

$$v(x,t) = \sum_{n\geq 1} c_n e^{-\frac{(2n-1)^2\pi^2}{4l^2}t} \cos\left(\frac{(2n-1)\pi}{2l}x\right)$$

where we took all terms for $2n-1<0$ together with the corresponding term $-(2n-1)$, since all functions are even in $(2n-1)$.

$\rightarrow$ (2p) *The initial condition will help us find, coefficients that depend on time*

$$v(x,t) = \sum_{n\geq 1} c_n \cos\left(\frac{(2n-1)\pi}{2l}x\right) \stackrel{!}{=} f(x)$$

This can be solved by a Fourier expansion:

$$c_n = \frac{\int_0^l f(x)\cos\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x}{\int_0^l \cos^2\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x} = \frac{l}{2}\int_0^l f(x)\cos\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x$$

By a double-angle formula,

$$\int_0^l \cos^2\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x = \int_0^l \frac{1}{2} + \frac{\cos\left(\frac{2(2n-1)\pi}{2l}x\right)}{2}\,\mathrm{d}x = \frac{l}{2}$$

$\rightarrow$ (1p) Now, we just take it all together, and note that $u(x,t) = v(x,t) + s(x)$:

$$\begin{aligned} u(x,t) = &\sum_{n\geq 1} \frac{l}{2}\int_0^l f(x)\cos\left(\frac{(2n-1)\pi}{2l}x\right)\,\mathrm{d}x \cdot e^{-\frac{(2n-1)^2\pi^2}{4l^2}t}\cos\left(\frac{(2n-1)\pi}{2l}x\right) \\ &- e^x + x + 20 + e^l \end{aligned}$$

4. Lös problemet: (Solve the following problem):

$$\begin{aligned} &u_t(r,\theta,t) = \Delta u(r,\theta,t), \quad 0<r<3,\ t>0,\ -\pi<\theta<\pi, \\ &u(3,\theta,t) = 0, \\ &u(r,\theta,0) = r\cos(\theta). \end{aligned}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated).

Rättningsmall

(1p) Variabelseparera $u_t = \Delta u$ i polära koordinater (r, θ) och tidsvariabeln t.

$$\frac{T'}{T} = \frac{R'' + r^{-1}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

Låt $\lambda = T'/T$ och skriv

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} = \lambda r^2$$

eller, ekvivalent,

$$\frac{r^2R'' + rR'}{R} - \lambda r^2 = -\frac{\Theta''}{\Theta}.$$

(1p) Eftersom höger- och vänsterled beror på θ respektive r är detta också en separerad ekvation och både sidor måste vara konstant.

(1p) Vi argumenterar för att Θ ska vara 2π-periodisk. Det betyder att Θ'' = konstant gång Θ och $\Theta(-\pi) = \Theta(\pi)$ och $\Theta'(-\pi) = \Theta'(\pi)$. Vi löser den här ekvation (se boken kapitel 1 exempel med Rings of Saturn).

(1p) Vi skriver ner vår slutsats:

$$\Theta_n(\theta) = e^{in\theta}, \quad \Theta_n'' = -n^2\Theta_n, \quad n \in \mathbb{Z}.$$

(3p) Vi går igenom fallen $\lambda > 0$, $\lambda = 0$ och $\lambda < 0$. Om $\lambda > 0$ vi får den modifierade Besselekvationen. Lösningarna K går mot oändligheten när $r \to 0$, och I saknar nollställen för $r > 0$. Alltså kan inte $\lambda > 0$ ge några fysikaliska lösningar som samtidigt uppfyller randvillkoret vid $r = 3$. Säg att $\lambda = 0$ i stället. Då gäller att $r^2R''(r) + rR'(r) - n^2R(r) = 0$, Eulers ekvation. Lösningarna är $ar^n + br^{-n}$ för konstanter a, b om $n > 0$, eller $R(r) = a + b\log(r)$ ifall $n = 0$. Alltså kan inte heller $\lambda = 0$ ge några fysikaliska lösningar som samtidigt uppfyller randvillkoret vid $r = 3$. Antag $\lambda < 0$. Låt $\lambda = -\nu^2$. Vi erhåller Besselekvationen $r^2R''(r) + rR'(r) + (\nu^2r^2 - n^2)R(r) = 0$, som har lösningarna J_n eller Y_n. Lösningarna Y_n går mot ∞ när $r \to 0$, så vi utesluter dem. Randvillkoret $u(3, \theta, t) = 0$ ger $R(3) = 0$ och alltså $J_n(3\nu) = 0$. Det betyder att det finns lösningar för alla λ sådana att

$$\lambda = -\left(\frac{\pi_{n,k}}{3}\right)^2,$$

där $\pi_{n,k}$ är nollställe nummer k till J_n. Så

$$R_{n,k}(r) = J_n(\pi_{n,k} r/3).$$

(1p) Vi löser för T funktionen:

$$T' = \lambda T, \quad \lambda = -\nu^2, \quad \nu = \frac{\pi_{n,k}}{3},$$

altså

$$T_{n,k}(t) = c_{n,k} e^{-\lambda_{n,k} t}, \quad \lambda_{n,k} = \frac{(\pi_{n,k})^2}{9}.$$

(1p) Superposition!

$$u(r, \theta, t) = \sum_{k>0, n \in \mathbb{Z}} T_{n,k}(t) \Theta_n(\theta) R_{n,k}(r)$$

(2p) För att uppfylla begynnelsevillkoret ska vi lösa

$$u(r, \theta, 0) = \sum_{k>0, n \in \mathbb{Z}} T_{n,k}(0) \Theta_n(\theta) R_{n,k}(r) = r \cos \theta.$$

Så vi har

$$T_{n,k}(0) = c_{n,k} = \frac{\int_{-\pi}^{\pi} \int_0^3 r \cos \theta \overline{R_{n,k}(r) \Theta_n(\theta)} r dr d\theta}{\int_{-\pi}^{\pi} \int_0^3 |R_{n,k}(r) \Theta_n(\theta)|^2 r dr d\theta}.$$

5. Beräkna: (Compute):

$$\sum_{n=1}^{\infty} \frac{1}{4n^4 - n^2}$$

Hint: Hitta a och b så att

$$\frac{1}{4n^4 - n^2} = \frac{a}{n^2} + \frac{b}{4n^2 - 1}$$

och sedan andvänd tabellen av Fourierserier.

Solution:. $\rightarrow$ (2p) We follow the hint and find that

$$\frac{1}{4n^4 - n^2} = -\frac{1}{n^2} + \frac{4}{4n^2 - 1}.$$

Our series becomes:

$$\sum_{n=1}^{\infty} \frac{1}{4n^4 - n^2} = -\sum_{n=1}^{\infty} \frac{1}{n^2} + 4\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

→ (4p) The first sum is the Basel problem, and we know it to be $\frac{\pi^2}{6}$.

→ (4p) For the other series, we look at at Table 5.2 and see that

$$|\sin(x)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1} \quad \forall x \in (-\pi; \pi)$$

Plugging in $x = 0$, we see that

$$0 = |\sin(0)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(0)}{4n^2 - 1} = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

⇒ Our series therefore has the value:

$$\sum_{n=1}^{\infty} \frac{1}{4n^4 - n^2} = 2 - \frac{\pi^2}{6}$$

(10p)

6. Sök en begränsad lösning till problemet (find a bounded solution to this problem)

$$\begin{aligned} &u_t = u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = \frac{x}{x^2 + 2}. \end{aligned}$$

Om det dyker upp väldigt besvärliga integraler behöver ni inte beräkna dem, utan bara motivera varför de dyker upp och skriva ner dem korrekt. (If horrible Dark Souls integrals appear you don't need to calculate them just explain why they appear and make sure they're correct.).

Rättningsmall

(3p) Det här är värmeledningsekvationen för $x \in \mathbb{R}$ med begynnelsevillkor $u(x,0) = f(x)$ där $f \in L^2(\mathbb{R})$. Det osar Fouriertransform! Vi transformerar ekvationen:

$$\hat{u}_t = -\xi^2 \hat{u},$$

där

$$\hat{u}(\xi, t) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x,t)\, dx.$$

(1p Fouriertransform, 1p x variabel, 1p får rätt Fouriertransform av u_{xx}.)

(2p) Löser vi denna nya ekvation får vi, med någon godtycklig funktion av ξ, som vi kallar a,

$$\hat{u}(\xi, t) = a(\xi) e^{-\xi^2 t}.$$

(1p) Begynnelsevillkoret kan Fouriertransformeras, eftersom

$$\|f\|^2 = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+2)^2} dx = \frac{\pi}{2\sqrt{2}}, \quad f(x) = \frac{x}{x^2+2}.$$

Altså

$$a(\xi) = \hat{f}(\xi) \implies \hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 t}.$$

(4p) Om man bara skriver

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{-\xi^2 t} e^{ix\xi} d\xi$$

får man 2p. Om man hitta funktionen som har Fouriertransform $e^{-\xi^2 t}$ och skriva som en faltning

$$u(x,t) = \int_{\mathbb{R}} f(x-y) e^{-y^2/(4t)} (4\pi t)^{-1/2} dy = \int_{\mathbb{R}} f(y) e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} dy$$

får man 4p.

7. Lös problemet: (Solve the following problem):

$$\begin{aligned}
&u(x,t) + \int_0^t (t-s) u_{xx}(x,s) ds = 2, \ x, t > 0 \\
&u(x,0) = 0, \\
&u(0,t) = \sin(t).
\end{aligned}$$

Solution:

We *try* to solve this by applying the Laplace transform in the t-variable. The integral is a convolution integral. The Laplace transform of a convolution is the product of the transformed functions, so the Laplace transform of the integral is the Laplace transform of $u_{xx}(x,t)$ times the Laplace transform of t, which is $1/z^2$. The Laplace transform of the integral is the Laplace transform of $u_{xx}(x,t)$ divided by the transform variable z. We obtain, with $U(x,z)$ denoting the Laplace transform of u with respect to t:

$$U(x,z) + U_{xx}(x,t)/z^2 = 2/z$$

which we can write as

$$z^2 U(x,z) + U_{xx}(x,t) = 2z$$

This is an inhomogeous linear ordinary differential equation in x. Solve it with your favourite method to obtain that

$$U(x,z) = A(z)e^{izx} + B(z)e^{-izx} + \frac{2}{z}.$$

At this point we are stuck. The Laplace transform of anything transformable vanishes as the imaginary part of z tends to both ∞ and $-\infty$. So both of these exponentials are problematic in their own way. The coefficient functions can't help them because $x > 0$ can tend to infinity. So the only part that is okay is the particular solution $\frac{2}{z}$. Hence - sorry about this - but at least the point distribution is generous.

In case you are curious, this is how I first thought to grade the problem, given that it's messed up.

(a) (5p) Choosing to use Laplace transform methods. It doesn't matter in which variable or if you do it right or whatever. Just naming Laplace transform methods and/or writing some fancy hats gets you these points.

(b) 1p) for saying to do the transform in the t variable.

(c) (2p) for correctly transforming the pde under the LT (if it is partially correct you get 1 out of 2 points).

(d) (2p) if you found a particular solution to the ode (the $2/z$ part).

However, then I thought more about it, and consulted with some students, and decided that some people might have realized it was not working, and then tried other methods and/or eventually given up. So I decided that since it's unsolvable anyone who tried *anything whatsoever* gets 10 points. At least, the problem SCREAMS that you should try Laplace transform (that's what was intended), so it is reasonable that one would at least *try* that. So, it's just 10 points for trying, since the mistake was on our side.

8. Determine the *maximum* value of

$$\int_{-1}^{1} x^3 f(x) dx$$

among all continuous bounded real-valued functions in $[-1, 1]$ that satisfy

$$\int_{-1}^{1} |f(x)|^2 dx = 1, \quad \int_{-1}^{1} x f(x) dx = 0.$$

(Hint: expand f using the Legendre polynomial base and use Hilbert space facts about orthogonal bases of Hilbert spaces.).

Solution.

We can expand f in terms of Legendre polynomials, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

Then, we note that since $P_1(x) = x$ and $P_3(x) = (5x^3 - 3x)/2$, we have that $x^3 = 2P_3(x)/5 + 3P_1(x)/5$. Therefore, what we are asked to maximize is equal to

$$\int_{-1}^{1} x^3 f(x) \mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} \left(\frac{2P_3(x)}{5} + \frac{3P_1(x)}{5} \right) P_n(x) \mathrm{d}x$$

$$= \frac{2a_3 \|P_3(x)\|^2}{5} + \frac{3a_1 \|P_1(x)\|^2}{5} = \frac{4a_3}{35} + \frac{6a_1}{15}$$

where we used the orthogonality of the Legendre polynomials and that their norm is given by $\int_{-1}^{1} P_n(x)^2 \mathrm{d}x = 2/(2n+1)$.

Now, we use the constraints. First, note that

$$\int_{-1}^{1} x f(x) \mathrm{d}x = \int_{-1}^{1} P_1(x) f(x) \mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_1(x) P_n(x) \mathrm{d}x = \frac{2}{3} a_1 = 0,$$

meaning that we must set $a_1 = 0$.

Secondly,

$$\int_{-1}^{1} |f(x)|^2 \mathrm{d}x = \sum_{n=0}^{\infty} a_n^2 \int_{-1}^{1} P_n(x)^2 \mathrm{d}x = \sum_{n=0}^{\infty} a_n^2 \frac{2}{2n+1} = 1$$

We see that the only thing affecting the maximum value of the integral is a_3 (and a_1, but the first constraint takes care of it!), so all the other coefficients must be chosen in a way that makes a_3 as big as possible while keeping

$$\sum_{n=0}^{\infty} a_n^2 \frac{2}{2n+1} = 1.$$

The way to achieve this is to put everything on a_3 and put the rest of the remaining coefficients to 0. Indeed, if we opt to choose, say, a_4 to be something other than 0 we end up with the condition $\frac{2}{7}a_3^2 + \frac{2}{8}a_4^2 = 1$, meaning that we must make a_3 smaller to make sure that the condition hold. If a_3 is made smaller, the integral we are asked to *maximize* becomes... you know it, smaller, which we do not want! Therefore, we set all other coefficients than a_3 to 0, and get

$$\sum_{n=0}^{\infty} a_n^2 \frac{2}{2n+1} = a_3^2 \frac{2}{7} = 1$$

meaning that $a_3 = \sqrt{7/2}$, so that the maximum value of the integral is achieved by the function $f(x) = \sqrt{7/2}P_3(x)$ and is given by

$$\int_{-1}^{1} x^3 \sqrt{7/2} P_3(x) \mathrm{d}x = \frac{4}{35}\sqrt{\frac{7}{2}}.$$

(a) (1p) Writing down the expansion of $f(x)$ in the Legendre polynomial basis. All or nothing.

(b) (2p) Writing x^3 in terms of Legendre polynomials. 1 point if you realize that this is the way forward, but get the wrong expansion.

(c) (2p) Use the orthogonality of the Legendre polynomials as well as their norm to rewrite the problem as the problem of maximizing the coefficients only. Small mistakes, e.g., wrong norm of the Legendre polynomials but the correct reasoning with orthogonality gets you 1 point.

(d) (1p) Use the first constraint to get rid of a_1.

(e) (2p) Use the second constraint to set all other coefficients (everything except a_3) to 0. All or nothing.

(f) (1p) Use the second constraint to determine a_3.

(g) (1p) Deduce (you must write it down explicitly!) the maximal value of the integral and write down $f(x)$.

Lycka till! May the FourierForce be with you! ♡ Julie, Carl-Joar, Björn, Erik, & Kolya

2 Fun facts!

2.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

2.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with u the unknown function and p and g specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called μ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the Mμthod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital C. The solution is:

$$u(t) = \frac{\int_0^t (\mu(s)g(s)ds) + C}{\mu(t)}.$$

2.3 How to solve second order linear ODEs

Theorem 1 (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function u that depends on one variable, with constants b and c given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:

1. ***If** $b^2 \neq 4ac$**,** then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$**,** then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

Theorem 2 (Particular solution to linear second order ODEs)**.** *Assume that y_1 und y_2 are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

Then, if y_p is a particular solution of the inhomogeneous ODE, so that

$$L(y_p) = g,$$

then all solutions to $L(y) = g$ can be expressed as

$$c_1 y_1 + c_2 y_2 + y_p,$$

for y_1 and y_2 as above, for coefficients c_1 and c_2. One way to find a particular solution to the ODE

$$L(y) = g(t)$$

is to calculate

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

The Wronskian of y_1 and y_2, denoted by $W(y_1, y_2)$ above, is defined to be

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t).$$

2.4 Bessel facts

Definition 1 (The Bessel function J of order ν)**.** The Bessel function J of order ν is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Bessel function satisfies the Bessel equation:

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0.$$

The modified Bessel equation is satisfied by I_ν and K_ν

$$x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x).$$

For real values of ν the function $I_\nu(x) \neq 0$ for all $x > 0$. The function $K_\nu(x)$ tends to ∞ as $x \to 0$. The Γ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{1}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of ν, it is an entire function of $x \in \mathbb{C}$.

Theorem 3 (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the m^{th} positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n} r/L)\}_{m \geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0, L)$. Recall that this is the weighted $\mathcal{L}^2$ space on the interval $(0, L)$ with respect to the weight function r, so the scalar product

$$\langle f, g \rangle = \int_0^L f(r) \overline{g(r)} r dr.$$

More generally, for any $\nu \in \mathbb{R}$, and for any $L > 0$, the functions

$$\{J_\nu(\pi_m r/L)\}_{m \geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0, L)$, where above π_m denotes the m^{th} zero of the Bessel function J_ν. They have norms equal to

$$\int_0^L |J_\nu(\pi_m r/L)|^2 r dr = \frac{L^2}{2} \left(J_{\nu+1}(\pi_m)\right)^2.$$

Corollary 1 (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n} r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

are an orthogonal basis for $\mathcal{L}^2$ on the disk of radius L.

Theorem 4 (Bessel functions as bases in some other cases)**.** *Assume that $L > 0$. Let the weight function $w(x) = x$. Fix $\nu \in \mathbb{R}$. Then J'_ν has infinitely many positive zeros. Let*

$$\{\pi'_k\}_{k\geq 1}$$

be the positive zeros of J'_ν. Then we define

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu > 0, \quad k \geq 1.$$

In case $\nu = 0$, define further $\psi_0(x) = 1$. (If $\nu \neq 0$, then this case is omitted.) Then $\{\psi_k\}_{k\geq 1}$ for $\nu \neq 0$ is an orthogonal basis for $\mathcal{L}^2_w(0, L)$. For $\nu = 0$, $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w(0, L)$. Moreover the norm

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \geq 1, \quad ||\psi_0||_w^2 = \frac{L^{2\nu+2}}{2\nu+2}.$$

Next, fix a constant $c > 0$. Then there are infinitely many positive solutions of

$$\mu J'_\nu(\mu) + c J_\nu(\mu) = 0,$$

that can be enumerated as $\{\mu_k\}_{k\geq 1}$. Then

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}^2_w(0, L)$.

2.5 Orthogonal polynomials

Definition 2. The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right). \tag{2}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1, 1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

The first few Legendre polynomials are $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2 - 1)$, and $P_3 = \frac{1}{2}(5x^3 - 3x)$.

Definition 3. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}_2^2(\mathbb{R})$ with respect to the weight function e^{-x^2}. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

Definition 4. The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x}).$$

The Laguerre polynomials $\{L_n^\alpha\}_{n\geq 0}$ are an orthogonal basis for $\mathcal{L}_\alpha^2$ on $(0, \infty)$ with the weight function $\alpha(x) = x^\alpha e^{-x}$. Their norms squared,

$$||L_n^\alpha||^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

2.6 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

1.	$f(x) = x$	$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}$.
2.	$f(x) = \lvert x\rvert$	$\frac{\pi}{2} + \sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$
3.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$	$\frac{\pi}{4} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\frac{(-1)^{n+1}}{2in} + \frac{(-1)^n-1}{2\pi n^2}\right] e^{inx}$
4.	$f(x) = \sin^2(x)$	$\frac{1}{2} - \frac{1}{4}\left(e^{2ix} + e^{-2ix}\right)$
5.	$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{2}{i\pi} \sum_{n\geq 1} \frac{e^{(2n-1)ix} - e^{-(2n-1)ix}}{2n-1}$
6.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{1}{2} + \sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{i\pi(2n-1)}$
7.	$f(x) = \lvert\sin(x)\rvert$	$\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1}$
8.	$f(x) = \lvert\cos(x)\rvert$	$\frac{2}{\pi} - \frac{2}{\pi}\sum_{n\geq 1} \frac{(-1)^n\left[e^{inx} + e^{-inx}\right]}{4n^2-1}$
9.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$	$\frac{1}{\pi} - \frac{1}{\pi}\sum_{n\geq 1} \frac{e^{2inx} + e^{-2inx}}{4n^2-1} + \frac{1}{4i}\left(e^{ix} - e^{-ix}\right)$
10.	$f(x) = x^2$	$\frac{\pi^2}{3} + 2\sum_{n\geq 1} \frac{(-1)^n(e^{inx} + e^{-inx})}{n^2}$
11.	$f(x) = x(\pi - \lvert x\rvert)$	$\frac{4}{i\pi}\sum_{n\geq 1} \frac{e^{i(2n-1)x} - e^{-i(2n-1)x}}{(2n-1)^3}$
12.	$f(x) = e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b-in} e^{inx}$
13.	$f(x) = \sinh x$	$\frac{\sinh(\pi)}{i\pi}\sum_{n\geq 1} \frac{(-1)^{n+1} n}{n^2+1}\left[e^{inx} - e^{-inx}\right]$

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

1.	$f(x) = x$	$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx).$
2.	$f(x) = \|x\|$	$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2}$
3.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$	$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2} + \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sin(nx)$
4.	$f(x) = \sin^2(x)$	$\frac{1}{2} - \frac{1}{2}\cos(2x)$
5.	$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{4}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
6.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$	$\frac{1}{2} + \frac{2}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
7.	$f(x) = \|\sin(x)\|$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1}$
8.	$f(x) = \|\cos(x)\|$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1}\frac{(-1)^n\cos(2nx)}{4n^2-1}$
9.	$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin(x), & 0 < x < \pi. \end{cases}$	$\frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1} + \frac{1}{2}\sin(x)$
10.	$f(x) = x^2$	$\frac{\pi^2}{3} + 4\sum_{n\geq 1}\frac{(-1)^n\cos(nx)}{n^2}$
11.	$f(x) = x(\pi - \|x\|)$	$\frac{8}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{(2n-1)^3}$
12.	$f(x) = e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\left(\frac{1}{b} + \sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2}\left[2b\cos(nx) - 2n\sin(nx)\right]\right)$
13.	$f(x) = \sinh x$	$\frac{2\sinh(\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}n}{n^2+1}\sin(nx)$

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1, \cos(nx), \sin(nx)\}_{n\geq 1}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x \notin (-\pi,\pi)$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\lvert\xi\rvert}$
$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \lvert x\rvert < a \\ 0 & \lvert x\rvert > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \lvert\xi\rvert < a \\ 0 & \lvert\xi\rvert > a \end{cases}$

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a > 0$ and $c \in \mathbb{R}$.

1.	$\Theta(t)f(t)$	$\widetilde{f}(z)$
2.	$\Theta(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
3.	$e^{ct}\Theta(t)f(t)$	$\widetilde{f}(z-c)$
4.	$\Theta(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
5.	$\Theta(t)f'(t)$	$z\widetilde{f}(z)-f(0)$
6.	$\Theta(t)f^{(k)}(t)$	$z^k\widetilde{f}(z)-\sum_0^{k-1} z^{k-1-j}f^{(j)}(0)$
7.	$\Theta(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
8.	$\Theta(t)tf(t)$	$-\widetilde{f}'(z)$
9.	$\Theta(t)t^{-1}f(t)$	$\int_z^\infty \widetilde{f}(w)dw$
10.	$\Theta f * \Theta g(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
11.	$\Theta(t)t^\nu e^{ct}$	$\Gamma(\nu+1)(z-c)^{-\nu-1}$
12.	$\Theta(t)(t+a)^{-1}$	$e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$
13.	$\Theta(t)\sin(ct)$	$\frac{c}{z^2+c^2}$
14.	$\Theta(t)\cos(ct)$	$\frac{z}{z^2+c^2}$

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a>0$ is constant and $c \in \mathbb{C}$.

15.	$\Theta(t)\sinh(ct)$	$\frac{c}{z^2-c^2}$
16.	$\Theta(t)\cosh(ct)$	$\frac{z}{z^2-c^2}$
17.	$\Theta(t)\sin(\sqrt{at})$	$\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$
18.	$\Theta(t)t^{-1}\sin(\sqrt{at})$	$\pi\,\mathrm{erf}(\sqrt{a/(4z)})$
19.	$\Theta(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$
20.	$\Theta(t)\,\mathrm{erf}(at)$	$z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$
21.	$\Theta(t)\,\mathrm{erf}(\sqrt{t})$	$(z\sqrt{z+1})^{-1}$
22.	$\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$	$((z-1)\sqrt{z})^{-1}$
23.	$\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$	$z^{-1}e^{-a\sqrt{z}}$
24.	$\Theta(t)t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$
25.	$\Theta(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
26.	$\Theta(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
27.	$\Theta(t)t^{\nu}J_{\nu}(t)$	$2^{\nu}\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$
28.	$\Theta(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.