Fourieranalys MVE030 och Fourier Metoder MVE290 2023.augusti.22

Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.

Maximalt antal poäng: 80.

Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & miniräknare som helst.

Examinator: Julie Rowlett.

Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

1 Uppgifter

1. Plancharel Theorem Please! Antar att f och g är i $\mathcal{L}^2(\mathbb{R})$. Bevisa att gäller:

$$\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle.$$

Observera att $\langle f, g \rangle$ är skalarprodukten i Hilbertrummet $\mathcal{L}^2(\mathbb{R})$.

(10p)

- 2. Good things come in threes! Antar att $\{\phi_n\}_{n\geq 0}$ är ortonormala i ett Hilbertrum *H*. Bevisa att följande är äquivalenta:
 - (a) Om $f \in H$ sedan gäller att $f = \sum_{n>0} \langle f, \phi_n \rangle \phi_n$.
 - (b) $||f||^2 = \sum_{n \ge 0} |\langle f, \phi_n \rangle|^2$.
 - (c) Om $v \in H$ och $\langle v, \phi_n \rangle = 0 \forall n$ måste v = 0.

(10 p)

3. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(2t)\cos(2x), & 0 < t, -\pi < x < \pi \\ u(-\pi,t) = u(\pi,t), & t > 0, \\ u_x(-\pi,t) = u_x(\pi,t), & t > 0, \\ u(x,0) = |x| - \pi, & x \in [-\pi,\pi]. \end{cases}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration but need not be calculated). (10 p) 4. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t = \Delta u, & 0 < z < H, -\pi < \theta < \pi, \, 0 < r < L, \, 0 < t, \\ u_z(r, 0, \theta, t) = 0, & u(r, H, \theta, t) = 0, \\ u(L, z, \theta, t) = 0, & \\ u(r, z, \theta, 0) = 45, \end{cases}$$

där $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$. (Hint: you may wish to use cylindrical coordinates for which $\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}$. As with the previous exercise, certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.) (10 p)

5. Beräkna: (Compute):

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{\pi + n^2}.$$

(10p)

6. Lös problemet: (Solve the following problem):

$$\begin{cases} u_{xx}(x,y) - u_{yy}(x,y) = 0, & x \in \mathbb{R}, \ y > 0, \\ u(x,0) = f(x) \in \mathcal{L}^2(\mathbb{R}), \\ u_y(x,0) = 0. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

(10 p)

7. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t, x > 0, \\ u(0,t) = e^{7t}, & t > 0, \\ u(x,0) = 0, & x > 0. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

(10 p)

8. Hitta et polynomet p(x) av högst grad 45 som minimeras (Find the polynomial p(x) of at most degree 45 which minimizes the following integral):

$$\int_{-4}^{4} |e^{2\cos(x)} - p(x)|^2 dx.$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration but need not be calculated). (10p)

Lycka till! May the FourierForce be with you
! \heartsuit Julie, Carl-Joar, Jan, Erik, & Kolya

2 Fun and possibly helpful facts!

2.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

2.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with u the unknown function and p and g specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called μ known as the *integrating factor*,

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the $M\mu$ thod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital C. The solution is:

$$u(t) = \frac{\int_0^t (\mu(s)g(s)ds) + C}{\mu(t)}$$

2.3 How to solve second order linear ODEs

Theorem 1 (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs). Consider the ODE, for the unknown function u that depends on one variable, with constants b and c given in the equation:

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:

1. If $b^2 \neq 4ac$, then a basis of solutions is

$$\{e^{r_1x}, e^{r_2x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

2. If $b^2 = 4ac$, then a basis of solutions is

$$\{e^{rx}, xe^{rx}\}, with r = -\frac{b}{2a}.$$

Theorem 2 (Particular solution to linear second order ODEs). Assume that y_1 und y_2 are a basis of solutions to the ODE

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

Then a solution to the ODE

$$L(y) = g(t)$$

is given by

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

The Wronskian of y_1 and y_2 , denoted by $W(y_1, y_2)$ above, is defined to be

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

2.4 Definition of a regular SLP (by request)

A regular Sturm Liouville problem is to find all solutions to

$$L(f(x)) + \lambda w(x)f(x) = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$
(1)

The eigenvalues of the SLP are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy (1). The constituents in the problem in (1) must satisfy the following conditions in order for the problem to be a regular SLP:

- 1. The function w, known as a *weight function*, must be both positive and continuous on the interval [a, b].
- 2. The differential operator L must be of the form

$$L(f(x)) = (r(x)f'(x))' + p(x)f(x).$$

Above r and p are *specified* real valued functions. The functions r, r', and p must be continuous, and r must be positive on [a, b].

3. The boundary conditions must be equations of the form:

$$B_i(f) = \alpha_i f(a) + \alpha'_i f'(a) + \beta_i f(b) + \beta'_i f'(b) = 0, \quad i = 1, 2.$$
(2)

Above, the coefficients $\alpha_i, \alpha'_i, \beta_i, \beta'_i$ must be fixed complex numbers. Moreover, the boundary conditions must guarantee that for any unknown functions ϕ and ψ if one only knows that they both satisfy (2), that is enough to guarantee that

$$r(b)\left(\overline{\psi(b)}\phi'(b) - \overline{\psi'(b)}\phi(b)\right) - r(a)\left(\overline{\psi(a)}\phi'(a) - \overline{\psi'(a)}\phi(a)\right) = 0.$$
(3)

We note that the unknown functions ϕ and ψ in (3) are only assumed to satisfy (2); they need *not* necessarily solve the equation (1).

2.5 Bessel function facts

Definition 1 (The Bessel function J of order ν). The Bessel function J of order ν is defined to be the series

$$J_{\nu}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Γ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0.$$
(4)

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of ν , it is an entire function of $x \in \mathbb{C}$.

Theorem 3 (Recurrence Formulas). For all x and ν we have

$$(x^{-\nu}J_{\nu}(x))' = -x^{-\nu}J_{\nu+1}(x),$$

$$(x^{\nu}J_{\nu}(x))' = x^{\nu}J_{\nu-1}(x),$$

$$xJ'_{\nu}(x) - \nu J_{\nu}(x) = -xJ_{\nu+1}(x),$$

$$xJ'_{\nu}(x) + \nu J_{\nu}(x) = xJ_{\nu-1}(x),$$

$$xJ_{\nu-1}(x) + xJ_{\nu+1}(x) = 2\nu J_{\nu}(x),$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_{\nu}(x).$$

Theorem 4 (Bessel functions as an orthogonal base). Fix L > 0. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the m^{th} positive zero of the Bessel function $J_{|n|}$. Then the functions

$$\{J_{|n|}(\pi_{m,n}r/L)\}_{m\geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0,L)$. Recall that this is the weighted \mathcal{L}^2 space on the interval (0,L) with respect to the weight function r, so the scalar product

$$\langle f,g\rangle = \int_0^L f(r)\overline{g(r)}rdr.$$

More generally, for any $\nu \in \mathbb{R}$, and for any L > 0, the functions

$$\{J_{\nu}(\pi_m r/L)\}_{m\geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0,L)$, where above π_m denotes the m^{th} zero of the Bessel function J_{ν} . They have norms equal to

$$\int_0^L |J_{\nu}(\pi_m r/L)|^2 r dr = \frac{L^2}{2} \left(J_{\nu+1}(\pi_m) \right)^2.$$

Corollary 1 (Orthogonal base for functions on a disk). The functions

$$\{J_{|n|}(\pi_{m,n}r/L)e^{in\theta}\}_{m\geq 1,n\in\mathbb{Z}}$$

are an orthogonal basis for \mathcal{L}^2 on the disk of radius L.

Theorem 5 (Bessel functions as bases in some other cases). Assume that L > 0. Let the weight function w(x) = x. Fix $\nu \in \mathbb{R}$. Then J'_{ν} has infinitely many positive zeros. Let

$$\{\pi'_k\}_{k\geq 1}$$

be the positive zeros of J'_{ν} . Then we define

 $\psi_k(x) = J_{\nu}(\pi_k x/L), \quad \nu > 0, \quad k \ge 1.$

In case $\nu = 0$, define further $\psi_0(x) = 1$. (If $\nu \neq 0$, then this case is omitted.) Then $\{\psi_k\}_{k\geq 1}$ for $\nu \neq 0$ is an orthogonal basis for $\mathcal{L}^2_w(0,L)$. For $\nu = 0$, $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w(0,L)$. Moreover the norm

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \ge 1, \quad ||\psi_0||_w^2 = \frac{L^{2\nu+2}}{2\nu+2}.$$

Next, fix a constant c > 0. Then there are infinitely many positive solutions of

$$\mu J_{\nu}'(\mu) + c J_{\nu}(\mu) = 0,$$

that can be enumerated as $\{\mu_k\}_{k>1}$. Then

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k \ge 1}$$

is an orthogonal basis for $\mathcal{L}^2_w(0,L)$.

2.6 Orthogonal polynomials

Definition 2. The Legendre polynomials, are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2 - 1)^n \right).$$
(5)

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1,1)$, and

$$||P_n||^2 = \frac{2}{2n+1}$$

Definition 3. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}_2^2(\mathbb{R})$ with respect to the weight function e^{-x^2} . Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

Definition 4. The Laguerre polynomials,

$$L_n^{\alpha}(x) = \frac{x^{-\alpha}e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n}e^{-x}).$$

The Laguerre polynomials $\{L_n^{\alpha}\}_{n\geq 0}$ are an orthogonal basis for \mathcal{L}_{α}^2 on $(0,\infty)$ with the weight function $\alpha(x) = x^{\alpha}e^{-x}$. Their norms squared,

$$||L_n^{\alpha}||^2 = \frac{\Gamma(n+\alpha+1)}{n!}$$

2.7 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of f(x) on $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal f(x) for $x \notin (-\pi,\pi)$.

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1, \cos(nx), \sin(nx)\}_{n\geq 1}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of f(x) on $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does not equal f(x) for $x \notin (-\pi,\pi)$.

	â
f(x)	$\hat{f}(\xi)$
f(x-c)	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi - c)$
$\int f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
f'(x)	$i\xi \hat{f}(\xi)$
xf(x)	$i(\hat{f})'(\xi)$
(f * g)(x)	$\hat{f}(\xi)\hat{g}(\xi)$
f(x)g(x)	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2 + a^2)^{-1}$	$(\pi/a)e^{-a \xi }$
$e^{-a x }$	$2a(\xi^2 + a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & x < a \\ 0 & x > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi \chi_a(\xi) = \begin{cases} \pi & \xi < a \\ 0 & \xi > a \end{cases}$

Table 3: Above the function is on the left, its Fourier transform on the right. Here a > 0 and $c \in \mathbb{R}$.

1.	$\Theta(t)f(t)$	$\widetilde{f(z)}$
2.	$\Theta(t-a)f(t-a)$	$e^{-az}\widetilde{f(z)}$
3.	$e^{ct}\Theta(t)f(t)$	$\widetilde{f(z-c)}$
4.	$\Theta(t) f(at)$	$a^{-1}\widetilde{f(a^{-1}z)}$
5.	$\Theta(t) f'(t)$	$z\widetilde{f(z)} - f(0)$
6.	$\Theta(t) f^{(k)}(t)$	$\frac{1}{z^{k} \widetilde{f(z)} - \sum_{0}^{k-1} z^{k-1-j} f^{(j)}(0)}$
7.	$\Theta(t) \int_{0}^{t} f(s) ds$	$z^{-1}\widetilde{f(z)}$
8.	$\Theta(t)tf(t)$	$-\widetilde{f}'(z)$
9.	$\Theta(t)t^{-1}f(t)$	$\int_{-\infty}^{\infty} \widetilde{f(w)} dw$
10.	$\Theta f * \Theta q(t)$	$\widetilde{f}(z)\widetilde{q}(z)$
11.	$\Theta(t)t^{\nu}e^{ct}$	$\Gamma(\nu+1)(z-c)^{-\nu-1}$
12.	$\Theta(t)(t+a)^{-1}$	$e^{az}\int_{az}^{\infty} \frac{e^{-u}}{u}du$
13.	$\Theta(t)\sin(ct)$	$\frac{c}{z^2+c^2}$
14.	$\Theta(t)\cos(ct)$	$\frac{z}{z^2+c^2}$

Table 4: Above, the function is on the left, its Laplace transform on the right. Here a > 0 is constant and $c \in \mathbb{C}$.

		1
15.	$\Theta(t)\sinh(ct)$	$\frac{c}{z^2-c^2}$
16.	$\Theta(t)\cosh(ct)$	$\frac{z}{z^2-c^2}$
17.	$\Theta(t)\sin(\sqrt{at})$	$\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$
18.	$\Theta(t)t^{-1}\sin(\sqrt{at})$	$\pi \operatorname{erf}(\sqrt{a/(4z)})$
19.	$\Theta(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\operatorname{erfc}(z/2a)$
20.	$\Theta(t) \operatorname{erf}(at)$	$z^{-1}e^{z^2/(4a^2)}\operatorname{erfc}(z/(2a))$
21.	$\Theta(t) \operatorname{erf}(\sqrt{t})$	$(z\sqrt{z+1})^{-1}$
22.	$\Theta(t)e^t \operatorname{erf}(\sqrt{t})$	$((z-1)\sqrt{z})^{-1}$
23.	$\Theta(t) \operatorname{erfc}(a/(2\sqrt{t}))$	$z^{-1}e^{-a\sqrt{z}}$
24.	$\Theta(t)t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}e^{a/(4z)}\operatorname{erfc}(\sqrt{a/(4z)})$
25.	$\Theta(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
26	$\Theta(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
20.	$\Theta(t)t^{\nu}I(t)$	$\frac{2\omega}{2^{\nu}\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}}$
28.	$\Theta(t)J_0(\sqrt{t})$	$\frac{z^{-1}e^{-1/(4z)}}{z^{-1}}$

Table 5: Above, the function is on the left, its Laplace transform on the right. Here a > 0 is constant and $c \in \mathbb{C}$.

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Observera att $\langle f,g\rangle$ är skalarprodukten i Hilbertrummet $\mathcal{L}^2(\mathbb{R}).$ (10p)

(a) (2p) for writing the scalar product correctly:

$$\langle f,g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

(b) (2p) for using the FIT to say that

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi.$$

(c) (2p) for inserting this into the scalar product

$$2\pi \langle f,g \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi \overline{g(x)} dx.$$

(d) (2p) for moving the complex conjugate out

$$= \int_{\mathbb{R}} \hat{f}(\xi) \overline{\int_{\mathbb{R}} e^{-ix\xi} g(x) dx} d\xi$$

(e) (2p) for recognizing the Fourier transform of g in this as well as the scalar product

$$= \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \langle \hat{f}, \hat{g} \rangle.$$

- 2. Good things come in threes! Antar att $\{\phi_n\}_{n\geq 0}$ är ortonormala i ett Hilbertrum *H*. Bevis att följande är äquivalenta:
 - (a) Om $f \in H$ sedan gäller att $f = \sum_{n \ge 0} \langle f, \phi_n \rangle \phi_n$.
 - (b) $||f||^2 = \sum_{n \ge 0} |\langle f, \phi_n \rangle|^2$.
 - (c) Om $v \in H$ och $\langle v, \phi_n \rangle = 0 \forall n$ måste v = 0.

(10 p)

- (a) (3p) Assume the first statement. Then use the infinite dimensional Pythagorean theorem to prove the second statement.
- (b) (3p) Assume the second statement. Then observe that for such a v, its norm is zero, so it must be zero.
- (c) (4p) Assume the third statement. Then show that for $g = \sum_{n\geq 0} \langle f, \phi_n \rangle \phi_n$ we have $\langle f - g, \phi_n \rangle = 0$ for all *n*. Use this to conclude by the third statement that f - g = 0 which is equivalent to f = g.
- 3. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(2t)\cos(2x), & 0 < t, -\pi < x < \pi \\ u(-\pi,t) = u(\pi,t), & t > 0, \\ u_x(-\pi,t) = u_x(\pi,t), & t > 0, \\ u(x,0) = |x| - \pi, & x \in [-\pi,\pi]. \end{cases}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration but need not be calculated). (10 p)

(a) (1p) SLPs are the keys to solving inhomogeneous pde's. Even if you do nothing else, this rhyme is worth one point. Equivalently, you get a point if you set up the SLP to solve

$$X'' + \lambda X = 0, \quad X(-\pi) = X(\pi), \quad X'(-\pi) = X'(\pi).$$

(or written equivalently).

(b) (2p) Solve this SLP. You should obtain (see the rings of Saturn example in Chapter 1 of the textbook for the derivation of these solutions)

$$X_n(x) = e^{inx}, \quad n \in \mathbb{Z}.$$

One point for the correct function and one point for the correct range of n. Note that you could also have $\{\sin(nx)\}_{n\geq 1}$ together with $\cos(nx)\}_{n\geq 0}$. This is equivalent and correct.

(c) (1p) Set up the solution you seek to be a series

$$u(x,t) = \sum_{n \in \mathbb{Z}} T_n(t) X_n(x),$$

where we will need to solve for the T_n functions using the inhomogeneous pde together with the initial condition.

(d) (2p) Expand the inhomogeneity in terms of the X_n base:

$$\sin(2t)\cos(2x) = \sum_{n \in \mathbb{Z}} \sin(2t) \frac{\langle \cos(2x), X_n \rangle}{||X_n||^2} X_n(x) = \sum_{n \in \mathbb{Z}} c_n \sin(2t) X_n(x),$$

with

$$\langle \cos(2x), X_n \rangle = \int_{-\pi}^{\pi} \cos(2x) \overline{X_n(x)} dx, \quad ||X_n||^2 = \int_{-\pi}^{\pi} |X_n(x)|^2 dx.$$

It is okay if you leave these integrals like this (and don't calculate them) as long as you have correctly defined the scalar product and the norm squared. Each of these correctly defined is worth one point. It is possible to simplify life by observing that

$$\cos(2x) = \frac{e^{i2x} + e^{-i2x}}{2},$$

and the functions e^{inx} are orthogonal. So, the coefficients

$$c_n = \begin{cases} \frac{1}{2}, & n = \pm 2, \\ 0, & n \neq \pm 2. \end{cases}$$

(e) (1p) Plug u into the heat equation (correctly) to obtain

$$u_t - u_{xx} = \sum_{n \in \mathbb{Z}} (T'_n(t) + n^2 T_n(t)) X_n(x).$$

(f) (1p) Set up the equation for T_n to solve

$$T'_{n}(t) + n^{2}T_{n}(t) = c_{n}\sin(2t).$$

(g) (1p) Set up the correct initial condition

$$\sum_{n \in \mathbb{Z}} T_n(0) X_n(x) = |x| - \pi = \sum_{n \in \mathbb{Z}} C_n X_n(x),$$

with

$$C_n = \frac{\langle |x| - \pi, X_n \rangle}{||X_n||^2}.$$

(h) (1p) Solve the ODE for $T_n(t)$. The method of integrating factor will give you

$$e^{-n^2t} \left[\int_0^t e^{n^2s} c_n \sin(2s) ds + C_n \right].$$

4. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t = \Delta u, & 0 < z < H, \ -\pi < \theta < \pi, \ 0 < r < L, \ 0 < t, \\ u_z(r, 0, \theta, t) = 0, & u(r, H, \theta, t) = 0, \\ u(L, z, \theta, t) = 0, & u(r, z, \theta, 0) = 45, \end{cases}$$

där $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$. (Hint: you may wish to use cylindrical coordinates for which $\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}$. As with the previous exercise, certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated.) (10 p)

- (a) (1p) Separate variables in polar coordinates.
- (b) (1p) Correctly apply the pde:

$$T'R\Theta Z = TR''\Theta Z + r^{-1}TR'\Theta Z + r^{-2}T\Theta''RZ + Z''R\Theta T.$$

(c) (1p) Separate variables to solve for either Z or Θ first. Either one works. For Theta you will get the SLP

$$\Theta'' = \text{ constant times } \Theta, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi).$$

Then as in the previous exercise the solutions are:

$$\Theta_n(\theta) = e^{in\theta}.$$

Note that you could also have $\{\sin(nx)\}_{n\geq 1}$ together with $\cos(nx)\}_{n\geq 0}$. This is equivalent and correct. (d) (1p) Next solve for Z. You will get the SLP

$$Z'' = \text{ constant times } Z, \quad Z'(0) = 0 = Z(H).$$

The solutions are (up to constant multiples)

$$Z_n(z) = \cos((2n+1)\pi z/(2H)), \quad n \ge 0, n \in \mathbb{Z}.$$

(e) (1p) Next solve for R. To do this we re-arrange to get

$$\frac{T'}{T} = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} \iff \frac{T'}{T} - \frac{Z''}{Z} = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} = \mu$$

for some constant μ . We then re-arrange the right side to get

$$r^2 \frac{R''}{R} + r \frac{R'}{R} - r^2 \mu = -\frac{\Theta''}{\Theta}.$$

For Θ_n this right side is n^2 so we have

$$r^{2}\frac{R''}{R} + r\frac{R'}{R} - r^{2}\mu = n^{2} \iff r^{2}R'' + rR' + (-r^{2}\mu - n^{2})R = 0.$$

- (f) (1p) Recognize that R satisfies this equation if and only if F(x) with $x = r\sqrt{|\mu|}$ satisfies either the modified Bessel equation if $\mu > 0$ or the Bessel equation if $\mu < 0$ or an Euler equation if $\mu = 0$.
- (g) (1p) Throw away the solutions to the modified Bessel and Euler equations because they either are not physical or cannot satisfy the boundary condition R(L) = 0.
- (h) (1p) Find the R part of the solution to be

$$R_{n,k}(r) = J_{|n|}(r\pi_{n,k}/L), \quad \mu_{n,k} = -\frac{\pi_{n,k}^2}{L^2}$$

where $\pi_{n,k}$ is the k^{th} positive zero of $J_{|n|}$. If you only do this in the case n = 0 it is correct, because in the end the only terms in the solution that will be non-zero are the terms with J_0 . So if you reduced to that case here and above it is fine. If you didn't though, it is also still correct.

(i) (1p) Use everything to now solve for the T function. Recall that Z_m satisfies $Z_m'' = (2m + 1)^2 - 2$

$$\frac{Z_m''}{Z_m} = -\frac{(2m+1)^2 \pi^2}{4H^2}$$

we have

$$\frac{T'}{T} - \frac{Z''}{Z} = \mu = -\frac{\pi_{n,k}^2}{L^2} \implies T'(t) = \lambda_{n,m,k} T(t), \quad \lambda_{n,m,k} = -\frac{\pi_{n,k}^2}{L^2} - \frac{(2m+1)^2 \pi^2}{4H^2}$$
so
$$T_{n,m,k}(t) = c_{n,m,k} e^{\lambda_{n,m,k} t}.$$

(j) (1p) Present your solution and define what the coefficients are:

$$u(r, z, \theta, t) = \sum_{n \in \mathbb{Z}, m \ge 0, k \ge 1} T_{n,m,k}(t) \Theta_n(\theta) Z_m(z) R_{n,k}(r),$$

with

$$c_{n,m,k} = \frac{\int_0^L \int_0^H \int_{-\pi}^{\pi} 45 \overline{\Theta_n(\theta) Z_m(z) R_{n,k}(r)} d\theta dz r dr}{\int_0^L \int_0^H \int_{-\pi}^{\pi} |\Theta_n(\theta) Z_m(z) R_{n,k}(r)|^2 d\theta dz r dr}.$$

5. Beräkna: (Compute):

$$\lim_{N \to \infty} \sum_{n = -N}^{N} \frac{1}{\pi + n^2}$$

We are rather lucky because we have been generously given a table that says that the trig Fourier series of the function e^{bx} in $\mathcal{L}^2(-\pi,\pi)$ is

$$\frac{\sinh(b\pi)}{\pi} \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{b - in} e^{inx}.$$

It is worth a whopping 5 points to identify a function whose trig Fourier series can be used to compute this series. This is pretty much hit or miss - either the function you choose *can* be used to calculate the series or it *cannot* (meaning there is no way to make the function you choose work).

Parseval method: With the function that I chose, I use the Parseval equality or equivalently the infinite dimensional Pythagorean theorem to get:

$$(1p) ||e^{bx}||^{2} = \int_{-\pi}^{\pi} e^{2bx} dx = \frac{e^{2b\pi} - e^{-2b\pi}}{2b} = \frac{\sinh(2b\pi)}{b}$$
$$(1p) = \sum_{n \in \mathbb{Z}} \left\| \frac{\sinh(b\pi)}{\pi} \frac{(-1)^{n}}{b - in} e^{inx} \right\|^{2}$$

$$(1p) = \sum_{n \in \mathbb{Z}} \frac{\sinh(b\pi)^2}{\pi^2} \frac{1}{b^2 + n^2} 2\pi$$
$$= \frac{2\sinh(b\pi)^2}{\pi} \sum_{n \in \mathbb{Z}} \frac{1}{b^2 + n^2}$$

One point for setting this equal to the norm on the other side:

$$(1p)\,\frac{\sinh(2b\pi)}{b} = 2\frac{\sinh(b\pi)^2}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{b^2+n^2}\iff \frac{\pi\sinh(2b\pi)}{2b\sinh(b\pi)^2} = \sum_{n\in\mathbb{Z}}\frac{1}{b^2+n^2}.$$

Setting $b = \sqrt{\pi}$ we get

$$(1p)\,\frac{\sqrt{\pi}\sinh(2\pi\sqrt{\pi})}{2\sinh(\pi\sqrt{\pi})^2} = \sum_{n\in\mathbb{Z}}\frac{1}{\pi+n^2}.$$

Pointwise convergence of trig Fourier series method:

(1p) For choosing the correct point and that is $x = \pi$ or $x = -\pi$. With either of these the series becomes

$$(1p) \frac{\sinh(b\pi)}{\pi} \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{b - in} e^{\pm in\pi} = \frac{\sinh(b\pi)}{\pi} \sum_{n \in \mathbb{Z}} \frac{1}{b - in}$$
$$\frac{\sinh(b\pi)}{\pi} \left[\frac{1}{b} + 2b \sum_{n \ge 1} \frac{1}{b^2 + n^2} \right].$$

Two points for using the theorem correctly to say that this is

$$(2p) \, \frac{e^{\pi b} + e^{-\pi b}}{2} = \cosh(b\pi).$$

Then one last point for doing the arithmetic to eek out the desired value with $b = \sqrt{\pi}$:

$$\frac{1}{b} \left(\cosh(b\pi) \frac{\pi}{\sinh(b\pi)} - \frac{1}{b} \right) = 2 \sum_{n \ge 1} \frac{1}{b^2 + n^2}$$
$$\implies \sum_{n \in \mathbb{Z}} \frac{1}{b^2 + n^2} = \frac{1}{b^2} + 2 \sum_{n \ge 1} \frac{1}{b^2 + n^2} = \frac{1}{b^2} + \frac{1}{b} \left(\cosh(\pi b) \frac{\pi}{\sinh(b\pi)} - \frac{1}{b} \right)$$
$$= \frac{\pi \cosh(b\pi)}{b \sinh(b\pi)}.$$

Setting $b = \sqrt{\pi}$ we get

$$=\frac{\sqrt{\pi}\cosh(\pi\sqrt{\pi})}{\sinh(\pi\sqrt{\pi})}.$$

If you are concerned that this doesn't look like the answer from the previous method, note that the doubling formula for the hyperbolic sine gives

$$\sinh(2\pi\sqrt{\pi}) = 2\sinh(\pi\sqrt{\pi})\cosh(\pi\sqrt{\pi}),$$

so our first answer

$$\frac{\sqrt{\pi}\sinh(2\pi\sqrt{\pi})}{2\sinh(\pi\sqrt{\pi})^2} = \frac{2\sqrt{\pi}\sinh(\pi\sqrt{\pi})\cosh(\pi\sqrt{\pi})}{2\sinh(\pi\sqrt{\pi})^2} = \frac{\sqrt{\pi}\cosh(\pi\sqrt{\pi})}{\sinh(\pi\sqrt{\pi})}$$

So indeed our answers match. I would be super impressed if anybody solved this BOTH ways just to be totally sure they are right... I have NEVER seen anyone do that - but hope springs eternal.

6. Lös problemet: (Solve the following problem):

$$\begin{cases} u_{xx}(x,y) - u_{yy}(x,y) = 0, & x \in \mathbb{R}, y > 0, \\ u(x,0) = f(x) \in \mathcal{L}^2(\mathbb{R}), \\ u_y(x,0) = 0. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

- (a) (4p) Recognizing to use the Fourier transform in the x variable. (2p) for Fourier transform and (2p) for x variable.
- (b) (1p) Using the properties of the Fourier transform to transform the pde:

$$\hat{u}_{xx}(\xi, y) - \hat{u}_{yy}(\xi, y) = 0 \iff \hat{u}_{yy}(\xi, y) = -\xi^2 \hat{u}(\xi, y).$$

(c) (1p) Solving this for

$$\hat{u}(\xi, y) = a(\xi)e^{iy\xi} + b(\xi)e^{-iy\xi}.$$

(d) (1p) Using the boundary condition to say that

$$\hat{u}(\xi, 0) = a(\xi) + b(\xi) = \hat{f}(\xi).$$

(e) (1p) Use the other boundary condition to say that

$$\hat{u}_y(\xi,0) = i\xi a(\xi) - i\xi b(\xi) = 0 \implies a(\xi) = b(\xi).$$

(f) (1p) Conclude that

$$\hat{u}(\xi, y) = \frac{1}{2}\hat{f}(\xi)(e^{iy\xi} + e^{-iy\xi}) = \cos(\xi y)\hat{f}(\xi).$$

(g) (1p) Invert the Fourier transform to conclude that

$$u(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \cos(\xi y) d\xi.$$
(10)

7. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t, x > 0, \\ u(0,t) = e^{7t}, & t > 0, \\ u(x,0) = 0, & x > 0. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

(10 p)

p)

- (a) (2p) Recognize that you should use the Laplace transform (1 point for that) and that the transform should be in the t variable (1 point for that).
- (b) (2p) Correctly Laplace transform the pde

$$\widetilde{u}_t(x,z) = \widetilde{u}_{xx}(x,z) = z\widetilde{u}(x,z) - u(x,0) = z\widetilde{u}(x,z).$$

(c) (2p) Solve this ode for

$$\widetilde{u}(x,z) = a(z)e^{x\sqrt{z}} + b(z)e^{-x\sqrt{z}}.$$

(d) (2p) One point for throwing away the $e^{x\sqrt{z}}$. The second point is for finding

$$\widetilde{u}(0,z) = \widetilde{e^{7t}}(z) = b(z) \implies \widetilde{u}(x,z) = \widetilde{e^{7t}}(z)e^{-x\sqrt{z}}.$$

(e) (2p) Use the table of Laplace transforms to obtain that the function

$$\Theta(t) \frac{x}{2\sqrt{\pi}t^{3/2}} e^{-x^2/(4t)} \longrightarrow e^{-x\sqrt{z}}$$

Hence

$$u(x,t) = \int_{\mathbb{R}} e^{7t - 7s} \Theta(t-s) \Theta(s) \frac{x}{2\sqrt{\pi}s^{3/2}} e^{-x^2/(4s)} ds.$$

If you just write that u(x,t) is the convolution of e^{7t} and this other $\frac{x}{2\sqrt{\pi}t^{3/2}}e^{-x^2/(4t)}$ function but you mess up the definition of convolution and/or forget the heavisides, you get one out of two points for this part.

(10 p)

8. Hitta et polynomet p(x) av högst grad 45 som minimeras (Find the polynomial p(x) of at most degree 45 which minimizes the following integral):

$$\int_{-4}^{4} |e^{2\cos(x)} - p(x)|^2 dx.$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration but need not be calculated). (10p)

Let $P_n(x)$ denote the Legendre polynomial of degree n. Then we know that these polynomials are an orthogonal base for $\mathcal{L}^2(-1,1)$. So, we do a little calculation by setting x/4 = t:

$$\int_{-4}^{4} P_n(x/4) P_m(x/4) dx = \int_{-1}^{1} P_n(t) P_m(t) (4dt)$$
$$= \begin{cases} 0, & n \neq m, \\ 4||P_n||^2 = \frac{8}{2n+1}, \& n = m. \end{cases}$$

So, the polynomials $\{P_n(x/4)\}_{n\geq 0}$ are an orthogonal base for $\mathcal{L}^2(-4, 4)$. For notational convenience let

$$\wp_n(x) := P_n(x/4), \quad f(x) := e^{\cos(x)}.$$

Consequently the polynomial we seek is

$$p(x) = \sum_{n=0}^{45} \frac{\langle f, \wp_n \rangle}{||\wp_n||^2} \wp_n(x),$$

with

$$\langle f, \wp_n \rangle = \int_{-4}^{4} f(x) \overline{\wp_n(x)} dx = \int_{-4}^{4} e^{\cos(x)} P_n(x/4) dx,$$
$$||\wp_n||^2 = \int_{-4}^{4} |\wp_n(x)|^2 dx = \frac{8}{2n+1}.$$

So the points breakdown is like this:

- (a) 2 points for using Legendre polynomials. As Beyonce would say, you need to SAY their name. (Not just write P_n without explaining what P_n is!!) If you write P_n but don't say their name, you get 1 out of 2 points.
- (b) 2 points for modifying them to $P_n(x/4)$. (this is pretty much all or nothing)
- (c) 2 points for the correct scalar product: $\langle f, \wp_n \rangle = \int_{-4}^4 e^{\cos(x)} P_n(x/4) dx$. (i don't really see how to do partial credit on this part either, probably all or nothing here too).
- (d) 2 points for the correct norm downstairs: $||\wp_n||^2 = \int_{-4}^4 |\wp_n(x)|^2 dx = \frac{8}{2n+1}$. (similarly, unclear how to do partial credit here?)
- (e) 2 points for putting it all together correctly. (I suppose if you do everything else right but goof this up in some minor way, then you could get 1 point instead of 2).

Please keep in mind that you do not actually have to compute any integrals here! Just write down what the correct integrals are!