Fourieranalys MVE030 och Fourier Metoder MVE290 2023.juni.7
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & miniräknare som helst.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

1 Uppgifter

1. (10 P) Låt f vara en 2π periodisk funktion och antar att $f \in \mathcal{C}^2(\mathbb{R})$. Bevisa att de Fourierkoefficienterna C_n av f och de Fourierkoefficienterna c_n av f' (f' är derivatan av f) uppfyller:

$$c_n = inC_n.$$

2. (10 P) (Samplingssatsen) Låt $f \in L^2(\mathbb{R})$ och $\hat{f}$ dess Fouriertransform. Antar att det finns $L > 0$ så att $\hat{f}(x) = 0 \ \forall |x| > L$. Bevisa att

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

3. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(x)\cos(t), & 0 < t,\ 0 < x < \pi \\ u(0,t) = 0, & t > 0, \\ u(\pi,t) = 0, & t > 0, \\ u(x,0) = x(x-\pi), & x \in [0,\pi]. \end{cases}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated). (10 p)

4. Hitta alla $\lambda > 0$ och funktioner f (icke 0-funktionen) så att i de polara koordinater (r, θ) gäller:

$$f_{rr} + r^{-1} f_r + r^{-2} f_{\theta\theta} = \lambda f, \quad f(1, \theta) = f(r, 0) = f(r, \pi) = 0.$$

(10 p)

5. Beräkna: (Compute):

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{e^2 + n^2}.$$

(10p)

6. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t = u_{xx} + x e^{-tx}, & t, x > 0, \\ u(0, t) = 0, \\ u(x, 0) = \frac{x}{\cosh(x)}. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

(10 p)

7. Lös problemet: (Solve the following problem):

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t), & t, x > 0, \\ u(x, 0) = 0, \ u_t(x, 0) = 0, & x > 0, \\ u(0, t) = (1 + t)^{3/2}. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

(10 p)

8. Låt $P_n(x)$ vara den Legendre polynom grad n. Beräkna (eller förklara varför gränsvärdet inte finns)

$$\lim_{N \to \infty} \sum_{n=0}^{N} (2n + 1) \left| \int_{-1}^{1} P_n(x) dx \right|^2.$$

(10p)

Lycka till! May the FourierForce be with you! ♡ Julie, Carl-Joar, Jan, Erik, & Kolya

2 Fun and possibly helpful facts!

2.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

2.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with u the unknown function and p and g specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called μ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the Mμthod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital C. The solution is:

$$u(t) = \frac{\int_0^t(\mu(s)g(s)ds) + C}{\mu(t)}.$$

2.3 How to solve second order linear ODEs

Theorem 1 (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function u that depends on one variable, with constants b and c given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:

1. ***If** $b^2 \neq 4ac$**,** then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$**,** then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

Theorem 2 (Particular solution to linear second order ODEs)**.** *Assume that y_1 und y_2 are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

Then a solution to the ODE

$$L(y) = g(t)$$

is given by

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

The Wronskian of y_1 and y_2, denoted by $W(y_1, y_2)$ above, is defined to be

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

2.4 Definition of a regular SLP (by request)

A regular Sturm Liouville problem is to find all solutions to

$$L(f(x)) + \lambda w(x) f(x) = 0, \quad B_i(f) = 0, \quad i = 1, 2. \tag{1}$$

The *eigenvalues* of the SLP are all numbers λ for which there exists a corresponding non-zero *eigenfunction* f so that together they satisfy (1). The constituents in the problem in (1) *must* satisfy the following conditions in order for the problem to be a regular SLP:

1. The function w, known as a *weight function,* must be both positive and continuous on the interval $[a, b]$.

2. The differential operator L must be of the form

$$L(f(x)) = (r(x)f'(x))' + p(x)f(x).$$

   Above r and p are *specified* real valued functions. The functions r, r', and p must be continuous, and r must be positive on $[a, b]$.

3. The boundary conditions must be equations of the form:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2. \tag{2}$$

   Above, the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ must be fixed complex numbers. Moreover, the boundary conditions must guarantee that for any unknown functions ϕ and ψ if one only knows that they both satisfy (2), that is enough to guarantee that

$$r(b)\left(\overline{\psi(b)}\phi'(b) - \overline{\psi'(b)}\phi(b)\right) - r(a)\left(\overline{\psi(a)}\phi'(a) - \overline{\psi'(a)}\phi(a)\right) = 0. \tag{3}$$

We note that the unknown functions ϕ and ψ in (3) are only assumed to satisfy (2); they need *not* necessarily solve the equation (1).

2.5 Bessel function facts

Definition 1 (The Bessel function J of order ν)**.** The Bessel function J of order ν is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Γ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{4}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of ν, it is an entire function of $x \in \mathbb{C}$.

Theorem 3 (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the m^{th} positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n}r/L)\}_{m\geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0,L)$*. Recall that this is the weighted* $\mathcal{L}^2$ *space on the interval* $(0,L)$ *with respect to the weight function* r*, so the scalar product*

$$\langle f,g\rangle = \int_0^L f(r)\overline{g(r)}rdr.$$

More generally, for any $\nu\in\mathbb{R}$*, and for any* $L>0$*, the functions*

$$\{J_\nu(\pi_m r/L)\}_{m\geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0,L)$*, where above* π_m *denotes the* m^{th} *zero of the Bessel function* J_ν*. They have norms equal to*

$$\int_0^L |J_\nu(\pi_m r/L)|^2 rdr = \frac{L^2}{2}\left(J_{\nu+1}(\pi_m)\right)^2.$$

Corollary 1 (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n}r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

are an orthogonal basis for $\mathcal{L}^2$ *on the disk of radius* L*.*

Theorem 4 (Bessel functions as bases in some other cases)**.** *Assume that* $L>0$*. Let the weight function* $w(x)=x$*. Fix* $\nu\in\mathbb{R}$*. Then* J'_ν *has infinitely many positive zeros. Let*

$$\{\pi'_k\}_{k\geq 1}$$

be the positive zeros of J'_ν*. Then we define*

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu>0, \quad k\geq 1.$$

In case $\nu=0$*, define further* $\psi_0(x)=1$*. (If* $\nu\neq 0$*, then this case is omitted.) Then* $\{\psi_k\}_{k\geq 1}$ *for* $\nu\neq 0$ *is an orthogonal basis for* $\mathcal{L}^2_w(0,L)$*. For* $\nu=0$*,* $\{\psi_k\}_{k\geq 0}$ *is an orthogonal basis for* $\mathcal{L}^2_w(0,L)$*. Moreover the norm*

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 xdx = \frac{L^2(\pi_k^2-\nu^2)}{2\pi_k^2}J_\nu(\pi_k)^2, \quad k\geq 1, \quad ||\psi_0||_w^2 = \frac{L^{2\nu+2}}{2\nu+2}.$$

Next, fix a constant $c>0$*. Then there are infinitely many positive solutions of*

$$\mu J'_\nu(\mu) + cJ_\nu(\mu) = 0,$$

that can be enumerated as $\{\mu_k\}_{k\geq 1}$*. Then*

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}^2_w(0,L)$*.*

2.6 Orthogonal polynomials

Definition 2. The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2-1)^n \right). \tag{5}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1,1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

Definition 3. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}^2_2(\mathbb{R})$ with respect to the weight function e^{-x^2}. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

Definition 4. The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x}).$$

The Laguerre polynomials $\{L_n^\alpha\}_{n \geq 0}$ are an orthogonal basis for $\mathcal{L}^2_\alpha$ on $(0, \infty)$ with the weight function $\alpha(x) = x^\alpha e^{-x}$. Their norms squared,

$$||L_n^\alpha||^2 = \frac{\Gamma(n+\alpha+1)}{n!}.$$

2.7 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

1.	$f(x)=x$	$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}.$
2.	$f(x)=\lvert x\rvert$	$\frac{\pi}{2}+\sum_{n\in\mathbb{Z},\text{ odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$
3.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ x, & 0<x<\pi.\end{cases}$	$\frac{\pi}{4}+\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\frac{(-1)^{n+1}}{2in}+\frac{(-1)^n-1}{2\pi n^2}\right]e^{inx}$
4.	$f(x)=\sin^2(x)$	$\frac{1}{2}-\frac{1}{4}\left(e^{2ix}+e^{-2ix}\right)$
5.	$f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{2}{i\pi}\sum_{n\geq 1}\frac{e^{(2n-1)ix}-e^{-(2n-1)ix}}{2n-1}$
6.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{1}{2}+\sum_{n\geq 1}\frac{e^{i(2n-1)x}-e^{-i(2n-1)x}}{i\pi(2n-1)}$
7.	$f(x)=\lvert\sin(x)\rvert$	$\frac{2}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{e^{2inx}+e^{-2inx}}{4n^2-1}$
8.	$f(x)=\lvert\cos(x)\rvert$	$\frac{2}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{(-1)^n\left[e^{inx}+e^{-inx}\right]}{4n^2-1}$
9.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin(x), & 0<x<\pi.\end{cases}$	$\frac{1}{\pi}-\frac{1}{\pi}\sum_{n\geq 1}\frac{e^{2inx}+e^{-2inx}}{4n^2-1}+\frac{1}{4i}\left(e^{ix}-e^{-ix}\right)$
10.	$f(x)=x^2$	$\frac{\pi^2}{3}+2\sum_{n\geq 1}\frac{(-1)^n(e^{inx}+e^{-inx})}{n^2}$
11.	$f(x)=x(\pi-\lvert x\rvert)$	$\frac{4}{i\pi}\sum_{n\geq 1}\frac{e^{i(2n-1)x}-e^{-i(2n-1)x}}{(2n-1)^3}$
12.	$f(x)=e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{b-in}e^{inx}$
13.	$f(x)=\sinh x$	$\frac{\sinh(\pi)}{i\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}n}{n^2+1}\left[e^{inx}-e^{-inx}\right]$
14.	$f(x)=e^{ibx}$, $b\notin\mathbb{Z}$	$\sum_{n\in\mathbb{Z}}\frac{\sin(b\pi)(-1)^n}{\pi(b-n)}e^{inx}$

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x\notin(-\pi,\pi)$.

1.	$f(x)=x$	$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$.
2.	$f(x)=\|x\|$	$\frac{\pi}{2}-\frac{4}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2}$
3.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ x, & 0<x<\pi.\end{cases}$	$\frac{\pi}{4}-\frac{2}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2}+\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sin(nx)$
4.	$f(x)=\sin^2(x)$	$\frac{1}{2}-\frac{1}{2}\cos(2x)$
5.	$f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{4}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
6.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{1}{2}+\frac{2}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
7.	$f(x)=\|\sin(x)\|$	$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1}$
8.	$f(x)=\|\cos(x)\|$	$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n\geq 1}\frac{(-1)^n\cos(2nx)}{4n^2-1}$
9.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin(x), & 0<x<\pi.\end{cases}$	$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$
10.	$f(x)=x^2$	$\frac{\pi^2}{3}+4\sum_{n\geq 1}\frac{(-1)^n\cos(nx)}{n^2}$
11.	$f(x)=x(\pi-\|x\|)$	$\frac{8}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{(2n-1)^3}$
12.	$f(x)=e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\left(\frac{1}{b}+\sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2}\left[2b\cos(nx)-2n\sin(nx)\right]\right)$
13.	$f(x)=\sinh x$	$\frac{2\sinh(\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}n}{n^2+1}\sin(nx)$
14.	$f(x)=e^{ibx}$, $b\notin\mathbb{Z}$	$\frac{\sin(b\pi)}{b\pi}+\frac{\sin(b\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{b^2-n^2}\left[2b\cos(nx)+2in\sin(nx)\right]$

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1,\cos(nx),\sin(nx)\}_{n\geq 1}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x\notin(-\pi,\pi)$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a > 0$ and $c \in \mathbb{R}$.

1.	$\Theta(t)f(t)$	$\widetilde{f(z)}$
2.	$\Theta(t-a)f(t-a)$	$e^{-az}\widetilde{f(z)}$
3.	$e^{ct}\Theta(t)f(t)$	$\widetilde{f(z-c)}$
4.	$\Theta(t)f(at)$	$a^{-1}\widetilde{f(a^{-1}z)}$
5.	$\Theta(t)f'(t)$	$z\widetilde{f(z)} - f(0)$
6.	$\Theta(t)f^{(k)}(t)$	$z^k\widetilde{f(z)} - \sum_0^{k-1} z^{k-1-j}f^{(j)}(0)$
7.	$\Theta(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f(z)}$
8.	$\Theta(t)tf(t)$	$-\widetilde{f}'(z)$
9.	$\Theta(t)t^{-1}f(t)$	$\int_z^\infty \widetilde{f(w)}dw$
10.	$\Theta f * \Theta g(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
11.	$\Theta(t)t^\nu e^{ct}$	$\Gamma(\nu+1)(z-c)^{-\nu-1}$
12.	$\Theta(t)(t+a)^{-1}$	$e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$
13.	$\Theta(t)\sin(ct)$	$\frac{c}{z^2+c^2}$
14.	$\Theta(t)\cos(ct)$	$\frac{z}{z^2+c^2}$

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.

15.	$\Theta(t)\sinh(ct)$	$\frac{c}{z^2-c^2}$
16.	$\Theta(t)\cosh(ct)$	$\frac{z}{z^2-c^2}$
17.	$\Theta(t)\sin(\sqrt{at})$	$\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$
18.	$\Theta(t)t^{-1}\sin(\sqrt{at})$	$\pi\,\mathrm{erf}(\sqrt{a/(4z)})$
19.	$\Theta(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$
20.	$\Theta(t)\,\mathrm{erf}(at)$	$z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$
21.	$\Theta(t)\,\mathrm{erf}(\sqrt{t})$	$(z\sqrt{z+1})^{-1}$
22.	$\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$	$((z-1)\sqrt{z})^{-1}$
23.	$\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$	$z^{-1}e^{-a\sqrt{z}}$
24.	$\Theta(t)t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$
25.	$\Theta(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
26.	$\Theta(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
27.	$\Theta(t)t^\nu J_\nu(t)$	$2^\nu\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$
28.	$\Theta(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.

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1 Uppgifter

1. (10 P) Låt f vara en 2π periodisk funktion och antar att $f \in \mathcal{C}^2(\mathbb{R})$. Bevisa att de trigonometriska Fourierkoefficienterna C_n av f i den ortogonal bas $\{e^{inx}\}_{n\in\mathbb{Z}}$ på Hilbertrummet $\mathcal{L}^2(-\pi,\pi)$ och de Fourierkoefficienterna c_n av f' (f' är derivatan av f) uppfyller:

$$c_n = inC_n.$$

Solution: We use the definitions of the Fourier series and coefficients of f and f' respectively with respect to the given orthogonal base in the given Hilbert space.

$$(2p) \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

and

$$(2p) \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx.$$

You could use either of these expressions and integrate by parts. So, just the idea to integrate by parts is worth 2 points, because it is the key idea. Next, actually integrating by parts correctly will be worth another 2 points. If you take the expression for c_n and integrate by parts, you get

$$(2p) \quad c_n = \frac{1}{2\pi}\left(f(x)e^{-inx}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-in)e^{-inx} \right).$$

Now you get two points (2p) by observing that the first term vanishes due to the 2π periodicity of both f and e^{-inx}, so that we end up with

$$c_n = inC_n.$$

If you instead take the expression for C_n and integrate by parts there, it's slightly tricky because you need to separate the cases $n = 0$ and $n \neq 0$. For $n = 0$, it is worth one point to show that

$$c_0 = 0$$

by the 2π periodicity of f. For $n \neq 0$, the idea to integrate by points is worth 1 point, and getting the following expression right for $n \neq 0$ is worth 2 points:

$$C_n = \frac{1}{2\pi}\left(f(x)\frac{e^{-inx}}{-in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}dx \right).$$

It is worth two points (2p) for observing that the first term vanishes due to the 2π periodicity of both f and e^{-inx}. So, we get

$$c_0 = 0,\ C_n = \frac{c_n}{in},\ n \in \mathbb{Z} \setminus \{0\}.$$

Thus we get $c_n = inC_n$ holds for all $n \in \mathbb{Z}$.

2. (10 P) (Samplingssatsen) Låt $f \in L^2(\mathbb{R})$ och $\hat{f}$ dess Fouriertransform. Antar att det finns $L > 0$ så att $\hat{f}(x) = 0\ \forall |x| > L$. Bevisa att

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{L}\right)\frac{\sin(n\pi - tL)}{n\pi - tL}.$$

Solution: This theorem is all about the interaction between Fourier series and Fourier coefficients and how to work with both simultaneously. Since the Fourier transform $\hat{f}$ has compact support, the following equality holds as elements of $\mathcal{L}^2([-L, L])$,

$$(2p)\quad \hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx.$$

So 1 point for the idea to expand $\hat{f}$ as a Fourier series on $\mathcal{L}^2(-L, L)$ with respect to the OB $e^{in\pi x/L}$ and 1 point for doing it right.

Next, 1 point for the idea to use the FIT and 1 point for doing it correctly to obtain

$$(2p)\quad f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt}\hat{f}(x)dx.$$

Another point for using the fact that $\hat{f}(x) = 0$ for $|x| > L$ to obtain that

$$(1p) \quad f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \hat{f}(x) dx.$$

Now, one point for substituting the Fourier expansion of $\hat{f}$ into this integral,

$$(1p) \quad f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} c_n e^{in\pi x/L} dx.$$

One point for applying the FIT onto the coefficients and one point for using that $\hat{f}(x) = 0$ for $|x| > L$ to obtain

$$(1p) \quad c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx = \frac{1}{2L} \int_{\mathbb{R}} e^{ix(-n\pi/L)} \hat{f}(x) dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

Thus we have at this point

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L} dx.$$

One point for correctly calculating the integral

$$(1p) \quad \int_{-L}^{L} e^{x(it - in\pi/L)} dx = \frac{e^{L(it-in\pi/L)}}{i(t - n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t - n\pi/L)} = \frac{2i}{i(t - n\pi/L)} \sin(Lt - n\pi).$$

Two more points for correctly putting it all together to get the final expression

$$(2p) \quad f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

3. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(x)\cos(t), & 0 < t,\ 0 < x < \pi \\ u(0,t) = 0, & t > 0, \\ u(\pi,t) = 0, & t > 0, \\ u(x,0) = x(x-\pi), & x \in [0,\pi]. \end{cases}$$

(Note that certain integrals do not need to be calculated - they must be correctly stated with correct integrand and limits of integration - but need not be calculated). (10 p)

Solution: This is not an easy problem, but at least it is *really* similar to the third problem in the March exam. So for everyone who didn't pass that exam, I hope you studied its solutions!

(a) (1p) SLPs are the keys to solving inhomogeneous pde's. Even if you do nothing else, this rhyme is worth one point. If you don't do the rhyme, you still get a point if you set up the SLP to solve

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(\pi) = 0.$$

(b) (2p) Solve this SLP. You should obtain (see the vibrating string example in Chapter 1 of the textbook for the derivation of these solutions)

$$X_n(x) = \sin(nx), \quad n \in \mathbb{N}_{\geq 1}.$$

One point for the correct function and one point for the correct range of n.

(c) (1p) Set up the solution you seek to be a series

$$u(x,t) = \sum_{n \in \mathbb{Z}} T_n(t) X_n(x),$$

where we will need to solve for the T_n functions using the inhomogeneous pde together with the initial condition.

(d) (2p) Expand the inhomogeneity in terms of the X_n base:

$$\sin(x)\cos(t) = \sum_{n \geq 1} \cos(t) \frac{\langle \sin(x), X_n \rangle}{||X_n||^2} X_n(x) = \sum_{n \geq 1} c_n \cos(t) X_n(x),$$

with

$$\langle \sin(x), X_n \rangle = \int_0^\pi \sin(x)\overline{X_n(x)}dx, \quad ||X_n||^2 = \int_0^\pi |X_n(x)|^2 dx.$$

It is okay if you leave these integrals like this (and don't calculate them) as long as you have correctly defined the scalar product and the norm squared. Each of these correctly defined is worth one point. It is possible to simplify life by observing that $\sin(x) = X_1$, and $\{X_n\}_{n\geq 1}$ are orthogonal. So, the coefficients

$$c_n = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

(e) (1p) Plug u into the heat equation (correctly) to obtain

$$u_t - u_{xx} = \sum_{n\geq 1} (T_n'(t) + n^2 T_n(t)) X_n(x).$$

(f) (1p) Set up the equation for T_n to solve

$$T_n'(t) + n^2 T_n(t) = c_n \sin(t).$$

(g) (1p) Set up the correct initial condition

$$\sum_{n\geq 1} T_n(0) X_n(x) = x(x-\pi) = \sum_{n\geq 1} C_n X_n(x),$$

with

$$C_n = \frac{\langle x(x-\pi), X_n \rangle}{||X_n||^2}.$$

(If you have correctly defined the scalar product and norm squared you do not need to write it out again).

(h) (1p) Solve the ODE for $T_n(t)$. The method of integrating factor will give you

$$e^{-n^2 t} \left[\int_0^t e^{n^2 s} c_n \cos(s) ds + C_n \right].$$

4. Hitta alla $\lambda > 0$ och funktioner f (icke 0-funktionen, som är alltid lika 0) så att i polara koordinater (r, θ) gäller:

$$f_{rr} + r^{-1} f_r + r^{-2} f_{\theta\theta} = \lambda f, \quad f(1, \theta) = f(r, 0) = f(r, \pi) = 0.$$

(10 p)

Solution: This is classical separation of variables using polar coordinates.

(a) (2p) Start by separating variables and looking for a solution to the pde of the form

$$R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' = -\lambda R\Theta.$$

It's one point for the idea to separate variables and one point for doing it correctly in the equation.

(b) (1p) Continue to re-arrange the equation until you separate out the variables fully:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda = -\frac{\Theta''}{\Theta} \implies \text{both sides are equal a constant.} \tag{1}$$

(c) (1p) Set up the SLP for Θ:

$$\Theta'' + \Lambda\Theta = 0, \quad \Theta(0) = \Theta(\pi) = 0.$$

(d) (1p) Aren't we lucky that I was so kind as to give you the exact same SLP in the previous problem? You literally just solved this(!) So, you know the solutions are

$$\Theta_n(\theta) = \sin(n\theta), \quad n \in \mathbb{N}_{\geq 1}.$$

(e) (1p) Calculate that

$$-\frac{\Theta_n''}{\Theta_n} = n^2.$$

(f) (1p) Substitute this back in to (1) to get the equation for R:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda = n^2 \iff r^2R'' + rR' + (r^2\lambda - n^2)R = 0.$$

(g) (1p) We only need to look for $\lambda > 0$, so the solution to this equation is $R_n(r) = J_n(r\sqrt{\lambda})$.

(h) (1p) To satisfy the boundary condition, we need $J_n(\sqrt{\lambda}) = 0$ (from $R_n(1) = 0$). So the values of λ are the positive zeros of the Bessel functions J_n for $n \geq 1$.

(i) (1p) The functions are $J_n(r\sqrt{\lambda})\sin(n\theta)$ where λ is as just described.

5. Beräkna: (Compute):

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{e^2+n^2}.$$

Solution: This is almost the same as the problem we had in the march exam! I have just changed the π to an e.

We are rather lucky because we have been generously given a table that says that the trig Fourier series of the function e^{bx} in $\mathcal{L}^2(-\pi,\pi)$ is

$$\frac{\sinh(b\pi)}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b-in} e^{inx}.$$

It is worth a whopping 5 points to identify a function whose trig Fourier series can be used to compute this series. This is pretty much hit or miss - either the function you choose *can* be used to calculate the series or it *cannot* (meaning there is no way to make the function you choose work).

Parseval method:. With the function that I chose, I use the Parseval equality or equivalently the infinite dimensional Pythagorean theorem to get:

$$\begin{aligned}
(1p)\, ||e^{bx}||^2 &= \int_{-\pi}^{\pi} e^{2bx}dx = \frac{e^{2b\pi}-e^{-2b\pi}}{2b} = \frac{\sinh(2b\pi)}{b} \\
(1p)\ &= \sum_{n\in\mathbb{Z}} \left\| \frac{\sinh(b\pi)}{\pi} \frac{(-1)^n}{b-in} e^{inx} \right\|^2 \\
(1p)\ &= \sum_{n\in\mathbb{Z}} \frac{\sinh(b\pi)^2}{\pi^2} \frac{1}{b^2+n^2} 2\pi \\
&= \frac{2\sinh(b\pi)^2}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{b^2+n^2}
\end{aligned}$$

One point for setting this equal to the norm on the other side:

$$(1p)\, \frac{\sinh(2b\pi)}{b} = 2\frac{\sinh(b\pi)^2}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{b^2+n^2} \iff \frac{\pi\sinh(2b\pi)}{2b\sinh(b\pi)^2} = \sum_{n\in\mathbb{Z}} \frac{1}{b^2+n^2}.$$

Setting $b=e$ we get

$$(1p)\, \frac{\pi\sinh(2e\pi)}{2e\sinh(e\pi)^2} = \sum_{n\in\mathbb{Z}} \frac{1}{e^2+n^2}.$$

Pointwise convergence of trig Fourier series method:
(1p) For choosing the correct point and that is $x = \pi$ or $x = -\pi$. With either of these the series becomes

$$(1p)\ \frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{b-in}e^{\pm in\pi} = \frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{b-in}$$

$$\frac{\sinh(b\pi)}{\pi}\left[\frac{1}{b} + 2b\sum_{n\geq 1}\frac{1}{b^2+n^2}\right].$$

Two points for using the theorem correctly to say that this is

$$(2p)\ \frac{e^{\pi b}+e^{-\pi b}}{2} = \cosh(b\pi).$$

Then one last point for doing the arithmetic to eek out the desired value with $b = \pi$:

$$\frac{1}{b}\left(\cosh(b\pi)\frac{\pi}{\sinh(b\pi)} - \frac{1}{b}\right) = 2\sum_{n\geq 1}\frac{1}{b^2+n^2}$$

$$\implies \sum_{n\in\mathbb{Z}}\frac{1}{b^2+n^2} = \frac{1}{b^2}+2\sum_{n\geq 1}\frac{1}{b^2+n^2} = \frac{1}{b^2}+\frac{1}{b}\left(\cosh(\pi b)\frac{\pi}{\sinh(b\pi)} - \frac{1}{b}\right)$$

$$= \frac{\pi\cosh(b\pi)}{b\sinh(b\pi)}.$$

Setting $b = e$ we get

$$(1p)\qquad = \frac{\pi\cosh(e\pi)}{e\sinh(e\pi)}.$$

If you are concerned that this doesn't look like the answer from the previous method, note that the doubling formula for the hyperbolic sine gives

$$\sinh(2e\pi) = 2\sinh(e\pi)\cosh(e\pi),$$

so our first answer

$$\frac{\pi\sinh(2e\pi)}{2e\sinh(e\pi)^2} = \frac{2\pi\sinh(e\pi)\cosh(e\pi)}{2e\sinh(e\pi)^2} = \frac{\pi\cosh(e\pi)}{e\sinh(e\pi)}.$$

So indeed our answers match. I would be super impressed if anybody solved this BOTH ways just to be totally sure they are right... I have NEVER seen anyone do that - but hope springs eternal.

(10p)

6. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t = u_{xx} + xe^{-tx}, & t, x > 0, \\ u(0,t) = 0, \\ u(x,0) = \frac{x}{\cosh(x)}. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

Solution: This is an inhomogeneous heat equation on a half space.

(a) (2p) One point for the idea to extend oddly to $x < 0$ based on the boundary condition, and one point for doing it correctly so that the functions are

$$xe^{-t|x|}, \quad \frac{x}{\cosh(x)}.$$

(The second function is already odd so it's already good! And yes I did that on purpose.)

(b) (2p) Two points for the idea to use the Fourier transform in the x variable. (1p for FT, 1p for correct variable).

(c) (1p) One point for actually doing the FT correctly to obtain

$$\hat{u}_t(\xi, t) = -\xi^2 \hat{u}(\xi, t) + \widehat{xe^{-t|x|}}.$$

(d) (2p) This is a first order ODE for the Fourier transform in the t variable, and it can be solved by the method of integrating factor. The solution is:

$$\hat{u}(\xi, t) = e^{-t\xi^2} \left(\int_0^t e^{s\xi^2} \widehat{xe^{-s|x|}} ds + \hat{u}(\xi, 0) \right).$$

(e) (1p) Use the initial data to obtain

$$\hat{u}(\xi, t) = e^{-t\xi^2} \left(\int_0^t e^{s\xi^2} \widehat{xe^{-s|x|}} ds + \widehat{\frac{x}{\cosh(x)}} \right).$$

(f) (2p) Use the fact that FT turns products into convolutions so the solution is

$$u(x,t) = \int_{\mathbb{R}} \int_0^t (4\pi(t-s))^{-1/2} e^{-(x-y)^2/(4(t-s))} ye^{-s|y|} ds dy$$

$$+\int_{\mathbb{R}}(4\pi t)^{-1/2}e^{-(x-y)^2/(4t)}\frac{y}{\cosh(y)}dy.$$

In this last step, if you just used the FIT instead, to say

$$u(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ix\xi}e^{-t\xi^2}\left(\int_0^t e^{s\xi^2}\widehat{xe^{-s|x|}}ds+\widehat{\frac{x}{\cosh(x)}}\right)d\xi$$

I'll still give you these last two points, just cause.

(10 p)

7. Lös problemet: (Solve the following problem):

$$\begin{cases} u_{tt}(x,t)=u_{xx}(x,t), & t,x>0, \\ u(x,0)=0,\ u_t(x,0)=0, & x>0, \\ u(0,t)=(1+t)^{3/2}. \end{cases}$$

(Note that it is okay if your answer is in the form of an integral, but preferably it should not be given as an inverse-transform.)

Solution:

(a) (2p) Recognize that you should use the Laplace transform (1 point for that) and that the transform should be in the t variable (1 point for that).

(b) (2p) Correctly Laplace transform the pde

$$\widetilde{u}_{tt}(x,z)=\widetilde{u}_{xx}(x,z)=z^2\widetilde{u}(x,z)-zu_t(x,0)-u(x,0)=z^2\widetilde{u}(x,z).$$

(c) (2p) Solve this ode for

$$\widetilde{u}(x,z)=a(z)e^{xz}+b(z)e^{-xz}.$$

(d) (2p) The boundary condition says that

$$\widetilde{u}(0,z)=\widetilde{(1+t)^{3/2}}.$$

So, $a(z)+b(z)$ is equal to that right side. Now, note that the Laplace transform of anything Laplace transformable should vanish as the real part of z tends to infinity. Since our range of x is $x>0$, this motivates us to seek a solution involving only e^{-xz}. So, we have

$$\widetilde{u}(x,z)=e^{-xz}\widetilde{(1+t)^{3/2}}.$$

(e) (2p) By the table, the Laplace transform of $\Theta(t-x)f(t-x)$ is $e^{-xz}\widetilde{f}(z)$. So, using this with $f(t)=(1+t)^{3/2}$, this means that

$$u(x,t)=\Theta(t-x)(1+(t-x))^{3/2}.$$

(10 p)

8. Låt $P_n(x)$ vara den Legendre polynom grad n. Beräkna (eller förklara varför gränsvärdet inte finns)

$$\lim_{N\to\infty}\sum_{n=0}^{N}(2n+1)\left|\int_{-1}^{1}P_n(x)dx\right|^2.$$

Solution: If you just write that the limit is 4, I'll give you all 10 points because I'm cool like dat.

(2p) Two points for saying that P_n are an orthogonal basis for $\mathcal{L}^2$ on the interval $[-1,1]$.

(2p) Observe that

$$\int_{-1}^{1}P_n(x)dx=\langle 1,P_n\rangle.$$

The function 1 is an element of the Hilbert space $\mathcal{L}^2(-1,1)$. So we can expand it with respect to the orthogonal base $\{P_n(x)\}_{n\geq 0}$,

$$(2p)\quad \sum_{n\geq 0}\frac{\langle 1,P_n\rangle}{||P_n||^2}P_n(x).$$

Two points for applying Parseval's equality to say that

$$(2p)\quad \sum_{n\geq 0}\frac{|\langle 1,P_n\rangle|^2}{||P_n||^4}||P_n||^2=||1||^2=\int_{-1}^{1}1^2dx=2.$$

So, we get that

$$\sum_{n\geq 0}\frac{|\langle 1,P_n\rangle|^2}{||P_n||^2}=2.$$

At the end of the exam we have the nice formula for

$$||P_n||^2=\frac{2}{2n+1}.$$

So we get that

$$(1p) \quad \sum_{n \geq 0} \frac{(2n+1)|\langle 1, P_n \rangle|^2}{2} = 2.$$

Consequently

$$(1p) \quad \lim_{N \to \infty} \sum_{n=0}^{N} (2n+1) \left| \int_{-1}^{1} P_n(x) dx \right|^2 = 4.$$

Alternative method for the smartypants: I say this because I initially solved this exercise by the above clumsy yet effective method. Then I realized that there is a *much* smarter method, and knowing the nice young people who take my class, I bet some of you are so clever that you immediately saw it (unlike me). The scalar products

$$(8p) \quad \langle 1, P_n \rangle = \begin{cases} 2, & n = 0, \\ 0, & n > 0. \end{cases}$$

This is because $P_0 = 1$, and the Legendre polynomials are orthogonal on $\mathcal{L}^2(-1, 1)$. So, haha, the first term in the sum is

$$(1p) \quad (2 * 0 + 1) * |2|^2 = 4,$$

and (1p) all the other terms in the sum are ZERO. So of course the sum converges to four. YAYAY!

(10p)

Lycka till! May the FourierForce be with you! ♡ Julie, Carl-Joar, Jan, Erik, & Kolya

2 Fun and possibly helpful facts!

2.1 The Laplace operator

The Laplace operator in two and three dimensions is respectively

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

In polar coordinate in two dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

In cylindrical coordinates in three dimensions

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

2.2 How to solve first order linear ODEs

If one has a first order linear differential equation, then it can always be arranged in the following form, with u the unknown function and p and g specified in the ODE:

$$u'(t) + p(t)u(t) = g(t).$$

We compute in this case a function traditionally called μ known as the *integrating factor,*

$$\mu(t) := \exp\left(\int_0^t p(s)ds\right).$$

For this reason we call this method the Mμthod. When computing the integrating factor the constant of integration can be ignored, because we will take care of it in the next step. We compute

$$\int_0^t \mu(s)g(s)ds = \int_0^t \mu(s)g(s)ds + C.$$

Don't forget the constant here! That's why we use a capital C. The solution is:

$$u(t) = \frac{\int_0^t(\mu(s)g(s)ds) + C}{\mu(t)}.$$

2.3 How to solve second order linear ODEs

Theorem 1 (Basis of solutions for linear, constant coefficient, homogeneous second order ODEs)**.** *Consider the ODE, for the unknown function u that depends on one variable, with constants b and c given in the equation:*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

A basis of solutions is one of the following pairs of functions depending on whether $b^2 \neq 4ac$ or $b^2 = 4ac$:

1. ***If** $b^2 \neq 4ac$**,** then a basis of solutions is*

$$\{e^{r_1 x}, e^{r_2 x}\}, \text{ with } r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. ***If** $b^2 = 4ac$**,** then a basis of solutions is*

$$\{e^{rx}, xe^{rx}\}, \text{ with } r = -\frac{b}{2a}.$$

Theorem 2 (Particular solution to linear second order ODEs)**.** *Assume that y_1 und y_2 are a basis of solutions to the ODE*

$$L(y) = y'' + q(t)y' + r(t)y = 0.$$

Then a solution to the ODE

$$L(y) = g(t)$$

is given by

$$Y(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

The Wronskian of y_1 and y_2, denoted by $W(y_1, y_2)$ above, is defined to be

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

2.4 Definition of a regular SLP (by request)

A regular Sturm Liouville problem is to find all solutions to

$$L(f(x)) + \lambda w(x) f(x) = 0, \quad B_i(f) = 0, \quad i = 1, 2. \tag{2}$$

The *eigenvalues* of the SLP are all numbers λ for which there exists a corresponding non-zero *eigenfunction* f so that together they satisfy (2). The constituents in the problem in (2) *must* satisfy the following conditions in order for the problem to be a regular SLP:

1. The function w, known as a *weight function,* must be both positive and continuous on the interval $[a, b]$.

2. The differential operator L must be of the form

$$L(f(x)) = (r(x)f'(x))' + p(x)f(x).$$

   Above r and p are *specified* real valued functions. The functions r, r', and p must be continuous, and r must be positive on $[a, b]$.

3. The boundary conditions must be equations of the form:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2. \tag{3}$$

   Above, the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ must be fixed complex numbers. Moreover, the boundary conditions must guarantee that for any unknown functions ϕ and ψ if one only knows that they both satisfy (3), that is enough to guarantee that

$$r(b)\left(\overline{\psi(b)}\phi'(b) - \overline{\psi'(b)}\phi(b)\right) - r(a)\left(\overline{\psi(a)}\phi'(a) - \overline{\psi'(a)}\phi(a)\right) = 0. \tag{4}$$

We note that the unknown functions ϕ and ψ in (4) are only assumed to satisfy (3); they need *not* necessarily solve the equation (2).

2.5 Bessel function facts

Definition 1 (The Bessel function J of order ν)**.** The Bessel function J of order ν is defined to be the series

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n}.$$

The Γ (Gamma) function in the expression above is defined to be

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt, \quad s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0. \tag{5}$$

For $\nu \in \mathbb{C}$, the Bessel function is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, while for integer values of ν, it is an entire function of $x \in \mathbb{C}$.

Theorem 3 (Bessel functions as an orthogonal base)**.** *Fix $L > 0$. Fix any integer $n \in \mathbb{Z}$. Let $\pi_{m,n}$ denote the m^{th} positive zero of the Bessel function $J_{|n|}$. Then the functions*

$$\{J_{|n|}(\pi_{m,n} r/L)\}_{m \geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0,L)$. Recall that this is the weighted $\mathcal{L}^2$ space on the interval $(0,L)$ with respect to the weight function r, so the scalar product

$$\langle f, g\rangle = \int_0^L f(r)\overline{g(r)} r dr.$$

More generally, for any $\nu \in \mathbb{R}$, and for any $L > 0$, the functions

$$\{J_\nu(\pi_m r/L)\}_{m\geq 1}$$

are an orthogonal base for $\mathcal{L}^2_r(0,L)$, where above π_m denotes the m^{th} zero of the Bessel function J_ν. They have norms equal to

$$\int_0^L |J_\nu(\pi_m r/L)|^2 r dr = \frac{L^2}{2}\left(J_{\nu+1}(\pi_m)\right)^2.$$

Corollary 1 (Orthogonal base for functions on a disk)**.** *The functions*

$$\{J_{|n|}(\pi_{m,n} r/L)e^{in\theta}\}_{m\geq 1, n\in\mathbb{Z}}$$

are an orthogonal basis for $\mathcal{L}^2$ on the disk of radius L.

Theorem 4 (Bessel functions as bases in some other cases)**.** *Assume that $L > 0$. Let the weight function $w(x) = x$. Fix $\nu \in \mathbb{R}$. Then J'_ν has infinitely many positive zeros. Let*

$$\{\pi'_k\}_{k\geq 1}$$

be the positive zeros of J'_ν. Then we define

$$\psi_k(x) = J_\nu(\pi_k x/L), \quad \nu > 0, \quad k \geq 1.$$

In case $\nu = 0$, define further $\psi_0(x) = 1$. (If $\nu \neq 0$, then this case is omitted.) Then $\{\psi_k\}_{k\geq 1}$ for $\nu \neq 0$ is an orthogonal basis for $\mathcal{L}^2_w(0,L)$. For $\nu = 0$, $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w(0,L)$. Moreover the norm

$$||\psi_k||_w = \int_0^L |\psi_k(x)|^2 x dx = \frac{L^2(\pi_k^2 - \nu^2)}{2\pi_k^2} J_\nu(\pi_k)^2, \quad k \geq 1, \quad ||\psi_0||^2_w = \frac{L^{2\nu+2}}{2\nu+2}.$$

Next, fix a constant $c > 0$. Then there are infinitely many positive solutions of

$$\mu J'_\nu(\mu) + c J_\nu(\mu) = 0,$$

that can be enumerated as $\{\mu_k\}_{k\geq 1}$. Then

$$\{\varphi_k(x) = J_\nu(\mu_k x/L)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}^2_w(0,L)$.

2.6 Orthogonal polynomials

Definition 2. The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2 - 1)^n \right). \tag{6}$$

The Legendre polynomials are an orthogonal base for $\mathcal{L}^2(-1,1)$, and

$$||P_n||^2 = \frac{2}{2n+1}.$$

Definition 3. The Hermite polynomials are defined to be

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The Hermite polynomials are an orthogonal base for $\mathcal{L}^2_2(\mathbb{R})$ with respect to the weight function e^{-x^2}. Moreover, their norms squared are

$$||H_n||^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2} dx = 2^n n! \int_{\mathbb{R}} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

Definition 4. The Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x}).$$

The Laguerre polynomials $\{L_n^\alpha\}_{n \geq 0}$ are an orthogonal basis for $\mathcal{L}^2_\alpha$ on $(0, \infty)$ with the weight function $\alpha(x) = x^\alpha e^{-x}$. Their norms squared,

$$||L_n^\alpha||^2 = \frac{\Gamma(n + \alpha + 1)}{n!}.$$

2.7 Tables of trig Fourier series, Fourier transforms, and Laplace transforms

1.	$f(x)=x$	$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n e^{inx}}{-in}.$
2.	$f(x)=\lvert x\rvert$	$\frac{\pi}{2}+\sum_{n\in\mathbb{Z},\ \text{odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right)$
3.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ x, & 0<x<\pi.\end{cases}$	$\frac{\pi}{4}+\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\frac{(-1)^{n+1}}{2in}+\frac{(-1)^n-1}{2\pi n^2}\right]e^{inx}$
4.	$f(x)=\sin^2(x)$	$\frac{1}{2}-\frac{1}{4}\left(e^{2ix}+e^{-2ix}\right)$
5.	$f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{2}{i\pi}\sum_{n\geq 1}\frac{e^{(2n-1)ix}-e^{-(2n-1)ix}}{2n-1}$
6.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{1}{2}+\sum_{n\geq 1}\frac{e^{i(2n-1)x}-e^{-i(2n-1)x}}{i\pi(2n-1)}$
7.	$f(x)=\lvert\sin(x)\rvert$	$\frac{2}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{e^{2inx}+e^{-2inx}}{4n^2-1}$
8.	$f(x)=\lvert\cos(x)\rvert$	$\frac{2}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{(-1)^n\left[e^{inx}+e^{-inx}\right]}{4n^2-1}$
9.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin(x), & 0<x<\pi.\end{cases}$	$\frac{1}{\pi}-\frac{1}{\pi}\sum_{n\geq 1}\frac{e^{2inx}+e^{-2inx}}{4n^2-1}+\frac{1}{4i}\left(e^{ix}-e^{-ix}\right)$
10.	$f(x)=x^2$	$\frac{\pi^2}{3}+2\sum_{n\geq 1}\frac{(-1)^n(e^{inx}+e^{-inx})}{n^2}$
11.	$f(x)=x(\pi-\lvert x\rvert)$	$\frac{4}{i\pi}\sum_{n\geq 1}\frac{e^{i(2n-1)x}-e^{-i(2n-1)x}}{(2n-1)^3}$
12.	$f(x)=e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{b-in}e^{inx}$
13.	$f(x)=\sinh x$	$\frac{\sinh(\pi)}{i\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}n}{n^2+1}\left[e^{inx}-e^{-inx}\right]$
14.	$f(x)=e^{ibx}$, $b\notin\mathbb{Z}$	$\sum_{n\in\mathbb{Z}}\frac{\sin(b\pi)(-1)^n}{\pi(b-n)}e^{inx}$

Table 1: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{e^{inx}\}_{n\in\mathbb{Z}}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x\notin(-\pi,\pi)$.

1.	$f(x)=x$	$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$.
2.	$f(x)=\|x\|$	$\frac{\pi}{2}-\frac{4}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2}$
3.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ x, & 0<x<\pi.\end{cases}$	$\frac{\pi}{4}-\frac{2}{\pi}\sum_{n\geq 1}\frac{\cos((2n-1)x)}{(2n-1)^2}+\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sin(nx)$
4.	$f(x)=\sin^2(x)$	$\frac{1}{2}-\frac{1}{2}\cos(2x)$
5.	$f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{4}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
6.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases}$	$\frac{1}{2}+\frac{2}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{2n-1}$
7.	$f(x)=\|\sin(x)\|$	$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1}$
8.	$f(x)=\|\cos(x)\|$	$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n\geq 1}\frac{(-1)^n\cos(2nx)}{4n^2-1}$
9.	$f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin(x), & 0<x<\pi.\end{cases}$	$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n\geq 1}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$
10.	$f(x)=x^2$	$\frac{\pi^2}{3}+4\sum_{n\geq 1}\frac{(-1)^n\cos(nx)}{n^2}$
11.	$f(x)=x(\pi-\|x\|)$	$\frac{8}{\pi}\sum_{n\geq 1}\frac{\sin((2n-1)x)}{(2n-1)^3}$
12.	$f(x)=e^{bx}$	$\frac{\sinh(b\pi)}{\pi}\left(\frac{1}{b}+\sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2}\left[2b\cos(nx)-2n\sin(nx)\right]\right)$
13.	$f(x)=\sinh x$	$\frac{2\sinh(\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}n}{n^2+1}\sin(nx)$
14.	$f(x)=e^{ibx}$, $b\notin\mathbb{Z}$	$\frac{\sin(b\pi)}{b\pi}+\frac{\sin(b\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{b^2-n^2}\left[2b\cos(nx)+2in\sin(nx)\right]$

Table 2: Here is a small collection of trigonometric Fourier expansions for functions in $\mathcal{L}^2(-\pi,\pi)$ in terms of the orthogonal base $\{1,\cos(nx),\sin(nx)\}_{n\geq 1}$. The series on the right are all 2π periodic functions, so the graph of these functions looks like the graph of $f(x)$ *on* $(-\pi,\pi)$. On the rest of the real line, outside of the interval $(-\pi,\pi)$ the graph of the series is copy-pasted repeatedly over the rest of the real line, so the series does *not* equal $f(x)$ for $x\notin(-\pi,\pi)$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x)=\begin{cases}1 & \|x\|<a\\ 0 & \|x\|>a\end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi)=\begin{cases}\pi & \|\xi\|<a\\ 0 & \|\xi\|>a\end{cases}$

Table 3: Above the function is on the left, its Fourier transform on the right. Here $a>0$ and $c\in\mathbb{R}$.

1.	$\Theta(t)f(t)$	$\widetilde{f(z)}$
2.	$\Theta(t-a)f(t-a)$	$e^{-az}\widetilde{f(z)}$
3.	$e^{ct}\Theta(t)f(t)$	$\widetilde{f(z-c)}$
4.	$\Theta(t)f(at)$	$a^{-1}\widetilde{f(a^{-1}z)}$
5.	$\Theta(t)f'(t)$	$z\widetilde{f(z)}-f(0)$
6.	$\Theta(t)f^{(k)}(t)$	$z^k\widetilde{f(z)}-\sum_0^{k-1}z^{k-1-j}f^{(j)}(0)$
7.	$\Theta(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f(z)}$
8.	$\Theta(t)tf(t)$	$-\widetilde{f}'(z)$
9.	$\Theta(t)t^{-1}f(t)$	$\int_z^\infty \widetilde{f(w)}dw$
10.	$\Theta f*\Theta g(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
11.	$\Theta(t)t^\nu e^{ct}$	$\Gamma(\nu+1)(z-c)^{-\nu-1}$
12.	$\Theta(t)(t+a)^{-1}$	$e^{az}\int_{az}^\infty \frac{e^{-u}}{u}du$
13.	$\Theta(t)\sin(ct)$	$\frac{c}{z^2+c^2}$
14.	$\Theta(t)\cos(ct)$	$\frac{z}{z^2+c^2}$

Table 4: Above, the function is on the left, its Laplace transform on the right. Here $a>0$ is constant and $c\in\mathbb{C}$.

15.	$\Theta(t)\sinh(ct)$	$\frac{c}{z^2-c^2}$
16.	$\Theta(t)\cosh(ct)$	$\frac{z}{z^2-c^2}$
17.	$\Theta(t)\sin(\sqrt{at})$	$\sqrt{\pi a}(4z^3)^{-1/2}e^{-a/(4z)}$
18.	$\Theta(t)t^{-1}\sin(\sqrt{at})$	$\pi\,\mathrm{erf}(\sqrt{a/(4z)})$
19.	$\Theta(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\,\mathrm{erfc}(z/2a)$
20.	$\Theta(t)\,\mathrm{erf}(at)$	$z^{-1}e^{z^2/(4a^2)}\,\mathrm{erfc}(z/(2a))$
21.	$\Theta(t)\,\mathrm{erf}(\sqrt{t})$	$(z\sqrt{z+1})^{-1}$
22.	$\Theta(t)e^t\,\mathrm{erf}(\sqrt{t})$	$((z-1)\sqrt{z})^{-1}$
23.	$\Theta(t)\,\mathrm{erfc}(a/(2\sqrt{t}))$	$z^{-1}e^{-a\sqrt{z}}$
24.	$\Theta(t)t^{-1/2}e^{-\sqrt{at}}$	$\sqrt{\pi/z}e^{a/(4z)}\,\mathrm{erfc}(\sqrt{a/(4z)})$
25.	$\Theta(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
26.	$\Theta(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
27.	$\Theta(t)t^{\nu}J_{\nu}(t)$	$2^{\nu}\pi^{-1/2}\Gamma(\nu+1/2)(z^2+1)^{-\nu-1/2}$
28.	$\Theta(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$

Table 5: Above, the function is on the left, its Laplace transform on the right. Here $a > 0$ is constant and $c \in \mathbb{C}$.