Fourieranalys MVE030 och Fourier Metoder MVE290 2023.mars.17
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA (highlights and sticky notes okay as long as no writing on them) & miniräknare som helst.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

1 Uppgifter

1. (The Bessel functions can be generated by a function that's exponentiated!). Bevisa att de Bessel funktioner uppfyller: (Prove that the Bessel functions satisfy):

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}, \quad \forall x \in \mathbb{R}, \quad z \in \mathbb{C} \setminus \{0\}.$$

(10p)

We begin by writing out the power series expansion for the exponential functions

$$(2p) \quad e^{xz/2} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$(2p) \quad e^{-x/(2z)} = \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

Each of these is worth 2 points as you probably figured out by the notation above. If you fudge up but get it partly right, you'd get one out of two points.

These series converge beautifully, absolutely and uniformly for z in compact subsets of $\mathbb{C} \setminus \{0\}$. So, since we presume that $z \neq 0$, we can multiply these series:

$$(1p) \quad e^{xz/2} e^{-x/(2z)} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k\geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}.$$

Correctly combining the two series is worth 1 point. We then introduce a new variable,

$$(1p) \quad n := j - k.$$

This is worth one point. We then compute that

$$(1p) \quad j = n + k,$$

$$(1p) \quad j + k = n + 2k.$$

So, we have that

$$(1p) \quad e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{n+2k}\frac{z^n}{\Gamma(n+k+1)k!}.$$

It is okay if you have $(n+k)!$ downstairs instead of $\Gamma(n+k+1)$.

Now, we just need to observe that what we have sitting there is the Bessel function since

$$(1p) \quad J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(k+n+1)}.$$

Hence, we have indeed see that

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} J_n(x)z^n.$$

Here are a few notes about the grading in general. Each problem will be graded by either (1) following this point scheme or (2) if you've basically got a correct solution but make some mistakes here and there then we may instead calculate your score by subtracting the # of f-ups from 10. The result according to both ways is equivalent, but if there is any discrepancy then we take the higher value. ALSO: if you skip steps but obtain the right results you will get full credit. We are not pedantic about these steps, they are just to help decide how to give partial credit for stuff that is messed up. Right is right, good is good, steps or no steps. So, as long as it is a step that we in the teaching team could also skip ourselves (thus maybe you are just as clever as we are, why not?) then we give you full credit for the result (without taking the steps). If it is something that say JR could never in a million years possibly do in her head, then we might be skeptical that you did it in your head with zero steps (unless you use some magic beta formula in which case you should write that down so we know this was the miracle that occurred). We do NOT do half points. Our grading policy is strictly Diophantine.

2. (Fourier series pass the test, they can approximate the best!) Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara en ortogonal mängd i ett Hilbert-rum, H. Om $f \in H$ och

$$\sum_{n\in\mathbb{N}} c_n\phi_n \in H,$$

bevisa olikheten:

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle\phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n\phi_n||.$$

Bevisa att olikheten blir en likhet precis om:

$$c_n = \langle f, \phi_n\rangle \quad \forall n \in \mathbb{N}.$$

(10 p)

It is convenient to introduce a bit of terminology. Let

$$\widehat{f_n} = \langle f, \phi_n\rangle, \quad g = \sum_n \widehat{f_n}\phi_n, \quad \varphi := \sum_n c_n\phi_n.$$

We would like to compare $||f - g||$ and $||f - \varphi||$. You don't have to introduce this notation if you don't want to.

$$(1p) \quad ||f - g + g - \varphi||^2 = \langle f - g + g - \varphi, f - g + g - \varphi\rangle$$

$$(1p) \quad = ||f - g||^2 + ||g - \varphi||^2 + \langle f - g, g - \varphi\rangle + \langle g - \varphi, f - g\rangle$$

$$(1p) \quad = ||f - g||^2 + ||g - \varphi||^2 + 2\,\mathrm{Re}\langle f - g, g - \varphi\rangle.$$

The first two terms are clear, so we will work out the last term and then take its real part.

$$(1p) \quad \langle f - g, g - \varphi\rangle = \langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$

$$(2p) \quad = \sum_n \overline{\overline{\widehat{f_n}}}\langle f, \phi_n\rangle - \sum_n \overline{c_n}\langle f, \phi_n\rangle - \sum_n \widehat{f_n}\langle \phi_n, \sum_m \widehat{f_m}\phi_m\rangle + \sum_n \widehat{f_n}\langle \phi_n, \sum_m c_m\phi_m\rangle$$

$$(1p) \quad = \sum |\widehat{f_n}|^2 - \sum \overline{c_n}\widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n}\overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have therefore shown that

$$(1p) \quad ||f - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 \geq ||f - g||^2.$$

Equality holds if and only if

$$(1p) \quad ||g - \varphi||^2 = 0 \iff g = \varphi.$$

By their definitions this holds if and only if

$$(1p) \quad \widehat{f_n} = c_n \forall n.$$

3. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(t)\cos(x), & 0 < t,\ -\pi < x < \pi \\ u(-\pi,t) = u(\pi,t), & t > 0, \\ u_x(-\pi,t) = u_x(\pi,t), & t > 0, \\ u(x,0) = |x| - \pi, & x \in [-\pi,\pi]. \end{cases}$$

(10 p)

(a) (1p) SLPs are the keys to solving inhomogeneous pde's. Even if you do nothing else, this rhyme is worth one point. Equivalently, you get a point if you set up the SLP to solve

$$X'' + \lambda X = 0, \quad X(-\pi) = X(\pi), \quad X'(-\pi) = X'(\pi).$$

(or written equivalently).

(b) (2p) Solve this SLP. You should obtain (see the rings of Saturn example in Chapter 1 of the textbook for the derivation of these solutions)

$$X_n(x) = e^{inx}, \quad n \in \mathbb{Z}.$$

One point for the correct function and one point for the correct range of n. Note that you could also have $\{\sin(nx)\}_{n\geq 1}$ together with $\cos(nx)\}_{n\geq 0}$. This is equivalent and correct.

(c) (1p) Set up the solution you seek to be a series

$$u(x,t) = \sum_{n\in\mathbb{Z}} T_n(t) X_n(x),$$

where we will need to solve for the T_n functions using the inhomogeneous pde together with the initial condition.

(d) (2p) Expand the inhomogeneity in terms of the X_n base:

$$\sin(t)\cos(x) = \sum_{n\in\mathbb{Z}} \sin(t)\frac{\langle\cos(x), X_n\rangle}{||X_n||^2}X_n(x) = \sum_{n\in\mathbb{Z}} c_n \sin(t)X_n(x),$$

with

$$\langle\cos(x), X_n\rangle = \int_{-\pi}^{\pi}\cos(x)\overline{X_n(x)}dx, \quad ||X_n||^2 = \int_{-\pi}^{\pi}|X_n(x)|^2dx.$$

It is okay if you leave these integrals like this (and don't calculate them) as long as you have correctly defined the scalar product and the norm squared. Each of these correctly defined is worth one point. It is possible to simplify life by observing that

$$\cos(x) = \frac{e^{ix}+e^{-ix}}{2},$$

and the functions e^{inx} are orthogonal. So, the coefficients

$$c_n = \begin{cases} \frac{1}{2}, & n = \pm 1, \\ 0, & n \neq \pm 1. \end{cases}$$

(e) (1p) Plug u into the heat equation (correctly) to obtain

$$u_t - u_{xx} = \sum_{n\in\mathbb{Z}}(T_n'(t) + n^2T_n(t))X_n(x).$$

(f) (1p) Set up the equation for T_n to solve

$$T_n'(t) + n^2T_n(t) = c_n\sin(t).$$

(g) (1p) Set up the correct initial condition

$$\sum_{n\in\mathbb{Z}} T_n(0)X_n(x) = |x| - \pi = \sum_{n\in\mathbb{Z}} C_nX_n(x),$$

with

$$C_n = \frac{\langle|x|-\pi, X_n\rangle}{||X_n||^2}.$$

(If you have correctly defined the scalar product and norm squared you do not need to write it out again). It is possible that you used the table to calculate these coefficients in which case you would have found:

$$C_0 = \frac{3\pi}{2}, \quad C_{2n+1} = -\frac{2}{\pi(2n+1)^2}, \quad n \in \mathbb{Z}, \quad C_{\pm 2n} = 0, \quad n \geq 1.$$

However you do NOT have to calculate these coefficients. It is fine if you simply write out what they are and you correctly define the scalar product and norm squared!

(h) (1p) Solve the ODE for $T_n(t)$. The method of integrating factor will give you

$$e^{-n^2 t}\left[\int_0^t e^{n^2 s} c_n \sin(s) ds + C_n\right].$$

4. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t = \Delta u, & 0 < z < H,\ -\pi < \theta < \pi,\ 0 < r < L,\ 0 < t, \\ u_z(r,0,\theta,t) = 0, & u_z(r,H,\theta,t) = 0, \\ u(L,z,\theta,t) = 0, \\ u(r,z,\theta,0) = 20, \end{cases}$$

där $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$. (10 p)

(a) (1p) Separate variables in polar coordinates.

(b) (1p) Correctly apply the pde:

$$T' R\Theta Z = TR''\Theta Z + r^{-1} TR'\Theta Z + r^{-2} T\Theta'' RZ + Z'' R\Theta T.$$

(c) (1p) Separate variables to solve for either Z or Θ first. Either one works. For Theta you will get the SLP

$$\Theta'' = \text{ constant times } \Theta, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi).$$

Then as in the previous exercise the solutions are:

$$\Theta_n(\theta) = e^{in\theta}.$$

Note that you could also have $\{\sin(nx)\}_{n\geq 1}$ together with $\cos(nx)\}_{n\geq 0}$. This is equivalent and correct.

(d) (1p) Next solve for Z. You will get the SLP

$$Z'' = \text{ constant times } Z, \quad Z'(0) = 0 = Z'(H).$$

The solutions are (up to constant multiples)

$$Z_n(z) = \cos(n\pi z/H), \quad n \geq 0,\, n \in \mathbb{Z}.$$

Alternatively, you could skip both of these by arguing that nothing will change in the z direction and the IC is independent of theta so the solution will be also. You are correct. So if you do this you get these 2 points for the Z and Theta parts of the solution immediately.

(e) (1p) Next solve for R. To do this we re-arrange to get

$$\frac{T'}{T} = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} \iff \frac{T'}{T} - \frac{Z''}{Z} = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} = \mu$$

for some constant μ. We then re-arrange the right side to get

$$r^2\frac{R''}{R} + r\frac{R'}{R} - r^2\mu = -\frac{\Theta''}{\Theta}.$$

For Θ_n this right side is n^2 so we have

$$r^2\frac{R''}{R} + r\frac{R'}{R} - r^2\mu = n^2 \iff r^2R'' + rR' + (-r^2\mu - n^2)R = 0.$$

(f) (1p) Recognize that R satisfies this equation if and only if $F(x)$ with $x = r\sqrt{|\mu|}$ satisfies either the modified Bessel equation if $\mu > 0$ or the Bessel equation if $\mu < 0$ or an Euler equation if $\mu = 0$.

(g) (1p) Throw away the solutions to the modified Bessel and Euler equations because they either are not physical or cannot satisfy the boundary condition $R(L) = 0$.

(h) (1p) Find the R part of the solution to be

$$R_{n,k}(r) = J_{|n|}(r\pi_{n,k}/L), \quad \mu_{n,k} = -\frac{\pi_{n,k}^2}{L^2}$$

where $\pi_{n,k}$ is the k^{th} positive zero of $J_{|n|}$. If you only do this in the case $n = 0$ it is correct, because in the end the only terms in the solution that will be non-zero are the terms with J_0. So if you reduced to that case here and above it is fine. If you didn't though, it is also still correct.

(i) (1p) Use everything to now solve for the T function. Recall that Z_m satisfies

$$\frac{Z_m''}{Z_m} = -\frac{m^2\pi^2}{H^2}$$

we have

$$\frac{T'}{T} - \frac{Z''}{Z} = \mu = -\frac{\pi_{n,k}^2}{L^2} \implies T'(t) = \lambda_{n,m,k}T(t), \quad \lambda_{n,m,k} = -\frac{\pi_{n,k}^2}{L^2} - \frac{m^2\pi^2}{H^2}$$

so

$$T_{n,m,k}(t) = c_{n,m,k}e^{\lambda_{n,m,k}t}.$$

(j) (1p) Present your solution and define what the coefficients are:

$$u(r, z, \theta, t) = \sum_{n\in\mathbb{Z}, m\geq 0, k\geq 1} T_{n,m,k}(t)\Theta_n(\theta)Z_m(z)R_{n,k}(r),$$

with

$$c_{n,m,k} = \frac{\int_0^L \int_0^H \int_{-\pi}^{\pi} 20\overline{\Theta_n(\theta)Z_m(z)R_{n,k}(r)}d\theta dz r dr}{\int_0^L \int_0^H \int_{-\pi}^{\pi} |\Theta_n(\theta)Z_m(z)R_{n,k}(r)|^2 d\theta dz r dr}.$$

Here you can alternatively see that the only non-zero terms are

$$c_{0,0,k}$$

because $\{\Theta_n(\theta)\}_{n\in\mathbb{Z}} = \{e^{in\theta}\}_{n\in\mathbb{Z}}$ are an orthogonal base. So, since $\Theta_0 = 1$, it is orthogonal to all the other Θ_n for all $n \neq 0$, and we are integrating the constant function 20 times Θ_n. So this shows that non-zero terms all have $n = 0$. Similarly, $\{Z_m(x) = \cos(m\pi z/H)\}_{m\geq 0}$ are also an orthogonal base, and $Z_0 = 1$. So it is orthogonal to all the other Z_m for all $m \geq 0$, and we are integrating the constant function 20 times Z_m. So, this shows that non-zero terms all have $m = 0$. So, this is another way to see that we could just take the function to be constant in both z and θ from the start. However, even if you did not do this and dragged around your thetas and your zs, you could still get the right answer (you would just have a lot of zero terms sitting in your solution). If you did realize this, then you would have just had

$$u(r, z, \theta, t) = \sum_{k\geq 1} c_k e^{-\pi_{0,k}^2 t/L^2} J_0(r\pi_{0,k}/L), \quad c_k = \frac{\int_0^L 20 J_0(r\pi_{0,k}/L) r dr}{\int_0^L |J_0(r\pi_{0,k}/L)|^2 r dr}.$$

Although it is not required, it is possible to calculate these coefficients rather nicely using the Bessel function fun facts provided in the exam. For the numerator, we use the recurrence relation

$$(xJ_1(x))' = xJ_0(x).$$

So, we make a substitution

$$x = r\pi_{0,k}/L \implies dx = dr\pi_{0,k}/L$$

$$\implies \int_0^L 20 J_0(r\pi_{0,k}/L) r dr = 20 \int_0^{\pi_{0,k}} x J_0(x) \frac{L^2}{\pi_{0,k}^2} dx$$

$$\overset{(xJ_1)'=xJ_0}{=} \frac{20L^2}{\pi_{0,k}^2} xJ_1(x)|_0^{\pi_{0,k}} = \frac{20L^2}{\pi_{0,k}} J_1(\pi_{0,k}).$$

For the denominator, this is computed for us in the formula collection at the end of the exam:

$$\int_0^L |J_0(r\pi_{0,k}/L)|^2 r dr = \frac{L^2}{2} J_1(\pi_{0,k}).$$

So, we obtain that

$$c_k = \frac{40}{\pi_{0,k}}.$$

You didn't need to do that calculation, but just in case anybody did and wanted to check they got it right, here it is.

5. Beräkna: (Compute):

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\pi^2 + n^2}.$$

(10p)

We are rather lucky because we have been generously given a table that says that the trig Fourier series of the function e^{bx} in $\mathcal{L}^2(-\pi,\pi)$ is

$$\frac{\sinh(b\pi)}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b - in} e^{inx}.$$

It is worth a whopping 5 points to identify a function whose trig Fourier series can be used to compute this series. This is pretty much hit or miss - either the function you choose *can* be used to calculate the series or it *cannot* (meaning there is no way to make the function you choose work).

Parseval method:. With the function that I chose, I use the Parseval equality or equivalently the infinite dimensional Pythagorean theorem to get:

$$(1p)\ ||e^{bx}||^2 = \int_{-\pi}^{\pi} e^{2bx} dx = \frac{e^{2b\pi} - e^{-2b\pi}}{2b} = \frac{\sinh(2b\pi)}{b}$$

$$(1p)\ = \sum_{n\in\mathbb{Z}} \left\| \frac{\sinh(b\pi)}{\pi} \frac{(-1)^n}{b-in} e^{inx} \right\|^2$$

$$(1p)\ = \sum_{n\in\mathbb{Z}} \frac{\sinh(b\pi)^2}{\pi^2} \frac{1}{b^2+n^2} 2\pi$$

$$= \frac{2\sinh(b\pi)^2}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{b^2+n^2}$$

One point for setting this equal to the norm on the other side:

$$(1p)\ \frac{\sinh(2b\pi)}{b} = 2\frac{\sinh(b\pi)^2}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{b^2+n^2} \iff \frac{\pi\sinh(2b\pi)}{2b\sinh(b\pi)^2} = \sum_{n\in\mathbb{Z}} \frac{1}{b^2+n^2}.$$

Setting $b = \pi$ we get

$$(1p)\ \frac{\sinh(2\pi^2)}{2\sinh(\pi^2)^2} = \sum_{n\in\mathbb{Z}} \frac{1}{\pi^2+n^2}.$$

Pointwise convergence of trig Fourier series method:
(1p) For choosing the correct point and that is $x = \pi$ or $x = -\pi$. With either of these the series becomes

$$(1p)\ \frac{\sinh(b\pi)}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{b-in} e^{\pm in\pi} = \frac{\sinh(b\pi)}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{b-in}$$

$$\frac{\sinh(b\pi)}{\pi} \left[\frac{1}{b} + 2b \sum_{n\geq 1} \frac{1}{b^2+n^2} \right].$$

Two points for using the theorem correctly to say that this is

$$(2p)\ \frac{e^{\pi b} + e^{-\pi b}}{2} = \cosh(b\pi).$$

Then one last point for doing the arithmetic to eek out the desired value with $b = \pi$:

$$\frac{1}{b}\left(\cosh(b\pi)\frac{\pi}{\sinh(b\pi)} - \frac{1}{b}\right) = 2\sum_{n\geq 1}\frac{1}{b^2+n^2}$$

$$\implies \sum_{n\in\mathbb{Z}}\frac{1}{b^2+n^2} = \frac{1}{b^2}+2\sum_{n\geq 1}\frac{1}{b^2+n^2} = \frac{1}{b^2}+\frac{1}{b}\left(\cosh(\pi b)\frac{\pi}{\sinh(b\pi)} - \frac{1}{b}\right)$$

$$= \frac{\pi\cosh(b\pi)}{b\sinh(b\pi)}.$$

Setting $b = \pi$ we get

$$= \frac{\cosh(\pi^2)}{\sinh(\pi^2)}.$$

If you are concerned that this doesn't look like the answer from the previous method, note that the doubling formula for the hyperbolic sine gives

$$\sinh(2\pi^2) = 2\sinh(\pi^2)\cosh(\pi^2),$$

so our first answer

$$\frac{\sinh(2\pi^2)}{2\sinh(\pi^2)^2} = \frac{2\sinh(\pi^2)\cosh(\pi^2)}{2\sinh(\pi^2)^2} = \frac{\cosh(\pi^2)}{\sinh(\pi^2)}.$$

So indeed our answers match. I would be super impressed if anybody solved this BOTH ways just to be totally sure they are right... I have NEVER seen anyone do that - but hope springs eternal.

6. Lös problemet: (Solve the following problem):

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, & x > 0, \quad y > 0, \\ u(0,y) = f(y) & \in \mathcal{L}^2(0,\infty), \\ u(x,0) = g(x) & \in \mathcal{L}^2(0,\infty). \end{cases}$$

   (a) (2p) Split this into two separate problems each occurring in a half space. The problems are

$$p_{xx}(x,y)+p_{yy}(x,y) = 0, \quad x,y > 0, \quad p(0,y) = f(y), \quad p(x,0) = 0.$$

$$q_{xx}(x,y)+q_{yy}(x,y) = 0, \quad x,y > 0, \quad q(0,y) = 0, \quad q(x,0) = g(x).$$

   Then the solution will be $u = p + q$.

(b) (2p) Pick one of these to solve first. Doesn't matter which one you pick. I will solve first for p. The two points are for recognizing that we should (1p) extend oddly in the y variable due to the boundary condition at $y = 0$ and (1p) use the Fourier transform in the y variable. Each of these observations is worth one point. (Analogously if you chose to solve for the q function first).

(c) (2p) Solve this using the Fourier transform:

$$\widehat{p}_{xx}(x,\xi) + \widehat{p}_{yy}(x,\xi) = 0 = \widehat{p}_{xx}(x,\xi) - \xi^2 \widehat{p}(x,\xi) = 0$$

$$(1p) \qquad \implies \widehat{p}(x,\xi) = a(\xi)e^{\xi x} + b(\xi)e^{-\xi x}.$$

One point for obtaining this. Since the Fourier transform preserves parity, our function $\widehat{p}(x,\xi)$ is an odd function in ξ. Consider $\xi > 0$. Since $x > 0$ the term $e^{\xi x}$ is not the Fourier transform of anything contained in $\mathcal{L}^2$. So we try to solve using the other term. The BC says that

$$\widehat{p}(0,\xi) = \hat{f}_o(\xi),$$

the odd extension of f. So, we will try to solve using

$$\widehat{p}(x,\xi) = \hat{f}_o(\xi)e^{-\xi x}, \quad x > 0, \quad \xi > 0.$$

Note that the Fourier transform preserves oddness, so the left side is odd, and therefore the right side should be odd as well. The term $\hat{f}_o$ is because f_o is odd and Fourier transform preserves parity. So, to make the *whole* right side also odd, we therefore extend the function $e^{-\xi x}$ to the negative real axis to be even. A whopping one point for figuring this out:

$$(1p) \quad \widehat{p}(x,\xi) = \hat{f}_o(\xi)e^{-|\xi|x}, \quad x > 0, \, \xi \in \mathbb{R}.$$

(d) (2p) Now, we look on the table of Fourier transforms and find that the function whose Fourier transform is $e^{-|\xi|x}$. We see that the function

$$\frac{1}{y^2 + a^2} \xrightarrow{FT} \frac{\pi}{a} e^{-a|\xi|}$$

where the transform is done in the y variable and $a > 0$ is a constant. So, take $a = x$ and re-arrange to get that

$$(1p) \quad \frac{x}{\pi}(y^2 + x^2)^{-1} \xrightarrow{FT} e^{-|\xi|x}.$$

The second point is for then using the table of Fourier transforms once more to get that:

$$p(x,y)$$

$$(1p) \;= \int_{\mathbb{R}} f_o(y-z)\frac{x}{\pi}(z^2+x^2)^{-1}dz = \int_{\mathbb{R}} f_o(z)\frac{x}{\pi}((y-z)^2+x^2)^{-1}dz$$

$$= \int_{-\infty}^{0} -f(-z)\frac{x}{\pi}((y-z)^2+x^2)^{-1}dz + \int_0^{\infty} f(z)\frac{x}{\pi}((y-z)^2+x^2)^{-1}dz$$

$$= \int_{\infty}^{0} f(s)\frac{x}{\pi}((y+s)^2+x^2)^{-1}ds + \int_0^{\infty} f(z)\frac{x}{\pi}((y-z)^2+x^2)^{-1}dz$$

$$= \int_0^{\infty} f(s)\frac{x}{\pi}\left[((y-s)^2+x^2)^{-1} - ((y+s)^2+x^2))^{-1}\right] ds.$$

If you just write that it is a convolution but don't get the convolution correctly defined you lose this point. You do NOT have to do all the unravelling that I have done here to make it pretty. The top line is correct and sufficient.

(e) (2p) The solution q is found in very much the same way. So two more points for getting the correct

$$q(x,y) = \int_0^{\infty} g(s)\frac{y}{\pi}\left[((x-s)^2+y^2)^{-1} - ((x+s)^2+y^2))^{-1}\right] ds.$$

If you did not make it pretty and simply left it as

$$q(x,y) == \int_{\mathbb{R}} g_o(x-z)\frac{y}{\pi}(z^2+y^2)^{-1}dz = \int_{\mathbb{R}} g_o(z)\frac{y}{\pi}((x-z)^2+y^2)^{-1}dz$$

you get the full 2 points. That is because it's nice to be pretty, but the bottom line is mathematical correctness and the less pretty solution is also correct and concise.

(10 p)

7. Lös problemet: (Solve the following problem):

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t, x > 0, \\ u(0,t) = e^t, & t > 0, \\ u(x,0) = 0, & x > 0 \end{cases}$$

(a) (2p) Recognize that you should use the Laplace transform (1 point for that) and that the transform should be in the t variable (1 point for that).

(b) (2p) Correctly Laplace transform the pde

$$\widetilde{u}_t(x,z) = \widetilde{u}_{xx}(x,z) = z\widetilde{u}(x,z) - u(x,0) = z\widetilde{u}(x,z).$$

(c) (2p) Solve this ode for

$$\widetilde{u}(x,z) = a(z)e^{x\sqrt{z}} + b(z)e^{-x\sqrt{z}}.$$

(d) (2p) One point for throwing away the $e^{x\sqrt{z}}$. The second point is for finding

$$\widetilde{u}(0,z) = \widetilde{e^t}(z) = b(z) \implies \widetilde{u}(x,z) = \widetilde{e^t}(z)e^{-x\sqrt{z}}.$$

(e) (2p) Use the table of Laplace transforms to obtain that the function

$$\Theta(t)\frac{x}{2\sqrt{\pi}t^{3/2}}e^{-x^2/(4t)} \longrightarrow e^{-x\sqrt{z}}.$$

Hence

$$u(x,t) = \int_{\mathbb{R}} e^{t-s}\Theta(t-s)\Theta(s)\frac{x}{2\sqrt{\pi}s^{3/2}}e^{-x^2/(4s)}ds.$$

If you just write that $u(x,t)$ is the convolution of e^t and this other $\frac{x}{2\sqrt{\pi}t^{3/2}}e^{-x^2/(4t)}$ function but you mess up the definition of convolution and/or forget the heavisides, you get one out of two points for this part.

(10 p)

8. Hitta et polynomet $p(x)$ av högst grad 17 som minimeras (Find the polynomial $p(x)$ of at most degree 17 which minimizes the following integral):

$$\int_{-4}^{4} |e^{\cos(x)} - p(x)|^2 dx.$$

(10p)

Let $P_n(x)$ denote the Legendre polynomial of degree n. Then we know that these polynomials are an orthogonal base for $\mathcal{L}^2(-1,1)$. So, we do a little calculation by setting $x/4 = t$:

$$\int_{-4}^{4} P_n(x/4)P_m(x/4)dx = \int_{-1}^{1} P_n(t)P_m(t)(4dt)$$

$$= \begin{cases} 0, & n \neq m, \\ 4||P_n||^2 = \frac{8}{2n+1}, \&n = m. \end{cases}$$

So, the polynomials $\{P_n(x/4)\}_{n\geq 0}$ are an orthogonal base for $\mathcal{L}^2(-4,4)$. For notational convenience let

$$\wp_n(x) := P_n(x/4), \quad f(x) := e^{\cos(x)}.$$

Consequently the polynomial we seek is

$$p(x) = \sum_{n=0}^{17} \frac{\langle f, \wp_n \rangle}{||\wp_n||^2} \wp_n(x),$$

with

$$\langle f, \wp_n \rangle = \int_{-4}^{4} f(x)\overline{\wp_n(x)}dx = \int_{-4}^{4} e^{\cos(x)} P_n(x/4)dx,$$

$$||\wp_n||^2 = \int_{-4}^{4} |\wp_n(x)|^2 dx = \frac{8}{2n+1}.$$

So the points breakdown is like this:

(a) 2 points for using Legendre polynomials. As Beyonce would say, you need to SAY their name. (Not just write P_n without explaining what P_n is!!) If you write P_n but don't say their name, you get 1 out of 2 points.

(b) 2 points for modifying them to $P_n(x/4)$. (this is pretty much all or nothing)

(c) 2 points for the correct scalar product: $\langle f, \wp_n \rangle = \int_{-4}^{4} e^{\cos(x)} P_n(x/4)dx$. (i don't really see how to do partial credit on this part either, probably all or nothing here too).

(d) 2 points for the correct norm downstairs: $||\wp_n||^2 = \int_{-4}^{4} |\wp_n(x)|^2 dx = \frac{8}{2n+1}$. (similarly, unclear how to do partial credit here?)

(e) 2 points for putting it all together correctly. (I suppose if you do everything else right but goof this up in some minor way, then you could get 1 point instead of 2).

Please keep in mind that you do not actually have to compute any integrals here! Just write down what the correct integrals are!