Re-Exam 2 in English

ⓘ This is a preview of the published version of the quiz

Started: 24 Aug at 15:01

Quiz instructions

Here is the zoom link: https://chalmers.zoom.us/j/68731694655
Solutions can be directly entered into Canvas, or if you have problems with that, please email: julie.rowlett@chalmers.se
You can also ask questions to Julie by phone: +46 31 772 34 19.
Please do NOT send text message solutions via phone as this will not work!
All study aids are allowed, but please do not communicate with anyone other than the proctor and/or instructor. There are 40 problems total, each worth 2 points. Betygsgränser för 3: 40 poäng, för 4: 53 poäng, för 5: 67 poäng.
You may write in English, Swedish (German and French are also fine if you want to have even more fun). You are free to switch between these languages as you wish. You may submit your exam in any readable format as well as using a combination of formats (hand-written for some parts, typed for other parts), just do what works best for you and make sure it's readable!
You got this!!! May the mathematical force be with you ♥

Question 1 — 2 pts

We expand e^{3x} = exp(3x) in a Fourier series using the orthogonal basis $\{e^{in\pi x/4}\}_{n\in\mathbb{Z}}$ on the Hilbert space $\mathcal{L}^2(-4,4)$. What is the value of the series when x=16?

- sinh(48)
- exp(6)
- sinh(6)
- cosh(12)
- cosh(48)
- cosh(6)

- exp(48)
- 1

Question 2 2 pts

Evaluate the same series as in the preceding exercise at x=-12.

- exp(-36)
- cosh(12)
- 1
- sinh(-36)

Question 3 2 pts

Let $\sum_{n\in\mathbb{Z}} c_n e^{in\pi x/4}$ be the series from the preceding two questions. Evaluate $\sum_{n\in\mathbb{Z}} |c_n|^2$.

- cosh(12)/12
- sinh(24)/24
- sinh(12)/12
- cosh(24)/24

Question 4 2 pts

Are the following boundary conditions self-adjoint for the SLP:
$f''(x) + \lambda f(x) = 0, \quad 0 < x < 1, \quad f'(0) = 0, \quad f'(1) = 0.$

- No.
- Yes.

Question 5

2 pts

You are solving a regular SLP of the form

$L(f) + \lambda f = 0$

on an interval (a,b). If you obtain [Select] , then you know you have made a mistake.

Question 6

2 pts

Assume that $\{f_n\}_{n\geq 1}$ are all of the solutions to

$f''(x) + \lambda f(x) = 0, x \in (0, 1), \quad f(0) = f(1) = 0,$

with corresponding eigenvalues $\{\lambda_n\}_{n\geq 1}$.

Can you use these to solve the following problem?

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < 1,\ t > 0 \\ u(x,0) = \sin(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

- No.
- Yes.

Maybe.

Question 7

2 pts

Consider the problem:

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < 5,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(5,t) = 5 & t > 0 \\ u(x,0) = e^x. \end{cases}$$

What should we do first to solve this problem?

- Apply the Fourier transform in x.
- Solve a regular SLP.
- Find a steady state solution.
- Separate variables.
- Apply the Laplace transform in t.

Question 8

2 pts

The solution to the preceding problem will involve:

- Both sines and cosines.
- An inverse Fourier transform.
- A linear function.
- An inverse Laplace transform.

- Cosines.

Question 9

2 pts

Evaluate the full solution at t=0 and x=10. The closest approximation is:

- 0
- 1
- 10
- 22026

Question 10

2 pts

For the solution u(x,t) to the problem in the preceding 3 questions, does $\lim_{t \to \infty} u(x,t)$ exist?

- Yes, but only for x between 0 and 5.
- Yes for all x.
- Yes, and it is 0 for all x.
- No.

Question 11

2 pts

Consider the problem $\begin{cases} u_{tt} - u_{xx} = 2xt, & 0 < x < 5,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(5,t) = 0 & t > 0 \\ u(x,0) = F(x) & \\ u_t(x,0) = G(x), & \end{cases}$

assuming that F and G are continuous functions on [0, 5].

Can any part of the solution to the previous PDE be recycled here?

- Yes, the steady state solution.
- No.
- Yes, the solutions of a regular SLP.
- Yes, the Laplace transform.
- Yes, the Fourier transform.
- Yes, the time dependent part of the solution.

Question 12 **2 pts**

Let the solution to the preceding problem be v(x,t). Does

$\lim_{t\to\infty} v(x,t)$ exist?

- No, there is no limit because the solution is unbounded.
- Yes, for some values of x but not all.
- Yes, for all values of x, but we cannot compute it without knowing F and G.
- No, there is no limit because the solution always oscillates.

Question 13

2 pts

A donut has just finished cooking and is starting to cool at room temperature. Mathematically this is described by:

$$\begin{cases} u_t - \Delta u = 0 & t > 0,\ 5 \le r \le 10,\ -\pi < \theta \le \pi,\ 0 \le z \le 5 \\ u(r,\theta,z,0) = 180 & 5 \le r \le 10,\ -\pi < \theta \le \pi,\ 0 \le z \le 5 \\ u(10,\theta,z,t) = 20 & t > 0,\ -\pi < \theta \le \pi,\ 0 \le z \le 5 \\ u(5,\theta,z,t) = 20 & t > 0,\ 5 \le r \le 10,\ -\pi < \theta \le \pi,\ 0 \le z \le 5 \\ u(r,\theta,0,t) = 20 & t > 0,\ 5 \le r \le 10,\ -\pi < \theta \le \pi,\ 0 \le z \le 5 \\ u(r,\theta,5,t) = 20 & t > 0,\ -\pi < \theta \le \pi,\ 0 \le z \le 5 \end{cases}$$

What should we do first to solve this problem?

- Apply the Laplace transform.
- Separate variables.
- Solve Bessel's equation.
- Apply the Fourier transform.
- Find a steady state solution.

Question 14

2 pts

Does

$\lim_{\theta \to \infty} u(r,\theta,z,t)$ exist?

- No, the solution is unbounded.
- Yes.
- No, the solution oscillates.
- Yes, and it is 20.

Question 15

2 pts

The solution u to the preceding problem contains what ingredients?

- an inverse Laplace transform
- an inverse Fourier transform
- cosines and real exponentials
- sines and real exponentials

Question 16

2 pts

Consider the vibrating string with fixed ends:

$$\begin{cases} u_{tt} = 4u_{xx} & 0 < t,\ 0 < x < \pi \\ u(x,t) = 0 & x = 0, x = \pi, \\ u(x,0) = \sin(x), & \\ u_t(x,0) = 0. & \end{cases}$$

Solve this. The solution contains:

- an inverse Fourier transform
- sine and cosine
- hyperbolic sine and hyperbolic cosine
- an inverse Laplace transform

Question 17

2 pts

Evaluate the solution at $x = \pi/2, t = \pi/2$. The closest value is:

- 3
- -3
- 0
- -1
- 1

Question 18 2 pts

Evaluate the solution at $x = \pi/6, t = 100\pi$. The closest value is

- 628
- -3
- 0.87
- 0.5

Question 19 2 pts

For the solution to the preceding couple of problems, does $\lim_{t\to\infty} u(x,t)$ exist?

- No, the solution is unbounded.
- Yes, and it is 0.

- No, the solution oscillates for all x.
- For certain values of x, yes.

Question 20

2 pts

Which technique can we use to solve

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < t,\ -\pi < x < \pi \\ u(x,0) = \frac{\sin(x)}{x} \\ u_t(x,0) = 0 \\ u(\pm\pi, t) = 0 \end{cases}$$

- A Laplace series.
- A Fourier series.
- The Fourier transform in x.
- The Laplace transform in t.

Question 21

2 pts

How would you solve the following equation for u?

$$u(x) + \int_{-\infty}^{\infty} e^{-|x-t|} u(t)dt = \frac{\sin(x)}{x}.$$

- Solve a Bessel equation.
- Apply the Fourier transform in x.
- Separate variables.

- Compute an inverse Laplace transform.

Question 22 2 pts

Let u be the solution to the preceding problem. The closest value to u(0) is:

- 0.5
- 0
- -1
- 1

Question 23 2 pts

For u in the preceding problem, does the following limit exist?

$\lim_{x \to \infty} u(x)$

- Yes, and it is 1.
- No, because the solution is unbounded.
- Yes, and it is zero.
- No, because the function oscillates.

Question 24 2 pts

Determine a and b that minimize

$\int_{-\pi}^{\pi} |x - a\sin(x) - b\cos(x)|^2 dx.$

- a=2, b=0.
- a=2*pi, b=0.
- b=2*pi, a=0.
- b=2, a=0.
- a=b=0.

Question 25

2 pts

Consider the following problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 \qquad 0 < x < 1,\ 0 < y \\ u_y(x,0) = 0 \\ u(1,y) = \frac{\sin(y)}{y} \\ u(0,y) = 0. \end{cases}$$

What technique will solve this problem?

- Fourier transform in x and odd extension.
- Laplace transform in x.
- Laplace transform in y.
- Fourier transform in x and even extension.
- Fourier transform in y and odd extension.
- Fourier transform in y and even extension.

- Fourier series in x.

Question 26 **2 pts**

The solution is of the form:

- A finite integral involving cosine and hyperbolic sines.
- An inverse Laplace transform.
- A finite integral involving sine and hyperbolic cosines.
- An integral over the whole real line.

Question 27 **2 pts**

Consider the following problem:

$$\begin{cases} u_t - u_{xx} = 0 \quad t, x > 0 \\ u(0,t) = e^t \\ u(x,0) = 0. \end{cases}$$

What technique will solve this problem?

- Laplace transform in t.
- Fourier transform in x and odd extension.
- Fourier transform in x and even extension.
- A Laplace series.

Question 28 **2 pts**

The solution to the preceding problem can be expressed as:

- an infinite integral with respect to the space variable
- a convolution involving hyperbolic trig functions
- a finite integral with respect to the time variable
- a Fourier series
- an inverse Fourier transform

Question 29 **2 pts**

Let u(x,t) be the solution from the preceding problem. Does

$\lim_{x \to \infty} u(x, t)$ exist?

- No.
- Only for t=0, in which case it is 0.
- Yes, and it is zero for any fixed t>0.

Question 30 **2 pts**

Choose to answer one of the following:

1. What topic did you most enjoy learning about in this course and why?

OR

2. What topic did you least enjoy learning about in this course and why?

(don't worry, I won't take it personal :)

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Question 31

2 pts

Assume that $\{\phi_n\}_{n\geq 1}$ is an orthonormal set in a Hilbert space H. If f is an element of H, what do we know about:

$\sum_{n\geq 1} < f, \phi_n > \phi_n$?

- It is an element of the Hilbert space that is not longer than f.
- Nothing, not enough information.
- It is an element of the Hilbert space.
- It is equal to f.

Question 32

2 pts

Assume that f is a continuous function on the interval [0, 1]. What does it mean if

$\int_0^1 \sin(n\pi x) f(x) = 0$ for all n?

- f is a constant function.
- f is an even function.
- f is identically zero.
- There is not enough information to say anything further.

Question 33

2 pts

Define

$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x-inx} dx.$

What can we say about the following series as N tends to infinity?

$\sum_{n=-N}^{N} c_n e^{inx}$

- It converges for all x.
- It converges for some x.
- It converges to the exponential function for all x.
- It converges for all x, and it can be differentiated termwise.

Question 34 **2 pts**

For c_n from the preceding problem, what can we say about

$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n$

?

- It converges to one.
- It diverges.
- It converges to cosh(pi).
- It converges to something.

Question 35 **2 pts**

Consider

$\lim_{t\to 0} \int_{-\infty}^{\infty} \frac{e^{-|y|/t}}{t} \cos(\pi - y) dy.$

- The limit is 1.
- The limit does not exist, because it keeps oscillating.
- The limit is 0.
- The limit does not exist because it tends to infinity.
- The limit is -1.

Question 36 **2 pts**

What theorem critically requires a consequence of Bessel's inequality for its proof?

- The sampling theorem.
- The pointwise convergence of Fourier series.
- Plancharel's theorem.
- The convolution approximation theorem (big bad cat).

Question 37 2 pts

Let J_n denote the Bessel function of order n. Consider

$\sum_{n \in \mathbb{Z}} J_n(x)(-1)^n$

- It only converges when x=0.
- It is equal to 1 for all x.
- It converges for all x to something finite.
- It might or might not converge, depending on the value of x.

Question 38 2 pts

Assume that a bounded function f satisfies f(x) = 0 for |x| > 100. Can we use this to deduce something about the following series?

$\sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{100}\right) \frac{\sin(n\pi)}{n\pi}$

- It converges to the sum of the left and right limits of f.
- Yes, it converges, but we need more information about f to compute the limit.
- Yes, it converges to f(0).
- It might or might not converge, depending on f.

Question 39

2 pts

If we know that we can compute

$\int_{-\infty}^{\infty} |f(x)|^2 dx$

and the value is a real number, can we also compute

$\int_{-\infty}^{\infty} f(x) dx$

?

- No, not necessarily.
- Yes, using the Fourier transform of f.
- By the Sampling Theorem, it is finite, but we need to know more about f to compute it.
- By the Cauchy-Schwarz inequality it is finite, but we need to know more about f to compute it.

Question 40

2 pts

Please answer one of the following (your choice!):

1. What is your favorite theory item from this course and why?

OR

2. What is your least favorite theory item from this course and why?

I hope this went well for you, but even in worst case scenario and you gotta retake it, don't stress! I will always be happy to see you again if you decide to re-take the course and am open to any suggestions to help improve your learning experience!

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Solutions to re-exam August 2021

Julie Rowlett[1]

1

August 25, 2021

1. We expand e^{3x} using the basis $\{e^{in\pi x/4}\}$ on $(-4,4)$. The result is an 8 periodic function. So,
$$16 = 0 + 8 + 8,$$
hence the value of the series is the value of the original function at $0 \in (-4,4)$, that is one.

2. At $x = -12$, we again use the 8 periodicity. Note that $-12 = -4 - 8$, so the value is the same as whatever the series converges to at -4. This is an endpoint, so the Fourier series converges to the average of the left and right limits, by the Lagom Convergence Theorem. So, this is
$$\frac{e^{3*-4} + e^{3*4}}{2} = \cosh(12).$$

3. We use the Parseval equality that says
$$\sum_{\mathbb{Z}} |\gamma_n|^2 = \int_{-4}^{4} |e^{3x}|^2 dx = \frac{e^{24} - e^{-24}}{6} = \frac{\sinh(24)}{3}.$$
for Fourier coefficients γ_n corresponding to *normalized* basis functions. So, we just have to figure out how to relate the series with unnormalized coefficients to the one with normalized coefficients. Since the basis functions have norms $\sqrt{8}$, then
$$\sum c_n e^{in\pi x/4} = \sum \sqrt{8} c_n \frac{e^{in\pi x/4}}{\sqrt{8}}.$$
So, $\gamma_n = \sqrt{8} c_n$, hence
$$\sum |c_n|^2 = \frac{1}{8} \sum |\gamma_n|^2 = \frac{\sinh(24)}{24}.$$

4. Yes, check the definition, it's self adjoint.

5. Eigenvalues according to the cute theorem about SLPs will tend to $+\infty$ when the problem is written this way. The will *not* tend to $-\infty$, so that is the problem. Everything else is possible.

6. Yes. The BCs are self adjoint, the operator L is the same in both the SLP and the PDE, so indeed we can use the solutions of this regular SLP to build the solutions to the PDE.

7. There is an undesirable boundary condition. To be able to proceed with all the other steps of solving the problem, we need to get rid of that 5 at $x = 5$. So we search for a steady state solution. It should satisfy

$$-f''(x) = 0, \quad f(0) = 0, \quad f(5) = 5.$$

We integrate 0 twice obtaining a linear function of x. To make these BCs work out, we see that this function is just

$$f(x) = x.$$

Let's keep this around for future use.

8. See, we need that SS to complete the next exercise. We solve the same nice homogeneous PDE, but now with good BCs. BUT, we subtract the SS from the initial condition because we will add the solution v to the SS to get the full solution u.

$$\begin{cases} v_t - v_{xx} = 0 \\ v(0,t) = 0 \\ v(5,t) = 0 \\ v(x,0) = e^x - x \end{cases}.$$

The full solution will be $u = v + x$. Everything is beautiful and perfectly attuned for variable separation, so we should totally do that next:

$$T'X - X''T = 0 \iff \frac{T'}{T} = \frac{X''}{X} \implies \text{both sides are constant.}$$

So we call the constant λ and solve for X because the BCs now say

$$X(0) = X(5) = 0.$$

We solve this and obtain (do the cases for the constant if needed!)

$$X_n(x) = \sin(n\pi x/5).$$

Going back to see what T_n will be, it satisfies $T_n' = -\frac{n^2\pi^2}{5^2}T_n$, so up to constant factors $T_n = e^{-n^2\pi^2 t/5^2}$. The solution is

$$u(x,t) = \sum_{n\geq 1} c_n T_n(t) X_n(x) + f(x).$$

The c_n we get by expanding $e^x - f(x)$ in terms of the OB $\{X_n\}$. However to compute this exercise, we don't need to do that. We just look at what is in our solution. We have sines, real exponentials, coefficients, and the SS which is the function $f(x) = x$. As previously observed it is linear. So the only answer on the list that matches what we've got is 'a linear function.'

9. At $t = 0$, *and* $x = 10$ this is a little tricky. Also, I should have made this more precise and written $0 < x < 5$ for the IC. Sorry about that. For example, if $x = t = 0$ we cannot have both conditions $u(0, t) = 0$, and $u(x, 0) = e^x$. How the ... is that supposed to work? So, for problems like these, I will make it more clear in the future. To explain the answer here, just look at what we end up with

$$x + \sum_{n \geq 1} c_n e^{-n^2\pi^2 t/25} \sin(n\pi x/5).$$

If we stuff in $t = 0$ and $x = 10$, all the sines run away and hide, and we're left with just $x = 10$. That's the answer.

10. Now we look at what happens when $t \to \infty$. Well, those exponentials all run away to zero. Hence, we are left with just the SS part, and that has a well defined limit for any x.

11. We can recycle the X_n part of the solution here (solns of RSLP).

12. So, we know the method for solving problems like this, and it's going to give a solution of the form

$$\sum X_n(x)T_n(t),$$

for new T_n so that we get the PDE stuff on the right and the ICs. Let me write out how we do that. We start by expanding the right side of the PDE in terms of the basis $X_n(x) = \sin(n\pi x/5)$. The t just comes along for the ride, so it looks like:

$$\sum_{n \geq 1} a_n t X_n(x), \quad a_n = \frac{4}{5}\int_0^5 x \sin(n\pi x/5) dx.$$

Note that the factor in front comes from the 2 together with the fact that the norms of the X_n are $5/2$. (integrate the sine squared from 0 to 5, this is what you get). We then take the series with those unknown T_n and hit it with the PDE on the left:

$$\sum T_n''(t)X_n - T_n(t)X_n''(x) = \sum X_n(x)(T_n''(t) + n^2\pi^2/25T_n(t)).$$

Here we used the fact that $X_n''(x) = -n^2\pi^2/25X_n(x)$. We set this equal to the other series, and then equate the terms:

$$a_n t X_n(x) = X_n(x)(T_n''(t) + n^2\pi^2/25T_n(t)) \iff a_n t = T_n''(t) + n^2\pi^2/25T_n(t).$$

The solution T_n to the ODE will be a sum of a particular solution and a solution to the same ODE but homogeneous, that is

$$T_n''(t) + n^2\pi^2/25T_n(t) = 0 \iff T_n(t) = A_n \cos(n\pi t/5) + B_n \sin(n\pi t/5).$$

A particular solution is the function

$$\frac{25a_n t}{n^2\pi^2}.$$

So, we'll have

$$T_n(t) = \frac{25a_n t}{n^2\pi^2} + A_n \cos(n\pi t/5) + B_n \sin(n\pi t/5), \quad v(x,t) = \sum T_n(t)X_n(x).$$

The coefficients will be determined by the IC data, F and G. However, we can already see that this is going to have problems with $T_n(t)$ as $t \to \infty$ coming from that first part. BUT those $X_n(x)$ are multiplying everything. They vanish quite a lot. Specifically at all integer multiples of 5. So at all those points, the whole solution simply vanishes, and the limit as $t \to \infty$ exists.

13. This also has inhomogeneous boundary conditions, so we should start by finding a steady state solution.

14. The solution is independent of θ, so yeah, this limit exists.

15. The SS solution is just 20. After dealing with that, the BCs become all zero. So, in the z variable, we will just get sines. In the t variable, it's a heat equation, so we're going to get real exponentials. There will not be cosines due to the boundary conditions (the r part will be some horrible Bessel nightmare, by the way). Laplace and Fourier transforms are useless here. Hence the correct answer is sines and real exponentials. I spared you having to figure out that Bessel stuff.

16. Vibrating string on a bounded interval, nice boundary conditions, so we just separate variables right away.

$$\frac{T''}{T} = 4\frac{X''}{X} = \lambda \text{ constant.}$$

The solution vanishes at $x = 0$ and $x = \pi$, so we solve the X part first,

$$X'' = \lambda X, \quad X(0) = X(\pi) = 0.$$

So, the only solutions to this (go through the cases for λ if you need to!), are (up to constant factors that we'll figure out later)

$$X_n(x) = \sin(nx).$$

Returning to T we get

$$\frac{T_n''}{T_n} = 4(-n^2) \implies T_n(t) = a_n \cos(2nt) + b_n \sin(2nt).$$

The full solution is

$$\sum_{n \geq 1} (a_n \cos(2nt) + b_n \sin(2nt)) \sin(nx).$$

The coefficients are obtained by the initial conditions. The easiest condition is that $u_t(x, 0) = 0$, and for this it requires $b_n = 0$ for all n. Then we use the other IC, to get that

$$a_n = \frac{2}{\pi} \int_0^\pi \sin(x) \sin(nx) dx = \begin{cases} 1 & n = 1 \\ 0 & n > 1. \end{cases}$$

So the solution is just $\cos(2t) \sin(x)$. How nice!

17. Evaluate at $x = \pi/2$, $t = \pi/2$ and the sine is 1, while the cosine is -1. Hence answer -1.

18. Evaluate at $x = \pi/6$ and $t = 100$. The cosine part will just be 1, whereas $\sin(\pi/6) = 1/2$. Draw a unit disk if needed (hey that's what I do if I forget these). Or since everything is allowed, like google it.

19. When $t \to \infty$ that cosine will oscillate around. BUT as we have seen before, if the sine vanishes, then the whole thing is constant and zero for all t. So, at all the x points where the sine vanishes, i.e. integer multiples of π, the limit exists. So, the answer is for certain values of x, yes.

20. This is another bounded interval, so Fourier series is our friend.

21. Fourier transform in x, use the fact that Fourier transform turns that convolution into a product.

22. This is a lot of work for 2 points, but then again, the previous one was not so much work for 2 points, so it evens out. OR if you're clever, you figure out all the easy ones first, and THEN at the end work on these harder ones. That's a good strategy. So, anyways, let's get to it.

$$\hat{u}(\xi) + \hat{u}(\xi) \frac{2}{\xi^2 + 1} = \pi \chi(\xi) = \pi \begin{cases} 1 & |\xi| < 1 \\ 0 & |\xi| \geq 1. \end{cases}$$

We solve for $\hat{u}$,

$$\hat{u}(\xi) = \pi \chi(\xi) \frac{\xi^2 + 1}{\xi^2 + 3}.$$

I'll re-write this as

$$\pi \chi(\xi) \left(1 - \frac{2}{\xi^2 + 3} \right).$$

We need to go backwards from the Fourier transform. The first term came from the sine, so we get

$$u(x) = \frac{\sin(x)}{x} - \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\chi(\xi)\frac{1}{\xi^2+3}d\xi.$$

With the definition of χ this becomes

$$u(x) = \frac{\sin(x)}{x} - \int_{-1}^{1} e^{ix\xi}\frac{1}{\xi^2+3}d\xi.$$

At $x = 0$ the first part becomes one (okay, you gotta take a limit, but you learned this in the first analysis course). So it remains to calculate that integral. It's going to be some arctan stuff. Watch:

$$u(0) = 1 - 2\int_0^1 \frac{1}{\xi^2+3}d\xi = 1 - \frac{2}{3}\int_0^1 \frac{1}{(\xi/\sqrt{3})^2+1}d\xi.$$

We change variables to $y = \xi/\sqrt{3}$, $dy = d\xi/\sqrt{3}$ and get

$$1-\frac{2}{\sqrt{3}}\int_0^{1/\sqrt{3}} \frac{1}{y^2+1}dy = 1-\frac{2}{\sqrt{3}}\Big(\arctan(1/\sqrt{3}) - \arctan(0)\Big) = 1-\frac{2}{\sqrt{3}}\arctan(1/\sqrt{3}).$$

To evaluate I would just google that last stuff. It's about 0.6. So, the whole thing is about 0.4, and the closest answer is 0.5.

23. This is also a little tough. The first part of u, $\sin(x)/x$ that vanishes when $x \to \infty$. Look at the second part, it is

$$\int_{-\infty}^{\infty} e^{ix\xi}\chi(\xi)\frac{1}{\xi^2+3}d\xi = \hat{v}(-x),$$

that is the Fourier transform of the function

$$v(\xi) := \chi(\xi)\frac{1}{\xi^2+3}$$

evaluated at the point $-x$. Note that this function v is an $L1$ function, so the Riemann Lebesgue Lemma says that its Fourier transform tends to zero at both $\pm\infty$. This allows us to conclude that the limit is zero. Cool use of theory right?

24. Best approximation theorem and the fact that the cosine and sine are orthogonal on this interval tells us that

$$a = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)dx, \quad b = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(x)dx.$$

The second of these vanishes. The first one we do using integration by parts and come up with

$$\frac{2}{\pi}\left(-x\cos(x)\right)_0^{\pi} = 2.$$

25. Seriously, the best way to prepare for the exam is to do loads of problems. It is so much like martial arts. So, with that in mind we see this big region in the y variable, but it's not the whole line, only half. So we check the boundary conditions. They are NICE. The derivative wrt y at zero vanishes. This points to the even/odd extension method, and since it's the derivative, the extension should be even. So, the method is extend evenly and Fourier transform in y.

26. Now we actually have to solve this. Okay.

$$\hat{u}_{xx}(x,\xi) + \hat{u}_{yy}(x,\xi) = 0 \iff \hat{u}_{xx}(x,\xi) + (i\xi)^2\hat{u}(x,\xi) = 0,$$

thus

$$\hat{u}(x,\xi) = a(\xi)\cosh(x\xi) + b(\xi)\sinh(x\xi).$$

We need $u(0,y) = 0$ and if we Fourier transform this we get

$$\hat{u}(0,\xi) = 0 \implies a(\xi) = 0.$$

The other condition gives us

$$\hat{u}(1,\xi) = \pi\chi(\xi),$$

that same stuff as before, remember? So,

$$\hat{u}(1,\xi) = b(\xi)\sinh(\xi) = \pi\chi(\xi) \implies b(\xi) = \pi\frac{\chi(\xi)}{\sinh(\xi)}.$$

Then, our solution is obtained by taking an inverse Fourier transform:

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iy\xi}\pi\frac{\chi(\xi)}{\sinh(\xi)}\sinh(x\xi)d\xi.$$

Well, that χ simplifies things to

$$\frac{1}{2}\int_{-1}^{1} e^{iy\xi}\frac{\sinh(\xi)}{\sinh(x\xi)}d\xi.$$

That ration of sinhs is an even function. $e^{iy\xi} = \cos(y\xi) + i\sin(y\xi)$. So only the cosine integral will survive, and we are left with

$$u(x,y) = \int_0^1 \cos(y\xi)\frac{\sinh(\xi)}{\sinh(x\xi)}d\xi.$$

That is a finite integral involving cosine and hyperbolic sines.

27. This problem is a typical Laplace transform problem. LT in t.

28. Well, now we just have to solve it. We hit the PDE with the LT in t,

$$\widetilde{u}_t(x,z) - \widetilde{u}_{xx}(x,z) = 0.$$

By the LT properties and the IC,

$$z\widetilde{u}(x,z) = \widetilde{u}_{xx}(x,z).$$

Hence,

$$\widetilde{u}(x,z) = a(z)e^{x\sqrt{z}} + b(z)e^{-x\sqrt{z}}.$$

Why did I use these instead of sinh and cosh you may ask? It's because it's easier to read off the decay properties with these two. Sinh and cosh both don't decay, so for it to work with them, I'd probably need to use both of them and get them to cancel off the non-decaying bits. If you think about how that would work, you'll realize that we'd need to get the exponentially growing stuff to go away, hence, only have the exponentially decaying stuff left. So, it's really all equivalent, but in this case we remember that in order for this to work, we need to be able to inverse LT, and for this the solution must decay when the real part of z tends to infinity. Since $x > 0$ in this problem, that means that the first term will not work. So we better hope the second one works. You see, sinh and cosh contain *both* terms in their definition. So, yeah it would be equivalent to use them, it would just take longer to come to this point. I jumped the gun based on - you guessed it - experience solving these problems. For the BC we need

$$\widetilde{u}(0,z) = \text{ LT of } e^t = \int_0^\infty e^t e^{-zt}dt = \frac{1}{1-z} = b(z).$$

Then, we'll obtain the solution by inverse Laplace transforming. The easiest way to do this is look for functions whose transforms are the things we end up with. Then we know that the product of LTs was obtained by transforming their convolution. We know where the b part came from, so we just need somebody whose LT is $e^{-x\sqrt{z}}$. That is provided by a table or internet search or whatever you use. Don't forget that the Laplace transform can be applied as long as we stay on the heavy side. So, the convolution has a heavyside for each function. The thing whose LT is that $e^{-x\sqrt{z}}$ is

$$\Theta(t)t^{-3/2}e^{-x^2/(4t)} \text{ has LT } \frac{2}{x}\sqrt{\pi}e^{-x\sqrt{z}}$$

$$\implies \Theta(t)\frac{x}{2\sqrt{\pi}t^{3/2}}e^{-x^2/(4t)} \text{ has LT } e^{-x\sqrt{z}}.$$

Remember, the integration (transform) is in the t variable, so x is just hanging out, coming along for the ride. Similar to the factors of 2 and $\sqrt{\pi}$. We can just juggle those around because we integrate wrt t. So, u is obtained by convolving the function whose LT gave us the $b(z)$, that is e^t BUT this function must vanish for $t < 0$ so it is really $\Theta(t)e^t$, together with the other function up there. The convolution is in the time variable, and it is precisely therefore

$$u(x,t) = \int_{-\infty}^{\infty} \Theta(t-s)e^{t-s}\Theta(s)\frac{x}{2\sqrt{\pi}s^{3/2}}e^{-x^2/(4s)}ds.$$

These heavysides kill off large chunks of the integral, leaving only

$$u(x,t) = \int_0^t e^{t-s}\frac{x}{2\sqrt{\pi}s^{3/2}}e^{-x^2/(4s)}ds.$$

It is a finite integral w.r.t. time. Maybe you could see that was going to happen from the beginning?

29. Now we look at the limit as $x \to \infty$ of the solution:

$$u(x,t) = \int_0^t e^{t-s}\frac{x}{2\sqrt{\pi}s^{3/2}}e^{-x^2/(4s)}ds.$$

Well, for any fixed $t > 0$, that exponential term decays super amazingly fast as $x \to \infty$, for all points in the integral except at $s = 0$. But that is just one point. So we can safely ignore it and conclude that the limit is zero for any $t > 0$.

30. Favorite or least favorite topic? I find it hard to convey Bessel functions, people seem to find them really confusing. So from a teaching perspective, this is a challenge.

31. This is an element in the Hilbert space whose length is at most that of f, by Bessel's inequality.

32. These sines are an OB for the interval $[0,1]$. The 3 equivalent conditions says that if the scalar product of f with all of these is zero, then f is zero.

33. This is the partial Fourier series for e^x. It converges for all x, but it cannot be differentiated termwise. (Try it! You'll run amok). Moreover, it only converges to e^x inside $(-\pi, \pi)$. Don't let your series go to waste, always remember to copy paste! Outside the interval the series stops being the function and is just that copy pasted 2π periodic version.

34. Well, if we stuff $x = 0$ into the series we get this thing. Since $x = 0$ is smack in the middle of the interval, and e^x is continuous there, the Fourier series converges to the value of the function $e^0 = 1$.

35. Convolution approximation theorem - big bad CAT - in action! The cosine is a bounded function, and the function $e^{-|y|}$ is in L^1. The variable t is playing the role of epsilon in that theorem. Note that

$$\int_{-\infty}^0 e^{-|y|}dy = 1 = \int_0^\infty e^{-|y|}dy, \quad \int_{-\infty}^\infty e^{-|y|}dy.$$

The function cosine is continuous at π, so the theorem says that the limit is

$$\cos(\pi) + \cos(\pi) = -2.$$

The closest was -1 in the list of answers, but nobody seemed to notice that this was off by a factor of 2. It should be -2.

36. Bessel's inequality is the result that allows us to conclude that the stuff we are trying to prove is zero actually is zero. The pointwise convergence of Fourier series is precisely the theorem that uses this. We subtract the stuff to which we want to show the series converges and then do a bunch of estimating/juggling around. In the end we prove that this stuff is none other than two Fourier coefficients of a piecewise C1 function, hence by Bessel's inequality as the big N tends to infinity they vanish.

37. Generating function for Bessel! This says that:

$$e^{x/1(z-1/z)} = \sum_{-\infty}^{\infty} J_n(x) z^n.$$

Set $z = -1$. Note that $-1 - (1/-1) = 0$. And $e^0 = 1$. Tada!

38. Sampling theorem in action! The function satisfies the hypotheses. The t part is missing in the sampling theorem statement, that is what happens if $t = 0$. Hence sampling theorem says this series converges to $f(0)$.

39. This means our function is in L^2 and thus Fourier transformable. Note that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} e^{i0x} f(x)dx = \hat{f}(0).$$

So we can make sense of this with help of the Fourier transform.

40. From a challenge perspective I like the Lagom convergence of Fourier series and the big bad cat. From a not freaking out perspective, I like cute facts about SLPs and Fourier coeffs of function and its derivative. I am a theoretical mathematician after all, so I love all theory items... Hopefully some of the fondness rubs off, even if you all are not such theory types.