Re-Exam in English

ⓘ This is a preview of the published version of the quiz

Started: 9 Jun at 18:27

Quiz instructions

Here is the zoom link: https://chalmers.zoom.us/j/63634430789

Solutions can be directly entered into Canvas, or if you have problems with that, please email: julie.rowlett@chalmers.se

You can also ask questions to Julie by phone: +46 31 772 34 19.

Please do NOT send text message solutions via phone as this will not work!

All study aids are allowed, but please do not communicate with anyone other than the proctor and/or instructor. There are 40 problems total, each worth 2 points. Betygsgränser för 3: 40 poäng, för 4: 53 poäng, för 5: 67 poäng.

You may write in English, Swedish (German and French are also fine if you want to have even more fun). You are free to switch between these languages as you wish. You may submit your exam in any readable format as well as using a combination of formats (hand-written for some parts, typed for other parts), just do what works best for you and make sure it's readable!

You got this!!! May the mathematical force be with you ♥

Question 1 — 2 pts

We expand e^{2x} = exp(2x) in a Fourier series using the orthogonal basis $\{e^{in\pi x/4}\}_{n\in\mathbb{Z}}$ on the Hilbert space $\mathcal{L}^2(-4,4)$. What is the value of the series when x=16?

- cosh(32)
- exp(8)
- sinh(8)
- exp(32)
- 1
- cosh(8)
- sinh(32)

Question 2

2 pts

Evaluate the same series as in the preceding exercise at x=4.

- cosh(8)
- exp(8)
- cosh(4)
- sinh(4)

Question 3

2 pts

Let $\sum_{n\in Z} c_n e^{in\pi x/4}$ be the series from the preceding two questions. Evaluate $\sum_{n\in Z} |c_n|^2$.

- sinh(16)/16
- cosh(16)/16
- sinh(4)
- cosh(4)

Question 4

2 pts

Are the following boundary conditions self-adjoint for the SLP:

$$f''(x) + \lambda f(x) = 0, \quad 0 < x < 1, \quad f(0) = 0, \quad f'(1) = 0.$$

- Yes.

No.

Question 5 **2 pts**

You are solving a regular SLP of the form

$L(f) + \lambda f = 0$

on an interval (a,b). If you obtain [Select] , then you know you have made a mistake.

Question 6 **2 pts**

Assume that $\{f_n\}_{n\geq 1}$ are the eigenfunctions of a regular SLP on the interval (0, 1),

$L(f) + \lambda f = 0$,

subject to self-adjoint boundary conditions,

with corresponding eigenvalues $\{\lambda_n\}_{n\geq 1}$.

Can you use these to help solve the following problem?

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < 1,\ t > 0 \\ u(x,0) = \sin(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

- Maybe.
- Yes.
- No.

Question 7 **2 pts**

Consider the problem:

$$\begin{cases} u_t - u_{xx} = 2x^2, & 0 < x < 5,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(5,t) = 5 & t > 0 \\ u(x,0) = e^x. \end{cases}$$

What should we do first to solve this problem?

- Separate variables.
- Apply the Laplace transform in t.
- Solve a regular SLP.
- Find a steady state solution.
- Apply the Fourier transform in x.

Question 8 **2 pts**

What should we do next to solve the problem in the preceding exercise?

- Apply the Fourier transform in x.
- Apply the Laplace transform in t.
- Compute a Laplace series.
- Apply the Fourier transform in t.
- Separate variables and solve a regular SLP.
- Separate variables and solve for the time dependent function and its coefficients.

Question 9 — 2 pts

Evaluate the full solution at t=0 and x=10. The closest approximation is:

- 0
- 200
- 22026
- -1448

Question 10 — 2 pts

For the solution u(x,t) to the problem in the preceding 3 questions, does $\lim_{t\to\infty} u(x,t)$ exist for x=1?

- No.
- Yes, and it is 0.
- Yes, and it is approximately 22.
- Yes, and it is approximately 13.

Question 11 — 2 pts

Consider the problem $\begin{cases} u_{tt} - u_{xx} = 2xt, & 0 < x < 5,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(5,t) = 0 & t > 0 \\ u(x,0) = F(x) \\ u_t(x,0) = G(x), \end{cases}$

assuming that F and G are continuous functions on [0, 5].

Can any part of the solution to the previous PDE be recycled here?

- Yes, the time dependent part of the solution.
- Yes, the solutions of a regular SLP.
- Yes, the steady state solution.
- No.

Question 12 **2 pts**

Let the solution to the preceding problem be v(x,t). Does

$\lim_{t\to\infty} v(x,t)$ exist for x=10?

- Yes, but we cannot compute it without knowing F and G.
- Yes, and it is 0.
- No, there is no limit because the solution oscillates.
- No, there is no limit because the solution is unbounded there.

Question 13 **2 pts**

We hit a disk shaped drum right in its center and listen to the resulting vibration. Mathematically this is described by:

$$\begin{cases} u_{tt} - \Delta u = 0 & t > 0,\ 0 \le r \le 10,\ -\pi < \theta \le \pi, \\ u(r,\theta,0) = r - 10 & 0 \le r \le 10,\ -\pi < \theta \le \pi, \\ u_t(r,\theta,0) = 0 & 0 \le r \le 10,\ -\pi < \theta \le \pi, \\ u(10,\theta,t) = 0 & t > 0,\ -\pi < \theta \le \pi. \end{cases}$$

What should we do first to solve this problem?

- Apply the Laplace transform.
- Find a steady state solution to deal with the initial conditions.
- Apply the Fourier transform.
- Solve Bessel's equation.
- Separate variables.

Question 14 — 2 pts

Solve the preceding problem to obtain the function $u(r, \theta, t)$. Does $\lim_{\theta\to\infty} u(0, \theta, 0)$ exist?

- No, the solution oscillates.
- Yes, and it is 0.
- No, the solution is unbounded.
- Yes, and it is -10.

Question 15 — 2 pts

The solution u to the preceding problem contains what ingredients?

- Bessel functions of orders 0, 1, ... and all positive integers together with sines..
- Bessel functions of orders 0, 1, and all positive integers together with sines and cosines.
- The zeroth order Bessel function, sines and hyperbolic trig functions.
- The zeroth order Bessel function and cosines.

Question 16

2 pts

We just got a piping hot hamburger at Wendy's. The temperature of the patty at time t is:

$$\begin{cases} u_t - \Delta u = 0 & t > 0,\ 0 < x < 4,\ 0 < y < 4, 0 < z \\ u(x, y, z, t) = u(x, y, z, t) = 20 & x = 0, x = 4, y = 0, y = 4, z = 0, z \\ u(x, y, z, 0) = 150. & \end{cases}$$

This means that it is 150 degrees throughout, and the surrounding room temperature is 20 degrees. What should we do first to solve this problem?

- Apply the Fourier transform.
- Separate variables.
- Find a steady state solution.
- Apply the Laplace transform.

Question 17

2 pts

To obtain the full solution of this problem, one must:

- Calculate integrals involving sines.
- Solve an inhomogeneous ODE.
- Solve an Euler equation.
- Solve a Bessel equation.

Question 18 2 pts

Key ingredients in the solution are:

- Bessel functions and cosines
- sines and real exponentials
- Bessel functions and sines
- hyperbolic sines and cosines (sinh and cosh)

Question 19 2 pts

Let u(x,y,z,t) be the solution from the preceding problem. Does $\lim_{t\to\infty} u(x, y, z, t)$ exist?

- Yes, and it is 20.
- No.
- Yes, and it is 150.
- Not always, because it depends on the variables x, y, and z.
- Yes, and it is 0.

Question 20 2 pts

Which technique is an important part of solving:

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < t, \ -\infty < x < \infty \\ u(x,0) = \frac{\sin(x)}{x} \\ u_t(x,0) = 0 \end{cases}$$

- The Laplace transform in x.
- The Fourier transform in x.
- A Fourier series.
- A regular SLP.

Question 21 2 pts

The solution of the preceding problem is obtained as:

- An integral involving sine and cosine.
- An inverse Laplace transform.
- A Fourier series.
- An integral involving two cosines.

Question 22 2 pts

What is the polynomial of at most degree 3 that minimizes
$\int_{-2}^{2} |\cos(2x) - p(x)|^2 dx$?

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Question 23

2 pts

Determine a and b that minimize

$\int_{-\pi}^{\pi} |\cos(x) - a\sin(x) - b\sin(2x)|^2 dx.$

- a=0, b=1/2.
- a=b=0.
- a=1, b=1/2.
- a=1/2, b=1.

Question 24

2 pts

For a and b as in the preceding problem, what is the approximate value of the integral?

- -3
- 9
- 3
- 6

Question 25 **2 pts**

Consider the following problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < 1,\ 0 < y \\ u_y(x, 0) = 0 & \\ u(1, y) = e^{-y} & \\ u(0, y) = 0. & \end{cases}$$

What technique will solve this problem?

- Fourier transform in x and odd extension.
- Laplace transform in y.
- Fourier transform in x and even extension.
- Fourier transform in y and odd extension.
- Fourier series in x.
- Laplace transform in x.
- Fourier transform in y and even extension.

Question 26 **2 pts**

The solution is of the form:

- An integral involving cosine and hyperbolic sines.
- A convolution.
- An inverse Laplace transform.
- An integral involving sine and hyperbolic cosines.

Question 27 **2 pts**

Consider the following problem for p(t) a polynomial function of t:

$$\begin{cases} u_t - u_{xx} = 0 & t, x > 0 \\ u(0,t) = p(t) \\ u(x,0) = 0. \end{cases}$$

What technique will solve this problem?

- Laplace transform in t.
- A regular SLP and a Fourier series.
- Fourier transform in x and odd extension.
- Fourier transform in x and even extension.

Question 28 **2 pts**

Assume that the polynomial in the preceding problem is just the constant function, 1. Solve it. The solution can be expressed as:

- a Fourier series
- a convolution involving hyperbolic trig functions
- an integral from 0 to infinity with integrand that is never zero
- a convolution involving a rapidly decaying exponential function

Question 29 — 2 pts

Let u(x,t) be the solution from the preceding problem. Does

$\lim_{t \to \infty} u(x,t)$ exist?

- No, the solution oscillates.
- Yes, and it tends to 0 for all x>0.
- Yes, and it is a linear function of x.
- No, the solution is unbounded.

Question 30 — 2 pts

Choose to answer one of the following:

1. What topic did you most enjoy learning about in this course and why?

OR

2. What topic did you least enjoy learning about in this course and why? (don't worry, I won't take it personal :)

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Question 31 2 pts

Assume that $\{\phi_n\}_{n\geq 1}$ is an orthogonal set in a Hilbert space H. If f is an element of H, what do we know about:

$\sum_{n\geq 1} < f, \phi_n >$?

- It is equal to f.
- Nothing, not enough information.
- It converges, but we cannot determine its limit without more information.
- It is an element of the Hilbert space.

Question 32 2 pts

Assume that $\{\phi_n\}_{n\geq 1}$ is an orthonormal set in a Hilbert space H. If f is an element of H, is

$$||f - \langle f, \phi_1 \rangle \phi_1|| \leq ||f - \phi_1||?$$

- No, not necessarily.
- Yes.

Question 33 2 pts

What can we say about this function?

$$\sum_{n=-N}^{N} e^{inx}$$

- It is even.
- It is odd.
- It converges to a Hilbert function as N goes to infinity.
- It is a Bessel function.

Question 34 2 pts

We expand a function in a trigonometric Fourier series as

$\sum_{n \in \mathbb{Z}} c_n e^{inx}$. Can we use this to obtain the Fourier series of the derivative?

- Not without further information.
- Yes, differentiate termwise, the derivative has coefficients inc_n.

Question 35 2 pts

In the convolution approximation theorem, what should we estimate first to prove the theorem?

- A Fourier series.
- The integral near zero.
- A best approximation.
- The integral near infinity.

Question 36 2 pts

Why is Plancharel's theorem useful?

- For computing Fourier series.
- For computing integrals.
- For computing Laplace series.
- For solving regular SLPs.

Question 37 2 pts

Let J_n denote the Bessel function of order n. Consider

$\sum_{n \in Z} J_n(x)$.

- It only converges when x=0.
- It converges for all x to something finite.
- It is equal to 1 for all x.
- It might or might not converge, depending on the value of x.

Question 38 2 pts

What is a key technique in the proof of the orthogonality of the Hermite polynomials?

- Integration by parts.
- A convolution.
- A Taylor series expansion.
- A best approximation.

Question 39 2 pts

Let H_n be the Hermite polynomial of degree n. What can you say about $\sum_{n \geq 0} H_n(1) \frac{2^n}{n!}$?

- It is equal to 1.
- It is equal to 2.
- It diverges.

Question 40 **2 pts**

Please answer one of the following (your choice!):

1. What is your favorite theory item from this course and why?

OR

2. What is your least favorite theory item from this course and why?

HAVE AN AWESOME SUMMER!!!

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Not saved Submit quiz

Solutions to re-exam June 2021

Julie Rowlett[1]

1

June 9, 2021

1. We expand e^{2x} using the basis $\{e^{in\pi x/4}\}$ on $(-4,4)$. The result is an 8 periodic function. So,
$$16 = 0 + 8 + 8,$$
hence the value of the series is the value of the original function at $0 \in (-4,4)$, that is one.

2. At $x = 4$, the Fourier series converges to the average of the left and right limits, by the Lagom Convergence Theorem. So, this is
$$\frac{e^8 + e^{-8}}{2} = \cosh(8).$$

3. We use the Parseval equality that says
$$\sum_{\mathbb{Z}} |\gamma_n|^2 = \int_{-4}^{4} |e^{2x}|^2 dx = \frac{e^{16} - e^{-16}}{4} = \frac{\sinh(16)}{2}.$$
for Fourier coefficients γ_n corresponding to *normalized* basis functions. So, we just have to figure out how to relate the series with unnormalized coefficients to the one with normalized coefficients. Since the basis functions have norms $\sqrt{8}$, then
$$\sum c_n e^{in\pi x/4} = \sum \sqrt{8} c_n \frac{e^{in\pi x/4}}{\sqrt{8}}.$$
So, $\gamma_n = \sqrt{8} c_n$, hence
$$\sum |c_n|^2 = \frac{1}{8} \sum |\gamma_n|^2 = \frac{\sinh(16)}{16}.$$

4. Yes, check the definition, it's self adjoint.

5. Eigenvalues cannot be complex, so that's the right answer, everything else is possible.

6. Maybe. We don't know what L is, it might not be ∂_{xx}. We also don't know what the self-adjoint BCs are, they could be different. If the L is ∂_{xx} and the BCs are the same, then we totally can use the solutions, otherwise we probably cannot.

7. EEEEK, get rid of that annoying but time independent inhomogeneity in the PDE. Find a steady state. At the same time, the SS can get rid of that annoying 5 in the boundary. Let's do it:

$$-f''(x) = 2x^2, \quad f(0) = 0, \quad f(5) = 5.$$

So, we get

$$f'(x) = -\frac{2x^3}{3} + A \implies f(x) = -\frac{x^4}{6} + Ax + B, \quad f(0) = 0 \implies B = 0.$$

The condition at 5 gives:

$$-\frac{5^4}{6} + 5A = 5 \implies A = 1 + \frac{5^3}{6}.$$

Let's keep this around for future use.

8. Next, we can solve for a solution v to

$$\begin{cases} v_t - v_{xx} = 0 \\ v(0,t) = 0 \\ v(5,t) = 0 \\ v(x,0) = e^x - f(x) \end{cases}.$$

The full solution will be $u = v + f$. Everything is beautiful and perfectly attuned for variable separation, so we should totally do that next:

$$T'X - X''T = 0 \iff \frac{T'}{T} = \frac{X''}{X} \implies \text{both sides are constant.}$$

So we call the constant λ and solve for X because the BCs now say

$$X(0) = X(5) = 0.$$

These are self-adjoint boundary conditions for the regular SLP $X'' = \lambda x$. In SLP style we could use $\Lambda = -\lambda$ to write it as $X'' - \Lambda X = 0$. So in any case, separating variables and solving a regular SLP is the way to go.

9. When we solve the regular SLP, we get the basis for L2,

$$X_n(x) = \sin(n\pi x/5).$$

Going back to see what T_n will be, it satisfies $T_n' = -\frac{n^2\pi^2}{5^2}T_n$, so up to constant factors $T_n = e^{-n^2\pi^2 t/5^2}$. The solution is

$$u(x,t) = \sum_{n\geq 1} c_n T_n(t) X_n(x) + f(x).$$

The c_n we get by expanding $e^x - f(x)$ in terms of the OB $\{X_n\}$. However to compute this exercise, we don't need to do that. When $x = 10$, all of those sines are zero. So, all we get is the SS. Hence the value $u(10, 0)$ is just $f(10)$.

$$-\frac{10^4}{6} + 10\left(1 + \frac{5^3}{6}\right) \approx -1448.$$

10. For $x = 1$, letting $t \to \infty$, the series part of the solution vanishes, so we are just left with the SS. Its value at $x = 1$ is

$$-\frac{1}{6} + 1 + \frac{5^3}{6} = \frac{5 + 5^3}{6} \approx 22.$$

11. We can recycle the X_n part of the solution here (solns of RSLP).

12. So, we know the method for solving problems like this, and it's going to give a solution of the form

$$\sum X_n(x)T_n(t),$$

for new T_n so that we get the PDE stuff on the right and the ICs. For $x = 10$ all of the X_n vanish. Hence this answer is zero.

13. This is nice and homogeneous so the first thing we should do is separate variables.

14. The whole problem is independent of θ, so in fact we just look at the IC at $r = 0$ and read off -10.

15. Okay, let's solve it. Separate variables, use polar coordinates:

$$\frac{T''}{T} = \frac{\Delta R}{R} = \lambda,$$

$$\Delta R = R'' + r^{-1}R = \lambda R \iff r^2 R'' + rR - \lambda r^2 R = 0, \quad R(10) = 0.$$

If $\lambda > 0$ this is a modified Bessel equation of order zero and has solutions I_0 and K_0. The problem with these is that K_0 tends to infinity at $r = 0$ and I_0 is never zero. So they won't work. If $\lambda = 0$, we'll get a linear combination of like $\log(r)$ and a constant, which won't work because the log blows up at zero, and well, the only constant that works is just zero. Hence $\lambda < 0$, and the solutions are

$$J_0(\sqrt{|\lambda|}r).$$

To get the boundary condition at $r = 10$, $\sqrt{|\lambda_n|} * 10$ should be a positive zero of J_0, hence

$$\lambda_n = -\frac{z_n^2}{10^2},$$

where z_n is the n^{th} positive zero of J_0. Then we take this back to solve for the time part, and we get $\cos(z_n t/10)$ and $\sin(z_n t/10)$. The solution is then

$$\sum J_0(z_n r/10)\left(a_n \cos(z_n t/10) + b_n \sin(z_n t/10)\right).$$

We get the coefficients from the IC. The derivative at $t = 0$ vanishes, which means there are no sines. Only cosines. The coefficients we get by expanding $r - 10$ in terms of the OB of Bessel functions. So, our solution contains zeroth order Bessel functions and cosines.

16. Does anybody actually go to Wendy's anymore? Do they exist in Sweden? Does anyone else think hamburgers are just gross? Sorry, let's solve this. The boundary conditions need to go, so we use a SS to get rid of them. It's just 20.

17. We next proceed to separate variables. Since it is a box, it's like three one dimensional problems. Watch:

$$\frac{T'}{T} = \frac{\Delta XYZ}{XYZ} = \Lambda.$$

Then move over to the $\Delta XYZ = X''YZ + Y''XZ + Z''XY$. We can do the variable separation again for each of these, and we get equations that look like $X'' = cX$ for a constant c, with $X(0) = X(4) = 0$. Hence it's a sine. The same thing happens for Y and Z. So the XYZ part is just a product of three sines. The t part looks like $T' = \Lambda T$ so up to constant factor $T = e^{\Lambda T}$. We find the constant factors by expanding the IC minus the SS using the XYZ basis. What's that gonna look like? Integrals with sines!

18. So, based on what we just did, the key ingredients are sines and real exponentials. Why real? Because it's a heat equation, so the Λ will be ≤ 0, since it's a hamburger that's cooling off.

19. Yes the limit exists as $t \to \infty$, and you don't even need math to know that. Let your hamburger sit around and it tends to room temperature (or some other person or animal eats it).

20. Wow, okay, this looks kind of hard. But let's stay calm. The $x \in \mathbb{R}$, and the function $u(x,0) = \frac{\sin(x)}{x}$ is in L^2 indicating Fourier transform in x should work.

21. All right, roll up our sleeves and solve this. Fourier transform the PDE in x. We get

$$\hat{u}_{tt}(\xi,t) - \hat{u}_{xx}(\xi,t) = 0.$$

Fourier transform magic says that the x derivatives can be swapped out for multiplication by $(i\xi)^2$, leaving

$$\hat{u}_{tt}(\xi,t) = (i\xi)^2\hat{u}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

This is an ODE for $\hat{u}$ in the t variable. Since $\xi \in \mathbb{R}$, $-\xi^2 \leq 0$ so the solutions are cos and sin, namely

$$\hat{u}(\xi, t) = a(\xi)\cos(\xi t) + b(\xi)\sinh(\xi t).$$

The derivative at $t = 0$ should vanish, hence the $b(\xi) = 0$. The other condition tells us that

$$a(\xi) = \widehat{\frac{\sin(x)}{x}}(\xi).$$

We look this up somewhere to get that the FT of that sine business is π for $|\xi| < 1$ and zero elsewhere. We now use the FIT:

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}a(\xi)\cos(\xi t)d\xi = \frac{1}{2}\int_{-1}^{1} e^{ix\xi}\cos(\xi t)d\xi.$$

Here we note that $e^{ix\xi} = \cos(x\xi) + i\sin(x\xi)$. The sine is odd. Cosine is even. We are integrating from -1 to 1. So all we get is

$$= \int_0^1 \cos(x\xi)\cos(\xi t)d\xi.$$

We can stop here.

22. We use the Legendre polynomials that are an OB on $[-1, 1]$. We need to tweak them to be an OB on $[-2, 2]$. So...

$$\int_{-2}^{2} P_n(x/2)P_m(x/2)dx \overset{t=x/2}{=} \int_{-1}^{1} P_n(t)P_m(t)2dt.$$

So, our coefficients are

$$c_n = \frac{\int_{-2}^{2}\cos(2x)P_n(x/2)dx}{\int_{-2}^{2}|P_n(x/2)|^2dx},$$

and polynomial $p(x) = \sum_{n=0}^{3} c_n P_n(x/2)$.

23. Well, it just so happens that when we try to project cosine onto $\sin(x)$ and $\sin(2x)$ it is orthogonal on this interval. Compute the integrals! So $a = b = 0$.

24. Since $a = b = 0$ we are integrating cosine squared from $-\pi$ to π. Use the double angle formula and some trig identities to get tat this is the length of the interval (2π) divided by 2. Hence $\pi \approx 3$.

25. Stay calm and carry on. We see e^{-y} which is very nice, and a homogeneous PDE. The BC at $y = 0$ wants the derivative to go away. Hence we should extend in y evenly and then use the FT in y.

26. Okay, let's do that:

$$\hat{u}_x x(x,\xi) + (i\xi)^2 \hat{u}(x,\xi) = 0 \iff \hat{u}_{xx}(x,\xi) = \xi^2 \hat{u}(x,\xi).$$

So this time it's positive and we therefore get

$$\hat{u}(x,\xi) = a(\xi)\cosh(\xi x) + b(\xi)\sinh(\xi x).$$

The condition $u(0,y) = 0 \implies \hat{u}(0,\xi) = 0$, so $a(\xi) = 0$. The other condition we need to have is

$$\hat{u}(1,\xi) = b(\xi)\sinh(\xi) = \widehat{e^{-|y|}}(\xi).$$

This tells us that (looking up what that FT is)

$$b(\xi) = \frac{2}{(\xi^2+1)\sinh(\xi)}.$$

Now we use the FIT:

$$u(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iy\xi} \frac{2\sinh(\xi x)}{(\xi^2+1)\sinh(\xi)} d\xi.$$

OBS! The stuff multiplying the $e^{iy\xi}$ is an even function of ξ. So, as before when we expand $e^{iy\xi} = \cos(y\xi) + i\sin(y\xi)$ the sine part of the integral will go away (odd times even is odd), leaving

$$u(x,y) = \frac{1}{\pi}\int_0^\infty \cos(y\xi)\frac{2\sinh(\xi x)}{(\xi^2+1)\sinh(\xi)} d\xi.$$

27. This just screams LAPLACE TRANSFORM in t.

28. Okay, let's just get this over with. We LT the PDE and get

$$\widetilde{u}_t(x,z) - \widetilde{u}_{xx}(x,z) = 0, \quad u(x,0) = 0 \implies z\widetilde{u}(x,z) = \widetilde{u}_{xx}(x,z).$$

Hence,

$$\widetilde{u}(x,z) = a(z)e^{\sqrt{z}x} + b(z)e^{-\sqrt{z}x}.$$

We also need $u(0,z) = \widetilde{1}$. To be inverse transformable, we need decay, so we therefore try to find a solution with just the second one. Laplace transforms convolutions to products, hence what we started with is

$$\int_{\mathbb{R}} \Theta(t-s)\frac{x}{2\sqrt{\pi}s^{3/2}}\Theta(s)e^{-x^2/4s} ds.$$

The reason is that we convolute the function that is 1 for $t > 0$ but 0 for $t < 0$ with the function

$$\frac{x}{2\sqrt{\pi}s^{3/2}e^{-x^2/4s}}\Theta(s),$$

because it has Laplace transform equal to $e^{-\sqrt{z}x}$, but it also has to be zero for $s < 0$. So, the resulting thing is

$$\int_0^t \frac{x}{2\sqrt{\pi}s^{3/2}} e^{-x^2/4s} ds.$$

A convolution with a rapidly decaying function, namely $e^{-x^2/4s}$.

29. If we let $t \to \infty$ in that function above, the integral spreads out over the whole real line, so since x is independent of t, the limit is just a constant times x, i.e. a linear function of x. OBS! This is ignoring the x in the exponential that we integrate... But in any case, it will converge to a function of x...
30. I found the previous problem pretty hard.
31. Nothing.
32. Yes, best approximation theorem in action.
33. It is even. (Did you google Hilbert function? I made it up, but later googled it to see if there is actually a thing by that name, and that is some weird sounding stuff!)
34. NO, not without more information.
35. One must start near zero, because you need to use the left and right sided limits to get a fixed little window about zero first. That window must be fixed before one can estimate the rest. (It does not work to do things the other way round).
36. For computing integrals of course!
37. Use the generating function for the Bessel functions, and you see it is equal to 1 no matter what x is.
38. Integration by parts!
39. Maybe you got the hint from that other problem to use the generating function. You get one again.
40. Big bad CAT, meow.