Final Exam in English

ⓘ This is a preview of the published version of the quiz

Started: 19 Mar at 13:33

Quiz instructions

The zoom room for the exam is the green room here: **join zoom room here**
https://chalmers.zoom.us/j/63589098687
Solutions can be directly entered into Canvas, if any trouble with that email:
julie.rowlett@chalmers.se
You can also ask questions to Julie by phone: +46 31 772 34 19.
Please do NOT send text message solutions via phone as this will not work!
All study aids are allowed, but please do not communicate with anyone other than the proctor and/or instructor. There are 40 problems total, each worth 2 points. Betygsgränser för 3: 40 poäng, för 4: 53 poäng, för 5: 67 poäng.

You may write in English, Swedish (German and French are also fine if you want to have even more fun). You are free to switch between these languages as you wish. You may submit your exam in any readable format as well as using a combination of formats (hand-written for some parts, typed for other parts), just do what works best for you and make sure it's readable!

You got this!!! May the mathematical force be with you ♥

Question 1 — 2 pts

We expand e^x = exp(x) in a Fourier series using the orthogonal basis $\{e^{in\pi x/2}\}_{n\in\mathbb{Z}}$ on the Hilbert space $\mathcal{L}^2(-2, 2)$. What is the value of the series when x=16?

- 0
- 1
- cosh(16)
- exp(16)

Question 2 — 2 pts

Evaluate the same series as in the preceding exercise at x=10.

- exp(10)
- cosh(2)
- 1
- cosh(10)

Question 3 — 2 pts

Let $\sum_{n\in\mathbb{Z}} c_n e^{in\pi x/2}$ be the series from the preceding two questions. Evaluate $\sum_{n\in\mathbb{Z}} |c_n|^2$.

- cosh(4)/2
- cosh(4)/4
- sinh(4)/4
- sinh(4)/2

Question 4 — 2 pts

Are the following boundary conditions self-adjoint for the SLP:
$f''(x) + \lambda f(x) = 0, \quad 0 < x < 1, \quad f(0) = 1, \quad f'(1) = 0.$

- Yes.

No.

Question 5

2 pts

When solving a regular SLP of the form $L(f) + \lambda f = 0$ which part of the regular SLP determines the sequence of eigenvalues one obtains?

- The weight function.
- The potential function.
- The operator L.
- The boundary conditions.

Question 6

2 pts

Assume that $\{f_n\}_{n\geq 1}$ are the eigenfunctions of a regular SLP with corresponding eigenvalues $\{\lambda_n\}_{n\geq 1}$ on an interval (a, b). Let $\phi(x), g(x)$ be continuous functions on [a,b].

Solve: $\begin{cases} u_{tt} - u_{xx} = 0 & a < x < b,\ t > 0 \\ u(x,0) = \phi(x) & a < x < b \\ u_t(x,0) = g(x) & a < x < b \end{cases}$ assuming the same boundary conditions hold for u as in the regular SLP.

We use separation of variables to obtain that T''/T = X''/X with X subject to the same BCs. Hence the constants for the equation X'' = constant X are -\lambda_n.

Then we get T_n(t) = a_n exp(sqrt(-\lambda_n) t) + b_n exp(-sqrt(-\lambda_n) t). The solution is the sum over all n of T_n(t) f_n(x). The coefficients are

The coefficients satisfy a_n+b_n = integral from a to b of phi(x) times complex conjugate of f_n(x) divided by the integral from a to b of |f_n(x)|^2.

a_n - b_n = integral from a to b of g(x) times complex conjugate of f_n(x) divided by the integral from a to b of |f_n(x)|^2 times sqrt(-\lambda_n).

Edit View Insert Format Tools Table

12pt Paragraph

p

0 words </>

Question 7

2 pts

Consider for two continuous functions f and g the problem:

$$\begin{cases} u_{tt} - u_{xx} = 2x - 3, & 0 < x < 5,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(5,t) = 0 & t > 0 \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x). & \end{cases}$$

What should we do first to solve this problem?

- Extend evenly in t.
- Extend oddly in x.
- Find a steady state solution to deal with the inhomogeneity in the boundary conditions.

- Find a steady state solution to deal with the inhomogeneity in the PDE.

Question 8

2 pts

What should we do next to solve the problem in the preceding exercise?

- Apply the Fourier transform in x.
- Apply the Fourier transform in t.
- Apply the Laplace transform in t.
- Separate variables and solve for the time dependent function and its coefficients.
- Separate variables and solve a regular SLP.

Question 9

2 pts

How do we complete the last step of the solution to the preceding problem?

- Apply the Fourier Inverse Theorem (FIT).
- Calculate a best approximation using the Legendre polynomials.
- Use an orthogonal basis for a Hilbert space.
- Apply the Laplace Inverse Theorem (LIT).

Question 10

2 pts

Consider the problem $\begin{cases} u_{tt} - u_{xx} = 2xt, & 0 < x < 5,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(5,t) = 0 & t > 0 \\ u(x,0) = F(x) & \\ u_t(x,0) = G(x), & \end{cases}$ where F and G are continuous functions on [0, 5].

Which part of the preceding three questions can be recycled here?

- The Fourier transform of the solution.
- The best approximation of the initial data.
- The solutions of a regular SLP.
- The steady state solution.

Question 11 — 2 pts

What will be an important part of the solution of the preceding problem?

- The solution to a second order ODE in t.
- Hyperbolic cosine (cosh)
- Hyperbolic sine (sinh)
- The Fourier Inverse Theorem.

Question 12 — 2 pts

Will the solution obtained to the preceding problem be of the form T(t) X(x)?

- No.
- Yes.

Question 13

2 pts

A cookie has just been taken from the oven. Its temperature is uniformly equal to 175 degrees C. The room temperature is 20 degrees C. Mathematically, the temperature in the cookie satisfies

$$\begin{cases} u_t - \Delta u = 0 & t > 0,\ 0 < r < 2,\ 0 < z < 1 \\ u(r, z, 0) = 175 & \\ u(2, z, t) = 20 & t > 0 \\ u(r, 0, t) = 20 & t > 0 \\ u(r, 1, t) = 20 & t > 0. \end{cases}$$

What should we do first to solve this problem?

- Find a steady state solution to deal with the boundary conditions.
- Apply the Fourier transform.
- Solve a regular SLP.
- Find a steady state solution to deal with the initial conditions.

Question 14

2 pts

In the preceding problem, what part of the solution should we solve for next?

- The part that depends on theta.
- The part that depends on r.
- The part that depends on z.

- The part that depends on t.

Question 15 2 pts

The solution to the cookie problem contains what ingredients?

- The zeroth order Bessel function, sines and exp.
- The zeroth order Bessel function, cosines and exp.
- Bessel functions of orders 0, 1, and all positive integers together with sines and hyperbolic cosines.
- Bessel functions of orders 0, 1, ... and all positive integers together with exp(i n theta) for all integers.

Question 16 2 pts

We are making a drum that is shaped like a box. Mathematically, this is described by

$$\begin{cases} u_{tt} - \Delta u = 0 & t > 0,\ 0 < x < 1,\ 0 < y < 1 \\ u(0, y, t) = u(1, y, t) = 0 \\ u(x, 0, t) = u(x, 1, t) = 0 \\ u(x, y, 0) = f(x, y) \\ u_t(x, y, 0) = g(x, y). \end{cases}$$

How should we start to solve this problem?

- Apply the Fourier transform.
- Find a steady state solution to deal with the initial conditions.
- Apply the Laplace transform.

- Separate variables.

Question 17 2 pts

An important part of the solution will be obtained by:

- Solving an Euler equation.
- Solving a regular SLP.
- Solving a Bessel equation.
- Solving an inhomogeneous ODE.

Question 18 2 pts

Key ingredients in the solution are:

- hyperbolic sines and cosines (sinh and cosh)
- sines and cosines
- Bessel functions and cosines
- Bessel functions and sines

Question 19 2 pts

As a side job, we deliver pizzas for Dominos. We are delivering a pizza in an insulated container. This is described by:

$$\begin{cases} u_t - \Delta u = 0 & t > 0,\ 0 \le r \le 10,\ 0 \le z \le 3 \\ u_r(10, \theta, z, t) = 0, & u_z(r, \theta, 0, t) = 0 = u_z(r, \theta, 3, t) = 0 \\ u(r, \theta, z, 0) = f(r, z). \end{cases}$$

Assume the initial temperature of the pizza, described by f(r,z) is a continuous function that ranges from 45 to 75 degrees C. Which part of the problem should we solve first?

- The theta part.
- The r part.
- The t part.
- The z part.

Question 20 2 pts

What will be an important ingredient in the solution to the pizza problem?

- hyperbolic cosines
- hyperbolic sines
- sines
- cosines

Question 21 2 pts

What happens to the solution to the pizza problem as t tends to infinity?

- It tends to zero.

- It tends to the Bessel function of order zero.
- It tends to a positive constant.
- It oscillates forever between positive and negative values.

Question 22

2 pts

What is the polynomial of at most degree 3 that minimizes
$\int_{-\pi}^{\pi} |\sin(x) - p(x)|^2 dx$?

Edit View Insert Format Tools Table

12pt Paragraph

Let p_n(x) denote the Legendre polynomial of degree n. Define coefficients a_n as the integral from -pi to pi of sin(x) times p_n(x/pi) divided by the integral from - pi to pi of |p_n(x/pi)|^2.

The polynomial we seek p(x) is equal to the sum from 0 to 3 of a_n p_n(x/\pi).

p 0 words </>

Question 23

2 pts

Determine a and b that minimize

$\int_{-\pi}^{\pi} |\sin(x) - a\cos(x) - b\cos(2x)|^2 dx.$

- a=0, b=1/2.
- a=1, b=1/2.
- a=1/2, b=1.
- a=b=0.

Question 24

2 pts

For a and b as in the preceding problem, what is the approximate value of the integral?

- 6
- 3
- -3
- 0

Question 25

2 pts

What technique will solve the problem:

$$\begin{cases} u_t - u_{xx} = G(x,t) & t > 0,\ x \in \mathbb{R} \\ u(x,0) = f(x), \end{cases}$$

assuming $\int_{\mathbb{R}} |f(x)|^2 dx < \infty, \quad \int_{\mathbb{R}} |G(x,t)|^2 dx < \infty, \quad \forall t \geq 0.$

- Fourier transform in x using an even extension.
- Laplace transform in t extending using the heavyside (heavyweight) function.

Fourier transform in x.

Laplace transform in t.

Question 26

2 pts

Consider the following problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad 0 < x < 1,\ 0 < y \\ u(x,0) = 0 \\ u(1,y) = f(y) \\ u(0,y) = 0. \end{cases}$$

Assuming $\int_0^\infty |f(y)|^2 dy < \infty$, what technique will solve this problem?

Fourier transform in y and even extension of f.

Fourier transform in y and odd extension of f.

A regular SLP and a Fourier series in x.

Laplace transform in y.

Question 27

2 pts

Having solved the preceding problem, what is an important part of the solution?

- a Fourier sine series
- a Fourier cosine series
- hyperbolic sine
- hyperbolic cosine

Question 28 2 pts

Consider the following problem:

$$\begin{cases} u_{tt} - u_{xx} = 0 & t, x > 0 \\ u(0,t) = e^t \\ u(x,0) = 0, & u_t(x,0) = 0. \end{cases}$$

What technique will solve this problem?

- A Laplace series expansion.
- Fourier transform in t and odd extension.
- Fourier transform in x and even extension.
- Laplace transform in t.

Question 29 2 pts

In the preceding problem, does the solution vanish, and if so when?

- Only when t=0, otherwise it is never zero.
- No, it never vanishes.
- Yes for x>t.
- Yes for x<t.

Question 30 2 pts

Choose to answer one of the following:

1. What topic did you most enjoy learning about in this course and why?

OR

2. What topic did you least enjoy learning about in this course and why?

Edit View Insert Format Tools Table

12pt Paragraph

I like it all...
Curious what you
think here!

p 0 words

Question 31 2 pts

Assume that $\{\phi_n\}_{n\geq 1}$ is an orthogonal set in a Hilbert space H. What does the following statement guarantee?

$$f \in H \text{ and } \langle f, \phi_n \rangle = 0 \forall n \implies f = 0.$$

Edit View Insert Format Tools Table

12pt Paragraph

We can conclude that this orthogonal set is in fact an orthogonal basis.

p 0 words </>

Question 32 2 pts

Assume that $\{\phi_n\}_{n\geq 1}$ is an orthonormal set in a Hilbert space H. If f is an element of H, is the statement below guaranteed to be true?

$$||f - \langle f, \phi_1 \rangle \phi_1|| \leq ||f - c\phi_1|| \quad \forall c \in \mathbb{C}?$$

- True
- False

Question 33 2 pts

In the proof of the pointwise convergence of Fourier series, why do we introduce a new piecewise-defined function near the end of the proof?

- So that we can apply Bessel's inequality.
- So that we can apply Pareseval's Theorem.
- So that we can apply the Fourier Inverse Theorem.
- So that we can simplify the problem to a regular SLP.

Question 34 2 pts

What technique do we use to prove the relationship between the Fourier coefficients for f and for those of f'?

- The Taylor series of f.
- Integration by parts.
- The best approximation theorem.
- Termwise differentiation of the Fourier series of f.

Question 35 2 pts

What is the cat's tail in the BB Convolution Approximation Theorem?

- The tail of a convergent integral over the real line, so we can make it small.
- The tail of a convergent series, so we can make it small.
- The integral in a region really close to zero, that gets closer and closer to zero, so it's like one of those cats with a stubby tiny tail.
- An integral over a region whose length gets shorter and shorter (the cat's tail shrinks!)

Question 36 — 2 pts

What is a key ingredient in the proof of Plancharel's theorem?

- The Laplace Inverse Theorem.
- The BB Convolution Approximation Theorem (CAT).
- Bessel's inequality.
- The Fourier Inverse Theorem.
- The Sampling Theorem.

Question 37 — 2 pts

In both generating function theorems, for Bessel and Hermite, what is a key ingredient in the proof?

- Parseval's equality.
- Taylor series.

- Bessel's inequality.
- Integration by parts.

Question 38 2 pts

In the proof of the orthogonality of the Hermite polynomials we use integration by parts. What happens to the boundary terms?

- They vanish by canceling each other out in a telescoping way.
- They vanish due to super-exponential decay at infinity.
- The polynomial has a zero of high enough degree that it makes the boundary terms vanish.
- They vanish because of self-adjointness.

Question 39 2 pts

How do we obtain a series in the Sampling Theorem?

- We compute a contour integral and use the Residue theorem to obtain the series.
- The function in the theorem can be expanded as a Fourier series.
- We use a best approximation to obtain the series.
- The Fourier transform can be expanded as a Fourier series.

Question 40 2 pts

Please answer one of the following (your choice!):

1. What is your favorite theory item from this course and why?

OR

2. What is your least favorite theory item from this course and why?

Edit View Insert Format Tools Table

12pt Paragraph

I guess now my favorite is the BBCAT because it is named after a cat... It is a real beast to prove but then one can use it to solve the heat equation, and that is awesome.

p 0 words

Saved at 13:34 Submit quiz