Fourieranalys MVE030 och Fourier Metoder MVE290 20.mars.2020
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

Study aids: All study aids are allowed, but you are kindly requested not to communicated with anyone else in any form during the writing of this exam, except Julie in case of questions. Please write your name and personal number on each page of your exam to make sure that no part of your exam is lost.

The exam is a blended format and is presented here in both Swedish and English. You may write in English, Swedish (German and French are also fine if you want to have even more fun). You are free to switch between these languages as you wish. You may submit your exam in any readable format as well as using a combination of formats (hand-written for some parts, typed for other parts), just do what works best for you and make sure it's readable! Lycka till, and may the mathematical force be with you ♡

OBS! For some problems, you may need to select more than one choice to receive the points for the problem. (i.e. to receive the points one may need to select *both* a and b or both a, b, and c, etc).

1. English version

(1) (10 p total) We are faced with the following problem.

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(t)\cos(x) & 0 < t,\ -\pi < x < \pi \\ u(x,0) = |x| - \pi & x \in [-\pi, \pi] \\ u(-\pi, t) = u(\pi, t) & t \geq 0 \end{cases}$$

(a) (2p) What should we do first?
  (i) Apply the Fourier transform.
  (ii) Apply the Laplace transform.
  (iii) Solve the homogeneous PDE.
  (iv) Find a steady state solution.
  (v) None of these.

(b) (2p) Are the boundary conditions self adjoint?
  (i) Yes.
  (ii) No.

(c) (2p) Which technique can NOT be used to correctly solve this problem?
  (i) Separation of variables.
  (ii) Fourier series.
  (iii) A regular Sturm-Liouville problem.
  (iv) Fourier transform.

(d) (2p) How do we deal with the inhomogeneity in the PDE?
  (i) Express it as a Fourier series in t.
  (ii) Express it as a Fourier series in x.
  (iii) Apply the Laplace transform in t.
  (iv) Apply the Fourier transform in x.
  (v) None of these

(e) (2p) What form will the solution take?
  (i) A Fourier series.
  (ii) A Fourier transform.
  (iii) A convolution.
  (iv) A Laplace transform.
  (v) A distribution.
  (vi) None of these.

(2) (10 p total)

(a) (2p) What is the difference between a partial differential equation and an ordinary differential equation?

(b) (2p) Consider the following problem:

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 & x > 0, \quad y > 0 \\ u(0,y) = f(y) & \in \mathcal{L}^2(0,\infty) \\ u(x,0) = g(x) & \in \mathcal{L}^2(0,\infty) \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) The Fourier transform.
(ii) The Fourier sine transform.
(iii) The Fourier cosine transform.
(iv) The Laplace transform.
(v) A Fourier series.
(vi) A regular Sturm-Liouville problem.
(vii) None of these.

(c) (2p) Consider the following problem:

$$\begin{cases} u(0,t) = e^t & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = 0 & x > 0 \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) A steady state solution.
(ii) The Fourier transform.
(iii) The Fourier sine trasform.
(iv) The Fourier cosine transform.
(v) The Laplace transform.
(vi) A Fourier series.
(vii) A regular Sturm-Liouville problem.
(viii) None of these.

(d) (2p) Consider the following problem:

$$\begin{cases} \sqrt{1+t}u_{xx} = u_t & 0 < x < 1, \quad t > 0 \\ u(0,t) = 1, & u(1,t) = 0 \\ u(x,0) = 1 - x^1. \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) A steady state solution.
(ii) Separation of variables.
(iii) The Fourier transform.
(iv) The Fourier sine trasform.
(v) The Fourier cosine transform.
(vi) The Laplace transform.
(vii) A Fourier series. (Right answer)
(viii) A regular Sturm-Liouville problem.
(ix) None of these.

(e) (2p) For the problem in the preceding question, are the boundary conditions self-adjoint?

(i) Yes.
(ii) No.

(3) (10 p total)

(a) (2p) What is the Fourier series of the function $\phi(x) = 1 \forall x \in \mathbb{R}$?

(b) (4p) The Fourier series of the function $f(x) = x$ for $x \in (-\pi, \pi)$ is

$$2 \sum_{n \geq 1} \frac{(-1)^{n+1} \sin(nx)}{n}.$$

Differentiating the Fourier series we obtain

$$2 \sum_{n \geq 1} (-1)^{n+1} \cos(nx).$$

Is this the same as the Fourier series for the function which is equal to $f'(x)$ for $x \in (-\pi, \pi)$?

If yes, explain why it is.

If no, explain why it is not.

(c) (4p) Let

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x-inx} dx.$$

Compute

$$\sum_{n \in \mathbb{Z}} c_n e^{in\pi}.$$

(4) (10 p total)

(a) (5p) We wish to compute

$$\sum_{n\geq 1} \frac{1}{n^4}.$$

Find a function whose Fourier series you could use to compute this series, and explain how to use it to compute the series.

(b) (5p) Find a function $\varphi(x)$ whose Fourier coefficients satisfy

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)e^{-inx}dx \neq 0 \quad \forall n \in \mathbb{Z},$$

and

$$\lim_{|n|\to\infty} n^k c_n = 0, \quad \forall k \in \mathbb{N}.$$

(5) (10p total)

(a) (5p) What is the polynomial $p(x)$ of at most degree 17 that minimizes the following integral
$$\int_{-4}^{4} |e^{\cos(x)} - p(x)|^2 dx?$$

(b) (5p) In what types of geometric settings do Bessel functions arise in solving PDEs like the heat equation and the wave equation?

(6) (10 p total)

(a) (2p) Can you solve a regular Sturm-Liouville problem correctly and obtain $\sqrt{-1}$ as an eigenvalue?

(i) Yes.

(ii) No.

(b) (2p) You have found all the eigenfunctions f_n and corresponding eigenvalues λ_n to the regular Sturm-Liouville problem

$$L(f) + \lambda f = 0, \qquad \text{on the interval } (a, b),$$

subject to the boundary conditions

$$B_i(f) = 0, \quad i = 1, 2.$$

What happens to λ_n when $n \to \infty$?

(c) (2p) Assume that u and v are solutions to the aforementioned regular SLP and have eigenvalues 2 and 4, respectively. Compute

$$\int_a^b u(x)\overline{v(x)}dx.$$

(d) (4p) Use $\{f_n\}_{n\geq 1}$ and $\{\lambda_n\}_{n\geq 1}$ to obtain the solution $u(x,t)$ to the following problem

$$\partial_t u - L(u) = 0, \quad t > 0, \quad x \in (a, b),$$
$$B_i(u) = 0, \quad i = 1, 2,$$
$$u(x, 0) = \varphi(x) \text{ is a bounded, continuous function on } [a, b].$$

1.1. **Theory.**

(1) (15p) I've attempted to prove the BBC, but I keep getting stuck. Can you help me finish the proof?

Theorem 1.1. *Let $g \in L^1(\mathbb{R})$. Define*

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\varepsilon > 0$,

$$g_\epsilon(x) := \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0^+} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof: First, since this should hold for all $x \in \mathbb{R}$, let us fix the point x.

(a) **What is the meaning of $f(x+)$ and $f(x-)$? Explain what are these things? (1p)**

(b) **Why is the statement in the theorem equivalent to proving the following statement (1p)?**

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y)g_\varepsilon(y)dy - \alpha f(x+) - \beta f(x-) = 0.$$

So, now that you've explained why the statement in the theorem is equivalent to proving the statement above, this can be achieved by proving that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = 0$$

and also

$$\lim_{\varepsilon \to 0} \int_{0}^{\infty} f(x-y)g_\varepsilon(y)dy - \int_{0}^{\infty} f(x-)g(y)dy = 0.$$

(c) **In the next step, we just pick one of the above and say it's enough to show the limit is zero for one of them. Why can we do that??? What is that voodoo magic? (1p)**

We would therefore like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy$$

as small as we like. In particular, let $\delta > 0$ be arbitrarily small. We wish to show that by choosing ε sufficiently small, we can make

$$\left| \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy \right| < \text{ a constant multiple of } \delta.$$

(d) **Why will it complete the proof to show that the inequality above is true? How does that prove the limit below? (2p)**

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = 0?$$

(e) **How do we obtain the equation below? It looks like magic to me, please explain?!? (1p)**

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = \int_{-\infty}^{0} g_\varepsilon(y)\left(f(x-y) - f(x+)\right)dy.$$

(f) **How can we find $y_0 < 0$ to make the inequality below true? (1p)**

$$|f(x-y) - f(x+)| < \delta \forall y \in [y_0, 0).$$

(g) **How do we use that above to obtain the inequality below? (1p)**

$$\left| \int_{y_0}^{0} (f(x-y) - f(x+)) g_\varepsilon(y) dy \right| \leq \delta ||g||_{L^1(\mathbb{R})}.$$

(h) **What is $||g||_{L^1(\mathbb{R})}$? (1p)**

(i) **How do we use this to obtain the inequality below? (2p)**

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+)) g_\varepsilon(y) dy \right| \leq 2M \int_{-\infty}^{y_0/\varepsilon} |g(y)| dy.$$

(j) **How can we use ε to obtain the inequality below? (1p)**

$$2M \int_{-\infty}^{y_0/\varepsilon} |g(y)| dy < \delta?$$

(k) **Why is the proof complete in this case now? (1p)**

(l) **In the second case in the theorem, when g vanishes outside a bounded interval, how can we use ε to obtain the equation below 'for ε small enough?' (2p)**

$$\int_{-\infty}^{y_0} (f(x-y) - f(x+)) g_\varepsilon(y) dy = 0.$$

(2) (5p) Explain your favorite proof from all of the theory-item-proofs in this course. Why is that proof your favorite? What do you like about it?

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1. English version

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(a) (2p) What should we do first?
  (i) Apply the Fourier transform.
  (ii) Apply the Laplace transform.
  (iii) Solve the homogeneous PDE. (This one is right!)
  (iv) Find a steady state solution.
  (v) None of these.

(b) (2p) Are the boundary conditions self adjoint?
  (i) Yes. (Right answer!) Implicitly we assume solution should be 2π periodic so the second set of BCs is $u'(-\pi) - u'('\pi) = 0$. This is what I had in mind...
  (ii) No. (Also right answer!) However, if people don't assume that 'implicit' second BC it can seem not self-adjoint because there are not enough (2) equations given. So, this will also be accepted.

(c) (2p) Which technique can NOT be used to correctly solve this problem?
  (i) Separation of variables.
  (ii) Fourier series.
  (iii) A regular Sturm-Liouville problem.
  (iv) Fourier transform. (Right answer - because it is wrong!)

(d) (2p) How do we deal with the inhomogeneity in the PDE?
  (i) Express it as a Fourier series in t.
  (ii) Express it as a Fourier series in x. (Right answer!)
  (iii) Apply the Laplace transform in t.
  (iv) Apply the Fourier transform in x.
  (v) None of these

(e) (2p) What form will the solution take?
  (i) A Fourier series. (Right answer!)
  (ii) A Fourier transform.
  (iii) A convolution.
  (iv) A Laplace transform.
  (v) A distribution.
  (vi) None of these.

(2) (10 p total)

(a) (2p) What is the difference between a partial differential equation and an ordinary differential equation?

Difference is the number of variables on which the unknown function depends. Equivalently, PDE has partial derivatives in it while ODE has ordinary derivatives in it. Similar explanations accepted.

(b) (2p) Consider the following problem:

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 & x > 0, \quad y > 0 \\ u(0,y) = f(y) & \in \mathcal{L}^2(0,\infty) \\ u(x,0) = g(x) & \in \mathcal{L}^2(0,\infty) \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) The Fourier transform. (Right answer - but also need to answer FST to get points.)
(ii) The Fourier sine trasform. (Right answer - but also need to answer FT - need to get both of these!)
(iii) The Fourier cosine transform.
(iv) The Laplace transform.
(v) A Fourier series.
(vi) A regular Sturm-Liouville problem.
(vii) None of these.

(c) (2p) Consider the following problem:

$$\begin{cases} u(0,t) = e^t & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = 0 & x > 0 \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) A steady state solution.
(ii) The Fourier transform.
(iii) The Fourier sine trasform.
(iv) The Fourier cosine transform.
(v) The Laplace transform. (Right answer)
(vi) A Fourier series.
(vii) A regular Sturm-Liouville problem.
(viii) None of these.

(d) (2p) Consider the following problem:

$$\begin{cases} \sqrt{1+t}u_{xx} = u_t & 0 < x < 1, \quad t > 0 \\ u(0,t) = 1, & u(1,t) = 0 \\ u(x,0) = 1 - x^1. \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) A steady state solution. (Right answer)
(ii) Separation of variables. (Right answer)
(iii) The Fourier transform.
(iv) The Fourier sine trasform.
(v) The Fourier cosine transform.
(vi) The Laplace transform.
(vii) A Fourier series. (Right answer)
(viii) A regular Sturm-Liouville problem. (Right answer)
(ix) None of these.

(e) (2p) For the problem in the preceding question, are the boundary conditions self-adjoint?

(i) Yes.
(ii) No. (Right answer)

(3) (10 p total)

(a) (2p) What is the Fourier series of the function $\phi(x) = 1 \forall x \in \mathbb{R}$?

It's the series 1. (no partial credit).

(b) (4p) The Fourier series of the function $f(x) = x$ for $x \in (-\pi, \pi)$ is

$$2 \sum_{n \geq 1} \frac{(-1)^{n+1} \sin(nx)}{n}.$$

Differentiating the Fourier series we obtain

$$2 \sum_{n \geq 1} (-1)^{n+1} \cos(nx).$$

Is this the same as the Fourier series for the function which is equal to $f'(x)$ for $x \in (-\pi, \pi)$?

If yes, explain why it is.

If no, explain why it is not.

No. Many explanations will be accepted. We already know from the first part the series for $f'(x) = 1$. It's one. That thing up there isn't the same. So, differentiating termwise is rubbish. So, 2 points for saying no, and 2 points for some kind of reasonable explanation.

(c) (4p) Let

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x-inx} dx.$$

Compute

$$\sum_{n \in \mathbb{Z}} c_n e^{in\pi}.$$

Well, well, well, the convergence theorem says that since these are the Fourier coefficients of the function e^x the series converges to the average of the left and right limit of the function which is equal to e^x on $(-\pi, \pi)$ and is 2π periodic. To figure out these limits, draw the graph. The average of these limits is

$$\frac{e^{\pi} + e^{-\pi}}{2} = \cosh(\pi).$$

Most likely this will just be all points or zero unless someone has a very creative way of being partially correct. I cannot predict that, as people are surprisingly creative.... Which is a good thing!

(4) (10 p total)

(a) (5p) We wish to compute

$$\sum_{n\geq 1} \frac{1}{n^4}.$$

Find a function whose Fourier series you could use to compute this series, and explain how to use it to compute the series.

Well, I'd choose to use $f(x) = x^2$, because it's got Fourier series equal to

$$\frac{\pi^2}{3} + \sum_{n\geq 1} 4\frac{(-1)^n \cos(nx)}{n^2}.$$

Then I'd choose to use Parseval's equality (or the infinite Pythagorean theorem) to say that

$$||x^2||^2 = \sum_{n\geq 0} ||a_n \cos(nx)||^2.$$

Here

$$a_0 = \frac{\pi^2}{3}, \quad a_n = 4\frac{(-1)^n}{n^2}.$$

The norms are

$$||x^2||^2 = \int_{-\pi}^{\pi} x^4 dx, \quad ||\cos(nx)||^2 = \int_{-\pi}^{\pi} |\cos(nx)|^2 dx.$$

So, we would get the following equality:

$$||x^2||^2 = 2\frac{\pi^5}{5} = \frac{\pi^4}{9}(2\pi) + \sum_{n\geq 1} \frac{16}{n^4}(\pi).$$

The factor of 2π comes from $||\cos(0x)||^2 = 2\pi$, whereas the factors of π come from $||\cos(nx)||^2 = \pi$ for all $n \geq 1$. Then, one has simply to re-arrange

$$2\frac{\pi^5}{5} - \frac{2\pi^5}{9} = 16\pi \sum_{n\geq 1} \frac{1}{n^4}.$$

So I suppose the sum in question gives

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{(2(9) - 2(5))\pi^5}{5 * 9 * 16 * \pi} = \frac{4\pi^4}{5 * 9 * 8} = \frac{\pi^4}{5 * 9 * 2} = \frac{\pi^4}{90}.$$

To get credit here you need to find a function which would actually work (2p). Explain how to use it in some correct way (pointwise convergence of Fourier series or Parseval will both work, but explanation needs to be correct!) to obtain the sum. Calculations and the actual value of the sum does not matter and will not give any points. What matters here is that you know how to use functions to compute sums.

(b) (5p) Find a function $\varphi(x)$ whose Fourier coefficients satisfy

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(x) e^{-inx} dx \neq 0 \quad \forall n \in \mathbb{Z},$$

and

$$\lim_{|n|\to\infty} n^k c_n = 0, \quad \forall k \in \mathbb{N}.$$

This is meant to be challenging. I would be a sneaky person and define my function to be

$$\varphi(x) := \sum_{n\in\mathbb{Z}} e^{-(n)^2} e^{inx}.$$

Heheh. Doesn't need to be in closed form. Anything correct will be accepted. No partial credit because either the function satisfies these conditions or it does not. There is no in between.

(5) (10p total)

(a) (5p) What is the polynomial $p(x)$ of at most degree 17 that minimizes the following integral

$$\int_{-4}^{4} |e^{\cos(x)} - p(x)|^2 dx?$$

$$p(x) = \sum_{n=0}^{17} a_n P_n(x/4)$$

where P_n is the Legendre polynomial of degree n, and

$$a_n = \frac{\int_{-4}^{4} e^{\cos(x)} P_n(x/4) dx}{\frac{8}{2n+1}}.$$

(b) (5p) In what types of geometric settings do Bessel functions arise in solving PDEs like the heat equation and the wave equation?

In disks, circles, pieces of disks, cylinders. Anything of this general nature will be accepted here.

(6) (10 p total)

(a) (2p) Can you solve a regular Sturm-Liouville problem correctly and obtain $\sqrt{-1}$ as an eigenvalue?

(i) Yes.

(ii) No. (Right answer)

No. Theory says that the eigenvalues are real.

(b) (2p) You have found all the eigenfunctions f_n and corresponding eigenvalues λ_n to the regular Sturm-Liouville problem

$$L(f) + \lambda f = 0, \qquad \text{on the interval } (a, b),$$

subject to the boundary conditions

$$B_i(f) = 0, \quad i = 1, 2.$$

What happens to λ_n when $n \to \infty$?

$\lambda_n \to \infty$ as $n \to \infty$. It grows approximately like n^2, but you don't need to remember that.

(c) (2p) Assume that u and v are solutions to the aforementioned regular SLP and have eigenvalues 2 and 4, respectively. Compute

$$\int_a^b u(x)\overline{v(x)}dx.$$

It's 0. Theory says they are orthogonal because different eigenvalues.

(d) (4p) Use $\{f_n\}_{n\geq 1}$ and $\{\lambda_n\}_{n\geq 1}$ to obtain the solution $u(x, t)$ to the following problem

$$\partial_t u - L(u) = 0, \quad t > 0, \quad x \in (a, b),$$
$$B_i(u) = 0, \quad i = 1, 2,$$
$$u(x, 0) = \varphi(x) \text{ is a bounded, continuous function on } [a, b].$$

Well, we can separate variables and look for solutions TX to solve

$$T'X - TL(X) = 0 \implies \frac{T'}{T} = \frac{L(X)}{X}.$$

Ergo both sides are constant. Call it ν. Look at the L side. It requires

$$L(X) = \nu X \implies L(X) - \nu X = 0.$$

Also

$$B_i(X) = 0, \quad i = 1, 2.$$

Hence the only solutions from SLP theory are $X_n = f_n$, and $\nu_n = -\lambda_n$. Hence up to presently unknown constant factors,

$$T_n(t) = c_n e^{-\lambda_n t}.$$

We use superposition since the PDE is homogeneous to write

$$u(x, t) = \sum_{n\geq 1} c_n e^{-\lambda_n t} f_n(x).$$

By the SLP theory the f_n are an orthogonal basis for $L^2(a, b)$ so we can expand the initial data:

$$c_n = \frac{\int_a^b \varphi(x)\overline{f_n(x)}dx}{\int_a^b |f_n(x)|^2 dx}.$$

1.1. **Theory.**

(1) (15p) I've attempted to prove the BBC, but I keep getting stuck. Can you help me finish the proof?

Theorem 1.1. *Let $g \in L^1(\mathbb{R})$. Define*

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\varepsilon > 0$,

$$g_\epsilon(x) := \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0^+} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof: First, since this should hold for all $x \in \mathbb{R}$, let us fix the point x.

(a) **What is the meaning of $f(x+)$ and $f(x-)$? Explain what are these things? (1p)**

Right and left sided limits, respectively, at the point x. One point for each.

(b) **Why is the statement in the theorem equivalent to proving the following statement (1p)?**

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y)g_\varepsilon(y)dy - \alpha f(x+) - \beta f(x-) = 0.$$

Because if we subtract the right side of what the limit is supposed to equal, then that limit should be zero.

So, now that you've explained why the statement in the theorem is equivalent to proving the statement above, this can be achieved by proving that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = 0$$

and also

$$\lim_{\varepsilon \to 0} \int_{0}^{\infty} f(x-y)g_\varepsilon(y)dy - \int_{0}^{\infty} f(x-)g(y)dy = 0.$$

(c) **In the next step, we just pick one of the above and say it's enough to show the limit is zero for one of them. Why can we do that??? What is that voodoo magic? (1p)**

The argument works in the same way in both cases. That's why.

We would therefore like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy$$

as small as we like. In particular, let $\delta > 0$ be arbitrarily small. We wish to show that by choosing ε sufficiently small, we can make

$$\left| \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy \right| < \text{ a constant multiple of } \delta.$$

(d) **Why will it complete the proof to show that the inequality above is true? How does that prove the limit below? (2p)**

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = 0?$$

We want to prove that

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy \to 0 \text{ as } \varepsilon \to 0.$$

This is true if and only if

$$\left|\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy\right| \to 0 \text{ as } \varepsilon \to 0.$$

The above is true if and only if for any arbitrary $\delta > 0$ we can make

$$\left|\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy\right| < \text{ some fixed constant times } \delta.$$

(e) **How do we obtain the equation below? It looks like magic to me, please explain?!? (1p)**

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = \int_{-\infty}^{0} g_\varepsilon(y)\left(f(x-y) - f(x+)\right)dy.$$

It's done by substituting variables in the second integral. For $y = \frac{z}{\varepsilon}$ we have $dy = \frac{dz}{\varepsilon}$ and the limits of integration don't change so

$$\int_{-\infty}^{0} g(y)dy = \int_{-\infty}^{0} g(z/\varepsilon)dz/\varepsilon = \int_{-\infty}^{0} g_\varepsilon(z)dz.$$

Now we can rename the integrating variable to y cause the name don't matter. Yo.

(f) **How can we find $y_0 < 0$ to make the inequality below true? (1p)**

$$|f(x-y) - f(x+)| < \delta \forall y \in [y_0, 0).$$

By definition of the right sided limit since for $y \in [y_0, 0)$ we have $x - y > 0$ and as $y \to 0$ this tends to x from the right. By definition of limit we can make the above $< \delta$ by choosing y_0 sufficiently close to 0 and negative.

(g) **How do we use that above to obtain the inequality below? (1p)**

$$\left|\int_{y_0}^{0}(f(x-y) - f(x+))g_\varepsilon(y)dy\right| \le \delta ||g||_{L^1(\mathbb{R})}.$$

We move the absolute values inside the integral. That gives us the δ factor in front. Then we use that

$$\int_{-\infty}^{y_0} |g_\varepsilon(y)|dy \le \int_{\mathbb{R}} |g_\varepsilon(y)|dy = \int_{\mathbb{R}} |g(z)|dz = ||g||_{L^1(\mathbb{R})}.$$

This also explains the next question.

(h) **What is $||g||_{L^1(\mathbb{R})}$? (1p)**

Next, there are two cases. In the first case, f is bounded, which means that there exists $M > 0$ such that $|f(x)| \le M$ holds for all $x \in \mathbb{R}$.

(i) **How do we use this to obtain the inequality below? (2p)**

$$\left|\int_{-\infty}^{y_0}(f(x-y) - f(x+))g_\varepsilon(y)dy\right| \le 2M \int_{-\infty}^{y_0/\varepsilon} |g(y)|dy.$$

Same idea as above. Move the absolute value inside the integral. $|\int ...| \le \int |...|$. This is always true. Then if $f(x)| \le M$ for all x we get that $|f(x-y) - f(x+)| \le |f(x-y)| + |f(x+)| \le 2M$. So we can put that out in front. In the last step we change variables in the integral for g.

(j) **How can we use ε to obtain the inequality below? (1p)**

$$2M\int_{-\infty}^{y_0/\varepsilon} |g(y)|dy < \delta?$$

Note that $y_0/\varepsilon \to -\infty$ when $\varepsilon \to 0$. So, we are looking at the tail of a convergent integral since g is in $\mathcal{L}^1$. So, we can make this tail as small as we want, in particular less than δ.

In this case, we therefore have the estimate

$$\left|\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy\right|$$

$$\leq \left|\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| + \left|\int_{y_0}^{0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| \leq \delta + \delta||g||_{L^1(\mathbb{R})}.$$

(k) **Why is the proof complete in this case now? (1p)**

Now we got the estimate we wanted because

$$\delta + ||g||_{L^1(\mathbb{R})}\delta = (1 + ||g||)\delta$$

is of the form a constant times δ.

(l) **In the second case in the theorem, when g vanishes outside a bounded interval, how can we use ε to obtain the equation below 'for ε small enough?' (2p)**

$$\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy = 0.$$

We just need to choose ε so that

$$0 < \varepsilon < \frac{y_0}{-L}, \text{ if } g \equiv 0 \text{ outside of } [-L, L].$$

(2) (5p) Explain your favorite proof from all of the theory-item-proofs in this course. Why is that proof your favorite? What do you like about it?

Oh geez, I like them all. How to choose??? Curious to hear what y'all think. This might be like free points, but that is to compensate for the BBC which is a hard theory item. Moreover, you had to explain all the steps. That's asking a lot. So it seems fair to be a little generous with this last one. Plus in addition to studying you gotta deal with a pandemic which is stressful. So why not end on a nice uplifting positive note? Speaking of which, do you like punk rock music? Are you curious what punk rock music sounds like? Check out my friends' band! They are super awesome. `https://middleagedqueers.bandcamp.com/releases` Their music helped keep me in a good mood while grading your exams!