Fourieranalys MVE030 och Fourier Metoder MVE290 16.mars.2018
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
Examinator: Julie Rowlett.
Telefonvakt: Olof Gieselsson, ankn 5325.

1. Bevisa att Besselfunktionerna, J_n, uppfyller:
$$\sum_{n \in \mathbb{Z}} J_n(x) z^n = e^{\frac{x}{2}(z - 1/z)}.$$
(10 p)

2. Bevisa att de Hermitska polynomen,
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$
uppfyller:
$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$
(10 p)

3. Beräkna den komplexa Fourierserien till den 2π-periodiska funktion $f(x)$ som är lika med e^{-x} i $(-\pi, \pi)$. Vad är seriens summa i punkten 3π? (10 p)

4. Hitta polynomet, p, av högst grad tre som minimera
$$\int_{-3}^{3} |\cosh(x) - p(x)|^2 dx.$$
(10 p)

5. Lös problemet:
$$\begin{aligned} u_{tt} - u_{xx} &= 20, \quad 0 < x < 4, \quad t > 0 \\ u(x,0) &= v(x), \\ u(0,t) &= 0, \\ u_t(x,0) &= 0, \\ u_x(4,t) &= 0. \end{aligned}$$
(10 p)

6. Lös problemet:

$$u_t - u_{xx} = G(x,t), \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x,0) = v(x).$$

(10 p)

7. Lös problemet:

$$\begin{cases} u_t - u_{xx} = 0, \quad x \in (0,\infty), \quad t \in (0,\infty) \\ u(0,t) = f(t) \\ u(x,0) = 0 \end{cases}$$

(10 p)

8. Låt

$$I_0(x) = \sum_{n\geq 0} \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}.$$

Visa att:

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos(\theta)} d\theta.$$

(Tips: Taylorutvecklingen av exponentialfunktionen och beräkna $\int_0^\pi cos^n(\theta) d\theta$.)
(10 p)

Lycka till! May the force be with you! ♡ Julie Rowlett.

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See the proofs of the theory items!

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Let us compute the Fourier coefficients:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}dx = \frac{1}{-2\pi(1+in)}\left(e^{-x(1+in)}\right)_{x=-\pi}^{\pi}$$

$$= \frac{-1}{2\pi(1+in)}\left(e^{-\pi(1+in)} - e^{\pi(1+in)}\right) = \frac{1}{2\pi(1+in)}\left(e^{\pi}e^{i\pi n} - e^{-\pi}e^{-i\pi n}\right)$$

$$= \frac{1}{\pi(1+in)}(-1)^n \sinh(\pi).$$

So, the Fourier series is

$$\sum_{n\in\mathbb{Z}} \frac{1}{\pi(1+in)}(-1)^n \sinh(\pi)e^{inx}.$$

At the point $3\pi = \pi + 2\pi$, since the function is 2π periodic, the series converges to whatever it converges to at the point π. Since we extend e^{-x} to be 2π periodic, denoting this function by $f(x)$,

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} e^{-x} = e^{\pi},$$

and

$$\lim_{x\to\pi^+} f(x) = \lim_{x\to-\pi^+} f(x) = \lim_{x\to-\pi^+} e^{-x} = e^{\pi}.$$

Above, we are using the fact that we extend to x outside of the interval $(-\pi, \pi)$ to make the function 2π periodic (we use this to go from the first limit to the second one). So, we see that the left and right limits of $f(x)$ as $x \to \pi$ are different. The theorem on the pointwise convergence of Fourier series says that the Fourier series of f converges to the average of the left and right sided limits. Thus the Fourier series converges to

$$\frac{e^{\pi} + e^{-\pi}}{2} = \cosh(\pi).$$

That wasn't so tough, was it?

4. Hitta polynomet, p, av högst grad tre som minimera

$$\int_{-3}^{3} |\cosh(x) - p(x)|^2 dx.$$

(10 p)

We see that above, the interval is from -3 to 3. The Legendre polynomials will not be orthogonal on that interval, but we can change the variable to remedy this. You see, Legendre polynomials $P_n(t)$ are pairwise orthogonal if t goes from -1 to 1. So, if x is from -3 to 3, then we define

$$t := x/3.$$

Then we compute

$$\int_{-3}^{3} P_n(x/3)P_m(x/3)dx = \int_{-1}^{1} P_n(t)P_m(t)(3dt) = \begin{cases} 0 & n \neq m \\ \frac{6}{n+1} & n = m \end{cases}$$

To find this calculation if you don't already have it burned into your brains, see β-12.2. However, I hope your brain is not too badly burned, so that you do not mindlessly pass to the formula below the one which gives the above calculation (once we have changed the variable to go from -1 to 1)!!! We are working on the interval $(-3, 3)$. We computed above that

$$\{P_n(x/3)\}_{n\geq 0}$$

is an orthogonal set *on this interval,* $(-3, 3)$. By the usual magical spectral theorem for adults, they are an orthogonal basis for $\mathcal{L}^2$ on the interval $(-3, 3)$. We expand the given function *on this interval* using this basis. The coefficients are thus

$$c_n = \frac{\int_{-3}^{3} \cosh(x) P_n(x/3) dx}{6/(n+1)}.$$

The denominator comes from the norm calculation we did above (using β):

$$||P_n||^2 = \int_{-3}^{3} P_n(x/3)^2 dx = \frac{6}{n+1}.$$

Hence, the polynomial we seek is

$$\sum_{n=0}^{3} c_n P_n(x/3).$$

Aren't you glad you don't actually have to compute those integrals

$$\int_{-3}^{3} \cosh(x) P_n(x/3) dx?$$

Yes, you're very welcome ♡

5. Lös problemet:

$$\begin{aligned} u_{tt} - u_{xx} &= 20, \quad 0 < x < 4, \quad t > 0 \\ u(x, 0) &= v(x), \\ u(0, t) &= 0, \\ u_t(x, 0) &= 0, \\ u_x(4, t) &= 0. \end{aligned}$$

(10 p)

Feeling a bit of déjà vu? Oui, je sais. Allons-y! Oh, joy of joys, the inhomogeneity in the PDE is *time independent!* So, we can deal with it by finding a steady state solution. So, we are looking for f to satisfy

$$\begin{cases} -f''(x) = 20 \\ f(0) = 0 \\ f'(4) = 0 \end{cases}$$

Start integrating the ODE:

$$-f''(x) = 20 \implies f'(x) = -20x + c \implies f(x) = -10x^2 + cx + b.$$

We find the constants using the boundary conditions:

$$f(0) = b = 0, \quad f'(4) = -20(4) + c \implies c = 80.$$

So, our steady state solution is

$$f(x) = -10x^2 + 80x.$$

Now, if we were to solve for the homogeneous PDE and just add it to the steady state solution, we'd screw up the IC. So, we instead look for a solution to:

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad 0 < x < 4, \quad t > 0 \\ u(x,0) &= v(x) - f(x), \\ u(0,t) &= 0, \\ u_t(x,0) &= 0, \\ u_x(4,t) &= 0. \end{aligned}$$

Then, our full solution will be

$$u(x,t) + f(x) = u(x,t) - 10x^2 + 80x.$$

So, let's get to finding u, shall we? We are on a bounded interval, and we see really nice self-adjoint boundary conditions (things equal to zero evokes self adjointness). So, we will use separation of variables. Consider:

$$T''X - X''T = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \text{ constant } = \lambda.$$

Above we have named the constant λ. The BCs in x are super nice, so we shall solve for X first. There are three cases. First, what if $\lambda = 0$? Then,

$$X'' = \lambda X \implies X'' = 0 \implies X(x) = ax + b.$$

To satisfy the BCs, we need

$$X(0) = b = 0, \quad X'(4) = a = 0 \implies X = 0.$$

That's not a legit solution. (Eigenfunctions shallt not be the constant zero function). So this is out. What about $\lambda > 0$? Then,

$$X(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x).$$

We need

$$X(0) = 0 = a \implies X(x) = b\sinh(\sqrt{\lambda}x).$$

We also need

$$X'(4) = \sqrt{\lambda}b\cosh(4\sqrt{\lambda}) = 0.$$

The function

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 0 \iff e^x = -e^{-x}.$$

That is impossible for $x \in \mathbb{R}$. So, the only way for $X'(4) = 0$ is if $b = 0$. We end up at the illegit zero solution. So, we know that $\lambda < 0$ is the only possibility. Then,

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To guarantee that $X(0) = 0$ we need $a = 0$. So, for now, let us drop the constant factor b, and determine it later. Then, we need

$$\cos(\sqrt{|\lambda|}4) = 0 \implies \sqrt{|\lambda|}4 = (n + 1/2)\pi, \quad n \in \mathbb{N}.$$

Thus

$$\sqrt{|\lambda|} = \frac{(n+1/2)\pi}{4}, \quad X_n(x) = \sin((n+1/2)\pi x/4), \quad \lambda = -\frac{(n+1/2)^2\pi^2}{16}.$$

We can now find the partner function, and we let it absorb the constant factors:

$$T_n'' = \lambda_n T_n \implies T_n(t) = a_n\cos(\sqrt{|\lambda_n|}t) + b_n\sin(\sqrt{|\lambda_n|}t).$$

We then smash them together because the PDE is homogenous, writing

$$u(x,t) = \sum_{n\geq 0} T_n(t)X_n(x)$$

$$= \sum_{n\geq 0}\left(a_n\cos(\sqrt{|\lambda_n|}t) + b_n\sin(\sqrt{|\lambda_n|}t)\right)\sin((n+1/2)\pi x/4).$$

We need

$$u_t(x,0) = 0.$$

So, differentiating termwise we get

$$\sum_{n\geq 0} b_n\sqrt{|\lambda_n|}X_n(x) = 0.$$

By the SLP theory, the set $\{X_n\}$ makes an orthogonal basis for the Hilbert space $\mathcal{L}^2$ on the interval $(0,4)$. The 0 function has an expansion: all of the coefficients are zero. So, $b_n\sqrt{|\lambda_n|} = 0$ for all n. This means we need $b_n = 0$ for all n. So, our solution looks like

$$u(x,t) = \sum_{n\geq 0} a_n \cos(\sqrt{|\lambda_n|}t)X_n(x).$$

To ensure that

$$u(x,0) = v(x) - f(x),$$

we need

$$\sum_{n\geq 0} a_n X_n(x) = v(x) - f(x).$$

Hence the left side should be the Fourier expansion of $v-f$ in terms of the OB $\{X_n\}$. These Fourier coefficients are:

$$a_n = \frac{\int_0^4 (v(x) - f(x))X_n(x)dx}{\int_0^4 X_n^2(x)dx}.$$

You have no desire to compute these integrals (and the fact that v is not specified makes it rather um impossible?) That is not a problem because we are already done! Our solution is

$$\sum_{n\geq 0} a_n \cos(\sqrt{|\lambda_n|}t)X_n(x) + f(x),$$

where a_n, $\sqrt{|\lambda_n|}$, X_n, and $f(x)$ have already been specified above. If you want to be really persnickety about the details, you can give names to the equations where these are specified and refer to them here.

6. Lös problemet:

$$u_t - u_{xx} = G(x,t), \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x,0) = v(x).$$

(10 p)

Time to go back to the basics, because we have an *inhomogeneous* heat equation which depends on both time and space. Not a problem. We got the cute heat kernel style solution involving convolution, so let us just do the same thing here and now. We hit the PDE with the Fourier transform on $x \in \mathbb{R}$ variable:

$$\hat{u}_t(\xi,t) - \widehat{u_{xx}}(\xi,t) = \hat{G}(\xi,t).$$

We use β 13.2.F10 with $n = 2$ there:

$$\widehat{u_{xx}}(\xi,t) = (i\xi)^2\hat{u}(\xi,t).$$

So the equation is

$$\hat{u}_t(\xi, t) + \xi^2 \hat{u}(\xi, t) = \hat{G}(\xi, t).$$

Stay calm. This is just an ODE for u with respect to the variable t. We look it up in β. We find the solution is given in β 9.1.3. First, we compute

$$\exp(-\int \xi^2 dt) = e^{-\xi^2 t} \text{ don't need integration constant here according to } \beta.$$

Next, we compute the solution is

$$\hat{u}(\xi, t) = e^{-\xi^2 t} \left(\int_0^t e^{\xi^2 s} \hat{G}(\xi, s) ds + C \right).$$

We use the IC to determine C:

$$\hat{u}(\xi, 0) = \hat{v}(\xi) = C,$$

so

$$\hat{u}(\xi, t) = e^{-\xi^2 t} \left(\int_0^t e^{\xi^2 s} \hat{G}(\xi, s) ds + \hat{v}(\xi) \right) = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi, s) ds + e^{-\xi^2 t} \hat{v}(\xi).$$

We know (or look it up in β) that to get a product from the Fourier transformation, we start with a convolution. In the second term, we can look up already that:

$$e^{-x^2/(4t)} (4\pi t)^{-1/2} \text{ Fourier transforms to } e^{-\xi^2 t}.$$

We get this from β 13.2 F37. Well, then similarly, the same formula shows that

$$e^{-x^2/((4(t-s))} (4\pi (t-s))^{-1/2} \text{ Fourier transforms to } e^{-\xi^2 (t-s)}.$$

So, our solution is given by the sum of the convolutions:

$$u(x, t) = \int_{\mathbb{R}} \int_0^t e^{-(x-y)^2/(4(t-s))} (4\pi(t-s))^{-1/2} G(y, s) ds dy + \int_{\mathbb{R}} e^{-(x-y)^2/(4t)} (4\pi t)^{-1/2} v(y) dy.$$

7. Lös problemet:

$$\begin{cases} u_t - u_{xx} = 0, & x \in (0, \infty), \quad t \in (0, \infty) \\ u(0, t) = f(t) \\ u(x, 0) = 0 \end{cases}$$

(10 p)

The x variable is in a half line, and it has a weird BC at $x = 0$, which depends on time. This just SCREAMS Laplace transform. So, let's do it! We Laplace transform the PDE in the t variable. Since $u(x, 0) = 0$, we get

$$z\tilde{u}(x, z) - \partial_{xx} \tilde{u}(x, z) = 0.$$

This is just an ODE for the x variable. We solve it:

$$\tilde{u}(x,z) = a(z)e^{-x\sqrt{z}} + b(z)e^{x\sqrt{z}}.$$

The second of these will be impossible to invert Laplace transform, so we chuck it out. We determine $a(z)$ using the weird BC:

$$\tilde{u}(0,z) = \tilde{f}(z) \implies a(z) = \tilde{f}(z).$$

Really, above we mean

$$\widetilde{\Theta f}(z),$$

because the function f is only defined for $t > 0$, so we extend it to be identically zero for $t < 0$, or equivalently just write Θf. Take your pick. So, our Laplace transformed solution is:

$$\tilde{u}(x,z) = \tilde{f}(z)e^{-x\sqrt{z}}.$$

The Laplace transform also turns convolutions into products. So, we are looking for somebody whose Laplace transform is $e^{-x\sqrt{z}}$. We turn to β 13.5 L61. This says that

$$\Theta(t)\frac{x}{2\sqrt{\pi t^3}}e^{-x^2/(4t)} \text{ Laplace transforms in the } t \text{ variable to } e^{-x\sqrt{z}}.$$

The $\Theta(t)$ is the heavyside function, which we need to make the function be Laplace transformable. We can only Laplace transform functions which are zero for $t < 0$. So, this is actually kinda sorta missing in β, but it is a little bit implicit in 13.5 L4, but I still find that rather unsatisfying. At least I wrote about it in the lecture notes... So, the solution is given by the convolution:

$$u(x,t) = \int_{\mathbb{R}} \Theta(t-s)\frac{x}{2\sqrt{\pi(t-s)^3}}e^{-x^2/(4(t-s))}\Theta(s)f(s)ds.$$

Since

$$\Theta(t-s) = \begin{cases} 0 & s > t \\ 1 & s < t \end{cases},$$

and similarly

$$\Theta(s) = \begin{cases} 0 & s < 0 \\ 1 & 0 < s \end{cases}$$

$$u(x,t) = \int_0^t \frac{x}{2\sqrt{\pi(t-s)^3}}e^{-x^2/(4(t-s))}f(s)ds.$$

8. Låt

$$I_0(x) = \sum_{n \geq 0} \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}.$$

Visa att:

$$I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos(\theta)} d\theta.$$

(Tips: Taylorutvecklingen av exponentialfunktionen och beräkna $\int_0^\pi cos^n(\theta) d\theta$.) (10 p)

We follow the hint:

$$\frac{1}{\pi} \int_0^\pi \sum_{n \geq 0} \frac{x^n \cos^n(\theta)}{n!} d\theta.$$

So, we need to compute

$$\int_0^\pi \cos^n(\theta) d\theta.$$

We turn to our trusty β. We have formula 7.4.212:

$$\int \cos^n \theta d\theta = \frac{1}{n} \sin(\theta) \cos^{n-1}(\theta) + \frac{n-1}{n} \int \cos^{n-2}(\theta) d\theta.$$

Let us use this to compute our definite integral:

$$\int_0^\pi \cos^n(\theta) d\theta = \frac{1}{n} \sin(\theta) \cos^{n-1}(\theta) \Big|_0^\pi + \frac{n-1}{n} \int_0^\pi \cos^{n-2}(\theta) d\theta.$$

The first part vanishes because $\sin(0) = \sin(\pi) = 0$. So we get a recursive formula:

$$\int_0^\pi \cos^n(\theta) d\theta = \frac{n-1}{n} \int_0^\pi \cos^{n-2}(\theta) d\theta.$$

We first compute

$$\int_0^\pi \cos^0(\theta) d\theta = \pi.$$

We next compute

$$\int_0^\pi \cos^1(\theta) d\theta = \sin(\pi) - \sin(0) = 0.$$

Hence by the inductive formula

$$\int_0^\pi \cos^n(\theta) d\theta = \begin{cases} 0 & n \text{ is odd} \\ \frac{n-1}{n} \int_0^\pi \cos^{n-2}(\theta) d\theta & n \text{ is even.} \end{cases}$$

So, for the case $n = 2$,
$$\int_0^\pi \cos^2(\theta) d\theta = \frac{2-1}{2}\pi = \frac{\pi}{2}.$$
We then compute for $n = 4$,
$$\int_0^\pi \cos^4(\theta) d\theta = \frac{4-1}{4}\frac{\pi}{2} = \frac{3\pi}{4*2}.$$
Next for $n = 6$,
$$\int_0^\pi \cos^6(\theta) d\theta = \frac{6-1}{6}\frac{3\pi}{4*2} = \frac{5*3*\pi}{6*4*2}.$$
Do you see the pattern? For even numbers, like $2k$, we have
$$\int_0^\pi \cos^{2k}(\theta) d\theta = \frac{(2k-1)(2k-3)\ldots\pi}{2k(2k-2)(2k-4)\ldots 2}.$$
On the top, we got the odd factorials missing the even ones, on the bottom we just got the even ones. Look at the bottom first:
$$2k(2k-2)(2k-4)\ldots 4*2 = 2(k)2(k-1)2(k-2)\ldots 2(2)2(1) = 2^k k!.$$
Making similar considerations, the numerator is:
$$(2k-1)(2k-3)\ldots 3*1 = \frac{(2k)!}{(2k)(2k-2)(2k-4)\ldots 4*2} = \frac{(2k)!}{2^k k!}.$$
Hence, together we have that
$$\int_0^\pi \cos^{2k+1}(\theta) d\theta = 0, \quad \int_0^\pi \cos^{2k}(\theta) d\theta = \frac{(2k)!\pi}{2^{2k}(k!)^2}.$$
Therefore
$$\frac{1}{\pi}\int_0^\pi \sum_{n\geq 0} \frac{x^n \cos^n(\theta)}{n!} d\theta = \frac{1}{\pi}\sum_{k\geq 0} \frac{x^{2k}}{(2k)!}\frac{(2k)!\pi}{2^{2k}(k!)^2} = \sum_{k\geq 0} \frac{\left(\frac{x}{2}\right)^{2k}}{(k!)^2} = I_0(x).$$
Please don't be cross; there always needs to be some challenging and/or surprising problem on exams. Don't you want to test your limits? Also, this version is not as tough as the previous solution where we didn't use the recursion formula of β but did it all by hand using the binomial theorem. That method works too, and it is beautiful, so props to you in any case if you made some progress on this problem!

Figure 1: This is not actually from the current trip, but a previous trip to Oz. That's a baby wombat at a rescue center for wild animals. In general, you should NOT try to touch wildlife, but rescue babies actually need to snuggle in order to grow up healthy, so in that case, I was helping this little wombat. What's this got to do with math? Well, when I traveled to Oz that time, I was trying to solve some math involving geometric measure theory, notoriously difficult stuff. A famous geometric measure theorist, Brian White, Professor at Stanford, has an equally famous photo with a koala bear in Australia... You can google that... So I had a silly idea that maybe the Australian animals impart GMT intuition, because another famous GMT'er is Professor Leon Simon, originally from Oz. So, that's the scoop with this funny photo. Now let's just hope that modern technology allows me to upload these solutions from the other side of the world! Y'all have been a great class, and I hope that y'all did well on the exam!