1. (10 p) Låt $f$ vara en $2\pi$ periodisk funktion. Antar att $f$ är styvvis kontuerlig (piecewise continuous) och att $\forall x \in \mathbb{R}$, dess höger och vänster gränsvärde existerar:

$$\lim_{y \to x^+} f(y) = f(x^+) \in \mathbb{R}, \quad \lim_{y \to x^-} f(y) = f(x^-) \in \mathbb{R}.$$ 

Låt

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$ 

Bevisa att gäller:

$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} (f(x_-) + f(x_+)), \quad \forall x \in \mathbb{R}.$$ 

2. (10 p) Definera Fourier transformen och ge dess Inversion-Formel.

3. (10 p) Beräkna:

$$\sum_{n=1}^{\infty} \frac{1}{4 + n^2}.$$ 

(Hint: Utveckla $e^{2x}$ i Fourier-series i intervallet $(-\pi, \pi)$).

4. (10 p) Hitta siffrorna $a_0$, $a_1$, och $a_2 \in \mathbb{C}$ som minimerar

$$\int_{0}^{\pi} |\sin(x) - a_0 - a_1 \cos(x) - a_2 \cos(2x)|^2 dx.$$ 

5. (10 p) Lägg problemet:

$$u_t - u_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x, 0) = e^{-x^2}.$$
6. (10 p) Beräkna
\[ \int_0^\infty \frac{\sin(x)}{xe^x} \, dx. \]

7. (10 p) Lös problemet:
\[ u_{xx} + u_{yy} = -20u, \quad 0 < x < 1, \quad 0 < y < 1, \]
\[ u(0, y) = u(1, y) = 0, \]
\[ u(x, 0) = 0, \]
\[ u(x, 1) = x^2 - x. \]

8. (10 p) Lös problemet:
\[ u_t - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = t + 1, \]
\[ u(1, t) = 0, \]
\[ u(x, 0) = 1 - x. \]

Lycka till! May the force be with you! ♥ Julie Rowlett.
Fourieranalys MVE030 och Fourier Metoder MVE290 22.augusti.2017

Maximalt antal poäng: 80.
Hjälpmedel: BETA.
Examinator: Julie Rowlett.
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1. (10 p) Låt \( f \) vara en \( 2\pi \) periodisk funktion. Antar att \( f \) är styvvis kontuerlig (piecewise continuous) och att \( \forall x \in \mathbb{R} \), dess höger och vänster gränsvärde existerar:

\[
 \lim_{y \to x^+} f(y) = f(x^+) \in \mathbb{R}, \quad \lim_{y \to x^-} f(y) = f(x^-) \in \mathbb{R}.
\]

Låt

\[
 S_N(x) = \sum_{-N}^{N} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.
\]

Bevisa att gäller:

\[
 \lim_{N \to \infty} S_N(x) = \frac{1}{2} \left( f(x^-) + f(x^+) \right), \quad \forall x \in \mathbb{R}.
\]

Solution is in the theory-proof compendium!

2. (10 p) Definerar Fourier transformen och ger dess Inversion-Formel.

Solution is in the theory-proof compendium!

3. (10 p) Beräkna:

\[
 \sum_{n=0}^{\infty} \frac{1}{4 + n^2}.
\]

(Hint: Utveckla \( e^{2x} \) i Fourier-series i intervallet \((-\pi, \pi)\)).

We follow the hint. To do that, we compute the Fourier coefficients:

\[
 c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2x} e^{-inx} dx = \frac{e^{2\pi - inx} - e^{2\pi + inx}}{2\pi(2 - in)} \bigg|_{x=-\pi}^{\pi} = \frac{\sinh(2\pi)}{\pi(2 - in)}.
\]

Above, we use the fact that \( e^{\pm in\pi} = (-1)^n \) together with basic rules for exponentials, like \( e^{a+b} = e^a e^b \), and the definition of sinh.
So, now we know that
\[ e^{2x} = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad x \in (-\pi, \pi). \]

What happens for \( x = \pi \) or \( x = -\pi \)? The series on the right does NOT converge to the function on the left!!!! Remember Theorem 2.1! Even easier on this particular exam is THEORY QUESTION #1. It tells you (på svenska även!) what happens! When we do a Fourier expansion, we extend \( e^{2x} \) from the interval \((-\pi, \pi)\) to \( \mathbb{R} \) as a 2\( \pi \) periodic function. Doing this, the function jumps at odd-integer multiples of \( \pi \). The Fourier series converges to the average of this “jump” at these points, so
\[
\frac{e^{2\pi} + e^{-2\pi}}{2} = \sum_{n \in \mathbb{Z}} c_n e^{in\pi} = \sum_{n \in \mathbb{Z}} c_n (-1)^n = \sum_{n \in \mathbb{Z}} \frac{\sinh(2\pi)}{\pi(2 - in)}.
\]

The left side is none other than \( \cosh(2\pi) \) so we bring it together with its buddy \( \sinh \),
\[
\pi \coth(2\pi) = \sum_{n \in \mathbb{Z}} \frac{1}{2 - in}.
\]

Next, we take away the \( n = 0 \) term and pair up the \( \pm n \) terms, so that
\[
\pi \coth(2\pi) = \frac{1}{2} + \sum_{n \geq 1} \frac{1}{2 - in} + \frac{1}{2 + in} = \frac{1}{2} + \sum_{n \geq 1} \frac{4}{4 + n^2}.
\]

Re-arranging, we have
\[
\sum_{n \geq 1} \frac{1}{4 + n^2} = \pi \coth(2\pi) - \frac{2}{4}.
\]

4. (10 p) Hitta siffrorna \( a_0, a_1, \) och \( a_2 \in \mathbb{C} \) som minimerar
\[
\int_0^\pi \left| \sin(x) - a_0 - a_1 \cos(x) - a_2 \cos(2x) \right|^2 dx.
\]

This is just expanding the sine in terms of a cosine basis on \( L^2(0, \pi) \). You can probably find some stuff in \( \beta \), or you can just do it by hand. The first three basis vectors here are constant multiples of \( \cos(kx) \) for \( k = 0, 1, 2 \). These are already orthogonal, because
\[
\int_0^\pi \cos(jx) \cos(kx) dx = 0 \text{ if } k \neq j.
\]
So, they just need to get normalized. Thus, we compute the $L^2$ norm (squared)

$$
\int_0^\pi \cos^2(kx)dx = \pi, \quad k = 0; \quad \text{or} \quad \frac{\pi}{2} \text{ for } k = 1, 2.
$$

The trick to computing the integral for $k = 1, 2$ is to use the double angle formula for the cosine,

$$
\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1,
$$

where we use the identity $\cos^2 + \sin^2 = 1$. Now, we have our basis vectors:

$$
\frac{1}{\sqrt{\pi}}, \quad \frac{\cos(kx)}{\sqrt{\pi}}, \quad k = 1, 2.
$$

Next, we compute the coefficients by computing the inner product of $\sin(x)$ with the basis vectors. It suffices for this purpose to compute:

$$
\frac{1}{\sqrt{\pi}} \int_0^\pi \sin(x)dx = \frac{1}{\sqrt{\pi}} (-\cos(\pi) + \cos(0)) = \frac{2}{\sqrt{\pi}}
$$

$$
\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi \sin(x)\cos(x)dx = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi \frac{1}{2} \sin(2x)dx = 0.
$$

$$
\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi \sin(x)\cos(2x)dx = \frac{\sqrt{2}}{\sqrt{\pi}} \left( \frac{\sin(x)\sin(2x)}{2} \bigg|_0^\pi - \int_0^\pi \cos(x)\frac{\sin(2x)}{2} dx \right)
$$

$$
= -\frac{\sqrt{2}}{\sqrt{\pi}} \left( \int_0^\pi \cos^2(x)\sin(x)dx \right)
$$

$$
= \frac{\sqrt{2}}{\sqrt{\pi}} \left( \frac{\cos^3(x)}{3} \bigg|_0^\pi \right) = -\frac{2\sqrt{2}}{3\sqrt{\pi}}.
$$

Hence, the best approximation of $\sin(x)$ in terms of this basis is

$$
\frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} - \frac{2\sqrt{2}}{3\sqrt{\pi}} \frac{\cos(2x)}{\sqrt{\pi}} = \frac{2}{\pi} - \frac{4}{3\pi} \cos(2x).
$$

The siffror we seek are therefore

$$
a_0 = \frac{2}{\pi}, \quad a_1 = 0, \quad a_2 = -\frac{4}{3\pi}.
$$
5. (10 p) Läs problemet:

\[ u_t - u_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R}, \]

\[ u(x, 0) = e^{-x^2} \]

There’s nothing like the IVP for the heat equation. We use the heat kernel (Schwartz integral kernel of the fundamental solution to the heat equation - you can learn more about Schwartz integral kernels in the future :-) )

\[ u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} e^{-y^2} dy. \]

For extra fun: compute this! It isn’t too bad...

6. (10 p) Beräkna

\[ \int_0^{\infty} \frac{\sin(x)}{xe^x} dx. \]

We just need to put on our Plancharel/Parseval (I always forget which is which so just lump them together) glasses. We know that

\[ \int_{\mathbb{R}} f(x)g(x)dx = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x)\hat{g}(x)dx, \]

as long as the two functions are real valued. If they’re complex valued, we gotta include some complex conjugation up in there.

Well, what we’ve got is not an integral over \( \mathbb{R} \) dagnammit. That is rather annoying. However, we can modify the integral to get an integral over \( \mathbb{R} \) with a few observations. The function \( \sin(x)/x \) is even. We have the product of that with \( e^{-x} \). We can extend \( e^{-x} \) to be an even function, using \( e^{-|x|} \). So, in this way

\[ \int_0^{\infty} \frac{\sin(x)}{xe^x} dx = \frac{1}{2} \int_{\mathbb{R}} \frac{\sin(x)}{x} e^{-|x|} dx. \]

So, while I don’t really fancy doing the above integral, using the Parseval/Plancharel trick, we can replace those functions by the Fourier transforms:

\[ \frac{1}{2} \int_{\mathbb{R}} \frac{\sin(x)}{x} e^{-|x|} dx = \frac{1}{4\pi} \int_{\mathbb{R}} \pi \chi_{(-1,1)}(x) \frac{2}{x^2 + 1} dx = \frac{1}{2} \int_{-1}^{1} \frac{1}{x^2 + 1} dx \]

\[ = \frac{1}{2} \arctan(x) \big|_{-1}^{1} = \frac{1}{2} \left( \frac{\pi}{4} - \frac{\pi}{4} \right) = \frac{\pi}{4}. \]

How cute.
7. (10 p) Lös problemet:

\[ u_{xx} + u_{yy} = -20u, \quad 0 < x < 1, \quad 0 < y < 1, \]

\[
\begin{align*}
&u(0, y) = u(1, y) = 0, \\
&u(x, 0) = 0, \\
&u(x, 1) = x^2 - x.
\end{align*}
\]

We begin by separating variables, writing \( u = XY \). Then, we get

\[ \frac{X''}{X} + \frac{Y''}{Y} = -20. \]

This means that \( X''/X \) and \( Y''/Y \) must both be constant, and we write

\[ \frac{X''}{X} = -20 - \frac{Y''}{Y} = \mu. \]

The BCs for \( X \) are nicer, so we start with \( X \). We have

\[ X'' = \mu X, \quad X(0) = X(1) = 0. \]

You can show that the only \( \mu \) which have a non-trivial solution \( X \) are \( \mu < 0 \), specifically,

\[ X = X_n = \sin(n \pi x), \quad \mu = \mu_n = -n^2 \pi^2, \quad n \in \mathbb{N}, n \geq 1, \]

up to a constant factor. Then, this also specifies the partner solution, because we know that \( Y \) satisfies

\[ \frac{Y''}{Y_n} = -\frac{X''}{X} - 20 = n^2 \pi^2 - 20 = \lambda_n. \]

For \( n = 1 \), we note that \( \lambda_1 < 0 \). Thus, we have \( Y_1 \) is a linear combination of \( \sin(\sqrt{|\lambda_1|} y) \) and \( \cos(\sqrt{|\lambda_1|} y) \). For \( n \geq 2, \lambda_n > 0 \), so there \( Y_n \) is a linear combination of \( \sinh(\sqrt{\lambda_n} y) \) and \( \cosh(\sqrt{\lambda_n} y) \). To figure out the constant factors, we use the BCs. We need \( Y_n(0) = 0 \) for all \( n \). Thus,

\[ Y_1(y) = \sin(\sqrt{|\lambda_1|} y), \quad Y_n(y) = \sinh(\sqrt{\lambda_n} y), \quad n \geq 2, \]
up to multiplication by a constant factor. Our full solution is then given by summing
\[ u(x, y) = \sum_{n \geq 1} a_n \sin(n\pi x)Y_n(y). \]

We need
\[ u(x, 1) = \sum_{n \geq 1} a_n \sin(n\pi x)Y_n(1) = x^2 - x. \]

Hence, we need to expand the function \( x^2 - x \) in terms of the \( L^2(0, 1) \) OB (not yet normalized) \( \{\sin(n\pi x)\} \). We compute the \( L^2 \) norms of the sines to be \( 1/\sqrt{2} \). Hence, the Fourier coefficients shall be
\[ c_n = \frac{1}{2} \int_0^1 (x^2 - x) \sin(n\pi x) \, dx. \]

Then, the coefficients, \( a_n \) are given by
\[ a_n = \frac{c_n}{Y_n(1)}. \]

We note that \( Y_1(1) = \sin(\sqrt{20} - \pi/2) \neq 0 \), and that sinh has no zeros on the real line. So, phew, we aren’t dividing by zero.

8. (10 p) Lös problemet:
\[ u_t - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = t + 1, \]
\[ u(1, t) = 0, \]
\[ u(x, 0) = 1 - x. \]

Staring at those weird BCs, we see that
\[ (t + 1)(1 - x) = (t + 1) \text{ at } x = 0, \]
and
\[ (t + 1)(1 - x) = 0 \text{ at } x = 1, \]
and
\[ (t + 1)(1 - x) = (1 - x) \text{ at } t = 0. \]
What happens if we hit \((t + 1)(1 - x)\) with the heat equation? We get
\[
(\partial_t - \partial_x^2)(t + 1)(1 - x) = 1 - x.
\]
So, we look for a steady state solution to:
\[
-f''(x) = x - 1 \implies f(x) = -\frac{x^3}{6} + \frac{x^2}{2} + ax + b.
\]
Now, because \((t + 1)(1 - x)\) takes care of the BCs, we want \(f\) to vanish at the boundaries. So, we want
\[
f(0) = f(1) = 0 \implies b = 0 \text{ and } a = -\frac{1}{3}.
\]
However, now the function \(f\) is going to screw up the IC, so we gotta fix it by finding \(v\) which satisfies
\[
v_t - v_{xx} = 0, \quad 0 < x < 1, \quad t > 0,
\]
\[
v(0, t) = v(1, t) = 0,
\]
\[
v(x, 0) = -f(x),
\]
and our full solution will be
\[
u(x, t) = (t + 1)(1 - x) + f(x) + v(x, t).
\]
This is just an IVP for the standard heat equation! We can solve it using separation of variables and a Fourier series. When we do that, we get
\[
T'X - TX'' = 0 \implies X'' = \text{constant } X,
\]
with BCs
\[
X(0) = X(1) = 0.
\]
Hence,
\[
X_n(x) = \sin(n\pi x) \text{ up to constant factor.}
\]
We then also get
\[
T_n(t) = e^{-n^2\pi^2 t},
\]
up to constant factor. Our full solution is
\[
v(x, t) = \sum_{n \geq 1} a_n e^{-n^2\pi^2 t} \sin(n\pi x).
\]
To get the constants, we use the IC which says

\[ v(x, 0) = \sum_{n \geq 1} a_n \sin(n\pi x) = -f(x) = \frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3}. \]

The coefficients are therefore given by

\[ a_n = 2 \int_0^1 \left( \frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3} \right) \sin(n\pi x) \, dx. \]

The 2 in front comes from the fact that the \( L^2 \) norm of the basis vectors \( \sin(n\pi x) \) is \( 1/\sqrt{2} \).