Fouriermetoder

20/01-25

1. Partial differential equation
2. Theorem 1

Chapter 1 A vibrating string's movement is found by solving the equation of sound!

Def: An ordinary differential eqn. = unknown function for one variable

A partial differential eqn. = unknown function of ≥ 2 variables

A vibrating string's movement is found by solving the equation of sound:

$x = 0$ $x = l$

string $= [0, l]$

x = distance from left end

l = total length

ends do not move

$u(x,t)$ = height at position x, time t

height $= 0$ = not moving

Boundary condition: $u(0,t) = 0$, $u(l,t) = 0$

c^2 = positive constant

Wave equation: $u_{tt}(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) = c^2 u_{xx}(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t)$

$$\begin{cases} 1 \quad u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0<t \quad 0<x<l \quad \text{(pde)} \\ 2 \quad u(0,t) = 0, \quad u(l,t) = 0 & \text{(BC)} \\ 3 \quad u(x,0) = f(x) \text{ initial position} & u_t(x,0) = g(x) \text{ initial velocity (IC)} \end{cases}$$

initial condition

Separate variables to start finding u.

1. Look for solutions to the pde of the form

$T(t)X(x) \rightarrow$ sub into the pde

$$T_{tt}(t) \cdot X(x) = c^2 T(t) \cdot X_{xx}(x)$$

$$T''(t) X(x) = c^2 T(t) \cdot X''(x)$$

$$\frac{T''(t)}{T(t)} = \frac{c^2 X''(x)}{X(x)} \quad \Rightarrow \text{Both sides are constant}$$

$$\frac{T''(t)}{T(t)} = \text{constant} = c^2 \frac{X''(x)}{X(x)} \quad \text{call it } \lambda$$

2. The BC will help us to see what the value of λ needs to be!

-> Solve for X first

$$c^2 \frac{X''(x)}{X(x)} = \lambda \iff \begin{cases} X''(x) = \frac{\lambda}{c^2} X(x) \\ X(0) = 0, \quad X(l) = 0 \quad \text{BC} \end{cases}$$

Theorem 1 (Basis of solutions for 2nd order linear homog ODE's)

$aX''(x) + bX'(x) + cX(x) = 0$ a, b, c are constant $a \neq 0$

if $b^2 \neq 4ac$ then $\{e^{r_1 x}, e^{r_2 x}\}$ is a basis with $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

If $b^2 = 4ac$ then $\{e^{(-\frac{b}{2a})x}, xe^{(-\frac{b}{2a})x}\}$ is a basis.

For the equation $X''(x) + (0)X'(x) - \frac{\lambda}{c^2}X(x) = 0$ apply the theorem!

Two cases: $(0)^2 \neq 4(1)\left(\frac{-\lambda}{c^2}\right) \Leftrightarrow \lambda \neq 0$

or $(0)^2 = 4(1)\left(-\frac{\lambda}{c^2}\right) \Leftrightarrow \lambda = 0$

If $\lambda = 0 \Rightarrow X''(x) = 0 \Rightarrow X(x) = Ax + b$

BC $X(0) = 0 \Rightarrow B = 0$ $X(l) = 0 \Rightarrow A(l) = 0 \Rightarrow A = 0$ Only the 0 solution

$\lambda \neq 0$ $X''(x) = \frac{\lambda}{c^2}X(x)$ Apply theorem $X(x) = Ae^{\sqrt{\frac{\lambda}{c^2}}x} + Be^{-\sqrt{\frac{\lambda}{c^2}}x}$

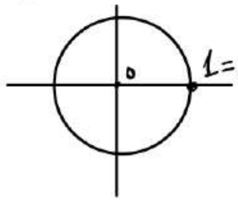
Constants for us are in $\mathbb{C}$ for $x \in \mathbb{R}$ $e^{ix} = \cos x + i\sin x$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

BC $X(0) = 0$ $X(0) = Ae^0 + Be^0 = 0$ $A + B = 0$ $B = -A$

$A\left(e^{\sqrt{\frac{\lambda}{c^2}}x} - e^{-\sqrt{\frac{\lambda}{c^2}}x}\right) = X(x)$ Next $X(l) = 0$ $Ae^{\sqrt{\frac{\lambda}{c^2}}l} = Ae^{-\sqrt{\frac{\lambda}{c^2}}l}$ $e^{2\sqrt{\frac{\lambda}{c^2}}\cdot l} = 1$

$e^{2\sqrt{\frac{\lambda}{c^2}}\cdot l} = 1$ holds when $2\sqrt{\frac{\lambda}{c^2}}\cdot l = 2\pi i \cdot k$ $\frac{\lambda}{c^2} = \left(\frac{\pi i \cdot k}{l}\right)^2$ $\lambda_k = \frac{-c^2\pi^2k^2}{l^2}$



$1 = e^0 = e^{2\pi i} = e^{k\pi i} = e^{-2\pi i}$

$X_k(x) = A\left(e^{i\pi kx/l} - e^{-i\pi kx/l}\right)$

$X_k(x) = \underbrace{2iA}_{}\sin\left(\frac{k\pi x}{l}\right)$

? find the coeffs later, can always scale a basis

$X_k(x) = \sin\left(\frac{k\pi x}{l}\right)$

$$\frac{T_k''(t)}{T_k(t)} = -\frac{c^2\pi^2k^2}{l^2} \Rightarrow T_k(t) = a_k e^{i\pi kt/l} + b_k e^{-i\pi kt/l}$$

$u_k(x,t) = X_k(x)T_k(t)$

Superposition just means addition and helps us to satisfy every condition!

$$u(x,t) = \sum_{k\in\mathbb{Z},\, k\neq 0} u_k(x,t)$$

The initial conditions will help us to find coefficients of functions that depend on time!

$u(x,0) = \sum_{k\in\mathbb{Z},\, k\neq 0} u_k(x,0) = f(x)$ $u_t(x,0) = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{\partial}{\partial t}u_k(x,0) = g(x)$

22/01-25

1. Two cases
2. Superposition

Solve an equation of diffusion to find the temperature's evolution!

Rings of saturn

1 Separation of variables

2 Superposition

Heat equation on a ring

$u(x,t)$

pde $\rightarrow u_t(x,t) = c^2 u_{xx}(x,t) \quad t>0 \quad -\pi < x < \pi$

BC $\rightarrow u(-\pi,t) = u(\pi,t) \quad u_x(-\pi,t) = u_x(\pi,t)$

IC $\rightarrow u(x,0) = f(x)$

1. Start with separation of variables: $u(x,t) = X(x)T(t)$

$$X(x)T'(t) = c^2 X''(x)T(t) \underset{\frac{1}{X(x)T(t)}}{\Rightarrow} \frac{T'(t)}{T(t)} = c^2\frac{X''(x)}{X(x)} = \text{constant} = \lambda$$

$$\begin{cases} T'(t) = \lambda T(t) \\ c^2 X''(x) = \lambda X(x) \Rightarrow X''(x) = \frac{\lambda}{c^2}X(x) \end{cases}$$

BC: $u(-\pi,t) = u(\pi,t) = X(-\pi)T(t) = X(\pi)T(t)$

$\Rightarrow X(-\pi) = X(\pi) \quad X'(-\pi) = X'(\pi)$

$$\begin{cases} X'' = \frac{\lambda}{c^2}X \\ X(-\pi) = X(\pi) \\ \Lambda = -\frac{\lambda}{c^2} \Rightarrow X'' + \Lambda X = 0 \end{cases}$$

Theorem 1: $af'' + bf' + df = 0 \quad a \neq 0$

$\underline{b^2 \neq 4ad} \quad \{e^{r_1x}, e^{r_2x}\} \quad r_1 = \frac{-b+\sqrt{b^2-4ad}}{2a} \quad r_2 = \frac{-b-\sqrt{b^2-4ad}}{2a}$

$\underline{b^2 = 4ad} \quad \{e^{rx}, xe^{rx}\} \quad r = -\frac{b}{2a}$

$a = 1 \quad b = 0 \quad d = \Lambda$

Two cases: $\Lambda = 0 \quad \Lambda \neq 0$

case I: $\Lambda = 0 \quad X'' = 0 \quad \{1, x\} \Rightarrow X(x) = Ax + B$

$X(-\pi) = -A\pi + B = X(\pi) = A\pi + B \Rightarrow A = 0$

$X'(-\pi) = A = 0 = X'(\pi) \Rightarrow X(x) = B$ is a solution

case II: $\Lambda \neq 0 \quad r_1 = \frac{\sqrt{-4\Lambda}}{2} = \sqrt{-\Lambda} = \frac{\sqrt{\lambda}}{c}$

$r_2 = -\frac{\sqrt{\lambda}}{c} \quad \{e^{\frac{\sqrt{\lambda}}{c}x}, e^{-\frac{\sqrt{\lambda}}{c}x}\}$

$X(x) = Ae^{\frac{\sqrt{\lambda}}{c}x} + Be^{-\frac{\sqrt{\lambda}}{c}x} \qquad X(\pi) = X(-\pi)$

$X(\pi) = Ae^{\frac{\sqrt{\lambda}\pi}{c}} + Be^{-\frac{\sqrt{\lambda}}{c}\pi} = Ae^{-\frac{\sqrt{\lambda}\pi}{c}} + B^{\frac{\sqrt{\lambda}\pi}{c}}$

$$(B-A)e^{\frac{\sqrt{\lambda}\pi}{c}} = (B-A)e^{-\frac{\sqrt{\lambda}\pi}{c}}$$

$B-A=0$ and/or $1 = \frac{e^{\frac{\sqrt{\lambda}\pi}{c}}}{e^{-\frac{\sqrt{\lambda}\pi}{c}}} = e^{2\pi\frac{\sqrt{\lambda}}{c}}$

$$X'(x) = \frac{A\sqrt{\lambda}}{c}e^{\frac{\sqrt{\lambda}}{c}x} - B\frac{\sqrt{\lambda}}{c}e^{-\frac{\sqrt{\lambda}}{c}x}$$

$$X'(-\pi) = \frac{A\sqrt{\lambda}}{c}e^{-\frac{\sqrt{\lambda}}{c}\pi} - B\frac{\sqrt{\lambda}}{c}e^{\frac{\sqrt{\lambda}}{c}\pi} =$$

$$X'(\pi) - \frac{A\sqrt{\lambda}}{c}e^{\frac{\sqrt{\lambda}}{c}\pi} - B\frac{\sqrt{\lambda}}{c}e^{-\frac{\sqrt{\lambda}}{c}\pi}$$

$$\frac{\sqrt{\lambda}}{c}(A+B)e^{-\frac{\sqrt{\lambda}}{c}\pi} = \frac{\sqrt{\lambda}}{c}(A+B)e^{\frac{\sqrt{\lambda}}{c}\pi}$$

$A+B=0$ and/or $e^{2\pi\frac{\sqrt{\lambda}}{c}} = 1$

i) $A=B$ if ~~$A=-B \Rightarrow A=B=0$~~

$\Rightarrow e^{2\pi\frac{\sqrt{\lambda}}{c}} = 1$ $\quad 2\pi\frac{\sqrt{\lambda}}{c} = 2\pi i n \quad n \in \mathbb{Z}$

$X_n(x) = e^{inx} + e^{-inx} = 2\cos(nx) \quad \lambda_n = -c^2n^2$

ii) $A=-B$ and $e^{2\pi\frac{\sqrt{\lambda}}{c}} = 1 \quad 2\pi\frac{\sqrt{\lambda}}{c} = 2\pi i n \quad n \in \mathbb{Z}$

$X_n(x) = e^{inx} - e^{-inx} = 2i\sin(nx)$

$\lambda_n = -c^2n^2$

iii) Say nothing about A and B take any linear combo of e^{2nx} and e^{-2nx} $\quad \lambda_n = -n^2c^2$

$X_n(x)T_n(t) = e^{-n^2c^2t}(c_n e^{2nx} + c_{-n}e^{-2nx}) \quad c_n, c_{-n} \in \mathbb{C}$

$T_n'(t) = \lambda_n T_n(t) = -c^2n^2T_n(t) \quad \frac{T_n'(t)}{T_n(t)} = \lambda_n \Rightarrow \frac{d}{dt}\log T_n(t) = \lambda_n$

$\log(T_n) = \lambda_n t + c \Rightarrow T_n(t) = e^{-c^2n^2t}$

2. Superposition

u_n, u_m both satisfy pde and satisfy BC

$$\frac{\partial}{\partial t}(u_n + u_m) = \frac{\partial}{\partial t}u_n + \frac{\partial}{\partial t}u_m = c^2\frac{\partial^2}{\partial x^2}u_n + c^2\frac{\partial^2}{\partial x^2}u_m = c^2\frac{\partial^2}{\partial x^2}(u_n + u_m)$$

$$u_n(\pi,t) + u_m(\pi,t) = u_n(-\pi,t) + u_m(-\pi,t)$$

$$u(x,t) = \sum_{n\in\mathbb{Z}} u_n(x,t) = \sum_{n\in\mathbb{Z}} c_n e^{-n^2c^2t}e^{inx}$$

$$u(x,0) = \sum_{n\in\mathbb{Z}} u_n(x,0) = \sum_{n\in\mathbb{Z}} c_n e^{inx} = f(x)$$

Fourier-series

24/01 - 25

1. Chapter 1 recap
2. Hilbert spaces
3. Dimension and orthogonal bases
4. Infinite dimensional vector space

Chapter 1 Recap:

2 Techniques for solving pde's

$$\begin{cases} \text{pde for } u(x,t) \text{ with } x \in \text{interval} \\ \text{Boundary conditions} \\ \text{Initial conditions} \end{cases}$$

1. Separate variables → Look for $X(x)T(t)$ to solve the p.d.e.
   Solve for X first because boundary conditions
   Find <u>all</u> $X_u(x)$ and λ_k ←
   Then find all $T_k(t)$ using now we know λ_k $T_k(t)X_u(x)$

2. Superposition just means addition and helps to satisfy every condition.
   $u(x,t)=\sum X_u(x)T_u(t)$

How to find the coefficients so that $u(x,0)=\sum X_u(x)\underset{\substack{\uparrow \\ \text{what should } \underline{\underline{\text{this}}} \text{ be?}}}{T_u(0)} = f(x) = \text{initial position}$

Need to represent $\vec{f}$ in terms of $\vec{X_u}$

$$\frac{\text{Scalar product of } \vec{f} \text{ with } \vec{X}}{(\text{Length of } \vec{X_u})^2}\vec{X_k} = \frac{\langle \vec{f}, \vec{X_u} \rangle}{\|X_u\|^2}\vec{X_u}$$

Chapter 2

You'll find your way through Hilbert spaces if you use orthogonal bases.

Hilbert spaces

Def: A Hilbert space is a complete normed vector space with a hermitian scalar product that defines the norm.

H = Hilbert space = $\{\overline{\text{vectors}}\}$
Scalar product takes $\vec{v}, \vec{w} \in H$ and $\langle v,w \rangle \in \mathbb{C}$
A hermitian scalar product must satisfy:

① $\langle v, w \rangle = \overline{\langle w, v \rangle}$
② $\langle v+w, u \rangle = \langle v, u \rangle + \langle w, u \rangle$
③ If $\lambda \in \mathbb{C}$ then $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$
④ For all $v \in H$ $\langle v,v \rangle \geq 0$, $\langle v,v \rangle = 0$ iff $v=0$ $\quad \|v\|^2 := \langle v, v \rangle$

OBS! Scalars are $\mathbb{C}$

Examples:

① $\mathbb{C} = H$ for $v, w \in \mathbb{C}$ How do we calculate $\langle v,w \rangle$? $\langle v,w\rangle = v\overline{w}$

② $\mathbb{C}^n = H = \{ v = (v_0, v_1, \dots, v_n)$ with each $v_n \in \mathbb{C}_n \}$

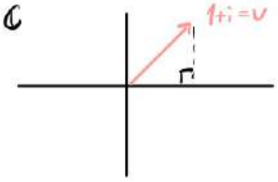
For $v, w \in \mathbb{C}^n$ $\langle v,w \rangle = \sum_{k=1}^{n} v_k \overline{w_k}$

Norm = length Every $v \in H$ must have a finite ≥ 0 length

$v \in \mathbb{C}$ $v = re^{i\theta}$ $r = \|v\|$ $\theta \in \mathbb{R}$

$v = r\cos\theta + i r \sin\theta$

$\|v\|^2 = v\overline{v} = re^{i\theta}\,\overline{re^{i\theta}} = re^{i\theta} \cdot re^{-i\theta} = r^2$



$\|v\| = \sqrt{2}$
$\|v\|^2 = 2$

Dimension and Orthogonal bases

Orthogonal: If v and w are non zero $v \perp w$ iff $\langle v,w \rangle = 0$

Orthogonal base: $\{\vec{v}_k\}$ that are mutually orthogonal and $\forall\ w \in H$

$$\vec{w} = \sum \frac{\langle \vec{w}, \vec{v}_k \rangle}{\|v_k\|^2} \vec{v}_k$$

(This is a definition)

An orthogonal base for $\mathbb{C}$ is any single nonzero $v \in \mathbb{C}$

For example 1. Then $\vec{w} \in \mathbb{C}$ is just $(\vec{w})1$

For $\mathbb{C}^n$ an orthogonal base is $(1,0,0,\dots) = \vec{e}_1$ $\vec{e}_2 = (0,1,0,0,\dots)$ $\vec{e}_n = (0,0,\dots,1)$

This is an OB since $\vec{w} \in \mathbb{C}^n$ $\vec{w} = (w_1, w_2, \dots, w_n)$

$$\sum_{k=1}^{n} \frac{\langle w, e_k \rangle}{\|e_k\|^2} \vec{e}_k = \sum_{k=1}^{n} \underbrace{w_k}_{\mathbb{C}\text{ scalar}} \vec{e}_k = (w_1, 0, \dots) + (0, w_2, 0, \dots) + (0, \dots, w_k, \dots) + (0,0,\dots,w_n)$$

$$= (w_1, w_2, \dots, w_n)$$

An infinite dimensional vector space

Def: Dimension of a Hilbert space is the # of vectors in an orthogonal space

"$\mathbb{C}^{\infty}$" $= \ell^2 =$ little ℓ^2 = span of $\vec{e}_k$ $k = 1,2,3,\dots = \{(v_1, v_2, v_3, \dots) : v_k \in \mathbb{C}$ for $k \in \mathbb{N}$ and $\sqrt{\sum_{k\geq 1} |v_k|^2} < \infty \}$

for $\vec{v}, \vec{w} \in \ell^2$ $\langle \vec{v}, \vec{w} \rangle = \sum_{k \geq 1} v_k \overline{w_k}$

$$\|\vec{v}\| = \sqrt{\sum_{k\geq 1} |v_k|^2} = \langle \vec{v}, \vec{v} \rangle$$

Useful facts about Hilbert spaces

Proposition: $u, v \in H \Rightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u, v\rangle$

Proof: $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle =$

$= \langle u,u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} =$

$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u, v\rangle$

Proposition (Cauchy and Schwarz inequality): $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$

Proposition: (Triangle inequality): $\|u+v\| \leq \|u\| + \|v\|$

Pythagorean Theorem:

$$\left.\begin{array}{l} \text{Assume that } \displaystyle\sum_{k\geq 1} \underset{\overset{\uparrow}{\text{also } \in H}}{\vec{u}_k} \in H \\ \\ \text{Assume that } \vec{u}_k \perp \vec{u}_j \quad \text{for } j \neq k \end{array}\right\} \left\|\sum_{k\geq 1} \vec{u}_k\right\|^2 = \sum_{k\geq 1} \|u_k\|^2$$

27/01-25

1. What is a vector?
2. Bessel's projection inequality
3. Does Orthogonal Good (DOG)

What is a vector? (Hilbert spaces)

- An element of a vector space
- Magnitude and direction
- A list of numbers

What can we do with vectors?

- Add them => another vector
- Scalar product => a scalar $s \in \mathbb{C}$
- multiply by scalars => another vector
- project one vector onto vector(s)
- Calculate its length $\in [0, \infty)$

Define a vector in a Hilbert space by:

Let $\{\bar{e}_k\}_{k \geq 1}$ be an orthonormal base.

An element in H, v, is $\sum_{k=1} \langle v, e_k \rangle \bar{e}_k$ $\quad v = (\langle v, e_1 \rangle + \langle v, e_2 \rangle + \dots + \langle v, e_k \rangle)$

The length of this vector is $\sqrt{\sum_{k \geq 1} |\langle v, e_k \rangle|^2} < \infty$

Theorem: Every (separable physical) infinite dimensional Hilbert space is equivalent to

$$\ell^2 = \left\{ (v_1, v_2, \dots) : v_k \in \mathbb{C} \text{ and } \sum_{k \geq 1} |v_k|^2 < \infty \right\} \qquad \langle v, w \rangle = \sum_{k \geq 1} v_k \bar{w}_k$$

In $\mathbb{C}^n$, some collection of mutually orthonormal vectors is a base if there are n of them

Example of ∞ many orthonormal vectors in ℓ^2 that are <u>not</u> a base?

$\bar{e}_k = (0, 0, \dots, \underset{k^{th}}{1}, 0, \dots)$ $\{\bar{e}_k\}_{k \geq 2}$ is not a base because $\bar{e}_1$ <u>cannot</u> be written as $\sum_{k \geq 2} \langle e_1, e_k \rangle \bar{e}_k = \bar{0} \neq \bar{e}_1$

How can we tell if it's the case, that an orthogonal set is also a base?

Bessel's projection inequality

Let $v \in H$ and $\{\bar{e}_k\}_{k \geq 1}$ be an ONS. Then $\sum_{k \geq 1} |\langle v, e_k \rangle|^2 < \infty$ and thus $\sum_{k \geq 1} \langle v, e_k \rangle \bar{e}_k \in H$

Need to be good at scalarproduct

Important: Project $\bar{v} \in H$ onto $\text{span}(\{\bar{e}_k\}_{k \geq 1})$ and the resulting vector $\sum_{k \geq 1} \langle v, e_k \rangle \bar{e}_k$ is shorter than (or same) length as $\bar{v}$.

Corollary: Let $\{\bar{\omega}_k\}_{k\geq1}$ be orthogonal in H. Then for $\bar{v}\in H$

$$\left\|\sum_{k\geq1}\frac{\langle v,w_k\rangle}{\|w_k\|^2}\bar{w}_k\right\|\leq\|v\|^2 \Rightarrow \left\|\sum_{k\geq1}\frac{\langle v,w_k\rangle}{\|w_k\|}\underbrace{\frac{\bar{w}_k}{\|w_k\|}}_{\bar{e}_k}\right\|\leq\|v\|^2$$

Proof: $V_N=\sum_{k=1}^{N}\frac{\langle v,w_k\rangle}{\|w_k\|}\frac{\bar{w}_k}{\|w_k\|}=\sum_{k=1}^{N}\left\langle v,\frac{w_k}{\|w_k\|}\right\rangle\frac{\bar{w}_k}{\|w_k\|}=\sum_{k=1}^{N}\langle v,e_k\rangle\bar{e}_k$

B. Proj. Ineq. $\Rightarrow \bar{V}_N\longrightarrow\sum_{k\geq1}\frac{\langle v,w_k\rangle}{\|w_k\|^2}\bar{w}_k\in H$ and its length is $\leq\|v\|^2$

Does Orthogonalbase Good?★ DOG

Assume that $\{\bar{e}_k\}_{k\geq1}$ is an orthonormal set in H.
The following are equivalent:

① If $\bar{v}\in H$, then $\bar{v}=\sum_{k\geq1}\langle v,e_k\rangle\bar{e}_k$ (Def of ONB)

② If $\bar{v}\in H$, then $\|v\|^2=\sum_{k\geq1}|\langle v,e_k\rangle|^2$

③ If $\bar{v}\in H$ and $\langle v,e_k\rangle=0\ \forall_k\Rightarrow\bar{v}=\bar{0}$

Proof:★

$$\begin{tikzcd} \textcircled{1} \arrow[rr, Rightarrow] & & \textcircled{2} \arrow[dl, Rightarrow] \\ & \textcircled{3} \arrow[ul, Rightarrow] & \end{tikzcd}$$

① ⇒ ②: Since $v=\sum_{k\geq1}\langle v,e_k\rangle\bar{e}_k$ by ① Pythagoras says that

$$\|v\|^2=\sum_{k\geq1}\|\langle v,e_k\rangle\bar{e}_k\|^2=\sum_{k\geq1}|\langle v,e_k\rangle|^2\|\bar{e}_k\|^2=\sum_{k\geq1}|\langle v,e_k\rangle|^2$$

② ⇒ ③: If $\langle v,e_k\rangle=0\ \forall k$, then by ② $\|v\|^2=0\Rightarrow\bar{V}=\bar{0}$

③ ⇒ ①: Idea: Prove $\bar{v}-\sum_{k\geq1}\langle v,e_k\rangle\bar{e}_k=\bar{0}$ (for an arbitray j)

Calculate: $\left\langle\bar{v}-\sum_{k\geq1}\langle v,e_k\rangle\bar{e}_k,\bar{e}_j\right\rangle=\langle\bar{v},\bar{e}_j\rangle+\left\langle-\sum_{k\geq1}\langle v,e_k\rangle\bar{e}_k,\bar{e}_j\right\rangle=$

$$=\langle v,e_j\rangle-\sum_{k\geq1}\underbrace{\langle\langle v,e_k\rangle e_k,e_j\rangle}=\langle v,e_j\rangle-\langle\langle v,e_j\rangle\bar{e}_j,\bar{e}_j\rangle=$$

$$=\langle v,e_j\rangle-\langle v,e_j\rangle\underbrace{\langle e_j,e_j\rangle}_{1}=\langle v,e_j\rangle-\langle v,e_j\rangle=0$$

$\langle v,e_k\rangle\bar{e}_k\perp\bar{e}_j$ if $j\neq k$
since $\langle v,e_k\rangle\in\mathbb{C}$ and $\bar{e}_k\perp\bar{e}_j$, $j\neq k$

Thus since $\langle v-\sum_{k\geq1}\langle v,e_j\rangle\bar{e}_k=\bar{0}\Rightarrow\bar{v}=\sum_{k\geq1}\langle v,e_k\rangle e_k$

29/01-25

1. Hilbert spaces
2. $\mathcal{L}^2$
3. Best Approximation Theorem (BAT)

Hilbert spaces

vector spaces = scalars = $\mathbb{C}$ = scalar product $\langle\, ,\rangle : H \times H \longrightarrow \mathbb{C}$

Def: Let $\{\bar{v}_k\}_{k\geq 1}$ be an orthogonal set in a Hilbert space H.
For $w \in H$ the fourier series of w with respect to $\{\bar{v}_k\}_{k\geq1}$ is:

$$\sum_{k\geq 1} \frac{\langle w, v_k\rangle}{\|v_k\|^2} v_k$$

Fourier is the (Projection onto the O.S)

Notation: $\hat{w}_k = \dfrac{\langle w, v_k\rangle}{\|v_k\|^2}$, k^{th} Fourier coefficient of w with respect to $\{v_k\}_{k\geq1}$

Def: Fix $0<l$. $\mathcal{L}^2(0,l) = \{$functions with $\int_0^l |f(x)|^2 dx < \infty\}$

Examples of functions $\underbrace{\text{in}}$ and $\underbrace{\underline{\text{not}}}$ in $\mathcal{L}^2(0,l)$:

$\|f\|^2 = \int_0^l |f(x)|^2 dx = \langle f, f\rangle$

$\langle f, g\rangle = \int_0^l f(x)\overline{g(x)}\,dx$

in: $\sin x$, $\cos x$, polynomials, e^x, $\log(x)$, any bounded fun

not: $\frac{1}{\sqrt{x}}, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$

Proposition: $X_k(x) = \sin\left(\frac{k\pi x}{l}\right)$ for $k\geq 1$ are in $\mathcal{L}^2(0,l)$. $\{X_k\}_{k\geq1}$ is an orthogonal set

Proof: $\int_0^l (X_k(x))^2 dx \leq \int_0^l 1\,dx = l$. So they are in $\mathcal{L}^2(0,l)$

$$\langle X_k, X_j\rangle = \int_0^l \sin\left(\frac{k\pi x}{l}\right)\overline{\sin\left(\frac{j\pi x}{l}\right)} = \int_0^l \left(\frac{e^{i\pi kx/l} - e^{-i\pi kx/l}}{2i}\right)\overline{\left(\frac{e^{i\pi jx/l} - e^{-i\pi jx/l}}{2i}\right)} =$$

$$= \frac{1}{4i}\int_0^l \left(e^{i\pi kx/l} - e^{-i\pi kx/l}\right)\cdot i \cdot \left(e^{-i\pi jx/l} - e^{i\pi jx/l}\right) dx = \frac{1}{4}\int_0^l e^{\frac{i\pi x}{l}(k-j)} - e^{\frac{i\pi x}{l}(k+j)} + e^{\frac{i\pi x}{l}(-k-j)} + e^{\frac{i\pi x}{l}(-k+j)} dx$$

$$\begin{cases} = 0 & \text{if } k \neq j \\ = \frac{l}{2} & \text{if } k = j \end{cases}$$

If $\{X_k\}_{k\geq1}$ is an orthogonal base then any bounded function on $(0,l)$ say $f(x)$ = height of string at point x at time $t=0$

$$f(x) = \sum_{k\geq1} \boxed{\frac{\langle f, X_k\rangle}{\|f_k\|^2}} X_k(x) \quad \hat{f}_k$$

Vibrating string solution: $\sum_{k\geq1} T_k(0)\, X_k(x) = f(x)$

$T_k(0) = \hat{f}_k$ with respect to X_k

Fourier series pass the test, they can approximate the best! **Best Approximation Theorem = BAT**

BAT$^{\bigstar}$= Let $\{X_k\}_{k\geq1}$ be an orthonormal set in H. $\left\{\begin{array}{l} c_k \in \mathbb{C} \text{ all } k \\ \text{and } \sum_{k\geq1} |c_k|^2 \end{array}\right.$

Then for any $v \in H$ $\|v-\sum_{k\geq1}\hat{v}_k X_k\| \leq \|v-\sum_{k\geq1} c_k X_k\|$ equalify iff $c_k=\hat{v}_k$ $\forall k$.

Proof$^{\bigstar}$: Compute $\left\|v-\sum_{k\geq1}\hat{v}_k X_k+\sum_{k\geq1}\hat{v}_k X_k-\sum_{k\geq1}c_k X_k\right\|^2=\left\|v-\sum_{k\geq1}c_k X_k\right\|^2$

$$\left\|v-\sum_{k\geq1}\hat{v}_k X_k+\sum_{k\geq1}\hat{v}_k X_k-\sum_{k\geq1}c_k X_k\right\|^2=$$

$$=\left\|v-\sum_{k\geq1}\hat{v}_k X_k\right\|^2+\left\|\sum_{k\geq1}\hat{v}_k X_k-\sum_{k\geq1}c_k X_k\right\|^2+2\,\mathrm{Re}\left\langle v-\sum_{k\geq1}\hat{v}_k X_k, \sum_{j\geq1}(\hat{v}-c_j)X_j\right\rangle$$

If we can show $\mathrm{Re}(\cdots)=0$, then $\|v-\sum_{k\geq1}c_k X_k\|^2\geq\|v-\sum_{k\geq1}\hat{v}_k X_k\|^2$ and equal iff

$$\underbrace{\left\|\sum_{k\geq1}(\hat{v}_k-c_k)X_k\right\|^2}_{=\sum_{k\geq1}|\hat{v}-c_k|^2}=0 \iff v_k=c_k \quad \forall k$$

$$\left\langle v,\sum_{j\geq1}(\hat{v}-c_j)X_j\right\rangle-\sum_{k\geq1}\hat{v}_k\left\langle X_k,\sum_{j\geq1}(\hat{v}_j-c_j)X_j\right\rangle=\sum_{j\geq1}\overline{(\hat{v}_j-c_k)}\underbrace{\langle v,X_j\rangle}_{=\hat{v}_j}-\sum_{j\geq1}\overline{(\hat{v}_j-c_j)\hat{v}_j}=0$$

31/01-25

1. Best Approximation Applications
2. Trigonometric Fourier Series

Best Approximation Applications:

Approximate $f(x) \in \mathcal{L}^2(0,l)$ by functions in $\mathcal{L}^2(0,l)$ that are orthogonal to each other.

Example: $X_k(x) = \sin\left(\frac{k\pi x}{l}\right)$ So if we want to approximate f by these we set $\hat{f}_k = \frac{\langle f, X_k\rangle}{\|X_k\|^2} = \frac{\int_0^l f(x)\overline{X_k(x)}\,dx}{\int_0^l |X_k(x)|^2 dx}$ and then $\sum_{k=1}^{10} \hat{f}_k X_k(x)$ is the

↳ 10 is arbitrary, could be anything depending on context

"best approximation of f(x) by any linear combo of $\left\{\sin\left(\frac{k\pi x}{l}\right)\right\}_{k=1}^{10}$

Minimizes $\left\|f - \sum_{k=1}^{10} c_k X_k\right\|^2 = \int_0^l \left|f(x) - \sum_{k=1}^{10} c_k \sin\left(\frac{k\pi x}{l}\right)\right|^2 dx$

Chapter 3 Trigonometric Fourier Series

$$\mathcal{L}^2(-\pi,\pi) = \{\underset{\uparrow\ \mathbb{C}\text{-valued}}{f(x)} : \underbrace{\int_{-\pi}^{\pi} |f(x)|^2 dx}_{=\|f\|^2_{\mathcal{L}^2(-\pi,\pi)}} < \infty\}$$

$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$

Proposition: The functions $\{e^{inx}\}_{n\in\mathbb{Z}}$ are an orthogonal set in $\mathcal{L}^2(-\pi,\pi)$

Proof: $\langle e^{inx}, e^{imx}\rangle = \int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}\,dx = \int_{-\pi}^{\pi} e^{ix(n-m)}dx = \begin{cases} 2\pi & n=m \\ \left.\frac{e^{ix(n-m)}}{i(n-m)}\right|_{-\pi}^{\pi} & n\neq m \end{cases}$ $e^{\pm i\pi(\text{odd})}$ | $e^{\pm i\pi(\text{even})}$ $\Rightarrow \left.\frac{e^{ix(n-m)}}{i(n-m)}\right|_{-\pi}^{\pi} = 0$

because same value at π & $-\pi$

Def: Assume $f \in \mathcal{L}^2(-\pi,\pi)$. Then its trig. Fourier Series with respect to $\{e^{inx}\}_{n\in\mathbb{Z}}$

is $\sum_{n\in\mathbb{Z}} \underset{\uparrow \text{ or call it } c_n}{\hat{f}_n} e^{inx}$ with $\hat{f}_n = \frac{\langle f, e^{inx}\rangle}{\|e^{inx}\|^2} = \frac{\int_{-\pi}^{\pi} f(x)\overline{e^{inx}}\,dx}{\int_{-\pi}^{\pi}|e^{inx}|^2 dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ since $\overline{e^{inx}} = e^{-inx}$

$$\left\{\begin{array}{ll} e^{inx} = \cos(nx) + i\sin(nx) & \cos(nx) = \frac{e^{inx}+e^{-inx}}{2} \\ e^{-inx} = \cos(nx) - i\sin(nx) & \sin(nx) = \frac{e^{inx}-e^{-inx}}{2i} \end{array}\right\}$$

Alternatively its tFZ with respect to $\{1, \cos(nx), \sin(nx)\}_{n\geq 1}$ is

↳ trigonometric Fourier Series

$C_0 + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$

$$C_0 = \hat{f}_0 = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,[\cos(0)]\,dx$$

$$a_n = \frac{\langle f, \cos(nx)\rangle}{\|\cos(nx)\|^2} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\overline{\cos(nx)}\,dx$$

$$b_n = \frac{\langle f, \sin(nx)\rangle}{\|\sin(nx)\|^2} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\overline{\sin(nx)}\,dx$$

$$a_n = \frac{C_n + C_{-n}}{2} \qquad b_n = \frac{C_n - C_n}{2i}$$

$\sum_{n \in \mathbb{Z}} \hat{f}_n e^{inx}$ for $x = \pi, -\pi$ $\qquad \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in(x+2\pi)} = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{inx}$

Properties of $tF\Sigma$

① If f is Real valued then so are a_n, $C_0 = a_0$, and b_n are also real.

② $tF\Sigma$ are 2π periodic Copy & paste!

③ For $x \in (-\pi, \pi)$ if f is continuous at x then the $tF\Sigma$ @ $x = f(x)$

④ At $\pm\pi$ $tF\Sigma$ converges to $\frac{f(\pi) + f(-\pi)}{2}$

⑤ Outside of $[-\pi, \pi]$ $tF\Sigma$ converges to the 2π periodic extension of f (copy-pasted from $(-\pi, \pi)$)

Proposition: Assume f is periodic with period p.
Then the value of $\int_a^{a+p} f(x)\,dx$ is the same $\forall a \in \mathbb{R}$

Proof: Let a and $b \in \mathbb{R}$. Consider $\int_b^{b+p} f(x)\,dx - \int_a^{a+p} f(x)\,dx = \int_0^{b+p} f(x)\,dx - \int_0^{b} f(x)\,dx - \int_0^{a+p} f(x)\,dx + \int_0^{a} f(x)\,dx$

$$= \int_{a+p}^{b+p} f(x)\,dx - \underbrace{\int_a^b f(x)\,dx}_{\text{sub } y = x+p} = 0$$

f is p-periodic
f(y-p) = f(y)

$$\int_0^b f(x)\,dx = \int_{a+p}^{b+p} f(y-p)\,dy = \int_{a+p}^{b+p} f(y)\,dy$$

$$\int_0^{b+p} - \int_0^{a+p} = \int_{a+p}^{b+p}$$

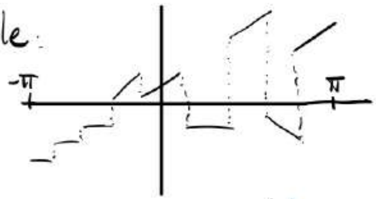
$$-\int_0^b + \int_0^a = -\left(\int_0^b - \int_0^a\right) = -\int_a^b$$

$\Rightarrow$ Zero! Thus $\int_0^{b+p} f(x)\,dx - \int_a^{a+p} f(x)\,dx = 0$ $\Rightarrow$ They are the same!

Examples of periodic: $e^{in(x+2\pi)} = e^{inx}$ $\quad \sin(n(x+2\pi)) = \sin(nx)$ $\quad \cos(n(x+2\pi)) = \cos(nx)$

Def: f is piecewise C^1 on $[-\pi, \pi]$ if f and f' are continuous and bounded on $[-\pi, \pi]$ and have at most finitely many "jump" discontinuities.
Equivalently the left and right limits of f and f' $\exists \forall x \in [-\pi, \pi]$, and at most finitely many points these limits are not equal.

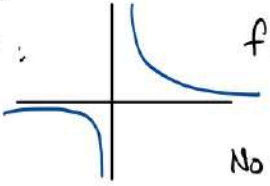
Example:



left limit @ x is $f(x_-) = \lim_{t \to x} f(t)$ $t<x$

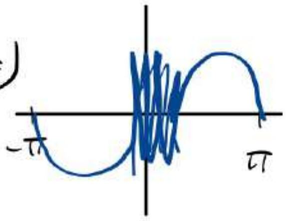
right limit @ x is $f(x_+) = \lim_{t \to x} f(t)$ $t>x$

Not piecewise $\mathcal{C}^1$:



$f(x) = \frac{1}{x}$

also not allowed: $\sin(\frac{1}{x})$



No $f(0_+)$ or $f(0_-)$

03/02-25

1 tF-C@T theorem + proof

2. tF-C@T corollaries

tF-CAT, it's corollaries and applications

Theorem (tF-C@T)*: Assume that f is piecewise C^1 on $[-\pi,\pi]$

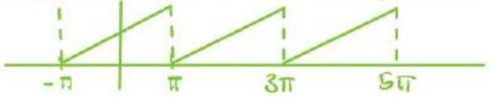
Then for any $x\in(-\pi,\pi)$ the tF∑@x:

$$\lim_{N\to\infty}\sum_{-N}^{N}\hat{f}_n e^{inx}=\frac{f(x_-)+f(x_+)}{2}$$

For $x\in\mathbb{R}\setminus(-\pi,\pi)$ the tF∑@x:

$$\lim_{N\to\infty}\sum_{-N}^{N}\hat{f}_n e^{inx}=\frac{f_{2\pi}(x_-)+f_{2\pi}(x_+)}{2}$$

← $f_{2\pi}$ is the 2π periodic extension of f to $\mathbb{R}\setminus(-\pi,\pi)$



Proof:* Fix $x\in(-\pi,\pi)$

$f(x_+)$ and $f(x_-)$ are <u>fixed</u> in $\mathbb{C}$

$$S_n=\sum_{-N}^{N}\hat{f}_n e^{inx}\qquad \text{Goal: }\lim_{N\to\infty}S_n-\left(\frac{f(x_+)+f(x_-)}{2}\right)=0$$

Use def. of $\hat{f}_n$: $S_n=\sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\,e^{-iny}dy\,)e^{inx}=\sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{in(x-y)}dy=$

$$=\left\{\begin{matrix}x-t=y\\ t=x-y\end{matrix}\right\}=\sum_{-N}^{N}\frac{1}{2\pi}\int_{\pi+x}^{-\pi+x}f(x-t)e^{int}(-dt)=\sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi+x}^{\pi+x}f(x-t)e^{int}dt=$$

$$=\sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-t)e^{int}dt=S_n=\int_{-\pi}^{\pi}f(x-t)\underbrace{\left(\frac{1}{2\pi}\sum_{-N}^{N}e^{int}\right)}_{\text{Prove some facts about this ① and ②}}dt$$

↑ Periodic Lemma

① $\frac{1}{2\pi}\sum_{-N}^{N}e^{int}=\frac{1}{2\pi}\left(1+2\sum_{-N}^{N}\cos(nt)\right)$

$\int_0^{\pi}\cos(nt)dt=\frac{\sin(nt)}{n}\Big|_0^{\pi}=0-0=0$

This is even $\Rightarrow\int_{-\pi}^{0}=\int_0^{\pi}\qquad\int_0^{\pi}\frac{1}{2\pi}\left(1+2\sum_{n=1}^{N}\cos(nt)\right)dt=\boxed{\frac{1}{2}}$

Formula for geometric series

② Geometric series - almost $\Rightarrow\sum_{-N}^{N}e^{int}=e^{-iNt}\sum_{n=0}^{2N}e^{(it)^n}=e^{-iNt}\left(\frac{e^{i(2N+1)t}-1}{e^{it}-1}\right)$

① $\Rightarrow\frac{f(x_+)}{2}=\left(\int_{-\pi}^{0}\frac{1}{2\pi}\sum_{-N}^{N}e^{int}dt\right)f(x_+)\qquad\frac{f(x_-)}{2}=\left(\int_0^{\pi}\frac{1}{2\pi}\sum_{-N}^{N}e^{int}dt\right)f(x_-)$

$f(x-t)$ if $t<0$ $x-t>x$ $\quad f(x-t)$ if $t>0$ $x-t<x$

Thus $S_n-\left(\frac{f(x_+)+f(x_-)}{2}\right)=\int_{-\pi}^{0}\frac{1}{2\pi}\sum_{-N}^{N}e^{int}\left(f(x-t)-f(x_+)\right)dt+\int_0^{\pi}\frac{1}{2\pi}\sum_{-N}^{N}e^{int}\left(f(x-t)-f(x_-)\right)dt$

$$\overset{②}{=} \int_{-\pi}^{0} \frac{1}{2\pi}\left(\frac{e^{i(N+1)t}-e^{-iNt}}{e^{it}-1}\right)\left(f(x-t)-f(x_+)\right)dt + \int_{0}^{\pi} \frac{1}{2\pi}\left(\frac{e^{i(N+1)t}-e^{-iNt}}{e^{it}-1}\right)\left(f(x-t)-f(x_-)\right)dt =$$

<u>Idea</u>: (Remember) Let $g(t) = \begin{cases} \dfrac{f(x-t)-f(x_+)}{e^{it}-1} & -\pi<t<0 \\ \dfrac{f(x-t)-f(x_-)}{e^{it}-1} & 0<t<\pi \end{cases}$

$$\lim_{t\to 0^-} g(t) = \lim_{t\to 0^-} \frac{-f'(x-t)}{ie^{it}} = \frac{-f'(x_+)}{i}$$ Ok!

$$\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+} \frac{-f'(x-t)}{ie^{it}} = \frac{-f'(x_-)}{i}$$ Ok!

Thus g is bounded on $[-\pi,\pi] \Rightarrow g \in \mathcal{L}^2(-\pi,\pi)$

Thus $\sum \hat{g}_n e^{inx} \in \mathcal{L}^2(-\pi,\pi)$ and Bessel's projektion inequality

$\Rightarrow \left\|\sum_{n\in\mathbb{Z}} \hat{g}_n e^{inx}\right\|^2 \le \|g\|^2 = \int_{-\pi}^{\pi} |g(t)|^2 dt < \infty$

Pythagoras $\Rightarrow \sum_{n\in\mathbb{Z}} \|\hat{g}_n e^{inx}\|^2 = \sum_{n\in\mathbb{Z}} |\hat{g}_n|^2 (2\pi) \le \|g\|^2 < \infty \Rightarrow$ converges

Thus $\lim_{n\to\pm\infty} \hat{g}_n = 0$

$$S_n - \frac{f(x_+)+f(x_-)}{2} = \hat{g}_{-N-1} + \hat{g}_N$$

Since $\hat{g}_{-N-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) e^{-(-N-1)it} dt$ } Thus $\hat{g}_{-N-1} \to 0$ as $N\to\infty$

$g_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) e^{-iNt} dt$ and $\hat{g}_N \to 0$ as $N\to\infty$

Thus $S_n = \hat{g}_{-N-1} - \hat{g}_N \to 0$ as $N\to\infty$ □

Corollary: $\{e^{inx}\}$ is an orthogonal base for $\mathcal{L}^2(-\pi,\pi)$

Proof: Idea: If f is in $\mathcal{C}^1$ and continuous on $(-\pi,\pi)$ then $\forall x\in(-\pi,\pi)$ $\sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx} = f(x)$

Since $f(x_+) = f(x) = f(x_-)$ $\forall x\in(-\pi,\pi)$ by continuity

For arbitrary $f\in\mathcal{L}^2(-\pi,\pi)$ we use the DOG to prove if $\langle f, e^{inx}\rangle = 0$ $\forall n \Rightarrow f=0$

□

Applications

$$\begin{cases} u_t = u_{xx} \text{ for } -\pi < x < \pi \quad 0 < t \\ u(-\pi, t) = u(\pi, t) \quad u_x(-\pi, t) = u_x(\pi, t) \\ u(x,0) = f(x) \end{cases}$$

$T'X = X''T \longleftrightarrow \frac{T'}{T} = \frac{X''}{X} = \lambda$ constant $\qquad X_n(x) = e^{inx} \quad n \in \mathbb{Z} \qquad T_n(t) = \text{coeff} \cdot e^{-n^2 t}$

Superposition: $u(x,t) = \sum_{n \in \mathbb{Z}} (\text{coeff}?) \, e^{-n^2 t} \cdot e^{inx}$

$u(x,0) = \sum_{n \in \mathbb{Z}} (\text{coeff}?) \, e^{inx} \underset{\substack{\uparrow \\ \text{Initial} \\ \text{condition}}}{=} f(x) \Rightarrow$ thus coeff $= \hat{f}_n = \frac{\langle f, e^{inx} \rangle}{\| e^{inx} \|^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

and $\{e^{inx}\}$ are an orthogonal base

$$u(x,t) = \sum_{n \in \mathbb{Z}} \hat{f}_n \, e^{-n^2 t} e^{inx}$$

07/02-25

Applications of the tf-C@T

Applications of the tf-C@T:

1. Solving pde's
2. Calculating Σ

1 tF-C@T finds OB's of Hilbert spaces $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an OB for $\mathcal{L}^2(-\pi,\pi)$

tF-C@T corollary 1: Let $a\in\mathbb{R}$ and $l>0$. Then $\left\{e^{in\pi(n-a)/l}\right\}$ is an OB for $\mathcal{L}^2(a-l, a+l)$

tF-C@T corollary 2: · $\left\{\sin\left(\frac{n\pi x}{l}\right)\right\}_{n\geq 1}$ are an OB for $\mathcal{L}^2(0,l)$

· $\left\{\cos\left(\frac{n\pi x}{l}\right)\right\}_{n\geq 1}$ are an OB for $\mathcal{L}^2(0,l)$

Recall:
$$\begin{array}{l} \text{pde} \\ \text{BC} \\ \text{IC} \end{array} \begin{cases} u_{tt} = u_{xx} \\ u(0,t)=0, \quad u(l,t)=0 \\ u(x,0)=f(x), \quad u_t(x,0)=g(x) \end{cases}$$

① Separate variables $\longrightarrow$ $X(x)T(t)$ into pde

Solve for X first. $\frac{T''}{T} = \frac{X''}{X} = \text{constant} = \lambda \underset{BC}{\Rightarrow} X_n(x) = \sin\left(\frac{nx\pi}{l}\right) \Rightarrow \lambda_n = \frac{-n^2\pi^2}{l^2}$

$$T_n(t) = a_n \cos\left(\frac{n\pi t}{l}\right) + b_n \sin\left(\frac{n\pi t}{l}\right)$$

$$u(x,t) = \sum_{n\geq 1} T_n(t) X_n(x)$$

$u(x,0) = \sum T_n(0) X_n(x) = f(x)$ $\quad$ X_n to be a base

$\{X_n\}_{n\geq 1}$ is an OB

To get $u(x,0)=f(x)$ we set $T_n(0) = \frac{\langle f, X_n\rangle}{\|X_n\|^2} = \frac{\int_0^l f(x)\overline{X_n(x)}\,dx}{\int_0^l |X_n(x)|^2 dx}$

$u_t(x,0) = \sum_{n\geq 1} T_n'(0) X_n(x) = g(x)$ $\quad$ $T_n'(0) = \frac{\langle g, X_n\rangle}{\|X_n\|^2} = b_n \frac{n\pi}{l}$

$\Rightarrow$ $b_n = \frac{\langle g, X_n\rangle}{\|X_n\|^2}\frac{l}{n\pi}$

2 New tool for computing $\sum$

$\sin(x) \Rightarrow \sin'(x) = \cos(x)$

- e^x related series
- telescoping Σ
- geometric sums
- $\sum_{n \geq 1} X^n$, $|X| < 1$
- taylor Σ of stuff we know

Example of sums we can compute now

Riemann zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ for $Re(s) > 1$

Extend to $\mathbb{C} \setminus \{1\}$ More generally "spectral zeta fcn" $\sum_{n \geq 1} \frac{1}{\lambda_n^s}$

Basel Problem: Compute $\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2}$ New trick

Assume $\{X_n\}$ is an OB for $\mathcal{L}^2$(interval) Then any $\mathcal{L}^2$: $f(x) = \sum \frac{\langle f, X_n \rangle}{\|X_n\|^2} X_n(x)$

$$\xRightarrow{OB} \|f\|^2 = \sum \left\| \frac{\langle f, X_n \rangle}{\|X_n\|^2} X_n \right\|^2 = \sum \frac{|\langle f, X_n \rangle|^2}{\|X_n\|^2}$$

$\|f\|^2 = \int |f(x)|^2 dx$ Uolla i tabell! $\|\text{coeffiecents}\|^2 \simeq \frac{1}{n^2}$

Method #1: $f(x) = x$ has with $\{X_n = e^{inx}\}_{n \in \mathbb{Z}}$ on $\mathcal{L}^2(-\pi, \pi)$

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^n e^{inx}}{-in} \underset{OB}{\Longrightarrow} \|f\|^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left\| \frac{(-1)^n e^{inx}}{-in} \right\|^2$$

$$\|f\|^2 = \int_{-\pi}^{\pi} |x|^2 dx = 2 \int_0^{\pi} x^2 dx = \frac{2x^3}{3} \Big|_0^{\pi} = \frac{2\pi^3}{3}$$

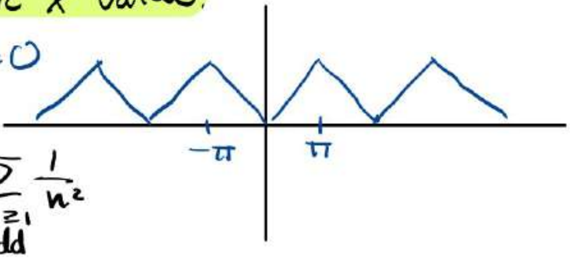
$$\left\| \frac{(-1)^n e^{inx}}{-in} \right\|^2 = \int_{-\pi}^{\pi} \left| \frac{(-1)^n e^{inx}}{-in} \right|^2 dx = \int_{-\pi}^{\pi} \frac{1}{n^2} dx = \frac{2\pi}{n^2}$$

$$\Rightarrow \frac{\pi^2}{3} = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} = 2 \sum_{n \geq 1} \frac{1}{n^2} = 2\, \zeta(2) \longrightarrow \frac{\pi^2}{6} = \zeta(2)$$

$\zeta(2k)$ for any $k \geq 1$ can be calculated

Method #2: Use tF-C@T to evaluate at a specific x value.

Example: $f(x) = |x| \xrightarrow[tF\Sigma]{} \frac{\pi}{2} + \sum_{n \in \mathbb{Z} \text{ odd}} e^{inx}\left(\frac{-2}{\pi n^2}\right)$ @ $x=0$



$$0 = \frac{\pi}{2} + \sum_{\substack{n\in\mathbb{Z} \\ \text{odd}}} \frac{-2}{\pi n^2} \Rightarrow \frac{\pi^2}{4} = \sum_{n \in \mathbb{Z}} \frac{1}{n^2} = 2\sum_{\substack{n\geq 1 \\ \text{odd}}} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{8} = \sum_{\substack{n\geq 1 \\ \text{odd}}} \frac{1}{n^2}$$

$$\sum_{n\geq 1} \frac{1}{n^2} = \sum_{k\geq 1} \frac{1}{(2k)^2} = \frac{1}{4}\sum_{k\geq 1} \frac{1}{k^2} = \frac{1}{4}\zeta(2)$$

$$\Rightarrow \zeta(2) = \sum_{\substack{n\geq 1 \\ \text{even}}} \frac{1}{n^2} + \sum_{\substack{n\geq 1 \\ \text{odd}}} \frac{1}{n^2} = \frac{\zeta(2)}{4} + \frac{\pi^2}{8} \Rightarrow \frac{3}{4}\zeta(2) = \frac{\pi^2}{8} \Rightarrow \zeta(2) = \frac{\pi^2}{6}$$

When a function is derived its tF coefficients get multiplied!

Theorem★: Assume f is piecewise $\mathcal{C}^1$ on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$ and continuous on $[-\pi, \pi]$. Then $(\hat{f'_n}) = in\hat{f}_n$

Proof★: Use the definition: $\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx - \widehat{(f')}_n$ integrate by parts!

$$= \frac{1}{2\pi}\left(f(x)e^{-inx}\right)\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(-in)e^{-inx}dx = in\hat{f}_n$$

$f(-\pi) = f(\pi)$ $e^{in\pi} = e^{-in\pi}$ $\Rightarrow 0$

OBS! When using pontwise evaluation of $tF\Sigma$ to compute

$$\sum \hat{f}_n e^{inx} = \frac{f(x_+) + f(x_-)}{2}$$

10/2 - 25

1 Sturm-Liouville Problems
2. The ABC's of SLP's theorem + proof

Chapter 4 Sturm-Liouville Problems

1. $X''(x) = \lambda X(x)$ for $0 < x < l$ and $X(0) = 0$ $X(l) = 0$ Dirichlet Boundary Condition
2. $X''(x) = \lambda X(x)$ for $-\pi < x < \pi$ and $X(-\pi) = X(\pi)$ $X'(-\pi) = X'(\pi)$ Periodic BC
3. $X''(x) = \lambda X(x)$ for $0 < x < l$ and $X'(0) = 0$ $X'(l) = 0$ Neumann BC
4. Robin BC is $X'(0) = \underset{\substack{\uparrow \\ \text{constant}}}{\alpha} X(0)$

Def: A (regular) SLP is to find all eigenfunctions $X(x)$ on (a,b) and eigenvalues λ that satisfy: $q'(x)X'(x) + q(x)X''(x) + \lambda w(x)X(x) = 0$ for $a < x < b$

and X satisfies periodic OR any combo of D, N, R BC

The function $q(x) \in C^1$, $w(x) \in C^0$ and $q(x) > 0$ $w(x) > 0$ are given in the prob

Often $q(x) = 1$, $w(x) = 1$ (Not always)

Proposition: Assume that L is a differential operator of the SLP type:

$$L(f) = q'(x)f'(x) + q(x)f''(x)$$

Then if f and g satisfy SLP BC's then

$$\langle L(f), g \rangle = \langle f, L(g) \rangle \leftarrow \mathcal{L}^2(a,b) \text{ scalar product}$$

Spectral Theorem for hermitian matrices:

- Hilbert space $\mathbb{C}^n$
- M = matrix (nxn with entries) then $Mv \in \mathbb{C}^n$ $\forall v \in \mathbb{C}^n$
- Eigenvectors satisfy $M\lambda - \underset{\leftarrow \text{scalar} \in \mathbb{C}}{\lambda} v = 0$ $v \neq 0$

If M is hermitian then there is an orthonormal base of eigenvectors for $\mathbb{C}^n$

Hermitian means $\forall v, w \in \mathbb{C}^n$ $\langle Mv, w \rangle = \langle v, Mw \rangle$

Theorem: ☆ The eigenfunctions of an SLP are an orthogonal base.

The ABC's of SLP

1. The eigenvalues of an SLP are real
2. Eigenfunctions with different eigenvalues are orthogonal in

$\mathcal{L}^2_w(a,b) = \{f : \int_a^b |f(x)|^2 w(x)\,dx < \infty\}$ so if f and g are eigenfunctions with different eigenvalues then $\int_a^b f(x)\overline{g(x)}\,w(x)\,dx = 0$

Proof:* Use the $\langle Lf,g\rangle=\langle f,Lg\rangle$ and the def of $\langle\,,\rangle$

1. If f is an eigenfunction then:

$$L(f)+\lambda wf=L(f(x))+\lambda w(x)f(x)=0$$

$$L(f(x))=-\lambda w(x)f(x)$$

$$\langle Lf,f\rangle=\langle-\lambda wf,f\rangle=-\lambda\int_a^b w(x)f(x)\overline{f(x)}dx=-\lambda\int_a^b w(x)|f(x)|^2dx$$

$$=\langle f,Lg\rangle=\langle f,-\lambda wf\rangle=\int_a^b f(x)\overline{(-\lambda w(x)f(x))}dx=-\bar{\lambda}\int_a^b \underbrace{w(x)}_{}|f(x)|^2dx$$

$w>0$ by defullt.

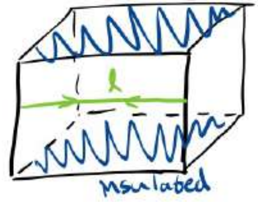
$\Rightarrow \lambda=\bar{\lambda}\Rightarrow$ Real

2. $$\langle Lf,g\rangle=\int_a^b-\lambda w(x)f(x)\overline{g(x)}dx=-\lambda\int_a^b w(x)f(x)\overline{g(x)}dx=$$

$$=\langle f,Lg\rangle=\int_a^b f(x)(\overline{-\mu w(x)g(x)})dx=\left\{\begin{matrix}\mu=\text{eigen value}\\ \text{for } g\end{matrix}\right\}=-\mu\int_a^b f(x)w(x)\overline{g(x)}dx$$

If $\lambda\neq\mu$ then $\Rightarrow\int_a^b f(x)w(x)\overline{g(x)}dx=0$ Note: Compute $\langle Lf,g\rangle$ because $Lf=-\lambda wf$

Heat diffusion Example:



$$\begin{cases}u_t=u_{xx}\\ u_x(0,t)=\alpha u(0,t)\quad u_x(l,t)=-\alpha u(l,t) & \text{Newtons law of cooling}\\ u(x,0)=f(x) & \text{initial temperature}\end{cases}$$

Separate variables in the pde $\to X(x)T(t)$ to solve $XT'=X''T\xrightarrow[XT]{\div}\frac{T'}{T}=\frac{X''}{X}=\lambda$

Solve for X first (BC): $X''=\lambda X\Leftrightarrow X''-\lambda X=0$ $\{\Delta=-\lambda\}$ $X''+\Delta X=0$

In SLP def. $q(x)=1$, $w(x)=1$ Robin BC. What λ and eigenfunctions X do we get?

$\underline{\lambda=0}$: $X''(x)=0\Rightarrow X(x)=ax+b$

Plug in the BC's @ $x=0$: $X'(0)=a=\alpha X(0)=\alpha b\Rightarrow\frac{a}{\alpha}=b$

@ $x=l$: $X'(l)=-\alpha X(l)\Leftrightarrow a=-\alpha(al+b)=-\alpha(al+\frac{a}{\alpha})=-\alpha a(l+\frac{1}{\alpha})$

$a=-\alpha a(l+\frac{1}{\alpha})$

if $a\neq0$ then $1=-\alpha(l+\frac{1}{\alpha})$ impossible since l,α are >0

$\Rightarrow$ No nonzero linear solutions

$\underline{\lambda\neq0}$: Then the solution is a linear combination of $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$

$\{e^{\pm\sqrt{\lambda}x}\}$ OR $\{e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x},e^{\sqrt{\lambda}x}-e^{-\sqrt{\lambda}x}\}$ convenient whenever we have boundary at $x=0$

Call $\varphi(x)=e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x}$ $\psi(x)=e^{\sqrt{\lambda}x}-e^{-\sqrt{\lambda}x}$ Solution is $a\varphi(x)+b\psi(x)$ for some $a,b\in\mathbb{C}$

$\varphi(x)=\sqrt{\lambda}\psi(x)$, $\psi'(x)=\sqrt{\lambda}\varphi(x)$

BC @ $x=0$: $a\varphi'(0)+b\psi'(0)=\alpha(a\varphi(0)+b\psi(0))$

$$a\sqrt{\lambda}\,\underbrace{\psi(0)}_{0}+b\sqrt{\lambda}\,\varphi(0)=\alpha a 2$$

$$2b\sqrt{\lambda}=2a\alpha \quad \Rightarrow \quad b=\frac{a\alpha}{\sqrt{\lambda}}$$

12/02-25

1. SLP's what are these?

→ 2 examples

SLP's - what are these?

Example 1. Newtons Law of Cooling $\quad x=0 \longleftrightarrow x=l$

$l>0$

$\alpha>0$

pde $\begin{cases} u_t = u_{xx} \quad 0<x<l \quad 0<t \end{cases}$

Robin BC $\quad u_x(0,t) = \alpha u(0,t) \qquad u_x(l,t) = -\alpha u(l,t)$

IC $\quad u(x,0) = f(x)$

Separate the variables: $T(t)X(x) \qquad T'X = X''T \xrightarrow{\div XT} \frac{T'}{T} = \frac{X''}{X} = \lambda$ constant

BC's => solve for X first $\quad X'' = \lambda X \quad$ SLP written $\quad X'' + \Lambda X = 0 \quad \{\Lambda = -\lambda\}$

$\lambda = 0$ OR $\lambda \neq 0$

↓ we only get the 0 function => No eigenfunction => No eigenvalue

$\lambda \neq 0$ Basis of solutions is $\{e^{\sqrt{\lambda}x}, e^{-\sqrt{\lambda}x}\}$

$\varphi(x) = e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x}$ and $\psi(x) = e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}$ $\quad \{\varphi, \psi\}$ are also a basis of solution.

$\varphi'(x) = \sqrt{\lambda}\,\psi(x) \qquad \psi'(x) = \sqrt{\lambda}\,\varphi(x)$ $\quad$ Solution to the $X'' = \lambda X$ is $a\varphi + b\psi \quad a,b \in \mathbb{C}$

BC @ $x=0$: $\quad a\varphi'(0) + b\psi'(0) = \alpha(a\varphi(0) + b\psi(0)) \qquad \psi(0) = 0 \quad \varphi(0) = 2$

$$a\sqrt{\lambda}\,\underbrace{\cancel{\psi(0)}}_{0} + b\sqrt{\lambda}\,\underbrace{\varphi(0)}_{2} = \alpha(a\underbrace{\varphi(0)}_{2} + b\underbrace{\cancel{\psi(0)}}_{0})$$

$\overbrace{a\varphi + b\psi \quad b = \frac{a\alpha}{\sqrt{\lambda}}}$

$2b\sqrt{\lambda} = 2a\alpha \quad \Rightarrow \quad b = \frac{a\alpha}{\sqrt{\lambda}} \quad \Rightarrow$ solution is of the form $\underset{\uparrow}{a}\left(\varphi(x) + \frac{\alpha}{\sqrt{\lambda}}\psi(x)\right)$

Solutions (X's) will only be unique up to multiplication by scalar => We can delete this

So now we have $\varphi(x) + \frac{\alpha}{\sqrt{\lambda}}\psi(x)$

BC @ $x=l$: $\quad \varphi'(l) + \frac{\alpha}{\sqrt{\lambda}}\psi'(l) = -\alpha\left(\varphi(l) + \frac{\alpha}{\sqrt{\lambda}}\psi(l)\right)$

$$\sqrt{\lambda}\,\psi(l) + \frac{\alpha}{\cancel{\sqrt{\lambda}}}\cancel{\sqrt{\lambda}}\,\varphi(l) = -\alpha\left(\varphi(l) + \frac{\alpha}{\sqrt{\lambda}}\psi(l)\right)$$

$$2\alpha\varphi(l) = -\sqrt{\lambda}\,\psi(l) - \frac{\alpha^2}{\sqrt{\lambda}}\psi(l) = -\left(\sqrt{\lambda} + \frac{\alpha^2}{\sqrt{\lambda}}\right)\psi(l)$$

$$2\alpha\varphi(l) = -\left(\sqrt{\lambda} + \frac{\alpha^2}{\sqrt{\lambda}}\right)\psi(l)$$

λ can be ⊕ or ⊖

If $\lambda \geq 0$ then $\varphi(l) = e^{\sqrt{\lambda} l} + e^{-\sqrt{\lambda} l} > 0$ $\Psi(l) = e^{\sqrt{\lambda} l} - e^{-\sqrt{\lambda} l} > 0$

Get positive = negative ↯

So $\lambda < 0$ is the only possibility.

$\lambda < 0 \Rightarrow \lambda = -|\lambda|$ $\varphi(x) = e^{\sqrt{\lambda} x} + e^{-\sqrt{\lambda} x} = e^{i\sqrt{|\lambda|} x} + e^{-i\sqrt{|\lambda|} x} = 2\cos(\sqrt{|\lambda|} x)$

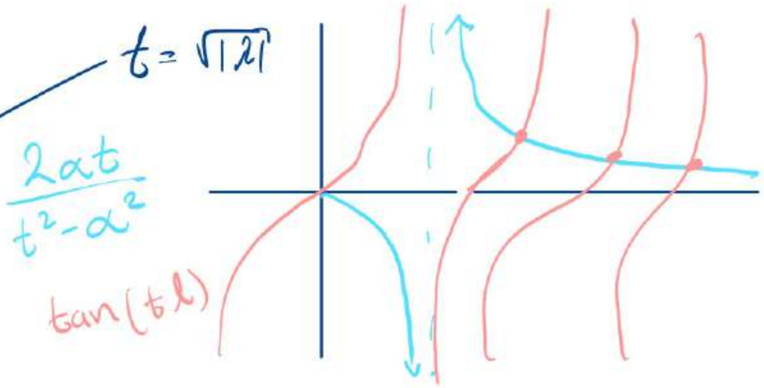
$\Psi(x) = e^{\sqrt{\lambda} x} - e^{-\sqrt{\lambda} x} = e^{i\sqrt{|\lambda|} x} - e^{-i\sqrt{|\lambda|} x} = 2\sin(\sqrt{|\lambda|} x)$

$$\underset{\lambda < 0}{\Rightarrow} 2\alpha\, 2\cos(\sqrt{|\lambda|}\, l) = -\left(i\sqrt{|\lambda|} + \frac{\alpha^2}{i\sqrt{|\lambda|}}\right) 2i\sin(\sqrt{|\lambda|}\, l)$$

$$\frac{4\alpha}{-2i\left(i\sqrt{|\lambda|} + \frac{\alpha^2}{i\sqrt{|\lambda|}}\right)} = \tan(\sqrt{|\lambda|}\, l)$$

$$\frac{-2\alpha}{-\sqrt{|\lambda|} + \frac{\alpha^2}{\sqrt{|\lambda|}}} = \tan(\sqrt{|\lambda|}\, l)$$

$$\frac{2\alpha\sqrt{|\lambda|}}{|\lambda| - \alpha^2} = \tan(\sqrt{|\lambda|}\, l)$$



So we have ∞ many $0 < \sqrt{|\lambda_1|} < \sqrt{|\lambda_2|} < \dots$ solutions to $\frac{2\alpha\sqrt{|\lambda|}}{|\lambda| - \alpha^2} = \tan(\sqrt{|\lambda|}\, l)$

$\varphi(x) + \frac{\alpha}{\sqrt{\lambda}}\phi(x) \Rightarrow X_n(x) = 2\cos(\sqrt{|\lambda_n|}\, x) + \frac{\alpha}{i\sqrt{|\lambda_n|}} 2i\sin(\sqrt{|\lambda_n|}\, x)$

$X_n(x) = 2\left(\cos(\sqrt{|\lambda_n|} + \frac{\alpha}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}\, x)\right) \xrightarrow[\text{delete 2}]{} X_n(x) = \cos(\sqrt{|\lambda_n|}\, x) + \frac{\alpha}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}\, x)$

$$\frac{T_n'}{T_n} = \frac{X_n''}{X_n} = \lambda_n = -\left(\sqrt{|\lambda_n|}\right)^2 \Rightarrow T_n(t) = c_n e^{\lambda_n t}$$

$$u(x,t) = \sum_{n \geq 1} T_n(t) X_n(x) \qquad u(x,0) = \sum_{n \geq 1} T_n(0) X_n(x)$$

$$T_n(0) = c_n = \frac{\langle f, X_n \rangle}{\|X_n\|^2} = \frac{\int_0^l f(x)\overline{X_n(x)}\,dx}{\int_0^l |X_n(x)|^2\,dx}$$

Example 2: $L(f) = (q f')' = q' f' + q f''$ $L(f) + \lambda w f = 0$ on interval with BC's

Arbitrary $q > 0$, $w > 0$ functions $\longrightarrow$ HARD (impossible?)

$q(x) = e^{2x} = w(x)$ (OBS: if $q(x) = w(x) = e^{x} \to f'' + \lambda f = 0$)

The SLP is:
$$\begin{cases} (e^{2x} f')' + \lambda e^{2x} f = 0 & 0 < x < 2 \\ f'(0) = 0 \\ f(2) = 0 \end{cases}$$

Neumann @ $x=0$ Dirichlet @ $x=2$

$f'' e^{2x} + 2e^{2x} f' + \lambda e^{2x} f = 0 \iff f'' + 2f' + \lambda f = 0$

Basis of solutions $\dfrac{-2 \pm \sqrt{4 - 4\lambda}}{2} = -1 \pm \sqrt{1 - \lambda}$

Let $\mu = \sqrt{1-\lambda}$
$\Rightarrow \mu > 0$ OR $\mu \in i\mathbb{R}$

If $\lambda = 1$ then Basis of solutions is $\{e^{-x}, xe^{-x}\}$ ...

If $\lambda \neq 1$ then Basis of solutions is $\{e^{(-1+\mu)x}, e^{(-1-\mu)x}\}$

$a e^{(-1+\mu)x} + b e^{(-1-\mu)x}$ for $a, b \in \mathbb{C}$

BC @ $x = 0$: $a(-1+\mu)(1) + b(-1-\mu)(1) = 0 \Rightarrow b = \dfrac{a(-1+\mu)}{1+\mu}$

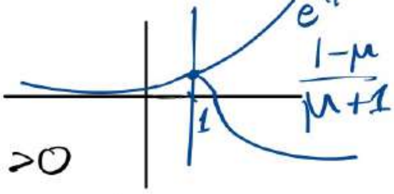
$\Rightarrow$ Solution is of the form $a\left(e^{(-1+\mu)x} + \dfrac{(-1+\mu)}{1+\mu} e^{(-1-\mu)x}\right)$ delete a!

$\Rightarrow e^{(-1+\mu)x} + \dfrac{(-1+\mu)}{1+\mu} e^{(-1-\mu)x}$

BC @ $x = 2$: $e^{(-1+\mu)2} + \dfrac{(-1+\mu)}{1+\mu} e^{(-1-\mu)2} = 0$ $\div$ by e^{-2}

$$e^{2\mu} + \frac{(-1+\mu)}{1+\mu} e^{-2\mu} = 0 \quad * e^{2\mu} \quad \Rightarrow \quad e^{4\mu} = \frac{1-\mu}{\mu+1}$$

$\mu > 0$ OR $\mu \in i\mathbb{R}$: $\mu > 0$



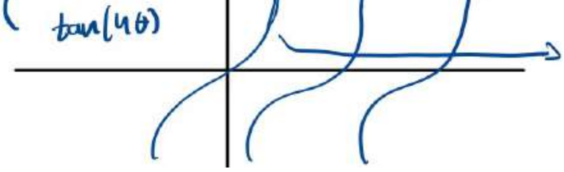
$\mu \in i\mathbb{R}$ $e^{4\mu} = e^{4i\theta}$ for $\theta > 0$

$e^{4i\theta} = \cos(4\theta) + i\sin(4\theta) = \dfrac{1 - i\theta}{1 + i\theta} = \dfrac{(1-i\theta)^2}{1+\theta^2} = \dfrac{1-\theta^2}{1+\theta^2} - \dfrac{2i\theta}{1+\theta^2}$

θ must satisfy BOTH equations simultaneously

$\cos(4\theta) = \dfrac{1-\theta^2}{1+\theta^2}$ and $\sin(4\theta) = \dfrac{-2\theta}{1+\theta^2}$

$\dfrac{\sin(4\theta)}{\cos(4\theta)} = \tan(4\theta) = \dfrac{-2\theta}{1-\theta^2}$



$\Rightarrow 0 < \theta_1 < \theta_2 < \dots$ ∞ many solutions $\theta_n \approx \frac{\pi n}{4}$ as $n \to \infty$

$\Rightarrow f_n(x) = e^{(-1+i\theta_n)x} + \dfrac{(-1+i\theta_n)}{1+i\theta_n} e^{(-1-i\theta_n)x}$ $\mu_n = i\theta_n$

can delete

$\Rightarrow f_n(x) = \dfrac{2ie^{-x}}{1+\theta_n}(\theta_n \cos(\theta_n x) + \sin(\theta_n x))$ $(i\theta_n)^2 = 1 - \lambda_n$

$\lambda_n = 1 + \theta_n^2$

13/02-25

SLP's solving the pde's

Chapter 5

Math is like a martial art, master it with efforts from the heart!

SLPs are the keys to solving inhomogeneous pde's

$$\begin{cases} \text{pde} & u_t = u_{xx} + F(x,t) \\ \text{pde} & u_t - u_{xx} = F(x,t) \\ \text{BC} & u_x(0,t)=0,\ u_x(l,t)=0 \\ \text{IC} & u(x,0)=f(x) \end{cases}$$

heat equation with source/sink $F(x,t)$

Neumann BC insulated @ $x=0$, $x=l$

$$\text{SLP}\begin{cases} X'' = \lambda X \\ X'(0)=0 \\ X'(l)=0 \end{cases}$$

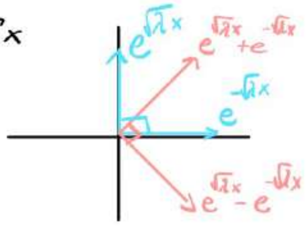
$X(x)T'(T) = X''(x)T(t) \Rightarrow \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = \lambda$ $\qquad X''(X) = \lambda X(x)$

$X''(X) - \lambda X(x) = 0$

$\underline{\lambda=0}$ $\quad X''=0$ and $X'(0)=0$ $X'(l)=0$ constant functions satisfy this $\lambda_0=0$ and $X_0(x)=1$

$\underline{\lambda \neq 0}$ Basis of solutions is $\{e^{\sqrt{\lambda}x}, e^{-\sqrt{\lambda}x}\}$

Since $x=0$ is an endpoint let's use $\varphi(x) = e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x}$ $\quad \psi(x) = e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}$ as basis of solutions



$\varphi'(x) = \sqrt{\lambda}e^{\sqrt{\lambda}x} - \sqrt{\lambda}e^{-\sqrt{\lambda}x} = \sqrt{\lambda}\ \psi(x)$

$\psi'(x) = \sqrt{\lambda}e^{\sqrt{\lambda}x} + \sqrt{\lambda}e^{-\sqrt{\lambda}x} = \sqrt{\lambda}\ \varphi(x)$

For a linear $a\varphi + b\psi$ for $a,b \in \mathbb{C}$

BC @ $x=0$: $\quad a\varphi'(0) + b\psi'(0) = 0$

$\sqrt{\lambda}(a\underbrace{\psi(0)}_{0} + b\underbrace{\varphi(0)}_{2}) = 0 \Rightarrow 2\sqrt{\lambda}b = 0 \quad b=0$

$\Rightarrow$ Only $a\varphi$ in the solution, unique only up to * by constants $\Rightarrow$ throw a away. $\varphi(x)$ only

BC @ $x=l$: $\varphi'(l) = 0 = \sqrt{\lambda}\psi(l)$ $\quad \psi(l) = e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l} = 0 \iff e^{\sqrt{\lambda}l} = e^{-\sqrt{\lambda}l} \iff e^{2\sqrt{\lambda}l} = 1$

$\Rightarrow$ Need $2\sqrt{\lambda}l = 2\pi i n$ $\quad \sqrt{\lambda_n} = \dfrac{\pi i n}{l}$ $\quad \lambda_n = -\dfrac{\pi^2 n^2}{l^2}$ $\quad n \in \mathbb{N}$

$$\varphi_n(x) = e^{\frac{i\pi x n}{l}} + e^{-\frac{i\pi x n}{l}} = 2\cos\left(\frac{n\pi x}{l}\right)$$

$$X_n(X) = \cos\left(\frac{n\pi x}{l}\right) \quad n \in \mathbb{N} \text{ and } X_0 = 1$$

$$\frac{X_n''}{X_n} = \frac{-n^2\pi^2}{l^2} = \lambda_n$$

STOP

Do superposition with unknown $T_n(t)$ functions

$u(x,t) = \sum_{n \geq 0} T_n(t) X_n(x) \rightarrow$ put in pde

$$u_t - u_{xx} = \sum_{n\geq 0} T_n'(t) X_n(x) - T_n(t) X_n''(x) = \sum_{n\geq 0} \left(T_n'(t) - \lambda_n T_n(t)\right) X_n(x) = F(x,t)$$

X_n are an OB $\Rightarrow F(x,t) = \sum_{n\geq 0} \frac{\langle F, X_n \rangle}{\| X_n \|^2} X_n(x)$ $\quad \frac{\langle F, X_n \rangle}{\| X_n \|^2} = \frac{\int_0^\ell F(x,t) \overline{X_n(x)}\, dx}{\int_0^\ell |X_n(x)|^2 dx} = \hat{F}_n(t)$

$$\sum_{n\geq 0} \underbrace{\left(T_n'(t) + \lambda_n T_n(t)\right)} X_n(x) = \sum_{n\geq 0} \underbrace{\hat{F}_n(t)} X_n(x)$$

$$T_n'(t) + \lambda_n T_n(t) = \hat{F}_n(t) \xrightarrow[\text{integration factor}]{} T_n(t) = e^{-\frac{n^2\pi^2 t}{\ell^2}} \left[\int_0^t e^{\frac{n^2\pi^2 s}{\ell^2}} \hat{F}_n(s)\, ds + T_n(0) \right]$$

FINNS på tentan

IC $\Rightarrow$ We need $u(x,0) = \sum_{n\geq 0} T_n(0) X_n(x) = f(x) \Rightarrow T_n(0) = \frac{\langle f, X_n \rangle}{\| X_n \|^2}$

BC that work for our methods are:

- Dirichlet (function=0)
- Neumann (derivative=0)
- periodic (same @ ends)
- Robin (please no)

Now what if we have something else?

$$\begin{cases} u_t - u_{xx} = F(x,t) \\ u_x(0,t) = \xi(t), \quad u_x(\ell,t) = \zeta(t) \\ u(x,0) = f(x) \end{cases}$$

Deal with the weird stuff <u>first</u>

Look for $L_1(x)$ = Linear and $L_2(x)$ linear and $A(t)$, $B(t)$ such that $L_1(x)A(t) + L_2(x)B(t)$ Satisfies the Boundary conditions

14/02-25

1. Coalitions solve tough boundary conditions
2. Rectangular drums

Question: What boundary conditions can we solve already (the SLP BC's):
$u(x,t)$ to solve a pde with BC's @ $x=0$ and @ $x=l$

Dirichlet: $u=0$ Neumann $u_x=0$ Robin $u_x=\alpha u$ periodic: $u(0,t)=u(l,t)$
$u_x(0,t)=u_x(l,t)$

If we have any other BC → Solve that first

$$\begin{cases} u_t-u_{xx}=F(x,t) & 0<x<l \quad 0<t \\ u_x(0,t)=\xi(t)\,, \quad u_x(l,t)=\zeta(t) \\ u(x,0)=f(x) \end{cases}$$

Questions: Can you find two functions $A(x)$ and $B(x)$ such that $\begin{cases} A'(0)=1 & A'(l)=0 \\ B'(0)=0 & B'(l)=1 \end{cases}$

Then $S(x,t)=A(x)\xi(t)+B(x)\zeta(t)$

satisfies $S_x(0,t)=A'(0)\xi(t)+B'(0)\zeta(t)=\xi(t)$

$S_x(l,t)=\zeta(t)$

$B(x)=\frac{x^2}{2l}$ $B'(0)=0$ $B'(l)=\frac{2l}{2l}=1$

$A(x)=x-\frac{x^2}{2l}$ $A'(x)=1-\frac{2x}{2l}$

$A'(0)=1-0=1$ $A'(l)=1-\frac{2l}{2l}=0$

Alt. $\sin\left(\frac{\pi x}{2l}\right)\cdot\frac{2l}{\pi}$ and $\cos\left(\frac{\pi x}{2l}\right)\cdot\frac{2l}{\pi}$

Does not matter! Set $S(x,t)=A(x)\xi(t)+B(x)\zeta(t)$

Now we look for v that solves

$$\begin{cases} v_t-v_{xx}=G(x,t) \\ v_x(0,t)=0\,, \quad u_x(l,t)=0 \\ v(x,0)=g(x) \end{cases}$$

yay

Then $u(x,t)=S(x,t)+v(x,t)$ the problem $G(x,t)=F(x,t)-(S_t-S_{xx})$

$g(x)=f(x)-S(x,0)$ $S_t(x,t)=A(x)\xi'(t)+B(x)\zeta'(t)$ $S_{xx}(x,t)=A''(x)\xi'(t)+B''(x)\zeta(t)$

To solve the problem for v → use the SLP method

1. Pretend the equation (pde) is homogeneous
2. Separate variables
3. Solve for the $X''=\lambda X$ function. We did that → $X_n(x)=\cos\left(\frac{n\pi x}{l}\right)$ $n\geq 0$

$\lambda_n=-\frac{\pi^2 n^2}{l^2}$ STOP Express G using these since they are an OB

$$G(x,t)=\sum_{n\geq 0}\hat{G}_n(t)X_n(x) \qquad \hat{G}_n(t)=\frac{\langle G, X_n\rangle}{\|X_n\|^2}=\frac{\int_0^l G(x,t)\overline{X_n(x)}\,dx}{\int_0^l |X_n(t)|^2\,dx}$$

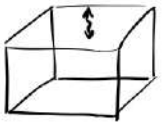
$V(x,t)=\sum_{n\geq 0}T_n(t)X_n(x)$ Put in pde $\sum_{n\geq 0}\left(T_n'(t)+\lambda_n T_n(t)\right)X_n(x)=\sum_{n\geq 0}G_n(t)X_n(x)$

Solve $T_n'(t)+\lambda_n T_n(t)=\hat{G}_n(t)$ Use Beta/Appendix/Ref

And $v(x,0)$ should be equal $g(x) \Rightarrow T_n(0)=\frac{\langle g, X_n\rangle}{\|X_n\|^2}$

Chapter 6 *Bessel functions are lots of fun - their zeros describe a vibrating drum!*

Drums



vibrating rectangular drumhead

$\Delta = \partial_{xx} + \partial_{yy}$ Homogeneous wave equation is $u_{tt} = \Delta u = u_{xx} + u_{yy}$

$$\begin{cases} u_t = \Delta u & \text{inside} \quad 0 < t \\ u(x,y,t) = S(x,y) & \text{BC is non-zero DO FIRST} \\ u(x,y,0) = f(x,y), & u_t(x,y,0) = g(x,y) \end{cases}$$

$$\left.\begin{aligned} u(0,y) &= S(0,y) = a(y) \\ u(l,y) &= S(l,y) = b(y) \\ u(x,0) &= S(x,0) = \alpha(x) \\ u(x,w) &= S(x,w) = \beta(x) \end{aligned}\right\} \text{Find } S(x,y) \text{ with } \Delta S(x,y) = 0 \text{ and with these boundary values}$$

Divide and conquer

$$\begin{cases} \Delta \emptyset = 0 \\ \emptyset(0,y) = a(y) \\ \emptyset(l,y) = b(y) \\ \emptyset(x,0) = 0 \\ \emptyset(x,w) = 0 \end{cases} \text{"time like"} \quad \text{and} \quad \begin{cases} \Delta \Psi = 0 \\ \Psi(0,y) = 0 \\ \Psi(l,y) = 0 \\ \Psi(x,0) = \alpha(x) \\ \Psi(x,w) = \beta(x) \end{cases} \text{unce} \quad \text{"time like"}$$

$S(x,y) = \emptyset + \Psi$ satisfies $\Delta S = 0$ and the BC

Solve for $\emptyset$ by first separating variables. Look for $\Delta(X(x)Y(y)) = 0$ $X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} \Rightarrow \text{constant} \quad \underbrace{Y(0)=0 \text{ and } Y(w)=0}_{\text{solve for it first}} \quad \underbrace{Y'' = -\text{constant} \cdot Y \text{ and } Y(0)=0 \;\; Y(w)=0}_{\text{Solved previously} \Rightarrow}$$

$$\Rightarrow \underbrace{Y_n(y) = \sin\left(\frac{n\pi y}{w}\right)}_{OB} \quad n \geq 1$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \frac{n^2\pi^2}{w^2} \Rightarrow X_n'' = \frac{n^2\pi^2}{w^2} X_n$$

$$X_n(x) = a_n \cosh\left(\frac{n\pi x}{w}\right) + b_n \sin\left(\frac{n\pi x}{w}\right) \quad \text{"time like"}$$

$$\emptyset(x,y) = \sum_{n\geq1} X_n(x) Y_n(y) \quad \text{Need} \quad \emptyset(0,y) = \sum_{n\geq1} X_n(0) Y_n(y) = a(y)$$

$$X_n(0) = \frac{\langle a, Y_n \rangle}{\| Y_n \|^2} = a_n = \frac{\int_0^w a(y) \overline{Y_n(y)} \, dy}{\int_0^w |Y_n(y)|^2 \, dy}$$

$$\emptyset(l,y) = \sum_{n\geq1} X_n(l) Y_n(y) = b(y) \Rightarrow X_n(l) = \frac{\langle b \; Y_n \rangle}{\| Y_n \|^2} = \frac{\int_0^w b(y) \overline{Y_n(y)} \, dy}{\int_0^w |Y_n(y)|^2 \, dy}$$

$$\Rightarrow \text{Need } a_n \cosh\left(\frac{n\pi l}{w}\right) + b_n \sinh\left(\frac{n\pi l}{w}\right) = \frac{\langle b, Y_n \rangle}{\| Y_n \|^2}$$

Do algebra (rearrange the equation to solve for b_n)

$$\psi(x,y) = \sum_{n\geq 1} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cosh\left(\frac{n\pi y}{l}\right) + B_n \sinh\left(\frac{n\pi y}{l}\right)\right]$$

$$A_n = \frac{\langle \alpha, \sin\left(\frac{n\pi x}{l}\right) \rangle}{\left\|\sin\left(\frac{n\pi x}{l}\right)\right\|^2} \qquad A_n \cosh\left(\frac{n\pi w}{l}\right) + B_n \sinh\left(\frac{n\pi w}{l}\right) = \frac{\langle \beta, \sin\left(\frac{n\pi x}{l}\right) \rangle}{\left\|\sin\left(\frac{n\pi x}{l}\right)\right\|^2}$$

Rearrange to get B_n

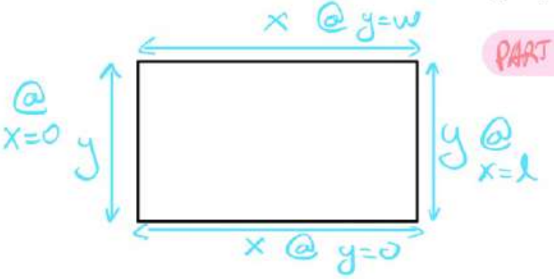
17/02-25

Drums:

1. Rectangular drumhead
2. Disk-shaped drumhead

1 Recall the problem $\begin{cases} u_{tt} = \Delta u \text{ inside} \\ \boxed{u(x,y,t) = S(x,y) \text{ on boundary}} \\ u(x,y,0) = f(x,y) \\ u_t(x,y,0) = g(x,y) \end{cases}$ PART 1

Solve first $\begin{cases} \Delta S(x,y) = 0 \text{ inside} \\ S(x,y) \text{ Boundary values} \end{cases}$



PART 1.1

$$\begin{cases} \Delta \emptyset = 0 \text{ inside} \\ \emptyset(0,y) = S(0,y) = a(y) \\ \emptyset(l,y) = S(l,y) = b(y) \\ \emptyset(x,0) = 0 \\ \emptyset(x,w) = 0 \end{cases}$$

PART 1.2

$$\begin{cases} \Delta \Psi = 0 \text{ inside} \\ \Psi(x,0) = S(x,0) = \alpha(x) \\ \Psi(x,w) = S(x,w) = \beta(x) \\ \Psi(0,y) = 0 \\ \Psi(l,y) = 0 \end{cases}$$

$\longrightarrow\longrightarrow$ Fast forward $\emptyset(x,y) = \sum_{n\geq 1} \sin\left(\frac{n\pi y}{w}\right)\left[a_n \cosh\left(\frac{n\pi x}{l}\right) + b_n \sinh\left(\frac{n\pi x}{l}\right)\right]$ $\boxed{S(x,y) = \emptyset + \Psi}$

$$a_n = \frac{\langle a, \sin\left(\frac{n\pi y}{w}\right) \rangle}{\|\sin\left(\frac{n\pi y}{w}\right)\|^2} \qquad a_n \cosh\left(\frac{n\pi l}{w}\right) + b_n \sinh\left(\frac{n\pi l}{w}\right) = \frac{\langle b, \sin\left(\frac{n\pi y}{w}\right) \rangle}{\|\sin\left(\frac{n\pi y}{w}\right)\|^2}$$

Analogously $\rightarrow$ $\Psi(x,y) = \sum_{n\geq 1} \sin\left(\frac{n\pi x}{l}\right)\left[\alpha_n \cosh\left(\frac{n\pi y}{l}\right) + \beta_n \sinh\left(\frac{n\pi y}{l}\right)\right]$

$$\alpha_n = \frac{\langle \alpha, \sin\left(\frac{n\pi x}{l}\right) \rangle}{\|\sin\left(\frac{n\pi x}{l}\right)\|^2} \qquad \alpha_n \cosh\left(\frac{n\pi w}{l}\right) + \beta_n \sinh\left(\frac{n\pi w}{l}\right) = \frac{\langle \beta, \sin\left(\frac{n\pi x}{l}\right) \rangle}{\|\sin\left(\frac{n\pi x}{l}\right)\|^2}$$

PART 2

Now solve for v: $\boxed{\begin{cases} V_{tt} = \Delta V \text{ inside} \\ V = 0 \text{ on boundary} \\ V(x,y,0) = f(x,y) - S(x,y) \\ V_t(x,y,0) = g(x,y) \quad (S_t = 0) \end{cases}}$

$u = v + S \leftarrow$

Seperate variables in the pde

$$\frac{T''}{T} = \frac{\Delta(XY)}{XY} = \text{constant} = \lambda$$

$\xrightarrow{BC's}$ solve $\frac{\Delta(XY)}{XY} = \lambda$ first $\frac{X''Y + XY''}{XY} = \lambda \iff \frac{X''}{X} + \frac{Y''}{Y} = \lambda$

$\iff \frac{X''}{X} = \lambda - \frac{Y''}{Y} = \text{constant} = \mu \iff X'' = \mu X$ and $X(0) = 0 \quad X(l) = 0$

(Done previously) $\Rightarrow \sin\left(\frac{n\pi x}{l}\right) = X_n(x) \quad \mu_n = \frac{-n^2\pi^2}{l^2} \quad \frac{X_n''}{X_n} = \mu_n = \frac{-n^2\pi^2}{l^2} = \lambda - \frac{Y''}{Y}$

$\iff \frac{Y''}{Y} = \lambda + \frac{n^2\pi^2}{l^2} \qquad Y'' = \underbrace{\left(\lambda + \frac{n^2\pi^2}{l^2}\right) Y \text{ and } Y(0) = 0 \quad Y(w) = 0}$

(Done previously) $\Rightarrow Y_m(y) = \sin\left(\frac{m\pi y}{w}\right)$ and $Y_m'' = -\frac{m^2\pi^2}{w^2} Y_m(y) = \left(\lambda + \frac{n^2\pi^2}{l^2}\right)$

$\Rightarrow \lambda = -\frac{m^2\pi^2}{w^2} - \frac{n^2\pi^2}{l^2}$ Now find T functions

$\frac{T''}{T} = \lambda_{m,n} = \frac{-m^2\pi^2}{w^2} - \frac{n^2\pi^2}{l^2}$ Basis of solutions to this ODE is $\left\{ e^{\pm\sqrt{\lambda_{mn}}t}, e^{\pm i\sqrt{|\lambda_{m,n}|}t} \right\}$

$\Leftrightarrow \{ \cos(\sqrt{|\lambda_{m,n}|}t), \sin(\sqrt{|\lambda_{m,n}|}t) \}$ $T_{m,n}(t) = a_{m,n}\cos(\sqrt{|\lambda_{m,n}|}t) + b_{m,n}\sin(\sqrt{|\lambda_{m,n}|}t)$

$V(x,y,t) = \sum_{m\geq 1}\sum_{n\geq 1} T_{m,n}(t) X_n(x) Y_m(y)$ $V(x,y,0) = f(x,y) - S(x,y) = \sum_{m\geq 1}\sum_{n\geq 1} a_{m,n} X_n(x) Y_m(y)$

$a_{m,n} = \frac{\langle f-S, X_n Y_n \rangle}{\| X_n Y_n \|^2} = \frac{\int_0^l \int_0^w (f-S)\overline{X_n Y_n}\,dy\,dx}{\int_0^l \int_0^w |X_n Y_m|^2\,dx\,dy}$ $v_t(x,y,0) = \sum_{m\geq 1}\sum_{n\geq 1} b_{m,n}\sqrt{|\lambda_{m,n}|} X_n Y_m = g$

$\Rightarrow b_{m,n}\sqrt{|\lambda_{m,n}|} = \frac{\langle g, X_n Y_m \rangle}{\| X_n, Y_m \|^2}$

Observe: Frequencies are $\sqrt{\frac{m^2\pi^2}{w^2} + \frac{n^2\pi^2}{l^2}}$

Not just integer multiples of "ground tone"

vs. a string  ground tone is $\frac{\pi}{L}$

Higher frequencies : $\frac{k\pi}{L}$ $k \geq 1$

Usual drum

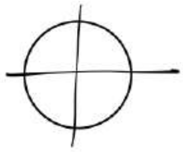
radius = 1

u = height of drumhead

boundary fixed at constant height = 0

Hit at the center

$$\begin{cases} u_{tt} = \Delta u & \text{inside the disk} \\ u = 0 & \text{on the boundary} \\ u(x,y,0) = f(x,y) & u_t(x,y,0) = g(x,y) \end{cases}$$

Boundary  $x^2+y^2=1$

OR $r=1$ $\Rightarrow$ use polar coordinates $\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}$

SV on the pde $\Rightarrow \frac{T''}{T} = \frac{\Delta(R\Theta)}{R\Theta} = \text{constant} = \lambda$ Solve for the $R\Theta$ part first

$$\frac{R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta''}{R\Theta} = \lambda \Leftrightarrow \left(\frac{R''}{R} + \frac{r^{-1}R'}{R} + \frac{R^{-2}\Theta''}{\Theta} = \lambda\right) * r^2$$

$$\frac{r^2R''}{R} + \frac{rR'}{R} - r^2\lambda = \overbrace{-\frac{\Theta''}{\Theta} = \text{constant}}^{\text{solve } \Theta \text{ first}}$$

Hit in the center $\Rightarrow f(x,y) = f(r)$ $g(x,y) = g(r)$ are rotationally symmetric (independent of θ)

$\Rightarrow$ Solution is also $\Rightarrow$ $\Theta = \text{constant} = 1$ $\Rightarrow \frac{\Theta''}{\Theta} = 0 \Rightarrow \left(\frac{r^2R''}{R} + \frac{rR}{R} - r^2\lambda = 0\right) * R$

$r^2 R'' + rR' - r^2 \lambda R = 0$ } Look this equation up

$\Rightarrow s = \sqrt{|\lambda|}\, r \Rightarrow F(s) = R(\sqrt{|\lambda|}\, r) \Rightarrow$ modified Bessel eqn. of order 0 if $\lambda > 0$

Bessel eqn of order 0 if $\lambda < 0$

$\underline{\lambda > 0} \Rightarrow$ Mod. Bessel eqn Basis of solutions I_0 & K_0 → Blows up @ $r=0$

never 0

BC require solution to be 0 when $r=1$ and not $\pm\infty$ at $r=0$

$\underline{\lambda < 0} \Rightarrow$ Bessel eqn. Basis of solutions is J_0 and W_0 ← Weber Bessel fcn (also called Y.)

Bessel order 0

$\Rightarrow R(r) = J_0(\sqrt{|\lambda|}\, r)$

$W_0 \to \infty$ @ zero

Need $R(1) = 0 \Rightarrow$ Need $J_0(\sqrt{|\lambda|}) = 0$

Theorem: $J_0(s) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{s}{2}\right)^{2n}}{(n!)^2}$ This has ∞ many positive zeros: $0 < \pi_1 < \pi_2 <$ π_n grows on the order n

$\{J_0(\pi_n r)\}_{n \geq 1}$ are an OB for $\mathcal{L}^2_r(0,1)$

This $R_n(r) = J_0(\underbrace{\pi_n}_{\sqrt{|\lambda|}\ \lambda<0} r)$ with $\lambda_n = -\pi^2 \Rightarrow \frac{T_n''}{T_n} = \lambda_n = -\pi_n^2 \Rightarrow T_n(t) = a_n \cos(\pi_n t) + b_n \sin(\pi_n t)$

$u(r,t) = \sum_{n \geq 1} T_n(t) R_n(r)$ IC $\Rightarrow a_n = \frac{\langle f, R_n \rangle}{\|R_n\|^2}$ with $\langle f, R_n \rangle = \int_0^1 f(r) \overline{R_n(r)}\, r\, dr$

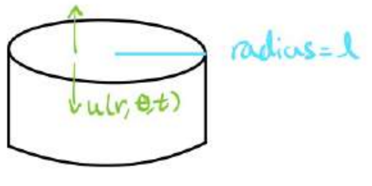
$\|R_n\|^2 = \int_0^1 |R_n(r)|^2 r\, dr$

$b_n \pi_n = \frac{\langle g, R_n \rangle}{\|R_n\|^2}$

18/02-25

1. Bessel functions
2. Generating function theorem + proof

Bessel functions (Drums and food)



$$\begin{cases} u_{tt} = \Delta u = u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta} \\ u(l, \theta, t) = 0 \quad \text{(Dirichlet BC)} \\ u(r, \theta, 0) = f(r, \theta) \quad u_t(r, \theta, 0) = g(r, \theta) \end{cases}$$

Separate variables and solve!

$$T''\Theta R = R''T\Theta + r^{-1}R'T\Theta + r^{-2}RT\Theta'' \Rightarrow \frac{T''}{T} = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} = \text{constant} = \lambda$$

BC's help us to see what λ needs to be $\Rightarrow R$ and Θ first

Θ: $\frac{r^2R''}{R} + \frac{rR'}{R} - r^2\lambda = -\frac{\Theta''}{\Theta} = \text{constant}$ BC for Θ are periodic because it's a disk

$\Rightarrow \Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$ (Look at day #2, Chapter 1 "Rings of saturn")

$\Theta_n(\theta) = e^{in\theta}$ for $n \in \mathbb{Z}$ Plug in $-\frac{\Theta_n''}{\Theta_n} = -(in)^2\frac{\Theta_n}{\Theta_n} = n^2$ put in R equ

n is the order

$\left(\frac{r^2R''}{R} + \frac{r^{-1}R'}{R} - r^2\lambda = n^2\right) * R \Rightarrow (★)\ r^2R'' + rR' \boxed{-r^2\lambda R} - \boxed{n^2}R = 0$ Look it up!

if $(-r^2\lambda)$ is negative => Modified B equ

if $(-r^2\lambda)$ is positive => B equ

This is (up to variable substitution) a Bessel eqn.

DLMF = Digital Library of Mathematical Functions (alt. Watson Treatise on the Theory of B. fcn)

References => Solutions to modified eqn. are linear combos of I_n and K_n ← blows up @ r=0

=> These do not give any physically feasible solutions that satisfy the BC ↑ not zero @ r=l

=> Must have $-r^2\lambda \geq 0 \Rightarrow \lambda \leq 0$

$\underline{\lambda = 0}$: $r^2R'' + rR' - n^2R = 0$ Look it up!

⇝ "Euler equation" A basis of solutions is $\{r^n, r^{-n}\}$ if $n \neq 0$, $\{1, \log(r)\}$ if $n = 0$

Will not vanish @ r=l

cannot be part of solution since $\to \infty$ as $r \to 0$

goes to $-\infty$ as $r \to 0$

does not vanish @ r=l

$\lambda < 0$: Let $s = r\sqrt{|\lambda|}$ then the eqn. for R becomes

♡ $s^2F''(s) + sF'(s) + (s^2 - n^2)F(s) = 0$ with $R(r) = F(r\sqrt{|\lambda|}) = F(s)$

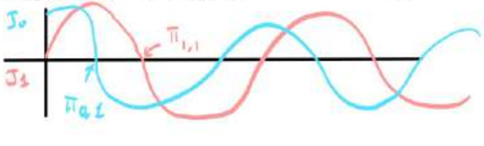
<=> R satisfies (★)

Theorem: A basis of solutions♡ are $J_n(s)$ and $Y_n(s)$

Bessel of order n≥0 Weber Bessel fcn.

$$J_n(s) = \sum_{k \geq 0} \frac{(-1)^k \left(\frac{s}{2}\right)^{2k+n}}{\Gamma(k+n+1) \cdot k!} \quad \cdot \quad |Y_n(s)| \xrightarrow[s \to 0]{} \infty \qquad \Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt \quad \begin{array}{l} Re(s) > 0 \\ \Gamma(1) = 1 \end{array}$$

$J_{-n}(s) = (-1)^n J_n(s)$ $J_n(s)$ has ∞ many positive zeros



$0 < \pi_{n,1} < \pi_{n,2} < \pi_{n,3} < \ldots \to \infty$

$\left\{J_n\left(\frac{\pi_{n,k}r}{l}\right)\right\}_{k \geq 1}$ is an OB for $\mathcal{L}^2_r(0, l)$

This $R_{n,k}(r) = J_n\left(\frac{\pi_{n,k} r}{\ell}\right)$ satisfies ★ and $R_{n,k}(\ell) = J_n(\pi_{n,k}) = 0$ for $n \geq 0$ $\sqrt{|\lambda|} = \frac{\pi_{n,k}}{\ell}$

$$\Rightarrow \lambda_{n,k} = \frac{-\pi^2_{n,k}}{\ell^2} = \frac{T''}{T} \quad \Rightarrow T_{n,k}(t) = a_{n,k}\cos\left(\frac{\pi_{n,k}}{\ell}\cdot t\right) + b_{n,k}\sin\left(\frac{\pi_{n,k}}{\ell}\cdot t\right)$$

$$u(r,\theta,t) = \sum_{k\geq1}\sum_{n\geq1} J_{n,k}\left(\frac{\pi_{n,k}\cdot r}{\ell}\right)\cdot\left(e^{in\theta} + e^{-in\theta}\right)T_{n,k}(t) + \sum_{k\geq1} J_0\left(\frac{\pi_{n,k} r}{\ell}\right)$$

The $a_{n,k}$ coefficients come from $u(r,\theta,0) = f(r,\theta)$

The $a_{n,k}$ coefficients are thus $\frac{\langle f, \theta_n R_{n,k}\rangle}{\|\theta_n R_{n,k}\|^2} = \frac{\int_0^\ell \int_{-\pi}^{\pi} f(r,\theta)\cdot\overline{\theta_n(\theta)R_{n,k}(r)}\cdot r\, dr\, d\theta}{\int_0^\ell \int_{-\pi}^{\pi} |\theta_n(\theta)R_{n,k}(r)|^2 \cdot r\, d\theta\, dr} = a_{n,k}$

$u_t(r,\theta,0) = g(r,\theta)$ $\qquad b_{n,k} = \underset{\text{Derivative}}{\boxed{\frac{\ell}{\pi_{n,k}}}}\frac{\langle g, \theta_n R_{n,k}\rangle}{\|\theta_n R_{n,k}\|^2} = \frac{\ell}{\pi_{n,k}}\frac{\int_0^\ell \int_{-\pi}^{\pi} g(r,\theta)\cdot\overline{\theta_n(\theta)R_{n,k}(r)}\cdot r\, dr\, d\theta}{\int_0^\ell \int_{-\pi}^{\pi} |\theta_n(\theta)R_{n,k}(r)|^2\cdot r\, d\theta\, dr}$

How were the Bessel functions discovered?

① To solve the ODE

② "generating function" $\Rightarrow e^{ix} = \cos(x) + i\sin(x)$ for $x \in \mathbb{R}$

Similarly $\underbrace{e^{\frac{x}{2}(z-\frac{1}{z})}}_{\text{"generating function"}} = \sum_{n\in\mathbb{Z}} J_n(x) z^n$ for $z \neq 0$

Theorem (★): Generating function

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{n\in\mathbb{Z}} J_n(x) z^n \leftarrow \sum_{n\in\mathbb{Z}}\sum_{k\geq0} \frac{(-1)^k\left(\frac{x}{2}\right)^{2k+n}}{\Gamma(n+k+1)k!} z^n$$

Proof★: $e^{\frac{x}{2}z} e^{-\frac{x}{2z}} = \sum_{k\geq0}\left(\frac{xz}{2}\right)^k \frac{1}{k!}\sum_{j\geq0}\left(-\frac{x}{2z}\right)^j\cdot\frac{1}{j!}$

Taylor expand

$$\sum_{k\geq0}\sum_{j\geq0}\frac{\left(\frac{xz}{2}\right)^k\left(\frac{-x}{2z}\right)^j}{k!\, j!} = \sum_{k\geq0}\sum_{j\geq0}\frac{(-1)^j\left(\frac{x}{2}\right)^{k+j} z^{k-j}}{k!\, j!}$$

Change from k & j to $n = k-j$ and j $\quad \Rightarrow e^{\frac{xz}{2} - \frac{x}{2z}} = \sum_{n\in\mathbb{Z}} \underbrace{\sum_{j\geq0}\frac{(-1)^j\left(\frac{x}{2}\right)^{n+2j}}{j!\,(n+j)!}}_{\text{This is the defn of } J_n(x)} z^n$

$k = n+j$

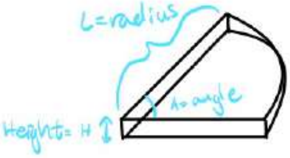
(Recall $(n+j)! = \Gamma(n+j+1)$)

21/02-25

Two examples that use Bessel

1. Pizza slice
2. Cake

Example of pizza slice



Heat (temperature) @ r, θ, z, t satisfies

$$\begin{cases} u_t = \Delta u & \text{inside } 0 < t \\ \boxed{u = 20 \quad \text{on boundary}} \\ u(r, \theta, z, 0) = 200 \end{cases}$$

Solve for $u(r, \theta, z, t)$. Do this first.

Look for S that only depends on (r, θ, z) with $\Delta S = 0$ and $S = 20$ on boundary

$S = 20$ does the job. Now solve for $V = u - 20 \Rightarrow \begin{cases} V_t = \Delta V \\ V = 0 \text{ on boundary} \\ V(r, \theta, z, 0) = 180 \end{cases}$

Seperate variables in the pde:

$$T' R \Theta Z = T R'' \Theta Z + r^{-1} T R' \Theta Z + r^{-2} T R \Theta'' Z + T R \Theta Z''$$

$$\Leftrightarrow \frac{T'}{T} = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} \Rightarrow \text{Both sides constant!}$$

$$\frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = \lambda$$

Solve for Θ or z first bcs they are simplest

T last

$$-\frac{Z''}{Z} = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} - \lambda = \text{constant}$$

(Deja vu!) hwp 1

And BC is $Z(0) = 0$, $Z(H) = 0$ $\Rightarrow$ $Z_n(z) = \sin\left(\frac{n z \pi}{H}\right)$ $n \geq 1$

Put in

$$\boxed{\frac{Z_n''}{Z_n} = -\frac{n^2\pi^2}{H^2}} \quad -\left(-\frac{n^2\pi^2}{H^2}\right) = \frac{R''}{R} + r^{-1}\frac{R'}{R} + r^{-2}\frac{\Theta''}{\Theta} - \lambda \Leftrightarrow \frac{r^2 R''}{R} + r\frac{R'}{R} - \lambda r^2 - \frac{n^2\pi^2}{H^2} r^2 = -\frac{\Theta''}{\Theta}$$

Put in

$\Theta(0) = 0$ $\Theta(A) = 0$ $\Rightarrow$ Deja vu! $\Rightarrow$ $\Theta_m(\theta) = \sin\left(\frac{m\pi\theta}{A}\right)$ and $\boxed{\frac{\Theta_m''}{\Theta_m} = -\frac{m^2\pi^2}{A^2}}$ $m \geq 1$

$$\Rightarrow \frac{r^2 R''}{R} + \frac{r R'}{R} - \lambda r^2 - \frac{n^2\pi^2}{H^2} r^2 = -\left(-\frac{m^2\pi^2}{A^2}\right) = \frac{m^2\pi^2}{A^2}$$

$$\Rightarrow r^2 R'' + r R' - \lambda r^2 R - \frac{n^2\pi^2}{H^2} r^2 R - \frac{m^2\pi^2}{A^2} R = 0$$

$$\Rightarrow r^2 R'' + r R' - \left(\frac{m^2\pi^2}{A^2} + \left(\lambda + \frac{n^2\pi^2}{H^2}\right) r^2\right) R = 0$$

Also $R(L) = 0$ Set $\Lambda = \left(\lambda + \frac{n^2\pi^2}{H^2}\right)$

$$\Rightarrow r^2 R'' + r R' - \left(\frac{m^2\pi^2}{A^2} + \Lambda r^2\right) R = 0$$

$s^2 f''(s) + s f'(s) - \left(\frac{m^2\pi^2}{A^2} + s^2\right) f(s) = 0$

If $R(r) = f(s)$ with $s = r\sqrt{|\Lambda|}$ then

$\Rightarrow$ Bessel equ for $f(s)$ with order $\frac{m\pi}{A}$. If Λ is positive $\Rightarrow$ modified BE

Solutions are $I_{\frac{m\pi}{A}}$ och $K_{\frac{m\pi}{A}}$ I is never zero and $K \to \infty$ @ 0

K is not physically feasible and I won't be zero @ $r=L$

If $\Lambda = 0 \Rightarrow$ Euler eqn $\Rightarrow r^{\pm\frac{m\pi}{A}}$ $r^{-\frac{m\pi}{A}} \xrightarrow[r\to 0]{} \infty$ $r^{\frac{m\pi}{A}}\Big|_{r=L} \neq 0$

$\Rightarrow \Lambda < 0$ and it's the Bessel eqn. Solutions are $J_{\frac{m\pi}{A}}$ and $Y_{\frac{m\pi}{A}}$ } Blows up@0

$\Rightarrow$ Only $J_{\frac{m\pi}{A}}(r\sqrt{\Lambda})$ BC $\Rightarrow J_{\frac{m\pi}{A}}(L\sqrt{|\Lambda|})$

$\Rightarrow$ Need $L\sqrt{|\Lambda|}$ to be a positive zero of $J_{\frac{m\pi}{A}}$ There are ∞ many positive zeros of $J_{\frac{m\pi}{A}}$

Enumerate as $0 < \pi_{m,1} \leq \pi_{m,2} < \dots$ k^{th} is $\pi_{m,k} \Rightarrow J_{\frac{m\pi}{A}}\left(\frac{r\,\pi_{m,n}}{L}\right) = R_{m,k}(r)$

Finally need T. $L\sqrt{|\Lambda|} = \pi_{m,k} \Rightarrow \sqrt{|\Lambda|} = \frac{\pi_{m,k}}{L} \underset{\Lambda<0}{\Rightarrow} \Lambda = \frac{-\pi^2_{m,k}}{L^2}$

and $\lambda = \Lambda - \frac{n^2\pi^2}{H^2} = -\frac{\pi^2_{m,k}}{L^2} - \frac{n^2\pi^2}{H^2} = \lambda_{n,m,k}$

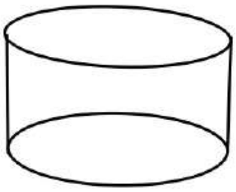
$\frac{T'}{T} = \lambda_{n,m,k} \Rightarrow T = C_{n,m,k}\, e^{\lambda_{m,n,k} t}$

$$v(r,\theta,z,t) = \sum_{n\geq 1}\sum_{m\geq 1}\sum_{k\geq 1} T_{n,m,k}(t)\, Z_n(z)\, \theta_m(\theta)\, R_{m,k}(r)$$

$$v(r,\theta,z,0) = 180 = \sum_{n,m,k\geq 1} T_{m,n,k}(0)\, Z_n(z)\, \theta_m(\theta)\, R_{m,k}(r)$$

$$\Rightarrow T_{n,m,k}(0) = C_{n,m,k} = \frac{\langle 180, z_n\theta_m R_{n,k}\rangle}{\|z_n\theta_m R_{n,k}\|^2} = \frac{\int_0^1\int_0^H\int_0^A 180\, \overline{z_n(z)\,\theta_m(\theta)\,R_{n,k}(r)}\, r\,dr\,d\theta\,dz}{\int_0^1\int_0^H\int_0^A |z_n\theta_m R_{n,k}|^2\, r\, dr\, d\theta\, dz}$$

Cake example



Baked and put it in insulated container

$\Rightarrow \begin{cases} u_t = \Delta u \\ \boxed{\text{Normal derivative of } u = 0 \text{ @ boundary}} \\ u(r,\theta,z,0) = f(r,\theta,z) \end{cases}$ → Neumann Nice! go to SV!

$\frac{T''}{T} = \frac{\Delta(R\theta Z)}{R\theta Z} = \text{constant} = \lambda$ $\overset{\text{Fast forward}}{\Longrightarrow\Rightarrow}$ $\frac{Z''}{Z} = \text{constant}$ $Z'(0) = 0$ $Z'(H) = 0$ $Z_n(z) = \cos\left(\frac{n\pi z}{H}\right)$ $n \geq 0$

$\overset{\text{Fast forward}}{\Rightarrow\Rightarrow}$ $\frac{\theta''}{\theta} = \text{constant}$ Whole cake $\Rightarrow$ periodic BC $\overset{\text{Kap 1}}{\Rightarrow}$ $\theta_m(\theta) = e^{im\theta}$ $m \in \mathbb{Z}$

$\frac{T'}{T} = \lambda = \frac{\Delta(R\theta Z)}{R\theta Z}$ $\overset{\text{algebra}}{\Longrightarrow\Rightarrow}$ $r^2R'' + rR' - \left(r^2\left(\frac{n^2\pi^2}{H^2} + \lambda\right) + m^2\right)R = 0$

n & $\lambda = 0 \Rightarrow$ constant func solves the ODE & BC: $R'(L) = 0$ & $m \Rightarrow R_0(r) = 1$ and $\lambda_0 = 0$

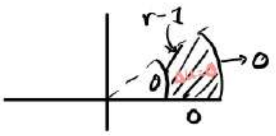
otherwise similar to last problem let $\Lambda = \lambda + \frac{n^2\pi^2}{H^2}$ $\Lambda < 0$ so that we get Bessel eqn

$\Rightarrow R_{m,k}(r\sqrt{|\Lambda|}) = J_m\left(\frac{r\,\pi'_{m,k}}{L}\right)$ with $\pi'_{m,k}$ the k^{th}

24/02-25

1. Example #3
2. French guy polynomials

#3 (Example) $\begin{cases} \Delta u = 0 & 0 < \theta < \frac{\pi}{4} \quad 1 < r < 2 \\ u(1,\theta) = 0 & u_r(2,\theta) = 0 \\ u(r,0) = 0 & \underbrace{u(r,\frac{\pi}{4}) = r-1}_{\theta \text{ will be "time like"}} \end{cases}$



Separate variables: $R''\Theta + r^{-1}R'\Theta + r^{-2}\Theta''R = 0 \quad \frac{\ast r^2}{R\Theta} \Rightarrow \frac{r^2R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = 0$

$\Leftrightarrow \frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \text{constant} = \lambda$

BCs $\Rightarrow$ Solve for R first $\quad \frac{r^2R''}{R} + \frac{rR'}{R} = \lambda \Leftrightarrow r^2R'' + rR' - \lambda R = 0$ Euler

$\underline{\lambda = 0}$ Basis of solutions is $\{1, \log(r)\}$ $\quad R(r) = a + b\log(r)$

@ $r=1$: $a=0$ $\quad$ @ $r=2$: $b\log(2) = 0 \Rightarrow b = 0 \quad \Rightarrow$ No nonzero solutions

$\underline{\lambda \neq 0}$: Basis of solutions is $\{r^{\pm\sqrt{\lambda}}\}$ $\quad ar^{\sqrt{\lambda}} + br^{-\sqrt{\lambda}}\Big|_{r=1} = 0 \Rightarrow a+b=0 \quad b=-a$

$\Rightarrow a(r^{\sqrt{\lambda}} - r^{-\sqrt{\lambda}})'\Big|_{r=2} = 0 \Rightarrow a(\sqrt{\lambda}\,2^{\sqrt{\lambda}-1} - (-\sqrt{\lambda})2^{-\sqrt{\lambda}-1}) = 0$

Assume $a \neq 0$: $2^{\sqrt{\lambda}-1} + 2^{-\sqrt{\lambda}-1} = 0 \quad 2^{\sqrt{\lambda}} = -2^{-\sqrt{\lambda}} \Leftrightarrow 2^{2\sqrt{\lambda}} = -1 = e^{i\pi(2k+1)} \Rightarrow$

$e^{\log(2)\cdot 2\sqrt{\lambda}} = e^{i\pi(2k+1)} \Rightarrow 2\log(2)\sqrt{\lambda} = i\pi(2k+1) \Rightarrow \sqrt{\lambda} = \frac{i\pi(2k+1)}{2\log(2)} \quad \lambda_k = \frac{-\pi^2(2k+1)^2}{4\log(2)^2}$

Up to constant factor $R_k(r) = r^{\frac{i\pi(2k+1)}{2\log(2)}} - r^{-\frac{i\pi(2k+1)}{2\log(2)}}$

$R_k(r) = \exp\left(\frac{i(2k+1)\pi \cdot \log(r)}{2\log(2)}\right) - \exp\left(\frac{-i(2k+1)\pi\log(r)}{2\log(2)}\right) \overset{\text{Eulers formula}}{=} 2i\sin\left(\frac{(2k+1)\pi\log(r)}{2\log(2)}\right)$

$R_k(r) = \sin\left(\frac{(2k+1)\pi\cdot\log(r)}{2\log(2)}\right)$

$-\frac{\Theta''}{\Theta} = \lambda_k = \frac{-\pi(2k+1)^2}{4\log(2)^2} \Rightarrow$ A basis of solutions is $\left\{e^{\frac{\pi(2k+1)\theta}{2\log(2)}}, e^{-\frac{\pi(2k+1)\theta}{2\log(2)}}\right\}$

$\underline{\text{OR}}$ $\left\{\sinh\left(\frac{\pi(2k+1)\theta}{2\log(2)}\right), \cosh\left(\frac{\pi(2k+1)\theta}{2\log(2)}\right)\right\}$

Condition @ $\theta = 0 \Rightarrow$ Use sinh and ~~cosh~~ $\quad \Theta(0) = 0$

$\Theta_k(\theta) = a_k\sinh\left(\frac{\pi(2k+1)\theta}{2\log(2)}\right) \qquad u(r,\theta) = \sum_{k\geq 0}\Theta_k(\theta)R_k(r)$

Need $u(r,\frac{\pi}{4}) = \sum_{k\geq 0}\Theta_k(\frac{\pi}{4})R_k(r) = r-1 \Rightarrow \sum_{k\geq 0}\underbrace{a_k\sinh\left(\frac{\pi^2(2k+1)}{8\log(2)}\right)}_{\text{These should be } \frac{\langle r-1, R_k\rangle}{\|R_k\|^2}}R_k(r) = r-1$

OBS: $r^2R'' + rR' - \lambda R = 0$ is $\underline{\text{not}}$ an SLP

But $rR'' + R' - r^{-1}\lambda R = 0$ $\underline{\text{is an SLP}}$ weight $w(r) = r^{-1}$ $\quad \underline{\text{if}}$ $\{R_k\}_{k\geq 0}$ are an OB

$\Rightarrow \langle r-1, R_k\rangle = \int_1^2 (r-1)\overline{R_k(r)}\,r^{-1}dr$

$\|R_k\|^2 = \int_1^2 |R_k(r)|^2 r^{-1}dr$

Chapter 7 *Orthogonal polynomials are special cases of orthogonal bases for Hilbert spaces!*

Legendre - problems in spheres
Hermite - quantum harmonic oscillator
Laguerre - radial coords (spherical)

Orthogonal polynomial expansions

Assume $-\infty < a < b < \infty$ $\quad \int_a^b |f(x)|^2 dx < \infty$

Then $f(x) = \sum_{n \geq 0} \frac{\langle f, \mathcal{P}_n \rangle}{\|\mathcal{P}_n\|^2} \mathcal{P}_n(x)$ for $\mathcal{P}_n$ polynomial of degree n with $\mathcal{P}_n \perp_{\mathcal{L}^2} \mathcal{P}_m$ for $n \neq m$

Moreover if $q(x) = \sum_{n=0}^{N} \frac{\langle f, \mathcal{P}_n \rangle}{\|\mathcal{P}_n\|^2} \mathcal{P}_n(x)$ is the best polynomial approximation of f by polynomials of degree $\leq N$

i.e $\int_a^b |f(x) - p(x)|^2 dx \geq \int_a^b |f(x) - q(x)|^2 dx \quad \forall p(x)$ of degree $\leq N$ and equality holds if $p = q$

Goal #1: Orthogonal polynomial bases & best polynomial approximation in $\mathcal{L}^2(a,b)$

Proposition 14: Assume $\{p_n\}_{n \geq 0}$ are polynomials with $\deg(p_n) = n$ $p_0 \neq 0$

Then for each $k \in \mathbb{N}_0$, any polynomial of degree k is a linear combo of $p_0, p_1, \dots, p_k$

Proposition 15: Assume the same as in Prop 14 and that $p_n \perp p_m$ in $\mathcal{L}^2(a,b)$ for $n \neq m$

Then $\{p_n\}$ are an OB for $\mathcal{L}^2(a,b)$

One can build OP's on $\mathcal{L}^2(a,b)$ "from scratch", BUT much more efficient to outsource to Legendre

Def: The n^{th} Legendre polynomial is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left((x^2-1)^n\right) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(\sum_{k=0}^{n} (-1)^{n-k} (x^2)^k \binom{n}{k}\right)$$

Prop 16: $P_0(x) = 1 \qquad P_n(x) = \frac{1}{2^n n!} \sum_{k \geq \frac{n}{2}}^{k \leq n} (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k-j)$

Theorem: The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$ and $\|P_n\|^2 = \frac{2}{2n+1}$

Proof: Assume $n \neq m$ and $n > m$ $\langle P_n, P_m \rangle = \int_{-1}^{1} P_n(x) \overline{P_m(x)} dx = \int_{-1}^{1} \frac{1}{2^n n!} \frac{d^n}{dx^n}\left((x^2-1)^n\right) P_m(x) dx$

$$\underset{\text{P.I.}}{=} (-1)^{m+1} \int_{-1}^{1} \frac{1}{2^n n!} \frac{d^{n-m-1}}{dx} (x^2-1)^n \underbrace{\cancel{\frac{d^{m+1}}{dx^{m+1}} P_m(x)}}_{0} dx = 0$$

For any $j < n$ $\left.\frac{d^j}{dx^j}\left((x^2-)^n\right)\right|_{x=-1}^{1} = 0$

m+1 times
show that
boundary terms = 0

25-02-25

1. Legendre polynomials
2. Hermite polynomials
   Two theorems + proofs

Legendre Polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)$$

Theorem: The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$ $\quad \|P_n\|^2 = \frac{2}{2n+1}$

Cor: The Legendre polynomials are an OB for $\mathcal{L}^2(-1,1)$

Thus any $f \in \mathcal{L}^2(-1,1)$ is equal to $\sum_{n\geq 0} \frac{\langle f, P_n\rangle}{\|P_n\|^2} P_n(x)$

We can transplant $P_n(x)$ to (a,b) to obtain an OB on $\mathcal{L}^2(a,b)$

Theorem 23: Let (a,b) be a bounded interval.

Then $\mathcal{P}_n(x) = P_n\left(\frac{x-m}{l}\right)$ $\quad m = \frac{a+b}{2}$ $\quad l = \frac{b-a}{2}$ are an OB for $\mathcal{L}^2(a,b)$

Cor: Let $f \in \mathcal{L}^2(a,b)$ Then for each $N \geq 0$ the unique polynomial of degree $\leq N$ that minimizes $\|f-q\|^2 = \int_a^b |f(x)-q(x)|^2 dx$ over all polynomials $q(x)$ is $\sum_{n=0}^{N} \frac{\langle f, \mathcal{P}_n\rangle}{\|\mathcal{P}_n\|^2}\mathcal{P}_n(x)$

To compute best polynomial approximations in $\mathcal{L}^2(a,b)$:

① Use theorem 23 & write the def. of
② Use corollary

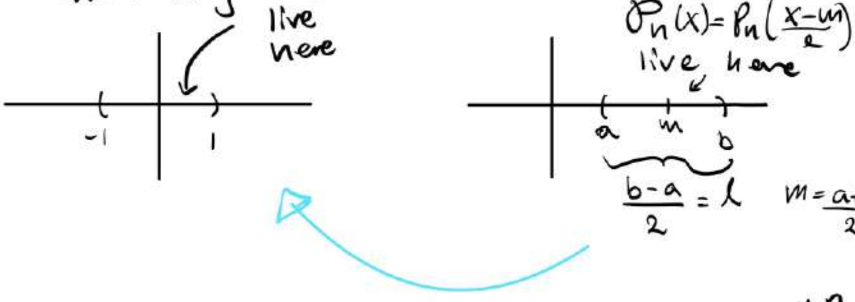
① $\mathcal{P}_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)$

② $\|\mathcal{P}_n\|^2 = \int_a^b |\mathcal{P}_n(x)|^2 dx = \int_a^b \mathcal{P}_n(x)\overline{\mathcal{P}_n(x)}dx$

③ $\langle f, \mathcal{P}_n\rangle = \int_a^b f(x)\overline{\mathcal{P}_n(x)}dx$

$P_n(x)$ = Legendre

$\mathcal{P}_n(x) = P_n\left(\frac{x-m}{l}\right) = P_n\left(\frac{x-\frac{a+b}{2}}{\frac{b-a}{2}}\right)$



$\frac{b-a}{2} = l \quad m = \frac{a+b}{2}$

$$\|P_n\|^2 = \int_{-1}^{1} |P_n(x)|^2 dx = \frac{2}{2n+1}$$

$$P_n\left(\frac{x-\frac{-1+1}{2}}{\frac{1--1}{2}}\right) = P_n(x)$$

Hermite Polynomials

Def: $\mathcal{L}^1(\mathbb{R}) = \{ f \text{ (measurable) such that } \int_{\mathbb{R}} |f(x)|\,dx < \infty \}/\sim$

$\mathcal{L}^2(\mathbb{R}) = \{ f \text{ (measurable) such that } \int_{\mathbb{R}} |f(x)|^2\,dx < \infty \}/\sim$

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx \qquad \|f\|^2 = \int_{\mathbb{R}} |f(x)|^2\,dx$$

In $\mathcal{L}^1(\mathbb{R})$ we have e^{-x^2}, also 0. These are also in $\mathcal{L}^2(\mathbb{R})$.

$\frac{1}{x}$ is in neither (0 is a problem) BUT $f(x) = \begin{cases} 0 & -1<x<1 \\ \frac{1}{x} & -\infty<x<-1 \\ & 1<x<\infty \end{cases}$ is in $\mathcal{L}^2(\mathbb{R})$, not in $\mathcal{L}^1(\mathbb{R})$

$g(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{for } 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$ is in $\mathcal{L}^1(\mathbb{R})$ but not $\mathcal{L}^2(\mathbb{R})$

$\mathcal{L}^2_{w(x)}(\mathbb{R}) = \{ f \text{ (measurable)} : \int_{\mathbb{R}} |f(x)|^2 w(x)\,dx < \infty \}$ Example: Set $w(x) = e^{-x^2}$

w(x) is real

Def: $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$ It is a polynomial of degree n

Theorem:★ $H_n(x)$ are orthogonal in $\mathcal{L}^2_{w(x)}(\mathbb{R})$ $w(x) = e^{-x^2}$

Proof:★ Assume $n>m$. Compute $\int_{\mathbb{R}} H_n(x)\overline{H_m(x)}\,e^{-x^2}\,dx = \int_{\mathbb{R}} (-1)^n \cancel{e^{x^2}} \frac{d^n}{dx^n}\left(e^{-x^2}\right)\overline{H_m(x)}\,\cancel{e^{-x^2}}\,dx =$

write it's def

$$= (-1)^n \int_{\mathbb{R}} \frac{d^n}{dx^n}\left(e^{-x^2}\right) H_m(x)\,dx \underset{P.I}{=} (-1)^n \left.\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right) H_m(x)\right|_{x\to-\infty}^{x\to\infty} - (-1)^n \int_{\mathbb{R}} \frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right) H_m'(x)\,dx$$

$\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right) = \text{polynomial} * e^{-x^2} \Rightarrow$

This is a polynomial $* \, e^{-x^2} \longrightarrow 0$

So fast it kills polynomial

Repeat this to get:

$$(-1)^{m+1} \int_{\mathbb{R}} \frac{d^{n-m-1}}{dx^{n-m-1}}\left(e^{-x^2}\right) \overset{0}{\cancel{\frac{d^{m+1}}{dx^{m+1}}\left(H_m(x)\right)}}\,dx = 0 \qquad \square$$

Theorem: $\{H_n\}$ are an OB for $\mathcal{L}^2_{w(x)}(\mathbb{R})$ with $w(x) = e^{-x^2}$

Thus any $f \in \mathcal{L}^2_w(\mathbb{R})$ is equal to $\sum_{n\geq 0} \frac{\langle f, H_n \rangle}{\|H_n\|^2} H_n(x)$

$\langle f, H_n \rangle = \int_{\mathbb{R}} f(x)\overline{H_n(x)}\,e^{-x^2}\,dx$

$\|H_n\|^2 = \int_{\mathbb{R}} |H_n(x)|^2 e^{-x^2}\,dx$

Cor: $\mathcal{H}_n(x) = H_n(\sqrt{a}\,x)$ are an OB for $\mathcal{L}^2_{e^{ax^2}}(\mathbb{R})$ for $a>0$.

Theorem: (Generating Hermite Polynomials) *

$$\sum_{n\ge 0} \underbrace{H_n(x)}_{\text{Put in its def!}} \frac{z^n}{n!} = e^{2xz-z^2}$$

Proof: * The left side is $\sum_{n\ge 0} (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{z^n}{n!} = e^{2xz-z^2}$

$$\sum_{n\ge 0} (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{z^n}{n!} = e^{-x^2+2xz-z^2} = e^{-(x-z)^2} = e^{-(z-x)^2}$$

The Taylor series of $e^{-(z-x)^2}$ @ $z=0$ is $\sum_{n\ge 0} \frac{z^n}{n!}\frac{d^n}{dz^n}\left(e^{-(z-x)^2}\right)\Big|_{z=0} = \sum_{n\ge 0}(-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{z^n}{n!}$

□

Laguerre polynomials

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(x^{\alpha+n}e^{-x}\right)$$

$\{L_n^\alpha\}_{n\ge 0}$ are an OB for $\mathcal{L}^2_{x^\alpha e^{-x}}(0,\infty)$

(α could be 0, $\alpha > -1$)

28/02-25

1. Mini-Review of course so far
2. Fourier Transform

Mini-Review

1. Solving pde, BC, IC $\begin{cases} u_t - u_{xx} = 0 \\ u(0,t) = 0 \quad u_x(l,t) = 0 \\ u(x,0) = f(x) \end{cases}$

① Seperate variables: $T'X = TX'' \Rightarrow \frac{T'}{T} = \frac{X''}{X} = \text{constant} = \lambda$

② Solve for X first using BC

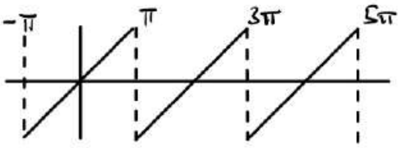
③ Solve for T

④ Superposition $\rightarrow \sum X_n(x) T_n(t)$

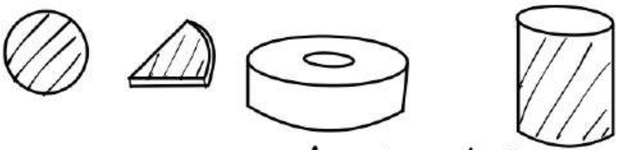
⑤ $T_n(0) = \frac{\langle f, X_n \rangle}{\|X_n\|^2} = \frac{\int_0^l f(x)\overline{X_n(x)}\,dx}{\int_0^l |X_n(x)|^2\,dx}$ IC

2. What kind of problem is $X'' + \lambda X = 0 \quad X(0) = 0 \quad X'(l) = 0$? An SLP!
So the solutions are an orthogonal base

3. How can we compute something like $\sum_{n \geq 1} \frac{1}{n^4}$?
Use a table of functions and their tFΣ
3.1 tF-C@T evaluate at a point Ex $f(x) = x$
tFΣ converges to 2π periodic extension of f
and average @ jump point



3.2 Parseval's method: $\|f\|^2 = \int_{\pi}^{\pi} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |c_n|^2 \|e^{inx}\|^2 = \sum_{n \in \mathbb{Z}} |c_n|^2 (2\pi)$

4. Solving problems in 

Use cylindrical coordinates! Be prepared for Bessel functions

5. Best polynomial approx. on (a,b) - use Legendre
on $\mathbb{R}$ w e^{-x^2} weight - use Hermite
on $(0,\infty)$ with $x^{\alpha} e^{-x}$ weight - use Laguerre

Chapter 8 Fourier Transform

Def: If it makes sense the Fourier transform of function f @ frequency ξ is $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$ (also $\mathcal{F}(f)(\xi)$)

Assume $f \in \mathcal{L}^1(\mathbb{R})$ Then $\hat{f}(\xi)$ is in $\mathbb{C}$ ($\lim_{R,M\to\infty} \int_{-M}^{R} f(x)e^{-ix\xi}\,dx$ exists)

If $f \in \mathcal{L}^1(\mathbb{R})$ then $\left|\int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx\right| \leq \int_{\mathbb{R}} |f(x)|\,dx < \infty$ finite

For example $f(x) = \frac{1}{x^2+1}$ $\hat{f}(\xi) = \int_{\mathbb{R}} \frac{1}{x^2+1} e^{-ix\xi}\,dx$

To compute this use Residue Theorem OR NAH

Use tables whenever possible!

Table 8.1

Function	Fourier transform
$(x^2+a^2)^{-1}$	$\frac{\pi}{a} e^{-a\|\xi\|}$

So for $a=1$ we get $\pi e^{-|\xi|}$

Fourier transform fact #2

If f and f' are Fourier transformable then $\hat{f'}(\xi) = i\xi\hat{f}(\xi)$

Def: The convolution (faltning) of two functions f and g is the function

$$f*g(x) = \int_{R} f(x-y)g(y)\,dy$$

OBS! $f*g(x) = g*f(x)$

Fourier transform fact #3: $\widehat{f\cdot g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

03/03-25

1. CAT theorem + proof
2. FIT - Fourier Inverse Theorem
3. Planchanel theorem + proof
4. Homogenuous heat eq. on $\mathbb{R}$
5. Inhomogenous heat eq. on $\mathbb{R}$

3/3-25 *A convolution could be the solution and the CAT can help us with that!*

CAT: Convolution Approximation Theorem: ★

Assume that $g' \in \mathcal{L}^1(\mathbb{R})$ and that $\int_{\mathbb{R}} g(x)dx = 1$

$\longrightarrow \int_{\mathbb{R}} |g(y)| dy < \infty$

Assume that f is continuous on $\mathbb{R}$

$g(x)=0 \; |x| \geq L$

Assume that either ② g has compact support - meaning $g(x) = 0 \; \forall \; |x| > L$ for some $L > 0$

OR ① f is bounded $\longrightarrow |f| \leq M$

For $\varepsilon > 0$ let $g_\varepsilon(x) = \dfrac{g(\frac{x}{\varepsilon})}{\varepsilon}$ Then $\lim\limits_{\varepsilon \to 0} f * g_\varepsilon(x) = f(x)$

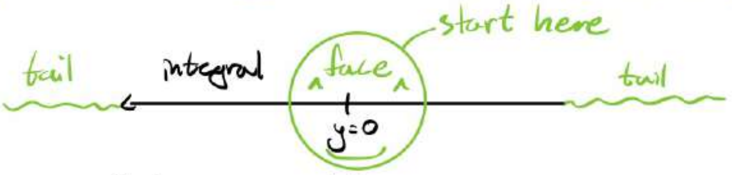
Proof:★ Fix the point $x \in \mathbb{R}$

$$f * g(x) = \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy \qquad \left(\int_{\mathbb{R}} g(y)dy = 1\right) f(x) \Rightarrow f(x) = \int_{\mathbb{R}} f(x) g(y) dy$$

$$\int_{\mathbb{R}} g(y) dy = \left\{ y = \frac{z}{\varepsilon} \right\} = \int_{\mathbb{R}} g\left(\frac{z}{\varepsilon}\right) \frac{dz}{\varepsilon} = \int_{\mathbb{R}} g_\varepsilon(z) dz$$

$$\Rightarrow |f * g_\varepsilon(x) - f(x)| = \left| \int_{\mathbb{R}} (f(x-y) - f(x)) g_\varepsilon(y) dy \right|$$

don't mess with the tail first!



Goal: $\lim\limits_{\varepsilon \to 0} \int_{\mathbb{R}} (f(x-y) - f(x)) g_\varepsilon(y) dy = 0$

Let $\delta > 0$ be given. Then $\exists \; y_0 > 0$ such that if $|y| < y_0$ then $|f(x-y) - f(x)| < \delta$ by continuity

Then $\left| \int_{-y_0}^{y_0} (f(x-y) - f(x)) g_\varepsilon(y) dy \right| \leq \int_{-y_0}^{y_0} |f(x-y) - f(x)| \, |g_\varepsilon(y)| dy < \delta \int_{\mathbb{R}} |g_\varepsilon(y)| dy = \delta \int_{\mathbb{R}} |g(z)| dz$

Thus $\left| \int_{-y_0}^{y_0} (f(x-y) - f(x)) g_\varepsilon(y) dy \right| \leq \delta \|g\|_{\mathcal{L}^1}$ with $\|g\|_{\mathcal{L}^1} = \int_{\mathbb{R}} |g(z)| dz$ is a constant

Goal is to prove that $\left| \int_{-y_0}^{y_0} (f(x-y) - f(x)) g_\varepsilon(y) dy \right| \leq \text{constant} \times \delta$ for all $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$

$|f * g_\varepsilon(x) - f(x)| < \text{constant} \times \delta$ for all $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$

$$\left| \int_{\mathbb{R}} (f(x-y) - f(x)) g_\varepsilon(y) dy \right| \leq \underbrace{\left| \int_{|y| < y_0} (f(x-y) - f(x)) g_\varepsilon(y) dy \right|}_{\text{THIS IS} \leq \|g\|_{\mathcal{L}^1} \delta} + \underbrace{\left| \int_{|y| \geq y_0} (f(x-y) - f(x)) g_\varepsilon(y) dy \right|}_{\text{tail}}$$

① Assume f is bounded. Then $|\text{tail}| \leq \int_{|y| \geq y_0} \underbrace{|f(x-y) - f(x)|}_{\leq 2M} |g_\varepsilon(y)| dy \leq 2M \int_{|y| > y_0} |g_\varepsilon(y)| dy$

Sub $\frac{y}{\varepsilon} = z \Rightarrow 2M \int_{\underbrace{|z| > \frac{y_0}{\varepsilon}}} |g(z)| dz$ Since $y_0 > 0$ is fixed, if ε is small, then $\frac{y_0}{\varepsilon} \longrightarrow \infty$

↳ tail of a convergent integral $\Rightarrow$ there is $\varepsilon_0 > 0$ such that $\int_{|z| > \frac{y_0}{\varepsilon}} |g(z)| dz < \delta \quad \forall \varepsilon < \varepsilon_0$

$\Rightarrow |tail| \leq 2M\delta$

② Next assume $|g(z)| = 0 \quad \underbrace{\forall |z| \geq L}_{\text{if } \varepsilon < \frac{y_0}{L}}$ $\quad |tail| \leq \int_{\underbrace{|z| > \frac{y_0}{\varepsilon}}_{|z| > L}} |f(x - \varepsilon z) - f(x)| |g(z)| dz$

$|z| > L \Rightarrow tail = 0$

Thus in both cases we have $|f * g_\varepsilon(x) - f(x)| < \delta \underbrace{(\|g\|_{\mathcal{L}^1} + 2M)}_{\text{constant}}$ $\square$

Theorem: The FIT = Fourier inverse theorem

Assume $f \in \mathcal{L}^2(\mathbb{R})$, then $\exists!$ $\hat{f} \in \mathcal{L}^2(\mathbb{R})$ and $f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi$

Theorem:☆ **Planchared** : $\forall f, g \in \mathcal{L}^2(\mathbb{R}) \quad \langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle$

Proof:★ Right side and use the FIT

$$2\pi \langle f, g \rangle = 2\pi \int_{\mathbb{R}} f(x) \overline{g(x)} dx \overset{FIT}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \underbrace{e^{ix\xi}}_{e^{-ix\xi}} d\xi \, \overline{g(x)} dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \overline{g(x) e^{-ix\xi}} d\xi = \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\xi) d\xi$$

Homogeneous heat equation on $\mathbb{R}$

$$\begin{cases} u_t = u_{xx}, \quad 0 < t, \quad x \in \mathbb{R} \\ u(x,t) \in \mathcal{L}^2(\mathbb{R}) \text{ for each } t > 0 \\ u(x,0) = f(x) \in \mathcal{L}^2(\mathbb{R}) \text{ and continuous} \end{cases}$$

CLUES $\Rightarrow$ F.T in x

$$\hat{u}_t(\xi, t) = \underbrace{\hat{u}_{xx}(\xi, t)}_{\text{Table \#5}} = (i\xi)^2 \hat{u}(\xi, t) = -\xi^2 \hat{u}(\xi, t)$$

$\partial_t \hat{u}(\xi, t) = -\xi^2 \hat{u}(\xi, t)$ Remember $T' = \lambda T \Rightarrow T(t) = Ce^{\lambda t}$

$T'' = \lambda T \Rightarrow T(t) = a_n e^{\sqrt{\lambda} t} + b_n e^{-\sqrt{\lambda} t}$

$\Rightarrow \hat{u}(\xi, t) = c(\xi) e^{-\xi^2 t}$ The IC tell us $c(\xi)$

$\hat{u}(\xi, 0) = \hat{f}(\xi) = c(\xi) e^0 = c(\xi)$

$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 t}$ In a hurry: $u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{-\xi^2 t} e^{ix\xi} d\xi$

If we have time, look for a fcn. whose F.T is $e^{-\xi^2 t} \Rightarrow u(x,t) = \int_{\mathbb{R}} f(x-y) g(y,t) dy$

Table p239 #9

$$e^{-ax^2/2} \xrightarrow{FT} \sqrt{\frac{2\pi}{a}}\, e^{-\xi^2/2a} \qquad \text{So } \frac{1}{2a} = t \qquad \Rightarrow e^{-x^2/4t} \xrightarrow{FT} \sqrt{2\pi(2t)}\, e^{-\xi^2 t}$$

$$\frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \xrightarrow{FT} e^{-\xi^2 t} \Rightarrow u(x,t) = \int_{\mathbb{R}} f(x-y) \frac{e^{-y^2/at}}{\sqrt{4\pi t}}\, dy$$

$$\text{CAT} \quad \varepsilon = \sqrt{t} \quad g(y) = \frac{e^{-y^2/4}}{\sqrt{4\pi}} \Rightarrow \lim_{t \to 0}$$

Inhomogeneous heat equation on $\mathbb{R}$

$$\begin{cases} u_t - u_{xx} = F(x,t) \in \mathcal{L}^2(\mathbb{R}) & \forall t > 0 \\ u(x,t) \in \mathcal{L}^2(\mathbb{R}) & \forall t > 0 \\ u(x,0) = f(x) \in \mathcal{L}^2(\mathbb{R}) \end{cases}$$

Clues F.T in X

$\hat{u}_t(\xi,t) + \xi^2 \hat{u}(\xi,t) = \hat{F}(\xi,t)$ Integrating factor method (μthod) ODE for $\hat{u}$ in t variable

$$\hat{u}(\xi,t) = e^{-\xi^2 t}\left(\int_0^t e^{\xi^2 s} \hat{F}(\xi,s)\, ds + \underbrace{\hat{u}(\xi,0)}_{\hat{f}(\xi)} \right)$$

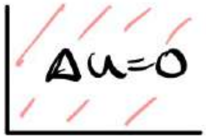
FIT

In a hurry: $u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left[e^{-\xi^2 t} \int_0^t e^{\xi^2 s} \hat{F}(\xi,s)\, ds + \hat{f}(\xi) \right] e^{ix\xi}\, d\xi$

07/03-25

1. Even/odd extensions - method of images problem in $x \in [0, \infty)$ with Dirichlet or Neumann @ $x = 0$

2. Dirichlet Problem in $\mathcal{L}^2(\mathbb{R})$



$\mathcal{L}^2(\mathbb{R})$ function

1. Method of images (even & odd extensions)

PDE $\begin{cases} u_t - u_{xx} = 0 & \text{for } 0<t \quad 0<x<\infty, \end{cases}$ CLUES that it is Fourier transform time

BC $u_x(0,t) = 0 \rightarrow$ Neumann

IC $u(x,0) = f(x) \leftarrow$ an $\mathcal{L}^2(\mathbb{R}_+)$ and $\mathcal{C}^0(\mathbb{R}_+)$ function

$x \in [0,\infty)$ and Neumann BC

↳ Extend evenly to $\mathbb{R}$ and use Fourier transform

Define $f_e(x) = \begin{cases} f(x), & x>0 \\ f(-x), & x<0 \end{cases}$ Even extensions of f

Solve now $\begin{cases} u_t - u_{xx} = 0 & 0<t \quad x\in\mathbb{R} \\ u(x,0) = f_e(x) \end{cases}$

$$u(x,t) = \int_{\mathbb{R}} \frac{f_e(y) e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy$$

⇐ Either learn this OR Fourier transform the PDE

Review from last lecture how to solve this: $\hat{u}_t(\xi,t) = -\xi^2 \hat{u}(\xi,t)$

$$\Rightarrow \hat{u}(\xi,t) = \hat{f}_e(\xi) e^{-\xi^2 t} \quad \Rightarrow \quad u(x,t) = f_e * \underset{\uparrow}{}$$

function whose FT is $e^{-\xi^2 t}$

Check the BC: $\underbrace{\int_{-\infty}^{0} \frac{f(-y) e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy}_{} + \int_0^{\infty} \frac{f(y) e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy$

Let $z=-y \Rightarrow \int_{\infty}^{0} \frac{f(z) e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} (-dz) = \int_0^{\infty} \frac{f(z) e^{-(x+z)^2/4t}}{\sqrt{4\pi t}} dz$

$$u(x,t) = \int_0^{\infty} \frac{f(y)}{\sqrt{4\pi t}} \left(e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t} \right) dy$$

Observe that $\frac{\partial}{\partial x} u(x,t) = \int_0^{\infty} \frac{f(y)}{\sqrt{4\pi t}} \left(\frac{-2(x-y)}{4t} e^{-(x-y)^2/4t} - \frac{2(x+y)}{4t} e^{-(x+y)^2/4t} \right) dy$ ← converges |beautifully| ⇒ $\int \partial_x = \partial_x \int$

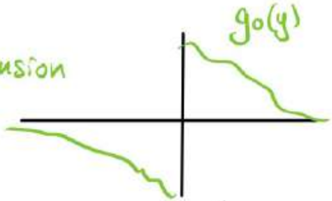
"beautifully" = absolutely and locally uniformly

So $u_x(0,t) = \int_0^{\infty} \frac{f(y)}{\sqrt{4\pi t}} \left(\cancel{\frac{2y}{4t} e^{-(x)^2/4t} - \frac{2y}{4t} e^{-(x)^2/4t}} \right) dy = 0$

0 yay!

↳ satisfies the BC

Next $\begin{cases} V_t - V_{xx} = 0 & 0<t,\ 0<x<\infty \\ \underline{V(0,t)=0} \xrightarrow{} x\in(0,\infty) \text{ and Dirichlet BC} \Rightarrow \text{Odd extension} \\ V(x,0) = g(x) \in \mathcal{L}^2(\mathbb{R}_+)\cap \mathcal{C}^0(\mathbb{R}_+) \end{cases}$



$\Rightarrow V(x,t) = \int_{\mathbb{R}} \frac{g_o(y)e^{-(x-y)^2/4t}}{\sqrt{4\pi t}}\,dy$ with $g_o(y) = \begin{cases} g(y) & y>0 \\ -g(-y) & y<0 \end{cases}$ $\Rightarrow$ Odd extension of g

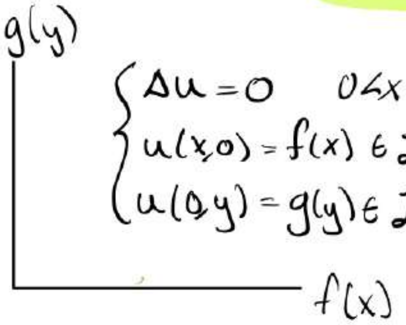
Analogous calculation $\longrightarrow$ $V(x,t) = \int_0^\infty \frac{g(y)}{\sqrt{4\pi t}}\left(e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right)dy$

Observe that $V(0,t)=0$

Summary: Enough to just use/write: Neumann $\Rightarrow u(x,t) = \int_0^\infty \frac{f(y)}{\sqrt{4\pi t}}\left(e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t}\right)dy$

Dirichlet $\Rightarrow v(x,t) = \int_0^\infty \frac{g(y)}{\sqrt{4\pi t}}\left(e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right)dy$

Dirichlet problem



$\begin{cases} \Delta u = 0 & 0<x,\ 0<y \\ u(x,0) = f(x) \in \mathcal{L}^2(\mathbb{R}_+) \\ u(0,y) = g(y) \in \mathcal{L}^2(\mathbb{R}_+) \end{cases}$

Idea Split into 2 problems, solve each and add

$\Rightarrow$ Solve for $\begin{cases} \Delta w = \\ w(x,0) = f(x) \\ w(0,y) = 0 \end{cases}$ & $\begin{cases} \Delta v = 0 \\ v(0,y) = g(y) \\ v(x,0) = 0 \end{cases}$ Dirichlet BC $\Rightarrow$ ODD extension!

Define $f_o(x) = \begin{cases} f(x) & 0<x \\ -f(-x) & x<0 \end{cases}$ odd

FT the PDE in x variable $\hat{w}_{xx}(\xi,y) + \hat{w}(\xi,y) = 0$

Recall $\hat{w}_x(\xi,y) = i\xi\hat{w}(\xi,y)$ $\quad \hat{w}_{xx}(\xi,y) = -\xi\hat{w}(\xi,y)$ $\quad \hat{w}_{yy}(\xi,y) = \xi^2\hat{w}(\xi,y)$

$\Rightarrow \hat{w}(\xi,y) = a(\xi)e^{\xi y} + b(\xi)e^{-\xi y}$

BC's are $w(x,0) = f(x)$ and $w(0,y) = 0$

$\hat{w}(\xi,0) = \hat{f}_o(\xi) \Rightarrow a(\xi) + b(\xi) = \hat{f}_o(\xi)$

In this problem $x\in\mathbb{R}$ and $\xi\in\mathbb{R}$ but $y>0$

Need to be able to Fourier transform and FIT

Problem with $e^{\xi y}$ for $y>0$ if $\xi>0$

Problem with $e^{-\xi y}$ for $y>0$ if $\xi<0$

To fix this => use $e^{-|\xi|y}\hat{f}_0(\xi) = \hat{w}(\xi, y)$ => $\hat{w}(\xi, 0) = \hat{f}_0(\xi)$

FIT: $w(x,y) = \frac{1}{2\pi}\int \underset{\uparrow}{e}^{-|\xi|y}\hat{f}_0(\xi)e^{ix\xi}\,d\xi \underset{\uparrow}{=} \int_0^\infty f(z)\cdot\left[\frac{y}{\pi((x-z)^2+y^2)} - \frac{y}{\pi((x+z)^2+y^2)}\right]dz$

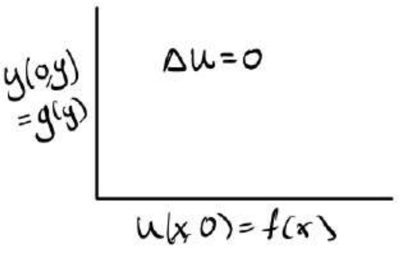
Function whose FT is $e^{-|\xi|y}$

use table and define of f_0

Anologously => $v(x,y) = \int_0^\infty g(z)\left[\frac{x}{\pi((y-z)^2+x^2)} - \frac{x}{\pi((y+z)^2+x^2)}\right]dz$

10/03-25

1. Dirichlet Problem - CAT
2. Sampling Theorem
3. Discrete Fourier Transform
4. Laplace transform



$$\longrightarrow \begin{cases} \Delta v=0 & 0<x \quad 0<y \\ v(0,y)=0 \\ v(x,0)=f(x) \end{cases} + \begin{cases} \Delta w=0 & 0<x \quad 0<y \\ w(0,y)=g(y) \\ w(x,0)=0 \end{cases}$$

$$\searrow u=v+w \swarrow$$

f and g are in $\mathcal{L}^2(\mathbb{R})$

⇒ Dirichlet BC and $\mathcal{L}^2(\mathbb{R})$ ⇒ ODDly extend the problem and FT the pde in x.

$$\rightsquigarrow \hat{v}(\xi,y)=\underline{A(\xi)}e^{-\xi y}+\underline{B(\xi)}e^{\xi y}$$

These are not in $\mathcal{L}^2(\mathbb{R})$ for

for $\xi<0$ for $\xi>0$

<u>BUT</u> $\left.\begin{matrix} e^{-\xi y}\in\mathcal{L}^2(\mathbb{R}_+) \\ e^{\xi y}\in\mathcal{L}^2(\mathbb{R}_-) \end{matrix}\right\} \Rightarrow \hat{v}(\xi,0)=C(\xi)e^{-|\xi|y}\in\mathcal{L}^2(\mathbb{R})$

$\hat{v}(\xi,0)=C(\xi)=\hat{f}_o(\xi) \Rightarrow \hat{v}(\xi,y)=\hat{f}_o(\xi)e^{-|\xi|y}$ } Find function whose FT is this from Table

$$v(x,y)=\int_{\mathbb{R}} f_o(x)\frac{y}{\pi[(x-z)^2+y^2]}dz$$

$\longrightarrow$ STOP here, full points on exam!

$V(x,0)=?$ Insted $\lim_{y\to0} v(x,y)=$ CAT limit!

Let $h(z)=\frac{1}{\pi(z^2+1)}$ Then $h\in\mathcal{L}^1(\mathbb{R})$ $\int_{\mathbb{R}} h(z)\,dz=1$

In CAT use $\varepsilon=y$ $h_\varepsilon(z)=\frac{h(\frac{z}{2})}{\varepsilon}$

$$\lim_{y\to0}\int_{\mathbb{R}} f_o(z)\frac{y}{\pi[(x-z)^2+y^2]}dz=\lim_{y\to0} f_o(z)\frac{dz/y}{\pi[\frac{(x-z)^2}{y}+1]}=\lim_{y\to0} f_o*h_y(x)\overset{\overset{CAT}{\downarrow}}{=}f_o(x)=f(x) \quad \forall x>0$$

$v(0,y)=\int_{\mathbb{R}} f_o(z)\frac{y}{\pi(z^2+y^2)}dz$ For

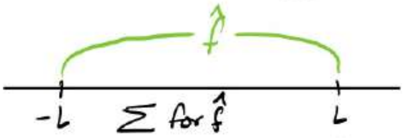
$\Rightarrow v(0,y)=0 \quad \forall\, y>0$ (actually $\forall\, y\neq0$)

Sampling Theorem*

Assume $f \in \mathcal{L}^2(\mathbb{R})$. Assume that $\hat{f}(\xi) = 0 \ \forall \, |\xi| > L$

Then $\forall \, t \in \mathbb{R}$ $f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}$

Proof*: FIT and



$\{e^{in\pi x/L}\}_{n \in \mathbb{Z}}$ are an orthogonal base for $\mathcal{L}^2(-L, L)$

$\Rightarrow$ Expand $\hat{f} \in \mathcal{L}^2(-L, L)$ in tFΣ $\quad \hat{f}(x) = \sum_{n \in \mathbb{Z}} c_n e^{in\pi x/L}$ with $c_n = \frac{1}{2L} \int_{-L}^{L} \hat{f}(x) e^{-in\pi x/L} dx$

$$\text{FIT: } f(t) = \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(x) e^{ixt} dx = \frac{1}{2\pi} \int_{-L}^{L} \left(\sum_{n \in \mathbb{Z}} c_n e^{in\pi x/L} \right) e^{ixt} dx = \frac{1}{2\pi} \int_{-L}^{L} \sum_{n \in \mathbb{Z}} \left(\frac{1}{2L} \int_{-L}^{L} \hat{f}(y) e^{-in\pi y/L} dy \right) e^{in\pi x/L} e^{ixt} dx =$$

$$= \frac{1}{2\pi} \int_{-L}^{L} \sum_{n \in \mathbb{Z}} \left(\frac{1}{2L} \int_{\mathbb{R}} \hat{f}(y) e^{-in\pi y/L} dy \right) e^{in\pi x/L} e^{ixt} dx \overset{\substack{\text{FIT} \\ \downarrow}}{=} \frac{1}{2\pi} \int_{-L}^{L} \sum_{n \in \mathbb{Z}} \frac{1}{2L} \cdot 2\pi \cdot f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L} \cdot e^{ixt} dx =$$

$$\underset{\substack{\uparrow \\ \hat{f} = 0 \text{ outside } [-L, L]}}{=} \frac{1}{2L} \int_{-L}^{L} \sum_{n \in \mathbb{Z}} f\left(-\frac{n\pi}{L}\right) e^{in\pi x/L} e^{ixt} dx \underset{\substack{\uparrow \\ \text{Beautiful} \\ \text{convergence}}}{=} \sum_{n \in \mathbb{Z}} f\left(-\frac{n\pi}{L}\right) \frac{1}{2L} \int_{-L}^{L} e^{\left(\frac{in\pi}{L} + it\right)x} dx =$$

$$= \sum_{n \in \mathbb{Z}} \frac{f\left(\frac{-n\pi}{L}\right)}{2L} \left[\frac{\sin\left(\left(\frac{in\pi}{L} + t\right)x\right)}{\frac{n\pi}{L} + t} \right]_{-L}^{L} = \sum f\left(\frac{n\pi}{L}\right) \left[\frac{\sin(n\pi + tL) - \sin(-n\pi - tL)}{2(n\pi + tL)} \right] =$$

$$= \sum_{n \in \mathbb{Z}} f\left(-\frac{n\pi}{L}\right) \frac{\sin(n\pi + tL)}{n\pi + tL} \underset{m = -n}{=} \sum_{m \in \mathbb{Z}} f\left(\frac{m\pi}{L}\right) \frac{\sin(-m\pi + tL)}{-m\pi + tL} \underset{\substack{\uparrow \\ \text{Sine=ODD}}}{=} \sum_{m \in \mathbb{Z}} f\left(\frac{m\pi}{L}\right) \frac{\sin(m\pi - tL)}{m\pi - tL}$$

Discrete Fourier Transform

Turn function into vector in $\mathbb{C}^n$

$\vec{f} = (f(n\tau))_{n=0}^{N-1}$ fix $\tau > 0$

Def: $\left(\hat{f}\left(\frac{2\pi k}{N\tau}\right) \right)_{k=0}^{N-1} = \vec{\hat{f}}$ $\qquad \hat{f}\left(\frac{2\pi k}{N\tau}\right) = \sum_{n=0}^{N-1} \frac{f(n\tau) e^{-2\pi ink/N}}{\sqrt{N}}$

DFIT: Discrete Fourier inverse theorem: $f(n\tau) = \sum_{k=0}^{N-1} \hat{f}\left(\frac{2\pi k}{N\tau}\right) e^{2\pi ink/N}$

Chapter 9 Laplace transform

The Laplace transform can be applied (and inverted) if we stay on the Heaviside

$\Theta(t)$ = Heaviside function $= \begin{cases} 1, \text{ if } t>0 \\ 0, \text{ if } t<0 \end{cases}$

To Laplace transform say $f(t)=t^2 \rightarrow$ instead $\mathcal{L}T$ $\Theta(t)t^2$

Def: Assume $f(t)=0$ for $t<0$ and $\exists$ $a>0$ $c>0$ such that $|f(t)| \leq Ce^{at}$ $\forall t>0$ Then

$$\tilde{f}(z) = \mathcal{L}f(z) = \int_0^\infty f(t)e^{-zt}dt \ \{\text{for } Re(z)>a\} = \int_{\mathbb{R}} \Theta(t)f(t)e^{i(iz)t}dt = \int_{\mathbb{R}} \Theta(t)f(t)e^{-i(-iz)t}dt = \widetilde{\Theta f}(-iz)$$

Properties of the Laplace transform

If f and g satisfy the definition of $\mathcal{L}T$ ($\mathcal{L}T$'able) then:

① $\hat{f}(x+iy) \longrightarrow 0$ as $|y| \longrightarrow \infty$ $\forall$ $x>a$ $(z=x+iy)$

② $\hat{f}(x+iy) \longrightarrow 0$ as $x \longrightarrow \infty$ $\forall y$

⑥ $\widetilde{f'}(z) = z\widetilde{f}(z) - f(0)$

12/03-25

1. The Laplace transform
2. Laplace inverse theorem ($\mathcal{L}$IT)
3. Example of q with $\mathcal{L}$T usage in solution

The $\mathcal{L}$aplace transform

Theorem (Properties of the $\mathcal{L}$T): The most important thing is how to use it!

Recall - need $f(t)=0$ $t<0$ and $|f(t)|\leq Ce^{at}$ $t\geq 0$ to $\mathcal{L}T$ $f(t)$

① $\tilde{f}(x+iy)\longrightarrow 0$ as $|y|\longrightarrow\infty$ for all $x>a$

② $\tilde{f}(x+iy)\longrightarrow 0$ as $x\longrightarrow\infty$ $\forall y$

$$\left.\begin{aligned} &③\ \mathcal{L}(\theta(t-b)f(t-b))(z)=e^{-bz}\hat{f}(z)\\ &④\ \mathcal{L}(e^{ct}f(t))(z)=\tilde{f}(z-c)\end{aligned}\right\}\ b,c\in\mathbb{R}$$

Do not need to memorize these, just were to find and use them.

⑤ $\mathcal{L}(f(bt))=\frac{1}{b}\tilde{f}\left(\frac{z}{b}\right)$

⑥ $\mathcal{L}(f')(z)=\hat{f}(z)-f(0)$

⑦ $\mathcal{L}\left(\int_0^t f(s)\,ds\right)(z)=\frac{\hat{f}(z)}{z}$

⑧ $\mathcal{L}(t\,f(t))(z)=-(\tilde{f})'(z)$

⑨ $\mathcal{L}(\theta f*\theta g)(z)=\widetilde{\theta f}*\widetilde{\theta g}(z)$

⑩ $\mathcal{L}\left(\frac{f(t)}{t}\right)(z)=\int_z^\infty\tilde{f}(w)\,dw$ (contour integral with $Re(w)\rightarrow\infty$ $Im(w)\longrightarrow$ Bounded

Def: $erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$, $erfc(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt$

Proposition: Assume $f, f', f'', \dots, f^{(k)}$ are $\mathcal{L}$-transformable.

Very useful!

Then $\mathcal{L}(f^{(k)})(z)=z^k f(z)-\underbrace{\sum_{j=1}^{k}f^{(k-j)}(0)\,z^{j-1}}$ This is a polynomial

Application to solve any linear constant coefficient ODE. (can be inhomogeneous)

An n^{th} order linear, constant coeff. ODE is $\sum_{k=0}^{n}c_k u^{(k)}(t)=f(t)$

$\mathcal{L}$ Transform the whole equation (it's nice if f is $\mathcal{L}T$-able)

$$\Rightarrow \underbrace{\sum_{k=0}^{n}c_k\,\underbrace{\widetilde{u^{(k)}}(z)}_{\text{use prop}}}=\tilde{f}(z)$$

$$\underbrace{\sum_{k=0}^{n}c_k z^k}_{\substack{\text{this is } p(z)\\ \text{polynomial}}}\tilde{u}(z)-\underbrace{\sum_{k=1}^{n}c_k\sum_{j=1}^{k}u^{(k-j)}(0)\,z^{j-1}}_{q(z)=\text{polynomial}}=\tilde{f}(z)$$

$$\underbrace{p(z)}\tilde{u}(z)-q(z)=\hat{f}(z) \qquad \tilde{u}(z)=\frac{\tilde{f}(z)+q(z)}{p(z)}\Rightarrow u(t)=\mathcal{L}^{-1}\left(\frac{\hat{f}(z)+q(z)}{p(z)}\right)(t)$$

$\mathcal{L}$aplace inverse theorem ($\mathcal{L}$IT): Assume that f is $\mathcal{L}$aplace transformable

For b > a $f(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \hat{f}(z)e^{zt}dz = \frac{1}{2\pi i}\int_{Re(z)=b\uparrow} \hat{f}(z)e^{zt}dz$

Could we solve such an ODE with Fourier transform?

$$\sum_{k=0}^{n} c_k u^{(k)}(x) = f(x)$$

Fourier transform ⇔ $x \in \mathbb{R}$

OR

$$\Downarrow$$

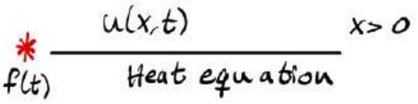
$$\sum_{k=0}^{n} c_k(i\xi)^k \hat{u}(\xi) = \hat{f}(\xi)$$

Dirichlet ⟷ $u(0)=0$
Neumann ⟷ $u'(0)=0$
BC

$$\Rightarrow \hat{u}(\xi) = \frac{\hat{f}(\xi)}{\sum_{k=0}^{n} c_k(i\xi)^k} \Rightarrow u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{f}(\xi)}{\sum_{k=0}^{q} c_k(i\xi)^z} e^{ix\xi}d\xi$$

Both a $\mathcal{L}$T and FT q
will be on exam.
Figure out which is which

Example:



$$\begin{cases} u_t = u_{xx} & \text{for } 0<t, x \\ u(0,t) = f(t) & \text{not } \mathcal{L}^2 \\ u(x,0) = 0 & \text{(constant initial temperature)} \end{cases}$$

when x can go to ±∞ transform.

⇒ CLUES $\mathcal{L}$T in t

$\mathcal{L}$T the PDE: $\underbrace{\tilde{u}_t(x,z)} = \tilde{u}_{xx}(x,z)$

Property #6 says $z\hat{u}(x,z) - \cancel{u(x,0)} = \tilde{u}_{xx}(x,z)$ → 0

$\underbrace{z\tilde{u}(x,z) = \tilde{u}_{xx}(x,z)}_{2^{nd} \text{ order ODE for } \tilde{u}}$ ← This is an ODE in x for $\tilde{u}(x,z)$

Basis of solutions is $e^{\sqrt{z}x}$ and $e^{-\sqrt{z}x}$

$\Rightarrow \tilde{u}(x,z) = a(z)e^{\sqrt{z}x} + b(z)e^{-\sqrt{z}x}$ STOP ⇒ THINK ⇒ LOOK @ prop of $\mathcal{L}$T

In this problem $0<x$ #2 $\tilde{u}(x,z) \longrightarrow 0$ when $Re(z) \longrightarrow \infty$

$$\left.\begin{array}{lll} e^{-\sqrt{z}x} \longrightarrow 0 & \text{when } Re(z) \longrightarrow \infty & \\ e^{\sqrt{z}x} \longrightarrow \infty & \text{when } Re(z) \longrightarrow \infty & \text{NO} \end{array}\right\} \Rightarrow \text{Hopefully } b(z)e^{-\sqrt{z}x} = \tilde{u}(x,z) \text{ solves the problem}$$

BC: $\tilde{u}(0,z) = b(z) = \hat{f}(z) \Rightarrow \tilde{u}(x,z) = \hat{f}(z)e^{-\sqrt{z}x}$

If short on time: $u(x,t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \hat{f}(z)e^{-\sqrt{z}x}e^{zt}dz$

If time permits look for function whose $\mathcal{L}T$ is $e^{-\sqrt{z}\,x}$

#26 in Table 9.1: $\theta(t)t^{-3/2}e^{-a^2/4t} \xrightarrow{\mathcal{L}T \text{ in } t} \frac{2}{a}\sqrt{\pi}\,e^{-a\sqrt{z}}$ for $a>0$

Since $x>0$ is "constant" from t perspective $\Rightarrow \theta(t)t^{-3/2}e^{-x^2/4t} \xrightarrow[\mathcal{L}T \text{ in } t]{} \frac{2}{x}\sqrt{\pi}\,e^{-x\sqrt{z}}$

Therefore: $\underbrace{\frac{x}{2\sqrt{\pi}}\theta(t)t^{-3/2}e^{-x^2/4t}}_{} \xrightarrow[\mathcal{L}T \text{ in } t]{} e^{-x\sqrt{z}}$

$\Rightarrow u(x,t) = \theta f * \text{this function} = \int_{\mathbb{R}} f(t-s)\theta(t-s)\frac{x}{2\sqrt{\pi}}\theta(s)s^{-3/2}e^{-x^2/4s}\,ds$

13/03-25

Exam review part 1

Exam 2009

1) $\int_0^\infty (e^{-2x} - P(x))^2 \, \underline{x \cdot e^{-x}} \, dx$

(0, ∞) and this weight => Laguerre

Laguerre Polynomial of degree n is L_n^α

They are an OB on $\mathcal{L}^2_{x^\alpha e^{-x}}(0,\infty)$. Here $\alpha = 1$

$$P(x) = \sum_{n=0}^{3} \frac{\langle e^{-2t}, L_n^\alpha(t) \rangle}{\| L_n^\alpha(t) \|^2} L_n^\alpha(x)$$

Here $\langle e^{-2t}, L_n^\alpha(t) \rangle = \int_0^\infty e^{-2t} \, \overline{L_n^\alpha(t)} \, t^\alpha e^{-t} \, dt$

$$\| L_n^\alpha \|^2 = \int_0^\infty |L_n^\alpha(t)|^2 t^\alpha e^{-t} \, dt$$

2) Funktionen $f(t)$ har Fourier transform

Clues to use FT method

$$\hat{f}(\omega) = \begin{cases} 0 & |\omega| < 1 \\ 1 & 1 < |\omega| < 4 \\ \omega^{-1} & |\omega| \geq 4 \end{cases}$$

For $\alpha > 0$ $\quad g_\alpha(t) = \frac{\sin(\alpha t)}{\pi t}$

Beräkna ① $\int_{\mathbb{R}} |g_\alpha * f(t)|^2 dt$ ② $\int_{\mathbb{R}} |g * f * f(t)|^2 dt$, ③ $\int_{\mathbb{R}} f(t) \cos(t) \, dt$

Solution:

① $\| g_\alpha * f \|^2 \underset{\uparrow}{=} \frac{1}{2\pi} \| \widehat{g_\alpha * f} \|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}_\alpha(\xi)|^2 |\hat{f}(\xi)|^2 d\xi = (*)$

Plancharel in fun fact

$g_\alpha(t) = \frac{\sin(\alpha t)}{\pi t}$ Table says $\frac{\sin(\alpha x)}{x} \xrightarrow{FT} \pi \chi_\alpha(\xi)$

$\hat{g}_\alpha(\xi) = \chi_\alpha(\xi)$ $\quad \frac{\sin(\alpha x)}{\pi x} \xrightarrow{FT} \chi_\alpha(\xi)$

$(*) = \frac{1}{2\pi} \int_{\mathbb{R}} |\chi_\alpha(\xi)|^2 |\hat{f}(\xi)|^2 d\xi = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} |\hat{f}(\xi)|^2 d\xi = \dots$ compute

② $\frac{1}{2\pi} \| \hat{g}_\alpha \hat{f} \hat{f} \|^2 = \frac{1}{2\pi} \int_{\alpha}^{\alpha} |\hat{f}(\xi)|^4 d\xi = \dots$ compute

③ $\int_{\mathbb{R}} f(t) \cos(t) dt \underset{\uparrow}{=} \int_{\mathbb{R}} \frac{f(t)}{2} [e^{it} + e^{-it}] dt = \frac{1}{2} [\hat{f}(-1) + \hat{f}(1)] = ?$ Since $\hat{f}$ not defined there

Euler formula

3) $\begin{cases} u_t = u_{xx} & 0 < t, x \\ u(0,t) = \sin(2t)e^{-t} \\ u(x,0) = \underbrace{\sin(x)}_{\text{Not FTable}} \end{cases}$ $\mathcal{L}T$ or ~~FT~~

No Dirichlet or Neumann BC

$$\mathcal{L}(u_t(x,t)) \underset{\substack{\uparrow \\ \text{Prop. of } \mathcal{L}T}}{=} z\tilde{u}(x,z) - u(x,0) = z\tilde{u}(x,z) - \sin(x)$$

$$z\tilde{u}(x,z) - \sin(x) = \hat{u}_{xx}(x,z)$$ Inhomog.

Homog. ODE is $z\tilde{u}(x,z) = \hat{u}_{xx}(x,z) \Rightarrow$ B of S $\{e^{\sqrt{z}x}, e^{-\sqrt{z}x}\}$

Need a particular solution to the ODE. Let's call it $y(x,z)$

Observe: $\tilde{u}(x,z) \longrightarrow 0 \quad Re(z) \to \infty$

Solution is $\tilde{u}(x,z) = a(z)e^{-x\sqrt{z}} + y(x,z)$

$$\tilde{u}(0,z) = \mathcal{L}(\underbrace{\sin(2t)e^{-t}}_{g(t)})(z) = \tilde{g}(z)$$

$$\Rightarrow a(z) + y(0,z) = \hat{g}(z) \Rightarrow a(z) = \tilde{g}(z) - y(0,z)$$

Short on time: $\mathcal{L}IT \Rightarrow u(x,t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} (a(z)e^{-x\sqrt{z}} + y(x,z))e^{zt}dt$

Solve completely: Look for soln. to $zy(x,z) - \sin(x) = y_{xy}(x,z)$

$$\Rightarrow y_{xx} - zy = -\sin(x) \qquad -c(z)\cancel{\sin(x)} - zc(z)\cancel{\sin(x)} = -\cancel{\sin(x)}$$

$$c(z) + zc(z) = 1 \qquad c(z) = \frac{1}{1+z}$$

$$\tilde{u}(x,z) = \frac{\sin(x)}{1+z} + \tilde{g}(z)\underset{\substack{\uparrow \\ \text{Table to find} \\ \text{whose } \mathcal{L}T \text{ this is}}}{e^{-x\sqrt{z}}}$$

Table #11 $\theta(t)t^{\nu}e^{ct} \longrightarrow \Gamma(\nu+1)(z-c)^{-\nu-1} \qquad \nu = 0 \quad \Gamma(1) = 1$

$$\theta(t)e^{-t} \longrightarrow \frac{1}{z+1}$$

$$u(x,t) = \sin(x)\theta(t)e^{-t} + \int_{\mathbb{R}} \sin(2(t-s))e^{-(t-s)}\theta(t-s)\frac{x}{2\sqrt{\pi}}e^{-\frac{x^2}{4s}}\theta(s)ds$$

4) $\begin{cases} \Delta u + u = 0, \quad 0<r<1, \; 0<z<1 \\ u_z(r,\theta,0) = 0 \quad u(r,\theta,1) = 0 \\ u(1,\theta,z) = 1 - z^2 \end{cases}$ Bounded => Seperate Variables and Σ

→ Dirichlet

→ Neumann

Indep. of θ (radially symmetriz) => Hejdå θ

Seperate varables in the PDE $\quad R''Z + r^{-1}R'Z + Z''R + RZ = 0$

Z has the BC's so solve for Z first

$$\underbrace{Z'(0)=0}_{\cos(\sim)} \quad Z(1) = 0 \qquad \frac{Z''}{Z} = \text{constant}$$

cos and sme

$$\Rightarrow \quad Z_n(z) = \cos\left(\pi\left(n+\tfrac{1}{2}\right)z\right) \; \substack{n\geq 0 \\ n\in\mathbb{Z}}$$ cosine is even => $n = 0, 1, 2 \ldots$

$$\frac{R''}{R} + \frac{r^{-1}R'}{R} + \frac{Z''}{Z} + 1 = 0 \quad \Rightarrow \quad \frac{R''}{R} + \frac{r^{-1}R'}{R} - \left(n+\tfrac{1}{2}\right)^2\pi^2 + 1 = 0$$

$$\Rightarrow \quad r^2R'' + rR' + \left(-\left(n+\tfrac{1}{2}\right)^2\pi^2 + 1\right)r^2R = 0$$

$$\Rightarrow \quad r^2R'' + rR' - \left(\left(n+\tfrac{1}{2}\right)^2\pi^2 - 1\right)r^2R = 0$$ Go to Fun Facts!

Mod. Bessel eqn. of order 0 with argument $\sqrt{\left(n+\frac{1}{2}\right)^2\pi^2 + 1}\cdot r$

Basis of solns. $\{I_0, \cancel{K_0}\}$

Since $\to \infty$ at 0

$$R_n(r) = I_0\left(\left(\sqrt{\left(n+\tfrac{1}{2}\right)^2\pi^2 - 1}\right)r\right)$$

$$u(r,z) = \sum_{n\geq 0} c_n Z_n(z) R_n(r)$$ Finally use "time like" BC

$$u(1,z) = \sum_{n\geq 0} c_n Z_n(z) R_n(1) = 1 - z^2 \quad \Rightarrow \quad c_n R_n(1) = \frac{\langle 1-z^2, Z_n \rangle}{\|Z_n\|^2}$$

with $\langle 1-z^2, Z_n \rangle = \int_0^1 (1-z^2)\overline{Z_n(z)}\,dz \qquad \|Z_n\|^2 = \int_0^1 |Z_n|^2\,dz$

6) $\begin{cases} u_{tt} = \Delta u & 0 < r < 1 \\ u(1, \theta, t) = \sin(\theta) \\ u(r, \theta, 0) = 0 \quad u_t(r, \theta, 0) = 0 \end{cases}$

Bounded disk => Sep vars and Σuperposition methods

θ BC is periodic BC BUT BC @ r=1 is a function ☹

Idea ! Find function f(r) sinθ such that it is 0 in the pde and f(r)=1

$$\text{PDE} \Rightarrow f''(r)\sin\theta + r^{-1}f'(r)\sin\theta - r^{-2}f(r)\sin\theta) = 0$$

$$\Updownarrow$$

$$f'' + r^{-1}f' - r^{-2}f = 0 \quad \times r^2 \iff r^2 f'' + r f' - f = 0 \Leftarrow \text{Euler eqn!}$$

⇒ A solution is $f(r) = r$

$u = v + r\sin\theta$

Now we have $r\sin\theta$ for BC and we solve: ♡ $\begin{cases} v_{tt} = \Delta v \\ v(1, \theta, t) = 0 \\ v(r, \theta, 0) = -r\sin\theta \quad v_t(r, \theta, 0) = 0 \end{cases}$

14/03-25

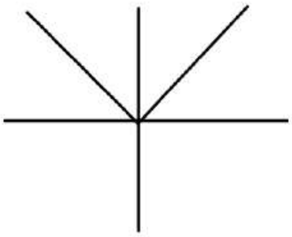
Exam review part 2

Last lecture ♡

tFΣ

1. Compute $\sum_{n\geq 1} \frac{1}{(2n-1)^2}$ => Table of tF-Series

$|x|$ has $\underbrace{\frac{\pi}{2} - \frac{4}{\pi} \sum_{n\geq 1} \frac{\cos((2n-1)x)}{(2n-1)^2}}_{\text{tF expansion on } [-\pi,\pi]}$ @ $x = 0$



tF-C@T says the Σ @ x=0 → 0

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\geq 1} \frac{1}{(2n-1)^2} \quad @\, x = 0$$

$$\frac{\pi^2}{8} = \sum_{n\geq 1} \frac{1}{(2n-1)^2}$$

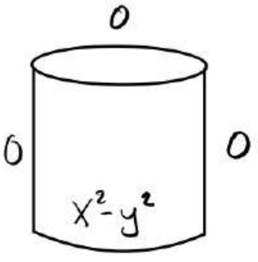
Julie's soln: $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} = \sum_{\text{even}} \frac{1}{n^2} + \sum_{\text{odd}} \frac{1}{n^2}$ $\qquad \frac{\pi^2}{6} = \frac{1}{4} \cdot \frac{\pi^2}{6} + \sum_{n\geq 1} \frac{1}{(2n+1)^2}$

$$\frac{\pi^2}{8} = \sum_{n\geq 1} \frac{1}{(2n+1)^2}$$

Both are correct to use

pde with pol. coord

2. $\begin{cases} \Delta u = 0 \\ u(r,\theta,5) = 0 \\ u(r,\theta,0) = 0 \\ u(5,\theta,z) = \end{cases}$



$0 < r < 5$

First turn $x^2 - y^2$ into polar coords:

$$r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r\cos(2\theta)$$

Hint: Try to solve with $v(r,z)\cos(2\theta) = u(r,\theta,z)$

Put $v(r,z)\cos(2\theta)$ into the PDE

$$V_{zz}\cos(2\theta) + V_{rr}\cos(2\theta) + r^{-1}V_r\cos(2\theta) - 4r^{-2}V\cos(2\theta) = 0$$

$V_{zz} + V_{rr} + r^{-1}V_r - 4r^{-2}V = 0$ $\qquad$ BC's: $V(r,5) = 0$ $\quad \underbrace{v(r,0) = r^2}_{Z \text{ is "time like"} \Rightarrow \text{Solve for } R \text{ first}}$ $\quad v(5,z) = 0$

Try SV here so look for solutions of the form $R(r)Z(z)$

$$Z''R + R''Z + r^{-1}R'Z - 4r^{-2}RZ = 0 \quad \div RZ$$

$$-\frac{Z''}{Z} = \frac{R''}{R} + \frac{R^{-1}}{R\cdot r} - \frac{4}{r^2} = \text{constant}$$

So solve for R <u>first</u>: $\frac{R''}{R} + \frac{R}{r\cdot R} - \frac{4}{r^2} = \lambda = \text{constant}$

$$r^2R''+rR'+(\underset{\text{order}=2}{-4}-\lambda r^2)R=0$$

This is a Bessel-type eqn. of order 2, unless $\lambda=0 \to$ Euler eqn

No physical solutions from Euler eqn. satisfy the BC

Solutions are $\underset{\text{Never zero}}{I_2}, \underset{\text{Blow up @}r=0 \Rightarrow \text{Not physical}}{K_2}, \boxed{J_2}, Y_2$ $J_2 \Rightarrow \lambda<0$

$R_n(r)=J_2(\dots r)$ need $R(5)=0 \Rightarrow \dots 5$ = zero of J_2

Let $\pi_n = n^{th}$ positive zero of J_2. Then $R_n(r)=J_2\left(\frac{\pi_n r}{5}\right)$ $\lambda_n=-\frac{\pi_n^2}{25}$

Now solve for Z: $\frac{Z''}{Z}=\lambda_n=\frac{-\pi_n^2}{25}$ Basis of solutions: $\left\{e^{\frac{\pi_n z}{5}}, e^{-\frac{\pi_n z}{5}}\right\}$

$$Z_n(z)=a_n\cosh\left(\frac{\pi_n z}{5}\right)+b_n\sinh\left(\frac{\pi_n z}{5}\right) \longrightarrow v(r,z)=\sum_{n\geq 1} Z_n(z)R_n(r)$$

Use BCs in z to find the coeffs:

$$v(r,0)=\sum_{n\geq 1} a_nR_n(r)=r^2 \Rightarrow a_n=\frac{\langle r^2,R_n\rangle}{\|R_n\|^2}=\frac{\int_0^5 r^2\overline{R_n(r)}\,r\,dr}{\int_0^5 |R_n(r)|^2\,r\,dr}$$

$$v(r,5)=\sum_{n\geq 1}\left(a_n\cosh(\pi_n)+b_n\sinh(\pi_n)\right)R_n(r)=0 \Rightarrow b_n=\frac{-a_n\cosh(\pi_n)}{\sinh(\pi_n)}$$

3. $Q(x)$ = polynomial of degree ≤ 3 that minimizes $\int_{\mathbb{R}}|Q(x)-e^x|^2e^{-x^2}dx$

french poly.

Hermite! $H_n(x)=n^{th}$ Hermite poly.

$$Q(x)=\sum_{n=0}^{3}\frac{\langle e^t,H_n\rangle}{\|H_n\|^2}H_n(x) \quad \text{with} \quad \langle e^t,H_n\rangle=\int_{\mathbb{R}}e^t\,\overline{H_n(t)}\,e^{-t^2}dt \qquad \|H_n\|^2=\int_{\mathbb{R}}|H_n(t)|^2dt$$

4. $$\begin{cases} u_{tt}=u_{xx} & 0<x<\pi \quad 0<t \\ \boxed{u_x(0,t)=1, \quad u(\pi,t)=1} & \text{Fix these first} \\ u(x,0)=0, \quad u_t(x,0)=0 & \end{cases}$$

Inhomo. pde

Can we find $S(x)$ with $S(x)$ with $S'(0)=1$ $S(\pi)=-1$ $S''(x)=0$?

$S(x) = ax+b \quad S'(x) = a \Rightarrow a = 1 \Rightarrow S(x) = x - \pi - 1$

Let $u = S + v$ with v solving: $\begin{cases} v_{tt} = v_{xx} \\ v_x(0,t) = 0 = v(\pi,t) \\ v(x,0) = -S(x) \quad v_t(x,0) = 0 \end{cases}$

Solve for v using Sep Vars & Σuperposition $\quad T''X = X''T \Longleftrightarrow \frac{T''}{T} = \frac{X''}{X} = \lambda$

$X'(0) = 0 \quad X(\pi) = 0 \Rightarrow X_n(x) = \cos((n+\frac{1}{2})x)$

$\frac{T''}{T} = -(n+\frac{1}{2})^2 \Rightarrow T_n(t) = a_n \cos((n+\frac{1}{2})t) + b_n \sin((n+\frac{1}{2})t)$

$v(x,t) = \sum_{n\geq 0} X_n(x) T_n(t) \quad v(x,0) = \sum_{n\geq 0} a_n X_n(x) = -S(x)$

$a_n = \frac{\langle -S, X_n\rangle}{\|X_n\|^2} = \frac{\int_0^\pi -S(x)\overline{X_n(x)}\,dx}{\int_0^\pi |X_n(x)|^2 dx} \qquad v_t(x,0) = \sum_{n\geq 0} X_n(x) b_n (n+\frac{1}{2}) = 0 \Rightarrow b_n = 0 \;\forall n$

5. $\hat{f}(\omega) = \frac{\omega\,\theta(\omega)}{(1+\omega^2)^2}$ Compute ① $\int_{\mathbb{R}} f(t) e^{-|2t|} \operatorname{sgn}(t)\,dt$ FT

② $\int_{\mathbb{R}} f(t)\,dt \Rightarrow \hat{f}(0) = 0$

③ $\int_{\mathbb{R}} f(t)\cos(3t)\,dt \Rightarrow \frac{\hat{f}(3) + \hat{f}(-3)}{2}$

$\langle f, g\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle \quad \hat{g}(\omega) = \int_{\mathbb{R}} g(t) e^{-it\omega} dt = \int_{-\infty}^{0} e^{2t} e^{-it\omega} dt + \int_0^\infty e^{-2t} e^{-it\omega} dt$

$$= -\frac{1}{2-i\omega} - \frac{1}{-2-i\omega} = \frac{2+i\omega-(2-i\omega)}{-(4+\omega^2)} = -\frac{2i\omega}{4+\omega^2}$$

$\Rightarrow \langle f, g\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}(\omega)}\,d\omega = \frac{1}{2\pi}\int_0^\infty \frac{\omega}{(1+\omega^2)^2}\cdot\frac{2i\omega}{(4+\omega^2)} = \frac{i}{\pi}\int_0^\infty \frac{\omega^2}{(1+\omega^2)(4+\omega^2)}\,d\omega$