FOURIER ANALYSIS & METHODS LECTURE NOTES

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2020.01.20

According to Gerry, Fourier Analysis is "A collection of related techniques for solving the most important partial differential equations of physics (and chemistry)." For example, we're going to be solving partial differential equations, abbreviated PDEs

- Δ Laplace equations (related to computing energy of quantum particles)
- $\square$ wave equations (describes the propagation of waves, hence also of light and electromagnetic waves)
- Ξ heat equation (describes the propagation of heat, is the quintessential diffusion equation)

What is a PDE?

Definition 1. A PDE is an equation for an unknown function (unsub) which depends on $n > 1$ independent real variables. Writing u for the unknown function,

$$u : \mathbb{R}^n \to \mathbb{C}.$$

The PDE for u is an equation that u is supposed to satisfy and contains u together with one or more partial derivatives of u. The PDE may also contain other, *specified* functions.

Example 1. *The Laplace equation for a function on* $\mathbb{R}^2$ *is:*

$$u_{xx} + u_{yy} = 0.$$

The Laplace operator on $\mathbb{R}^2$ *is:*

$$\Delta = \partial_{xx} + \partial_{yy},$$

so writing it this way the Laplace equation looks like

$$\Delta u = 0.$$

The wave equation for a function on $\mathbb{R}^3 \times [0, \infty)_t$ *is*

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}.$$

Sometimes there is a constant on one side or the other, but mathematicians often use interesting time units to be able to assume 'without loss of generality' this constant is 1. The heat equation for a function on $\mathbb{R} \times [0, \infty)_t$ *is*

$$u_t = u_{xx}.$$

Similarly, I like to assume the constant is 1.

1.1. **The sound check analogy.** Have you ever noticed that at a metal concert, even if the band has played thousands of concerts, even in the exact same venue, they *always* do a sound check? Do you know why? It's because the sound produced by the band obeys the wave equation. This equation is really hard to solve. Moreover, it is really sensitive to the geometry of the space where the band plays. Even if it's the same venue, the number of people inside is not the same, and these people are part of the geometry of the space. So, every time they play, the band has to do a sound check to see how the geometry of everything is affecting the solution of the wave equation which is basically how the band sounds.

The wave equation, and indeed all PDEs are HARD to solve. There is no single unifying theory to guide us to the solution of all PDEs. It's like the metal band: we have to do a sound check for each and every concert. There is no magic pre-set we can use for all our concerts. Similarly, we have to deal with each and every PDE individually and carefully. To solve them, we must study a variety of methods and learn how to use these methods and combine them when possible.

1.2. **The first method: Separation of variables (SV).** If you come to the (obligatory for Kf, option for TM and F) extra three lectures, you will learn how to classify every PDE on the planet. For the great majority of these, we have no hope to solve then analytically (that is, to write down a mathematical formula as the solution to the PDE).

In case you have forgotten, here is a reminder.

Definition 2. An ODE is an equation for an unknown function (unsub) which depends on *one* independent real variable. Writing u for the unknown function, an ODE for u is an equation that u is supposed to satisfy and contains u together with one or more derivatives of u. The ODE may also contain other, *specified* functions.

Question 3. *What is the difference between an ODE and a PDE?*[1]

So, to introduce the technique of separation of variables, let's think about a really down-to-earth example. A vibrating string, like the guitar or bass strings in our metal band. The ends of the string are held fixed, so they're not moving. You know this if you play or watch people play guitar. Let's mathematicize the string, by identifying it with the interval $[0, \ell] \subset \mathbb{R}$. The string length is ℓ. Let's define

$$u(x, t) := \text{ the height of the string at the point } x \in [0, \ell] \text{ at time } t \in [0, \infty[.$$

[1]Answer: the unknown function (unsub) in an ODE depends on only one variable, so the derivatives in the equation are 'ordinary derivatives.' The unknown function in a PDE depends on at least two variables, so we can no longer speak of ordinary derivatives, because the only derivatives that make sense when a function depends on two or more variables are *partial derivatives.* So, it's just a matter of how many variables does the unknown function in the equation depend on?

Then, let's just define the sitting-still height to be height 0. So, the fact that ends are sitting still means that

$$u(0,t) = u(\ell,t) = 0 \quad \forall t.$$

A positive height means above the sitting-still height, whereas a negative height means under the sitting-still height. The wave equation (I'm not going to derive it, but maybe you clever physics students can do that?) says that:

$$u_{xx} = \mathfrak{c}^2 u_{tt}.$$

The constant $\mathfrak{c}$ depends on how fast the string vibrates.

Question 4. *Is this equation a PDE or an ODE?*[2]

Technique 0 = Separation of Variables starts like this: we *assume* that

$$u(x,t) = X(x)T(t),$$

that is a product of two functions, each of which depends only on *one* variable. Why can we do this? Who knows, maybe it is rubbish! Maybe u is not of this form. Kind of like the sound check: we guess at the sound levels and then play a bit to see if it sounds good. Same here. We just have to try.

Assuming that u is of this form, we put this into the PDE:

$$u_{xx} = \mathfrak{c}^2 u_{tt} \iff X''(x)T(t) = \mathfrak{c}^2 X(x)T''(t).$$

Now, we would like to *separate variables* by getting everything dependent on x to one side of the equation and everything dependent on t to the other side. To achieve this, we divide both sides by $X(x)T(t)$:

$$\frac{X''}{X}(x) = \mathfrak{c}^2 \frac{T''}{T}(t).$$

Stop. Think. The left side depends only on x, whereas the right side depends only on t.

Exercise 1. *Explain in your own words why if one side of an equation depends on x and the other side depends on t, then both sides must be constant.*

What should we solve for first? X or T? We've got more information on X than we do on T, because we know that the ends are still. This means that

$$X(0) = X(\ell) = 0.$$

So, the equation for just f is

$$\frac{X''}{X}(x) = \text{ constant },$$
$$X(0) = X(\ell) = 0.$$

Let's give the constant a name. Call it λ. Then write

$$X''(x) = \lambda X(x), \quad X(0) = X(\ell) = 0.$$

Well, we can solve this. There are three cases to consider:

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[2] Answer: it's a PDE because the function depends on two independent variables: position on the string x and time t.

$\lambda = 0$ This means $X''(x) = 0$. Integrating both sides once gives $X'(x) =$ constant $= m$. Integrating a second time gives $X(x) = mx + b$. Requiring $X(0) = X(\ell) = 0$, well, the first makes $b = 0$, and the second makes $m = 0$. So, the solution is $X(x) \equiv 0$. The 0 solution. The waveless wave. Not too interesting.

$\lambda > 0$ The solution here will be of the form

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

Exercise 2. *Show that it is equivalent to write the solution as $A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$, for two constants A and B. Determine the relationship between A and B and a and b. Show that in order to guarantee that $X(0) = X(\ell) = 0$ you need $a = A = B = b = 0$. You should do this exercise, because it I strongly suspect you can do it. Think of it as a warm-up for Folland's exercises.*

Thus, with our teamwork, (me providing hints and you doing the actual work by solving the exercise) we have gotten the 0 solution again. The waveless wave. No fun there.

$\lambda < 0$ Finally, we have solution of the form

$$a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To make $X(0) = 0$, we need $a = 0$. Uh oh... are we going to get that stupid 0 solution again? Well, let's see what we need to make $X(\ell) = 0$. For that we just need

$$b\sin(\sqrt{|\lambda|}\ell) = 0.$$

That will be true if

$$|\lambda| = \frac{k^2\pi^2}{\ell^2}, \quad k \in \mathbb{Z}.$$

Super! We still don't know what b ought to be, but at least we've found all the possible X's, up to constant factors.

Just to clarify the fact that we've now found *all* solutions, we recall here a theorem from your multivariable calculus class.

th:omc

Theorem 5 (Second order ODEs)**.** *Consider the second order linear homogeneous ODE,*

$$au'' + bu' + cu = 0, \quad a \neq 0.$$

If $b = c = 0$, then a basis of solutions is given by

$$\{x, 1\},$$

so that all solutions are of the form

$$u(x) = Ax + B, \quad A, B \in \mathbb{R}.$$

If $c = 0$, then a basis of solutions is $\{e^{-b/ax}, 1\}$ so that all real solutions are given by

$$u(x) = Ae^{-bx/a} + B.$$

If $c \neq 0$, then a basis of solutions is one of the following:

(1) $\{e^{r_1 x}, e^{r_2 x}\}$ if $b^2 \neq 4ac$, where

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(2) $\{e^{rx}, xe^{rx}\}$ *if* $b^2 = 4ac$*, with* $r = -\frac{b}{2a}$.

Exercise 3. *Our equation is*

$$X'' = \lambda X \iff X'' - \lambda X = 0.$$

So, in the language of the above theorem, $a = 1$, $b = 0$, *and* $c = \lambda$*. Use this to find all solutions which satisfy* $X(0) = X(\ell) = 0$.

The solutions we've found are, up to constant factors:

$$X_k(x) = \sin\left(\frac{k\pi x}{\ell}\right), \quad \lambda_k = -\frac{k^2\pi^2}{\ell^2}.$$

> Do not worry about the constant factors at this point in time. Save them for later.[3]

Now, let's find the friends of X, the time functions, T which depend only on time. These come in pairs, so that X_1 comes together with T_1. This is because the value of the constant λ_1, comes from X_1. However, we've also got X_2, and the value of the constant λ_2 is different. So, for each pair we have

$$\frac{X_k''}{X_k} = \lambda_k = -\frac{k^2\pi^2}{\ell^2} = \mathfrak{c}^2\frac{T_k''}{T_k}.$$

This is almost the same equation we had before. Here we have, re-arranging:

$$T_k'' = -\frac{k^2\pi^2}{\mathfrak{c}^2\ell^2}T_k.$$

Exercise 4. *Use Theorem* th:omc *5 to show that a basis of solutions is given by*

$$\left\{e^{\frac{ik\pi t}{\mathfrak{c}\ell}}, e^{-\frac{ik\pi t}{\mathfrak{c}\ell}}\right\}.$$

Show that it is equivalent to use

$$\left\{\cos\left(\frac{k\pi t}{\mathfrak{c}\ell}\right), \sin\left(\frac{k\pi t}{\mathfrak{c}\ell}\right)\right\}$$

as a basis. Hint: remember $e^{i\theta} = \cos\theta + i\sin\theta$ *for* $i = \sqrt{-1}$ *for any* $\theta \in \mathbb{R}$.

Let us pause to think about what this means. The physics students may recognize that the numbers

$$\{|\lambda_k|\}_{k\geq 1}$$

are the resonant frequencies of the string. Basically, they determine how it sounds. The number $|\lambda_1|$ is the fundamental tone of the string. The higher $|\lambda_k|$ for $k \geq 2$ are harmonics. It is interesting to note that they are all square-integer multiplies of λ_1. Here's a question: if you can "hear" the value of $|\lambda_1|$, then can you tell me how long the string is? Well, yes, cause

$$|\lambda_1| = \frac{1}{\ell^2}, \implies \ell = \frac{1}{\sqrt{|\lambda_1|}}.$$

So, you can hear the length of a string. A couple of famous unsolved math problems: can one hear the shape of a convex drum? Can one hear the shape of a smoothly bounded drum? We can talk about these problems if you're interested.

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[3] The reason we should do this is because the less baggage we are carrying around, (i.e. the fewer symbols we got to write), the less likely we are to screw something up. So, we should remember the patience principle and be patient, wait to get the constants later.

So, now that we've got all these solutions, what should we do with them? Good question...

1.3. **Superposition principle and linearity.** Superposition basically means adding up a bunch of solutions. You can think of it like adding up a bunch of solutions to get a super solution!

Definition 6. A second order linear PDE for an unknown function u of n variables is an equation for u and its mixed partial derivatives up to order two of the form

$$L(u) = f,$$

where f is a given function, and there are known functions $a(x)$, $b_i(x)$, $c_{ij}(x)$ for $x \in \mathbb{R}^n$ such that

$$L(u) = a(x)u(x) + \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + \sum_{i,j=1}^{n} c_{ij}(x) u_{ij}(x).$$

In this context, L is called a *second order linear partial differential operator.*

The reason it's called linear is because it's well, linear.

Exercise 5. *For two functions u and v, which depend on n variables, show that*

$$L(u+v) = L(u) + L(v).$$

Moreover, for any constant $c \in \mathbb{R}$, show that

$$L(cu) = \mathfrak{c}L(u).$$

Definition 7. The wave operator, $\square$, defined for $u(x,y)$ with $(x,y) \in \mathbb{R}^2$ is

$$\square(u) = -u_{xx} + \mathfrak{c}^2 u_{tt}.$$

Exercise 6. *Verify that the wave operator is a second order linear partial differential operator.*

We have shown that the functions

$$u_k(x,t) = X_k(x) T_k(t)$$

satisfy

$$\square u_k = 0 \forall k.$$

Hence, if we add them up this remains true:

$$\square(u_1 + u_2 + u_3 + \ldots) = 0.$$

OBS![4]

Exercise 7. *Show that the equations*

$$X_k'' = \lambda_k X_k \iff f_k'' - \lambda_k X_k = 0$$

do not *add up. In particular, show that just the first two of these equations do not add up,*

$$X_1'' + X_2'' - (\lambda_1 + \lambda_2)(X_1 + X_2) \neq 0.$$

[4] I love this Swedish expression. Nothing quite like it in the languages I know. Well, the closest is maybe 小心 which is also very cute.

The reason these equations do not add up is because it's not the same L. *The equation for* X_k *is*

$$X_k'' - \lambda_k X_k = 0.$$

This depends on k*, and since each* $\lambda_1 \neq \lambda_2 \neq \lambda_3, \ldots$*, the differential operator is*

$$L_k = \frac{d^2}{dx^2} + \lambda_k.$$

This exercise shows that one must take care when smashing solutions (i.e. superposing) together!

When we look at the different $u_k(x,t)$ in the *wave equation,* it's all good, because it's always the same wave operator. Hence, we may indeed smash all our solutions together, include the (to be determined) coefficients, and write

$$u(x,t) = \sum_{k \geq 1} u_k(x,t) = \sum_{k \geq 1} \sin\left(\frac{k\pi x}{\ell}\right)\left(a_k \cos\left(\frac{k\pi t}{\mathfrak{c}\ell}\right) + b_k \sin\left(\frac{k\pi t}{\mathfrak{c}\ell}\right)\right),$$

and it satisfies

$$\Box u(x,t) = 0, \quad u(0,t) = u(\ell,t) = 0.$$

We've still got some unanswered questions:

(1) What are the constants a_k and b_k?
(2) If we can figure out what the constants are, then we are still left with this thing:
$$\sum_{k \geq 1} \sin\left(\frac{k\pi x}{\ell}\right)\left(a_k \cos(k\pi t/\ell) + b_k \sin(k\pi t/\ell)\right).$$
Is this hot mess going to converge?

2. Exercises to be done by oneself

1.1.1 Show that $u(x,t) = t^{-1/2} e^{-x^2/(4kt)}$ satisfies the heat equation
$$u_t = k u_{xx}.$$

1.2.5(a) Show that for $n = 1, 2, 3, \ldots$ $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies
$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

1.3.5 By separation of variables, derive the solutions $u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ of
$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

1.3.7 Use separation of variables to find an infinite family of independent solutions to
$$u_t = k u_{xx}, \quad u(0,t) = 0, \quad u_x(\ell,t) = 0,$$
representing heat flow in a rod with one end held at temperature zero and the other end insulated.

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1. 2019.01.22

Let's look at another example. Consider a circular shaped rod, like a rod that's been bent into a circle. Let's mathematicize it! To specify points on the rod, we just need to know the angle at the point. For this reason, we use the real variable x for the position, where x gives us the angle at the point on the rod. We use the variable $t \geq 0$ for time. The function $u(x,t)$ is the temperature on the rod at position x at time t.

The heat equation (homogeneous, which means no sources or sinks) tells us that:

$$u_t = ku_{xx},$$

for some constant $k > 0$. At this point our only techniques are separation of variables and superposition. We first use separation of variables to find solutions. So, let us do the same first step as we did in solving the homogeneous wave equation. It's just a means to an ends, by writing

$$u(x,t) = X(x)T(t).$$

Plug it into the heat equation:

$$T'(t)X(x) = kX''(x)T(t).$$

We want to *separate variables,* so we want all the t-dependent bits on the left say, and all the x-dependent bits on the right. This can be achieved by dividing both sides by $X(x)T(t)$,

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)}.$$

We now know that both sides must be constant. Let us call the constant λ, so that

$$\frac{T'}{T} = \lambda = k\frac{X''}{X}.$$

Exercise 1. *In your own words, explain why both sides of the equation must be constant.*

Now, we need to pick a side to begin... We actually have some information which is hiding inside the *geometry* of the problem. The geometry is referring to the x variable. What can you say about the angle x on the rod and the angle $x + 2\pi$ on the rod? They are the same. This means that our temperature function must be the same at x and at $x+2\pi$. So, we must have

$$X(x+2\pi) = X(x).$$

We can repeat this, obtaining

$$X(x+2\pi n) = X(x) \quad \forall n \in \mathbb{Z}.$$

This means that X is a periodic function with period equal to 2π. So, we have a bit of extra information about it. The equation for X is:

$$X''(x) = \frac{\lambda}{k} X(x)$$

for a constant λ.

Exercise 2. ***Case 1:*** *Show that if $\lambda = 0$, there is no solution to $X''(x) = 0$ which is 2π periodic, other than the constant solutions.*

Case 2: If $\lambda > 0$, then a basis of solutions is,

$$\{e^{\sqrt{\lambda}x/\sqrt{k}}, e^{-\sqrt{\lambda}x/\sqrt{k}}\}.$$

So, we can write

$$X(x) = ae^{\sqrt{\lambda}x/\sqrt{k}} + be^{-\sqrt{\lambda}x/\sqrt{k}}.$$

For the 2π periodicity to hold, we need

$$X(0) = X(2\pi) \implies a+b = ae^{\sqrt{\lambda}2\pi/\sqrt{k}} + be^{-\sqrt{\lambda}2\pi/\sqrt{k}} \implies a(e^{\sqrt{\lambda}2\pi/\sqrt{k}}-1) = b(1-e^{-\sqrt{\lambda}2\pi/\sqrt{k}})$$

$$\implies a = b\frac{(1-e^{-\sqrt{\lambda}2\pi/\sqrt{k}})}{e^{\sqrt{\lambda}2\pi/\sqrt{k}}-1}.$$

We also need

$$X(-2\pi) = X(0) \implies a+b = ae^{-\sqrt{\lambda}2\pi/\sqrt{k}} + be^{\sqrt{\lambda}2\pi/\sqrt{k}} \implies a(e^{-\sqrt{\lambda}2\pi/\sqrt{k}}-1) = b(1-e^{\sqrt{\lambda}2\pi/\sqrt{k}})$$

$$\implies a = b\frac{1-e^{\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{-\sqrt{\lambda}2\pi/\sqrt{k}}-1}.$$

So, we have two equations for a, therefore they should be equal:

$$a = b\frac{1-e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{\sqrt{\lambda}2\pi/\sqrt{k}}-1} = b\frac{1-e^{\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{-\sqrt{\lambda}2\pi/\sqrt{k}}-1}.$$

If $b=0$ then $a=0$ so the whole solution is the zero solution. If $b \neq 0$ then we must have

$$\frac{1-e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{\sqrt{\lambda}2\pi/\sqrt{k}}-1} = \frac{1-e^{\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{-\sqrt{\lambda}2\pi/\sqrt{k}}-1}.$$

Changing the sign of the top and bottom on the right side, this is equivalent to:

$$\frac{1-e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}{e^{\sqrt{\lambda}2\pi/\sqrt{k}}-1} = \frac{e^{\sqrt{\lambda}2\pi/\sqrt{k}}-1}{1-e^{-\sqrt{\lambda}2\pi/\sqrt{k}}}.$$

Call the left side $\star$. Then the right side is $\frac{1}{\star}$. So the equation is

$$\star = \frac{1}{\star} \implies \star^2 = 1 \implies \star = \pm 1.$$

Exercise 3. *Show that $\star > 0$.*

If
$$\star = 1 \implies 1 - e^{-\sqrt{\lambda}2\pi/\sqrt{k}} = e^{\sqrt{\lambda}2\pi/\sqrt{k}} - 1 \implies 2 = e^{\sqrt{\lambda}2\pi/\sqrt{k}} + e^{-\sqrt{\lambda}2\pi/\sqrt{k}}.$$

I don't like the negative exponent thing (it is really a fraction), so I am going to multiply by $e^{\sqrt{\lambda}2\pi/\sqrt{k}}$. Also, doing this turns it into a quadratic equation:

$$2e^{\sqrt{\lambda}2\pi/\sqrt{k}} = e^{4\pi\sqrt{\lambda}/\sqrt{k}} + 1 \iff e^{4\pi\sqrt{\lambda}/\sqrt{k}} - 2e^{2\pi\sqrt{\lambda}/\sqrt{k}} + 1 = 0$$

Now we can factor this equation because the left side is

$$(e^{2\pi\sqrt{\lambda}/\sqrt{k}} - 1)^2 = 0 \implies e^{2\pi\sqrt{\lambda}/\sqrt{k}} = 1 \iff 2\pi\sqrt{\lambda}/\sqrt{k} = 0 \lightning.$$

That $\lightning$ indicates a contradiction. Therefore, in the case where $\lambda > 0$, the only solution which is 2π periodic is the zero solution.

Hence, we are left with **Case 3:** $\lambda < 0$. Then, a basis of solutions is

$$\{\sin(\sqrt{|\lambda|}x/\sqrt{k}), \cos(\sqrt{|\lambda|}x/\sqrt{k}).$$

We need these solutions to be 2π periodic. They will be as long as $\sqrt{|\lambda|}/\sqrt{k}$ is an integer. So we need

$$\lambda < 0, \quad \frac{\sqrt{|\lambda|}}{\sqrt{k}} = n \in \mathbb{Z} \implies \lambda_n = -n^2 k.$$

Hence, our solution

$$X_n(x) = a_n \cos(nx) + b_n \sin(nx), \quad n \in \mathbb{N}_0.$$

Exercise 4. *Show that allowing complex coefficients, it is equivalent to use a basis of solutions*

$$\{e^{\pi inx}\}_{n \in \mathbb{Z}}.$$

Find A_n and B_n in terms of a_n and b_n so that

$$X_n(x) = A_n e^{inx} + B_n e^{-inx}.$$

Now, we can solve for the partner function, $T_n(t)$. Since

$$\frac{T_n'(t)}{T_n(t)} = \lambda_n = -n^2 k,$$

the equation for T_n is

$$T_n'(t) = -n^2 k T_n(t).$$

Consequently,

$$T_n(t) = e^{-n^2 kt} \text{ up to constant factor.}$$

So, we now have found the solutions

$$u_n(x,t) = X_n(x)T_n(t) = e^{-n^2 kt}(a_n \cos(nx) + b_n \sin(nx)).$$

These solutions satisfy the heat equation

$$\partial_t u_n - k\partial_{xx} u_n = 0.$$

Let us define the heat operator for functions of one real variable and one time variable,

$$\Xi := \partial_t - k\partial_{xx}.$$

Then we have

$$\Xi u_n(t) = 0 \forall n \in \mathbb{N}_0.$$

Consequently, we can use the superposition principle to smash all these solutions we have found into a super solution

$$u(x,t) = \sum_{n \geq 0} u_n(x,t) = \sum_{n \geq 0} e^{-n^2 tk}(a_n \cos(nx) + b_n \sin(nx)).$$

We do this because we do not know how many of the u_n functions we will need. In case we don't end up needing them all, then their coefficients will be zero, so they will just disappear on their own anyways. Let's think about the physics. The rod has some temperature function at time $t = 0$, which we call $u_0(x)$. Then $u_0(x)$ is also a 2π periodic function. We would like

$$u(x,0) = u_0(x) \iff \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx) = u_0(x).$$

So, given $u_0(x)$, can we find a_n and b_n so that this is true?

Fourier made the bold statement that we can do this. It took a long time to rigorously prove him right (like 100 years, because this whole theory about Hilbert spaces, measure theory, and functional analysis needed to get developed by Hilbert & his contemporaries).

1.1. **Introduction to Fourier Series of periodic functions.** If we have a finite one dimensional, connected set, then we can always mathematicize it as either (1) a bounded interval or (2) a circle. When we take a bounded interval of length 2ℓ, and we take any function whatsoever on that interval, we can always extend it to the rest of $\mathbb{R}$ to be 2ℓ periodic, by simply repeating its values from the interval. Hence, for both of these contexts we can do everything we desire with periodic functions.

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p iff for all $x \in \mathbb{R}$, $f(x + p) = f(x)$, and moreover, $p > 0$ is the smallest real number for which this is true.

For example, $\sin(x)$ is periodic with period 2π. Our heat equation examples, $f_n(x) = a_n \cos(nx) + b_n \sin(nx)$ are periodic with period $2\pi/n$. We shall prove a super useful little lemma about periodic functions and their integrals.

Lemma 2 (Integration of periodic functions lemma)**.** *If f is periodic with period p then for any $a \in \mathbb{R}$*

$$\int_a^{a+p} f(x)dx$$

is the same.

Exercise 5. *Give an example for how this fails to be true if the function f is* not *periodic. That is, take some non-periodic function and show that integrating it from say a to $a + p$ is not the same as integrating it from c to $c + p$.*

Proof: If we think about it, we want to show that the function

$$g(a) := \int_a^{a+p} f(x)dx$$

is a constant function. This looks awfully similar to the fundamental theorem of calculus. Now, this statement above is not true for non-periodic functions. So, we're going to need to use the assumption that f is periodic with period p. This tells us that f has the same value at both endpoints of the integral, so

$$f(a) = f(a+p) \implies f(a+p) - f(a) = 0.$$

Now, since we want to consider a as a variable, we don't want it at both the top and the bottom of the integral defining g. Instead, we can use linearity of integration to write

$$g(a) = \int_0^{a+p} f(x)dx - \int_0^a f(x)dx.$$

Then, using the fundamental theorem of calculus on each of the two terms on the right,

$$g'(a) = f(a+p) - f(a) = 0.$$

Above, we use the fact that f is periodic with period p. Hence, $g'(a) \equiv 0$ for all $a \in \mathbb{R}$. This tells us that g is a constant function, so its value is the same for all $a \in \mathbb{R}$.

So you survived a bit of theory, now let's return to our physical motivation! We wanted to find coefficients so that the $u(x,t)$ we found to solve the heat equation would match up with the initial data, $u_0(x)$. If it does, then (using some advanced PDE theory beyond the scope of this humble course), $u(x,t)$ is indeed THE UNIQUE solution to the heat equation with initial data $u_0(x)$. Hence, $u(x,t)$ actually tells us the temperature on the rod at position x at time t. Cool. So, setting $t=0$ in the definition of $u(x,t)$ we want

$$u_0(x) = \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx). \tag{1.1}$$

It is totally equivalent to work with complex exponentials, because

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}.$$

Exercise 6. *Show that we can write $u_0(x)$ as a series above in (1.1) if and only if we can write*

$$u_0(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Moreover, show that

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1, \quad c_n = \frac{1}{2}(a_n + ib_n), n \leq -1.$$

Finally, use this to show that

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n},\, n \geq 0, \quad b_n = i(c_n - c_{-n}),\, n \geq 0.$$

It is slightly more convenient for these purposes to do the calculation using the $\{e^{inx}\}_{n \in \mathbb{Z}}$ basis. This will be elucidated in a moment. The equation we want to obtain is:

$$u_0(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

The object on the right is a sum of coefficients $c_n \in \mathbb{C}$ times functions e^{inx}. It is simply a linear combination of the functions e^{inx}. If we could show that in a suitable sense these functions for a sort of "basis" then we should be able to expand our function u_0 in terms of this basis. Sure, the basis is infinite, so, you've graduated to "linear algebra for adults," in which your vectors are now infinite dimensional. [1] To continue with the linear algebra concept, we need a notion of dot product, in order to expand u_0 in terms of our basis functions e^{inx}. This is obtained using something called a scalar product, or dot product, or inner product: they all mean the same thing.

Definition 3. For two functions, f and g, which are real or complex valued functions defined on $[a, b] \subset \mathbb{R}$, we define their scalar product to be

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

We say that f and g are orthogonal if $\langle f, g\rangle = 0$. We define the $L^2([a, b])$ *norm* of a function to be

$$||f||_{L^2([a,b])} = \sqrt{\langle f, f \rangle}.$$

OBS! Learn this definition right now!!!! It is really important. Every detail:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx, \quad ||f||^2 = \langle f, f\rangle.$$

Now, if you wonder *why* it is defined this way, that is because defining things this way has the very pleasant consequence that it *works.* Meaning, when we define things this way, we are able to use the separation of variables technique to solve the PDEs.

2. Exercises to be done by oneself: Hints

1.1.1 Show that $u(x, t) = t^{-1/2}e^{-x^2/(4kt)}$ satisfies the heat equation

$$u_t = ku_{xx}.$$

Hint: Use the product rule when you're differentiating with respect to t. When you're differentiating with respect to x, remember that from x's perspective, t is just a constant.

1.2.5(a) Show that for $n = 1, 2, 3, \ldots$ $u_n(x, y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(1, y) = u(x, 0) = 0.$$

Hint: Use the product rule and remember that in the eyes of x, $\sinh(n\pi y)$ is constant. Similarly, in the eyes of y, $\sin(n\pi x)$ is constant.

1.3.5 By separation of variables, derive the solutions $u_n(x, y) = \sin(n\pi x)\sinh(n\pi y)$ of

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(1, y) = u(x, 0) = 0.$$

Hint: Start by writing $u(x, y) = X(x)Y(y)$. Plop it into the PDE. Get all the x dependent terms to one side of the equation and the y dependent terms to the other side. (probably do this by dividing by XY). Solve for X first. Use the conditions on $X(0) = X(1) = 0$. (Why?) Then once you have found your Xs (there will be many!) find their partner functions. Use the condition $Y(0) = 0$ (Why?) to help with this.

[1] Grigori Rozenblioum, who taught this class for many years, and is in general an awesome mathematician, used to say "If you can pass this course, then you've earned the right to buy Vodka at Systembolaget, regardless of your actual age."

1.3.7 Use separation of variables to find an infinite family of independent solutions to

$$u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(\ell, t) = 0,$$

representing heat flow in a rod with one end held at temperature zero and the other end insulated. **Hint:** Start by writing $u(x,t) = X(x)T(t)$. Follow the same type of procedure as for the preceding problem, but now you have the conditions on X that $X(0) = 0$, $X'(\ell) = 0$ (Why?) Find the X first (there will be many!), and then use these to find their partner functions. It will be kind of similar to the example from lecture today, but the boundary conditions are different, so this will change things.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.01.24

Proposition 1. *On the interval* $[-\pi, \pi]$, *the functions*

$$\phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$$

are an orthonormal set with respect to the scalar product,

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

Proof: By definition, we consider

$$\int_{-\pi}^{\pi} \frac{e^{inx}}{\sqrt{2\pi}} \frac{\overline{e^{imx}}}{\sqrt{2\pi}} dx.$$

We bring the constant factor out in front of the integral the constant factor, and we recall that $\overline{e^{imx}} = e^{-imx}$, so we are computing

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx.$$

Exercise 1. *Why is*

$$\overline{e^{imx}} = e^{-imx}?$$

Explain in your own words or prove it algebraically.

So, we compute,

$$\int_{-\pi}^{\pi} e^{ix(n-m)} dx = \begin{cases} 2\pi & m = n \\ \frac{e^{ix(n-m)}}{n-m}\Big|_{x=-\pi}^{\pi} & n \neq m \end{cases}.$$

Now, we know that

$$e^{i\pi(n-m)} = \begin{cases} 1 & n - m \text{ is even} \\ -1 & n - m \text{ is odd.} \end{cases}.$$

To see this, I just imagine where we are on the Liseberghjul... Or you can write this out as

$$e^{i\pi(n-m)} = \cos(\pi(n - m)) + i \sin(\pi(n - m)).$$

The sine term is always zero since n and m are integers, and the cosine is either 1 or -1. Similarly,

$$e^{-i\pi(n-m)} = \begin{cases} 1 & n-m \text{ is even} \\ -1 & n-m \text{ is odd.} \end{cases}.$$

So in all cases, when $n \neq m$,

$$e^{i\pi(n-m)} - e^{-i\pi(n-m)} = 0.$$

Hence,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = \begin{cases} \frac{2\pi}{2\pi} = 1 & n = m \\ 0 & n \neq m \end{cases}.$$

This is precisely what it means to be orthonormal!

So, now we know that $\{\phi_n(x)\}_{n \in \mathbb{Z}}$ are an orthonormal *set.* We want them to actually be an orthonormal *basis,* so that we can write for any $u_0(x)$,

$$u_0(x) = \sum_{n \in \mathbb{Z}} c_n \phi_n(x), \quad \phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}.$$

In analogue to linear algebra, we should expect the coefficients to be the scalar product of our function $u_0(x)$ with the basis functions (vectors), $\phi_n(x)$. More generally, for a 2π periodic function $v(x)$, we hope to be able to write it as

$$v(x) = \sum_{n \in \mathbb{Z}} c_n \phi_n(x), \quad c_n = \int_{-\pi}^{\pi} v(x) \overline{\phi_n(x)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} v(x) e^{-inx} dx,$$

so that

$$v(x) = \sum_{n \in \mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{-in\xi} d\xi \right) e^{inx}.$$

This motivates:

Definition 2. Assume f is defined $[-\pi, \pi]$. The Fourier coefficients of f are

$$c_n := \frac{1}{2\pi} \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The Fourier series of f is

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

1.1. **Computing Fourier series.** Let's start with the function $f(x) = |x|$. It satisfies $f(-\pi) = f(\pi)$. We will prove later that the Fourier series which is defined to be

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

converges to $f(x)$ for all points $x \in (-\pi, \pi)$. What happens at the endpoints $\pm\pi$? We must postpone this question for now. Looking at the series, we make the following observation

$$\sum_{n \in \mathbb{Z}} c_n e^{in(x+2\pi)} = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Consequently, the series is 2π periodic. So, although the series will converge to $f(x) = |x|$ for $x \in (-\pi, \pi)$, because we are going to prove that it does, once we leave this interval, the series will no longer converge to $f(x) = |x|$. The series will converge to the function which is equal to $f(x) = |x|$ inside the interval $(-\pi, \pi)$, and which is 2π periodic on the whole real line. So, the function to which the series converges has a graph that looks like a zig-zag. It's really important to keep this in mind.

So, now let's compute the Fourier coefficients:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|e^{-inx}dx, \quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|dx = \frac{2\pi^2}{2(2\pi)} = \frac{\pi}{2}.$$

Since

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

we compute:

$$\int_{-\pi}^{0} -xe^{-inx}dx, \quad \int_{0}^{\pi} xe^{-inx}dx.$$

We do substitution in the first integral to change it:

$$\int_{-\pi}^{0} -xe^{-inx}dx = \int_{0}^{\pi} xe^{inx}dx = \left.\frac{xe^{inx}}{in}\right|_0^{\pi} - \int_0^{\pi} \frac{e^{inx}}{in}dx$$

$$= \frac{\pi e^{in\pi}}{in} - \frac{e^{in\pi}}{(in)^2} + \frac{1}{(in)^2}.$$

Similarly we also use integration by parts to compute

$$\int_0^{\pi} xe^{-inx}dx = \left.\frac{xe^{-inx}}{-in}\right|_0^{\pi} - \int_0^{\pi} \frac{e^{-inx}}{(-in)}dx$$

$$= \frac{\pi e^{-in\pi}}{-in} - \frac{e^{-in\pi}}{(-in)^2} + \frac{1}{(-in)^2}.$$

Adding them up and using the 2π periodicity, we get

$$\frac{2e^{in\pi}}{n^2} - \frac{2}{n^2} = \frac{2(-1)^n - 2}{n^2}.$$

OBS! We need to divide by 2π to get

$$c_n = \frac{(-1)^n - 1}{\pi n^2}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

The Fourier series is therefore

$$\frac{\pi}{2} + \sum_{n \in \mathbb{Z},\ \text{odd}} e^{inx}\left(-\frac{2}{\pi n^2}\right).$$

Exercise 2. *Use these calculations to compute the series*

$$\sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx)$$

and to show that all of the b_n are equal to zero.

Now let's return to our example from Wednesday. We wish to solve the heat equation on a circular rod. Let

$$u(x,t) = \text{ the temperature at the point/angle } x \text{ and time } t.$$

Then the heat equation (physics!) dictates that

$$u_t - ku_{xx} = 0 \quad \forall x \in \mathbb{R}, \quad t > 0.$$

Above $k > 0$ is a constant which comes from - you guessed it - physics! There is some initial temperature along the rod as well,

$$u(x,0) = f(x).$$

Since the rod is circular,

$$u(x+2\pi,t) = u(x,t) \quad \forall x \in \mathbb{R},$$

so similarly,

$$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}.$$

When we solved the heat equation using separation of variables we obtained a solution which could be written either using complex exponentials or using sines and cosines. For simplicity, I am taking the complex exponentials,

$$u(x,t) = \sum_{n\in\mathbb{Z}} e^{-n^2kt} c_n e^{inx}.$$

So, we would like

$$u(x,0) = \sum_{n\in\mathbb{Z}} c_n e^{inx} = f(x).$$

Now we know how to find the coefficients,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

For the function, for example, $f(x) = |x|$ for $x \in (-\pi,\pi)$ which is defined on the rest of the real line to be 2π periodic, this is a function which makes sense as the initial temperature of the rod. We have computed these coefficients. The theory we will prove later will show that the Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$. Moreover, the theory will show that our solution $u(x,t)$ is the unique solution to the heat equation with initial condition given by f. Nice!

We are not limited to computing Fourier series of periodic functions, it's just that Fourier series will always be periodic functions themselves. For example, consider the function $f(x) = x$ defined on $(-\pi,\pi)$. By the theory we shall prove later, the Fourier series will converge to this function inside the interval $(-\pi,\pi)$. Outside this interval, the series will converge to a function which is 2π periodic, and is equal to x for $x \in (-\pi,\pi)$. So this will have little jumps at the points $(2n+1)\pi$ for $n \in \mathbb{Z}$. It will be discontinuous there. We don't need to worry about that, it's no problem whatsoever. For the moment we just are content that the Fourier series will converge to $f(x) = x$ for $x \in (-\pi,\pi)$. This is because in our applications, we will use these series to solve PDEs in bounded intervals. For now we are working with the bounded interval $(-\pi,\pi)$ but later we'll see that we can use the same techniques to handle any bounded interval.

Exercise 3. *Compute in the same way the Fourier coefficients*

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx \quad n \in \mathbb{Z}.$$

Use that calculation to show that $a_n = 0$ for all n, and then compute the Fourier sine series,

$$\sum_{n \geq 1} b_n \sin(nx).$$

Exercise 4. *Look at these two Fourier series, that is the series for $|x|$ and x. Do the series converge? Do they converge absolutely? Compare and contrast them!*

1.2. **Introducing Hilbert spaces.** A Hilbert space is a complete normed vector space whose norm is induced by a scalar product.

Definition 3. A Hilbert space, H, is a vector space. This means that H is a set which contains elements. If f and g are elements of H, then for any $a, b \in \mathbb{C}$ we have

$$af + bg \in H.$$

This is what it means to be a vector space. Moreover, Hilbert spaces have two other nice features: a scalar product and a norm. Let us write the scalar product as

$$\langle f, g \rangle : H \times H \to \mathbb{C}.$$

To be a scalar product it must satisfy:

$$\langle af, g \rangle = a \langle f, g \rangle \quad \forall a \in \mathbb{C},$$

$$\langle h + f, g \rangle = \langle h, g \rangle + \langle f, g \rangle,$$

and

$$\langle f, g \rangle = \overline{\langle g, f \rangle}.$$

The norm is defined through the scalar product via:

$$||f|| := \sqrt{\langle f, f \rangle}.$$

The norm must satisfy

$$||f|| = 0 \iff f = 0, \quad ||f + g|| \leq ||f|| + ||g||.$$

Finally, what it means to be complete is that if a sequence $\{f_n\} \in H$ is Cauchy, which means that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \varepsilon \quad \forall n, m \geq N,$$

then there exists $f \in H$ such that

$$\lim_{n \to \infty} f_n = f,$$

by which we mean that

$$\lim_{n \to \infty} ||f_n - f|| = 0.$$

Exercise 5. *As an example, we can take $H = \mathbb{C}^n$. For $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ the scalar product*

$$\langle z, w \rangle := \sum_{j=1}^{n} z_j \overline{w_j}.$$

Show that the scalar product defined in this way satisfies all the demands made upon it in the definition above. Why is $H = \mathbb{C}^n$ complete?

Now, let us fix a finite interval $[a, b]$. We shall be particularly interested in a Hilbert space known as $L^2([a, b])$ or once we have specified a and b, simply L^2. This is the actual grown-up mathematician definition of the Hilbert space, L^2. It can be gleefully ignored.

Definition 4 (The precise definition of L^2)**.** The Hilbert space $L^2([a, b])$ is the set of equivalence of classes of functions where f and g are equivalent if

$f(x) = g(x)$ for almost every $x \in [a, b]$ with respect to the one dimensional Lebesgue measure.

Moreover, for any f belonging to such an equivalence class, we require

l2finite
$$\int_a^b |f(x)|^2 dx < \infty. \tag{1.1}$$

If f and g are each members of equivalence classes satisfying (l2finite 1.1) the scalar product of f and g is then defined to be

l2sp
$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)} dx. \tag{1.2}$$

One can prove that with this definition we obtain a Hilbert space.

Theorem 5. *The space $L^2([a, b])$ for any bounded interval $[a, b]$ defined as above, with the scalar product defined as above, is a Hilbert space.*

This theorem is beyond the scope of this course. Moreover, the precise mathematical definition of L^2 is overkill for what we would like to do (solve PDEs). This is why I offer you:

Definition 6 (Our working-definition of L^2)**.** $L^2([a, b])$ is the set of functions which satisfy (l2finite 1.1), and is equipped with the scalar product defined in (l2sp 1.2).

Although we don't necessarily need it right now, you may be happy to know that the L^2 scalar product satisfies a Cauchy-Schwarz inequality,

$$|\langle f, g \rangle| \le ||f|| ||g||.$$

Exercise 6. *Use the Cauchy-Schwarz inequality to prove that for any $f \in L^2$ on the interval $[-\pi, \pi]$, the Fourier coefficients,*

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

satisfy

$$|c_n| \le \frac{||f||}{\sqrt{2\pi}}.$$

2. Exercises to be done by oneself: Answers

1.3.7 Use separation of variables to find an infinite family of independent solutions to

$$u_t = k u_{xx}, \quad u(0, t) = 0, \quad u_x(\ell, t) = 0,$$

representing heat flow in a rod with one end held at temperature zero and the other end insulated.

Answer:

$$u_n(x,t) = e^{-(2n+1)^2\pi^2 kt/(4l^2)} \sin\left(\frac{(2n+1)\pi x}{2l}\right).$$

FOURIER ANALYSIS & METHODS

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.01.27

The following proposition shows that any function that is bounded on a closed interval is an $\mathcal{L}^2$ function.

Proposition 1 (The Standard Estimate)**.** *Assume f is defined on some interval $[a, b]$. Assume that f satisfies a bound of the form $|f(x)| \leq M$ for $x \in [a, b]$.*[1] *Then,*

$$\left| \int_a^b f(x)dx \right| \leq (b-a)M.$$

Proof: Standard estimate!

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx \leq \int_a^b M dx = M(b-a).$$

Exercise 1. *Use The Standard Estimate to prove that any function which is continuous on a closed, bounded interval $[a, b]$ is in $\mathcal{L}^2$ on that interval.*

Example 1. So, it seems that a lot of functions will be in $\mathcal{L}^2$. What are some functions which are *not* in $\mathcal{L}^2$? Let's consider the interval $[-\pi, \pi]$. The function $f(x) = \frac{1}{x}$ is not in $\mathcal{L}^2$ on that interval, because

$$\int_{-\pi}^{\pi} \frac{1}{x^2} dx$$

is infinite. We could still have unbounded functions on this interval which *are* in $\mathcal{L}^2$, as long as their integrals can be defined. For example, let's define

$$f(x) := \begin{cases} x^{-1/3} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

[1] We actually only need this for "almost every" x, but to make that precise, we need some Lebesgue measure theory.

Then, we can integrate

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|x|^{5/3}3}{5}\Big|_{-\pi}^{\pi} = \frac{6\pi^{5/3}}{5}.$$

So, the function doesn't have to be bounded for the integral to be finite, but it also can't blow up too badly.

2. Bessel's Inequality (L^2 convergence of Fourier series)

Today we're going to investigate the issue of convergence of Fourier series. To move towards this question of convergence, we prove an important estimate known as the Bessel Inequality. Bessel's Theorem will be a very important ingredient in the proof of our first big theorem which is one of the *theory items,* which can appear on the exam.

Theorem 2 (Bessel Inequality)**.** *Assume that f is square-integrable on $[-\pi, \pi]$. Then the Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$ of f satisfy*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Proof: It is sufficient to show that

$$2\pi \sum_{n=-N}^{N} |c_n|^2 \leq ||f||^2 \quad \forall N \in \mathbb{N}.$$

Since on the right side we have the L^2 norm of a function, we would like to have the L^2 norm of a function. Recall the Pythagorean Theorem: when $a \perp b$ then the length of the vector $a + b = c$ is equal to $a^2 + b^2$. The same thing works in higher dimensions. In particular, since the functions e^{inx} are orthogonal for $n \neq m$, it is also true that $c_n e^{inx}$ are orthogonal for $n \neq m$, so we have

besselpythag

$$||\sum_{n=-N}^{N} c_n e^{inx}||^2 = \sum_{n=-N}^{N} ||c_n e^{inx}||^2 = \sum_{n=-N}^{N} 2\pi|c_n|^2. \tag{2.1}$$

Now, let's write

$$S_N(x) := \sum_{n=-N}^{N} c_n e^{inx}.$$

This is the partial Fourier expansion of f. Let us compare it to f using the L^2 norm:

$$\begin{aligned} 0 \leq ||S_N - f||^2 = \langle S_N - f, S_N - f\rangle &= \langle S_N, S_N - f\rangle - \langle f, S_N - f\rangle \\ &= \langle S_N, S_N\rangle - \langle S_N, f\rangle - \langle f, S_N\rangle + \langle f, f\rangle \\ &= ||S_N||^2 - \langle S_N, f\rangle - \langle f, S_N\rangle + ||f||^2. \end{aligned}$$

Let us compute the two terms in the middle:

$$\langle S_N, f\rangle = \int_{-\pi}^{\pi} \sum_{n=-N}^{N} c_n e^{inx}\overline{f(x)}dx = \sum_{n=-N}^{N} c_n \int_{-\pi}^{\pi} e^{inx}\overline{f(x)}dx = \sum_{n=-N}^{N} c_n \overline{\int_{-\pi}^{\pi} e^{-inx} f(x)dx}$$

$$= \sum_{n=-N}^{N} c_n 2\pi \overline{c_n}.$$

We compute:

$$\langle f, S_N \rangle = \int_{-\pi}^{\pi} f(x) \sum_{n=-N}^{N} \overline{c_n e^{inx}} dx = \sum_{n=-N}^{N} \overline{c_n} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \sum_{n=-N}^{N} \overline{c_n} 2\pi c_n.$$

Since

$$|c_n|^2 = c_n \overline{c_n}$$

we have

$$0 \le ||S_N - f||^2 = ||S_N||^2 - \langle S_N, f \rangle - \langle f, S_N \rangle + ||f||^2 = ||S_N||^2 - 2(2\pi) \sum_{n=-N}^{N} |c_n|^2 + ||f||^2.$$

besselpythag
By (2.1), we have

$$0 \le 2\pi \sum_{n=-N}^{N} |c_n|^2 - 2(2\pi) \sum_{n=-N}^{N} |c_n|^2 + ||f||^2 \implies 2\pi \sum_{n=-N}^{N} |c_n|^2 \le ||f||^2.$$

Corollary 3. *We have*

$$\sum_{n \in \mathbb{N}} |a_n|^2 + |b_n|^2 = 4|c_0|^2 + 2 \sum_{n \in \mathbb{Z} \setminus 0} |c_n|^2,$$

and

$$\lim_{|n| \to \infty} \star_n = 0, \quad \star = a, b, \text{ or } c.$$

Exercise 2. *The proof is an exercise. First, use the previous exercises where we expressed the a's and b's in terms of the c's. Next, what can you say about the terms of a non-negative, convergent series?*

2.1. **Pointwise convergence of Fourier Series.** By Bessel's inequality, we know that

$$\sum_{n \in \mathbb{Z}} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2.$$

Now, it's important to note that when the series of $|c_n|^2$ converges, then eventually $|c_n|^2 < 1$ so also $|c_n| < 1$. Then, $|c_n| > |c_n|^2$. So, just because the series of $|c_n|^2$ converges, the series with just c_n might not. For example,

$$\sum_{n \ge 1} \frac{1}{n^2} < \infty$$

whereas

$$\sum_{n \ge 1} \frac{1}{n} = \infty.$$

So Bessel's inequality doesn't tell us that the Fourier series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

always converges. This is a bit of a concern, because we want to use our method to solve PDEs. In fact, we will see that Fourier series always converge 'in norm,' meaning with respect to the L^2 norm. However, to solve PDEs, we would like the series to converge at specific points. To state the theorem which tells us when and how a Fourier series converges, we need the following definition.

Definition 4. A function is piecewise $\mathcal{C}^k$ on a bounded interval, I, if there is a finite set of points in the interval (possibly empty set) such that f is $\mathcal{C}^k$ on $I \setminus S$. Moreover, we assume that the left and right limits of $f^{(j)}$ exist at all of the points in S, for all $j = 0, 1, \ldots, k$.

Now we are going to prove the great big theorem about pointwise convergence of Fourier series.

Theorem 5 (Convergence of Fourier series)**.** *Assume that f is piecewise $\mathcal{C}^1$ on $[-\pi, \pi]$. Define f on the rest of $\mathbb{R}$ to be a 2π periodic function. Denote the left limit at x by $f(x_-)$ and the right limit by $f(x_+)$, so that for each $x \in \mathbb{R}$,*

$$f(x_{)}) := \lim_{t \to x, t < x} f(t), \quad f(x_+) := \lim_{t \to x, t > x} f(t).$$

Let

$$S_N(x) := \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Then

$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} \left(f(x_-) + f(x_+) \right), \quad \forall x \in \mathbb{R}.$$

Proof: This is a big theorem, because it requires several clever ideas in the proof. Smaller theorems can be proven by just "following your nose." So, to try to help with the proof, we're going to highlight the big ideas. To learn the proof, you can start by learning all the big ideas in the order in which they're used. Once you've got these down, then try to fill in the math steps starting at one idea, working to get to the next idea. The big ideas are like light posts guiding your way through the dark and spooky math.

Idea 1: Fix a point $x \in \mathbb{R}$. This first step is more getting into a frame of mind. Think of x as fixed. Then the numbers $f(x_-)$ and $f(x_+)$ are just the left and right limits of f at x, so these are also fixed. Our goal is to prove that:

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$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} \left(f(x_-) + f(x_+) \right). \tag{2.2}$$

Idea 2: Expand the series $S_N(x)$ using its definition.

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx}.$$

Now, let's move that lonely e^{inx} inside the integral so it can get close to its friend, e^{-iny}. Then,

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny+inx} dy.$$

We want to prove (2.2). Above we have $f(y)$ rather than $f(x)$. This leads us to…

Idea 3: Change the variable. Let $t = y - x$.

Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x)e^{-int}dt.$$

Remember that very first fact we proved for periodic functions? It said that the integral of a periodic function of period P from any point a to $a+P$ is the same, no matter what a is. Here $P = 2\pi$. This leads to...

Idea 4: Shift the integral

$$\int_{-\pi-x}^{\pi-x} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x)e^{-int}dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x) \frac{1}{2\pi} \sum_{-N}^{N} e^{int}dt.$$

Idea 4: Define the N^{th} Dirichlet kernel, $D_N(t)$.

$$D_N(t) = \frac{1}{2\pi} \sum_{-N}^{N} e^{int}.$$

Idea 5: Collect the even and odd terms of D_N to compute its integral.

Recall that

$$n \in \mathbb{N} \implies e^{int} + e^{-int} = 2\cos(nt), n > 0.$$

Hence, we can pair up all the terms ± 1, ± 2, etc, and write

$$D_N(t) = \frac{1}{2\pi} + \sum_{n=1}^{N} \frac{1}{\pi} \cos(nt).$$

So, $D_N(t)$ is an even function. Moreover, since $\cos(nt)$ is 2π periodic and even,

$$\int_{-\pi}^{\pi} \cos(nt)dt = 0 \quad \forall n \geq 1,$$

so

$$\int_{-\pi}^{\pi} D_N(t)dt = \int_{-\pi}^{\pi} \frac{1}{2\pi} dt = 1.$$

Since $D_N(t)$ is even, we also have:

$$\boxed{\texttt{dnint}} \quad (2.3) \qquad \int_{-\pi}^{0} D_N(t)dt = \frac{1}{2} = \int_{0}^{\pi} D_N(t)dt.$$

Idea 6: Go back to the original definition of $D_N(t)$ and re-write it to look like a geometric series.

As it stands, $D_N(t)$ looks almost like a geometric series, but the problem is that it goes from minus exponents to positive ones. We can fix that by factoring out the largest negative exponent, so

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \sum_{n=0}^{2N} e^{int}.$$

We know how to sum a partial geometric series. This gives

$$\boxed{\texttt{dngeo}} \quad (2.4) \qquad D_N(t) = \frac{1}{2\pi} e^{-iNt} \frac{1 - e^{i(2N+1)t}}{1 - e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})}.$$

Since
$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt,$$
fseriesconvg
(2.2) is equivalent to
$$\lim_{N\to\infty} \left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \frac{1}{2} \left(f(x_-) + f(x_+) \right) \right| = 0.$$
The S_N business has an integral, but the $f(x_\pm)$ don't. They have got a convenient factor of one half, so...

Idea 7: Use our calculation of the integral of D_N to write
$$\frac{1}{2} f(x_-) = \int_{-\pi}^{0} D_N(t) dt f(x_-), \quad \frac{1}{2} f(x_+) = \int_0^{\pi} D_N(t) dt f(x_+).$$

Hence we are bound to prove that
$$\lim_{N\to\infty} \left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \int_{-\pi}^{0} D_N(t) f(x_-) dt - \int_0^{\pi} D_N(t) f(x_+) dt \right| = 0.$$
It is quite natural now to split the integral into the left and right sides, so that we must prove
$$\lim_{N\to\infty} \left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_0^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right|.$$

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Idea 8: Use the second property (2.4) we proved about $D_N(t)$.
$$\left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_0^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right| =$$
$$\left| \int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_-)) dt + \int_0^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_+)) dt \right|.$$
Since there are these factors of e^{-iNt} and $e^{i(N+1)t}$, this sort of looks like some twisted version of a Fourier coefficient. This observation leads us to. . .

Idea 9: Define a new function
$$g(t) = \begin{cases} \frac{f(t+x) - f(x_-)}{1-e^{it}} & t \in [-\pi, 0) \\ \frac{f(t+x) - f(x_+)}{1-e^{it}} & t \in (0, \pi] \end{cases}.$$
The function g is well-defined on the interval $[-\pi, \pi] \setminus \{0\}$ because the denominator does not vanish there. Moreover, it has the same properties as f has on this interval. We extend g to all of $\mathbb{R}$ to be 2π periodic. What happens to g when $t \to 0$?
$$\lim_{t\to 0_-} \frac{f(t+x) - f(x_-)}{1-e^{it}} = \lim_{t\to 0_-} \frac{t(f(t+x) - f(x_-))}{t(1-e^{it})} = \frac{f'(x_-)}{-ie^{i0}} = if'(x_-).$$
For the other side, a similar argument shows that
$$\lim_{t\to 0_+} \frac{f(t+x) - f(x_-)}{1-e^{it}} = if'(x_+).$$
Therefore, g has finite left and right limits at $t = 0$, because f does. Hence, g is also a piecewise differentiable and piecewise continuous 2π periodic function. Consequently, g is bounded on $[-\pi, \pi]$ so it is in $L^2([-\pi, \pi])$ and Bessel's inequality holds.

Idea 10: Recognize the Fourier coefficients of the new function

$$\int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_-))dt + \int_{0}^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_+))dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iNt} g(t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(N+1)t} g(t)dt.$$

The first term above is by definition G_N, the N^{th} Fourier coefficient of g, whereas the second term above is by definition G_{-N-1}, the $-N-1$ Fourier coefficient of g. By Bessel's inequality,

$$\lim_{N\to\infty} G_N = 0 = \lim_{N\to\infty} G_{-N-1}.$$

2.2. Exercises for the week from [1].

2.2.1. *Exercises to be demonstrated in the large group.*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}.$$

(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := |\sin(x)|.$$

(3) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := \begin{cases} 1 & -a < x < a \\ -1 & 2a < x < 4a \\ 0 & \text{elsewhere in } (-\pi, \pi). \end{cases}.$$

Here one ought to assume that $0 < a < \pi$ for this to make sense.

(4) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = x^2.$$

2.2.2. *Exercises to be done by oneself (earlier in the week).*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := x(\pi - |x|).$$

(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = e^{bx}.$$

(3) Use the Fourier series for the function $f(x) = |\sin(x)|$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}.$$

(4) Use the Fourier series for the function $f(x) = x(\pi - |x|)$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

(5) Let $f(x)$ be the periodic function such that $f(x) = e^x$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on the rest of $\mathbb{R}$. Let

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

be its Fourier series. Therefore, by Theorem 2.1

$$e^x = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad \forall x \in (-\pi, \pi).$$

If we differentiate this series term-wise then we get $\sum inc_n e^{inx}$. On the other hand, we know that $(e^x)' = e^x$. So, then we should have

$$\sum inc_n e^{inx} = \sum c_n e^{inx} \implies c_n = inc_n \quad \forall n.$$

This is clearly wrong. Where is the mistake?

2.2.3. *Exercises to be demonstrated in the small groups.*

(1) Use the Fourier series of the function $f(x) = x(\pi - |x|)$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sums:

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(2) Use the Fourier series of the function $f(x) = e^{bx}$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sum:

$$\sum_{n \geq 1} \frac{1}{n^2 + b^2} = \frac{\pi}{2b} \coth(b\pi) - \frac{1}{2b^2}.$$

(3) Use the Fourier series of the function $f(x) = x^2$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sums:

$$x^2 - \pi^2 x = 12 \sum_{n \geq 1} \frac{(-1)^n \sin(nx)}{n^3}, \quad x \in (-\pi, \pi)$$

$$x^4 - 2\pi^2 x^2 = 48 \sum_{n \geq 1} \frac{(-1)^{n+1} \cos(nx)}{n^4} - \frac{7\pi^4}{15}$$

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2.2.4. *Exercises to be done by oneself (later in the week).*

(1) Determine the Fourier sine and cosine series of the function

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

(2) Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 2 \\ -1 & 2 < x < 4 \end{cases}$$

in a cosine series on $[0, 4]$.

(3) Expand the function e^x in a series of the form

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}, \quad x \in (0,1).$$

(4) Define

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 < t < 2 \\ 3-t & 2 \le t \le 3 \end{cases}$$

and extend f to be 3-periodic on $\mathbb{R}$. Expand f in a Fourier series. Determine, in the form of a Fourier series, a 3-periodic solution to the equation

$$y''(t) + 3y(t) = f(t).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.01.29

As a corollary to the theorem on the pointwise convergence of Fourier series we have

Corollary 1. *If f and g are 2π periodic and piecewise $\mathcal{C}^1$. Assume that at any point at which f is discontinuous, it satisfies*

$$f(x) = \frac{f(x_+) + f(x_-)}{2},$$

and the same is true for g. Then if f and g have the same Fourier coefficients, then $f = g$.

Proof: By assumption, f and g have the same Fourier series. Let us write the partial series

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx}.$$

By the theorem on the pointwise convergence of Fourier series,

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$$\lim_{N\to\infty} S_N(x) = \frac{f(x_+) + f(x_-)}{2} = \frac{g(x_+) + g(x_-)}{2}, \quad \forall x \in \mathbb{R}. \tag{1.1}$$

Now, at a point where f is continuous,

$$\frac{f(x_+) + f(x_-)}{2} = f(x).$$

Similarly, at a point where g is continuous

$$\frac{g(x_+) + g(x_-)}{2} = g(x).$$

So, by the assumptions on f and g, we have for all $x \in \mathbb{R}$

$$f(x) = \frac{f(x_+) + f(x_-)}{2}, \quad g(x) = \frac{g(x_+) + g(x_-)}{2}.$$

Thus, by (1.1),

$$f(x) = g(x) \quad \forall x \in \mathbb{R}.$$

1.1. **Fourier series to compute sums.** On an exam one may see the following:
Beräkna:
$$\sum_{n=0}^{\infty} \frac{1}{1+n^2}.$$
Tips: Utveckla e^x som en Fourier-serie på intervallet $(-\pi, \pi)$).

The best advice is to follow the hint. Moreover, if this Fourier series is contained in Beta, then begin by writing down the series contained in Beta. In case the series is not contained in Beta, we compute it:
$$\int_{-\pi}^{\pi} e^x e^{-inx} dx = \left. \frac{e^{x(1-in)}}{1-in} \right|_{x=-\pi}^{x=\pi} = \frac{e^{\pi} e^{-in\pi}}{1-in} - \frac{e^{-\pi} e^{in\pi}}{1-in} = (-1)^n \frac{2\sinh(\pi)}{1-in}.$$
Hence, the Fourier coefficients are
$$\frac{1}{2\pi}(-1)^n \frac{2\sinh(\pi)}{1-in},$$
and the Fourier series for e^x on this interval is
$$e^x = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)} e^{inx}, \quad x \in (-\pi, \pi).$$
We can pull out some constant stuff,
$$e^x = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{1-in}, \quad x \in (-\pi, \pi).$$
Now, we use the theorem which tells us that the series converges to the average of the left and right hand limits at points of discontinuity, like for example π. The left limit is e^{π}. Extending the function to be 2π periodic, means that the right limit approaching π is equal to $e^{-\pi}$. Hence
$$\frac{e^{\pi} + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\pi}}{1-in}.$$
Now, we know that $e^{in\pi} = (-1)^n$, thus
$$\frac{e^{\pi} + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{1}{1-in}.$$
We now consider the sum, and we pair together $\pm n$ for $n \in \mathbb{N}$, writing
$$\sum_{-\infty}^{\infty} \frac{1}{1-in} = 1 + \sum_{n \in \mathbb{N}} \frac{1}{1-in} + \frac{1}{1+in} = 1 + \sum_{n \in \mathbb{N}} \frac{2}{1+n^2}.$$
Hence we have found that
$$\frac{e^{\pi} + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\pi}}{1-in} = \frac{\sinh(\pi)}{\pi} \left(1 + \sum_{n \in \mathbb{N}} \frac{2}{1+n^2} \right).$$
The rest is mere algebra. On the left we have the definition of $\cosh(\pi)$. So, moving over the $\sinh(\pi)$ we have
$$\frac{\pi \cosh(\pi)}{\sinh(\pi)} = 1 + 2 \sum_{n \in \mathbb{N}} \frac{1}{1+n^2} \implies \left(\frac{\pi \cosh(\pi)}{\sinh(\pi)} - 1 \right) \frac{1}{2} = \sum_{n \in \mathbb{N}} \frac{1}{1+n^2}.$$
Wow.

1.1.1. *Caution.* To what does the Fourier series converge when x is not in the interval $(-\pi, \pi)$? When we build a Fourier series for a function defined on the interval $(-\pi, \pi)$, it is of the form:

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Each of the terms e^{inx} is a 2π periodic function. Hence the Fourier series is also a 2π periodic function. So, for $x = 2\pi$, the series does *not* converge to $e^{2\pi}$. Rather, it converges to e^0 because, writing

$$S(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad S(x + 2k\pi) = S(x) \quad \forall k \in \mathbb{Z}.$$

For $x \in (-\pi, \pi)$, by the Theorem we proved, we have that $S(x) = e^x$. However, for x outside this interval, the series converges to the function which is equal to e^x on $(-\pi, \pi)$ and is extended to be 2π periodic. Hence the series converges to the value at 0 since $2\pi = 0 + 2\pi$, and the series is 2π periodic. This is a really important subtlety.

Example: Use a Fourier series to compute

$$\sum_{n \geq 1} \frac{(-1)^n}{n^2 + b^2}.$$

Hint: Compute the Fourier series of the function which is equal to e^{bx} for $|x| < \pi$ and extended to be 2π periodic.

To do this, in case the series is not contained in Beta, we compute the coefficients directly:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{bx} e^{-inx} dx = \frac{1}{2\pi(b - in)} e^{(b-in)\pi} - \frac{1}{2\pi(b - in)} e^{(b-in)(-\pi)}.$$

To simplify things, let us note that

$$e^{\pm in\pi} = (-1)^n.$$

Thus

$$c_n = \frac{1}{2\pi(b - in)} (-1)^n e^{b\pi} - \frac{1}{2\pi(b - in)} (-1)^n e^{-b\pi} = \frac{(-1)^n}{2\pi(b - in)} \left(e^{b\pi} - e^{-b\pi}\right) = \frac{(-1)^n}{\pi(b - in)} \sinh(b\pi).$$

The Fourier series is therefore

$$\frac{1}{\pi} \sinh(b\pi) \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{b - in} e^{inx}.$$

Given the presence of the $(-1)^n$, which we also want, it makes sense to try computing with $x = 0$. The series is at this point

$$\frac{1}{\pi} \sinh(b\pi) \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{b - in}.$$

Let us re-arrange things a wee bit:

$$\frac{1}{\pi} \sinh(b\pi) \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{b - in} = \frac{\sinh(b\pi)}{\pi b} + \frac{1}{\pi} \sinh(b\pi) \sum_{n \geq 1} \frac{(-1)^n}{b - in} + \frac{1}{\pi} \sinh(b\pi) \sum_{n \leq 1} \frac{(-1)^n}{b - in}.$$

Let us re-write

$$\frac{1}{\pi}\sinh(b\pi)\sum_{n\leq 1}\frac{(-1)^n}{b-in} = \frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}\frac{(-1)^n}{b+in},$$

with the observation that

$$(-1)^n = (-1)^{-n}.$$

Consequently the series is:

$$\begin{aligned}
&\frac{\sinh(b\pi)}{\pi b} + \frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}\left(\frac{(-1)^n}{b-in} + \frac{(-1)^n}{b+in}\right)\\
&= \frac{\sinh(b\pi)}{\pi b} + \frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}(-1)^n\frac{b+in+b-in}{(b-in)(b+in)}\\
&= \frac{\sinh(b\pi)}{\pi b} + \frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}(-1)^n\frac{2b}{b^2+n^2}.
\end{aligned}$$

On the other hand, we use the theorem PWF$\sum$ to say that at the point $x = 0$ the Fourier series of this function converges to

$$\frac{f(0_+)+f(0_-)}{2}.$$

At the point 0, note that our function is defined to be e^{bx} for $|x| < \pi$ and certainly $|0| < \pi$, so in particular, the function is continuous and thus the left and right limits are both equal and equal to $f(0)$ which is 1. Thus the series converges to 1, and so

$$1 = \frac{\sinh(b\pi)}{\pi b} + \frac{1}{\pi}\sinh(b\pi)\sum_{n\geq 1}(-1)^n\frac{2b}{b^2+n^2}.$$

Re-arranging, we get

$$1 - \frac{\sinh(b\pi)}{\pi b} = \frac{2b\sinh(b\pi)}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2} \implies \frac{\pi}{2b\sinh(b\pi)} - \frac{1}{2b^2} = \sum_{n\geq 1}\frac{(-1)^n}{b^2+n^2}.$$

1.2. **Differentiating and Integrating Fourier series.** First, let us demonstrate a fact about the Fourier series of a function and its derivative. Note that this is a theory item, so you may be asked to prove this on the exam.

Theorem 2. *Assume that f is 2π periodic, continuous, and piecewise $\mathcal{C}^1$. Let a_n, b_n, and c_n be the Fourier coefficients as we have defined them previously, and let a_n', b_n', c_n' be the Fourier coefficients of f' according to the same definition. Then we have*

$$a_n' = nb_n, \quad b_n' = -na_n, \quad c_n' = inc_n.$$

Proof: DO NOT DIFFERENTIATE THE FOURIER SERIES TERMWISE. To do this, you would need to prove that the series can be differentiated termwise, which at this point we do not have the techniques to demonstrate. So, it will be an incomplete and incorrect proof. Not a good thing.

Instead, use the definition of Fourier coefficients and integration by parts:

$$\begin{aligned}
c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx &= \frac{1}{2\pi}\, f(x)e^{-inx}\Big|_{x=-\pi}^{x=\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(-ine^{-inx})dx\\
&= \frac{in}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = inc_n.
\end{aligned}$$

Above, we have used the fact that f is 2π periodic, and e^{-inx} is also 2π periodic so

$$\frac{1}{2\pi} f(x)e^{-inx}\Big|_{x=-\pi}^{x=\pi} = 0.$$

In the last step we use the definition of c_n. Recall that

$$a_n = c_n + c_{-n}, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad \forall n \in \mathbb{N}_{\geq 1},$$

and

$$b_n = i(c_n - c_{-n}), \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad \forall n \in \mathbb{N}_{\geq 1},$$

with

$$a_0 = c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

and the same relationship holds true for a_n', b_n', c_n'. We therefore compute

$$a_n' = c_n' + c_{-n}' = inc_n - inc_{-n} = in(c_n - c_{-n}) = nb_n,$$
$$b_n' = i(c_n' - c_{-n}') = i(inc_n + inc_{-n}) = -n(c_n + c_{-n}) = -na_n.$$

Now, using the theorem we have just proven, we obtain

Corollary 3. *Assume that f is 2π periodic, continuous, piecewise $\mathcal{C}^1$, and assume that f' is also piecewise $\mathcal{C}^1$. Then, if*

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

is the Fourier series for f, we have that

$$\sum_{n\in\mathbb{Z}} inc_n e^{inx}$$

is the Fourier series for f'.

Before demonstrating the results concerning integration of Fourier series, it shall be useful to introduce a certain Hilbert space known as "little ell two."

Definition 4. *Let*

$$\ell^2(\mathbb{C}) := \{(z_n)_{n\in\mathbb{Z}}, \quad z_n \in \mathbb{C}\forall n, \text{ and } \sum_{n\in Z} |z_n|^2 < \infty\}.$$

This is a Hilbert space with the scalar product

$$\langle z, w\rangle := \sum_{n\in\mathbb{Z}} z_n\overline{w_n}, \quad z = (z_n)_{n\in Z}, \quad w = (w_n)_{n\in\mathbb{Z}}.$$

The norm on the Hilbert space, $\ell^2 = \ell^2(\mathbb{C})$ is defined by

$$||z|| = \sqrt{\sum_{n\in\mathbb{Z}} |z_n|^2}.$$

We also have a Cauchy-Schwarz inequality:

$$|\langle z, w\rangle| \leq ||z||||w||.$$

We will use this together with the relationship between the Fourier coefficients for a piecewise $\mathcal{C}^1$ and continuous function, f, to prove

Theorem 5. *Assume that f is 2π periodic, continuous, and piecewise $\mathcal{C}^1$. Then the Fourier series of f converges absolutely uniformly to f on all of $\mathbb{R}$.*

Proof: By assumption, f' is piecewise continuous. Bessel's inequality tells us that

$$\sum_{\mathbb{Z}} |c_n'|^2 < \infty.$$

We use the preceding theorem to say that for all $n \neq 0$,

$$|c_n| = \left| c_n' \frac{1}{n} \right|.$$

Hence we can estimate

$$\sum_{n\in\mathbb{Z}} |c_n e^{inx}| = \sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c_n'|}{|n|}.$$

By Bessel's inequality

$$\sum_{n\in\mathbb{Z}} |c_n'|^2 < \infty,$$

and we know very well that

$$\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2} < \infty.$$

So, using the Cauchy-Schwarz inequality on ℓ^2, we have

$$\sum_{n\in\mathbb{Z}} |c_n| = |c_0| + \sum_{n\in\mathbb{Z}\setminus 0} \frac{|c_n'|}{|n|} \leq |c_0| + \sqrt{\sum_{n\in\mathbb{Z}\setminus 0} |c_n'|^2} \sqrt{\sum_{n\in\mathbb{Z}\setminus 0} |n|^{-2}} < \infty.$$

Therefore the Fourier series converges absolutely, and uniformly for all $x \in \mathbb{R}$, because we see that the convergence estimates are independent of the point x. Since the function is continuous, the limit of the series is, by the Theorem on the pointwise convergence of Fourier series

$$\frac{f(x_+) + f(x_-)}{2} = f(x).$$

We can repeat this idea to show that the more differentiable a function is, the faster its Fourier series converges.

Theorem 6. *Let f be 2π periodic, and assume that f is $\mathcal{C}^{k-1}$, and $f^{(k-1)}$ is piecewise $\mathcal{C}^1$, and f is piecewise $\mathcal{C}^k$. Then the Fourier coefficients of f satisfy*

$$\sum |n^k a_n|^2 < \infty, \quad \sum |n^k b_n|^2 < \infty, \quad \sum |n^k c_n|^2 < \infty.$$

If $|c_n| \leq c|n|^{-k-\alpha}$ for some $c > 0$ and $\alpha > 1$, for all $n \neq 0$, then $f \in \mathcal{C}^k$.

Proof: We apply the theorem relating the Fourier coefficients of f to those of the derivatives of f. Do it k times. We get

$$c_n^{(k)} = (in)^k c_n.$$

Next, we apply Bessel's inequality to conclude that since f is piecewise $\mathcal{C}^k$, $f^{(k)}$ is bounded on the interval hence it is in L^2 on the interval, and so

$$\sum |c_n^{(k)}|^2 < \infty.$$

Since

$$|c_n^{(k)}| = |n|^k |c_n|$$

this shows that

$$\sum |n^k c_n|^2 < \infty.$$

We have similar estimates for a_n and b_n using the same theorem, specifically

$$|a_n^{(k)}| = |n^k a_n|, \quad |b_n^{(k)}| = |n^k b_n|.$$

Hence,

$$\sum |n^k a_n| < \infty, \quad \sum |n^k b_n| < \infty.$$

Now we demonstrate the result which says that if the Fourier coefficients are sufficiently rapidly decaying, then the function f is actually in $\mathcal{C}^k$. Let

$$g(x) := f^{(k-1)}(x).$$

Then g is continuous and by assumption it is piecewise $\mathcal{C}^1$. Therefore, by the theorem on the pointwise convergence of Fourier series, the Fourier series of g converges to $g(x)$ for all x in $\mathbb{R}$. Next, we use the assumption and the fact that the Fourier coefficients of g are

$$c_n^{(k-1)} = (in)^{k-1} c_n.$$

Therefore

$$\sum_{n \in \mathbb{Z}} |c_n^{(k-1)} e^{inx}| = \left| c_0^{(k-1)} \right| + \sum_{n \neq 0} |n^{k-1}||c_n| \leq \left| c_0^{(k-1)} \right| + c \sum_{n \neq 0} |n|^{k-1-k-\alpha} < \infty.$$

Hence, the series converges absolutely and uniformly in $\mathbb{R}$. Moreover, differentiating the series termwise is legitimate, because the result

$$\sum_{n \in \mathbb{Z}} inc_n^{(k-1)} e^{inx}$$

also converges absolutely and uniformly in $\mathbb{R}$:

$$\sum_{n \in \mathbb{Z}} |inc_n^{(k-1)}| \leq \sum_{n \neq 0} |n||c_n^{(k-1)}| \leq c \sum_{n \neq 0} |n||n|^{k-1-k-\alpha} < \infty$$

because $\alpha > 1$. Since the series is equal to $g(x) = f^{(k-1)}(x)$ for all $x \in \mathbb{R}$, and the series is a differentiable function for all $x \in \mathbb{R}$, this shows that g is differentiable for all $x \in \mathbb{R}$. Moreover, g' is continuous on $\mathbb{R}$, because the series defines a continuous function.[1] This is the case because the series defining g' converges absolutely and uniformly for all of $\mathbb{R}$. Hence, $f^{(k-1)}$ is in $\mathcal{C}^1$ on all of $\mathbb{R}$, and therefore f is in $\mathcal{C}^k$ on all of $\mathbb{R}$.

We will prove a theorem about integrating Fourier series. To get warmed up, here is an exercise.

[1]This is true because the series should really be viewed as the limit of the partial series, and each partial series defines a smooth, thus also continuous, function. The uniform limit of continuous functions is itself a continuous function.

Exercise 1. *Show that if you compute the indefinite integrate*

$$\int e^{inx}dx, \quad n \in \mathbb{Z} \setminus \{0\},$$

the result is also a 2π periodic function. What happens in the case $n = 0$?

Theorem 7. *Let f be a 2π periodic function which is piecewise continuous. Define*

$$F(x) := \int_0^x f(t)dt.$$

If $c_0 = 0$, then

$$F(x) = C_0 + \sum_{n \neq 0} \frac{c_n}{in} e^{inx}, \quad C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)dx.$$

Similarly,

$$F(x) = \frac{1}{2} A_0 + \sum_{n \geq 1} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \cos(nx).$$

Proof: We first note that F is continuous and piecewise $\mathcal{C}^1$, because it is the integral of a piecewise continuous function. Moreover, assuming $c_0 = 0$, we see that

$$F(x+2\pi) - F(x) = \int_0^{x+2\pi} f(t)dt - \int_0^x f(t)dt = \int_x^{x+2\pi} f(t)dt = \int_{-\pi}^{\pi} f(t)dt = 2\pi c_0 = 0.$$

Above we have used the nifty lemma that allows us to slide around integrals of periodic functions. So, F satisfies the assumptions of the theorem on pointwise convergence of Fourier series. We therefore have pointwise convergence of the Fourier series of F. Moreover, applying the theorem relating the Fourier coefficients of $F' = f$ to those of F, we have

$$C_n = \frac{c_n}{in} \quad n \neq 0.$$

(That's because $c_n = C_n'$ and the theorem says $C_n' = inC_n$ which shows $c_n = inC_n$, which we can re-arrange as above). Of course, the formula for C_0 is just the usual formula for it, because we can't say anything more specific without knowing more information on f. The re-statement in terms of a and b follows from the relationship between these and the c_n.

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Remark 1. *If $c_0 \neq 0$, then define a new function*

$$g(t) := f(t) - c_0.$$

Since f is 2π periodic, so is g. Then, apply the theorem above to g. Note that

$$G(x) = \int_0^x g(t)dt = F(x) - c_0 x.$$

Moreover, the Fourier coefficients of g,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) - c_0) e^{-inx} dx = c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad \forall n \neq 0.$$

So, the series for $G(x)$ from the theorem is

$$\widetilde{C_0} + \sum_{n \neq 0} \frac{c_n}{in} e^{inx},$$

with

$$\widetilde{C_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (F(x) - c_0 x)\, dx = C_0.$$

So, in fact, it is the same C_0, where we have used the oddness of the function x above. Then, we get something of a corollary which says that in general, the series in the theorem,

$$C_0 + \sum_{n \neq 0} \frac{c_n}{in} e^{inx}, \quad C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) dx$$

converges to $F(x) - c_0 x$.

1.3. **Fourier sine and cosine series.** Let's say we are just looking at $[0, \pi]$. There are two ways to extend a function defined over there to all of $[-\pi, \pi]$. One way is oddly, and the other way is evenly. If we want to extend oddly, we define

$$f(x) := -f(-x), \quad x \in (-\pi, 0).$$

Then, we have computed in an exercise that the a_n coefficients are all zero, and the b_n coefficients are

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx.$$

Here we used the fact that sine is also an oddball. On the other hand, if we want to extend evenly, we define

$$f(x) := f(-x), \quad x \in (-\pi, 0).$$

Then, we have computed in an exercise that the b_n are all zero, because our function is even. Here we have the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx, \quad n \geq 0.$$

Above we used the fact that cosine is even. In this way, we may define Fourier sine and cosine series for functions on $[0, \pi]$. The Fourier sine series is defined to be

$$\sum_{n \geq 1} b_n \sin(nx), \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

whereas the Fourier cosine series is

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(nx), \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx, \quad \forall n \in \mathbb{N}.$$

Theorem 8. *Let f be a function which is piecewise $\mathcal{C}^1$ on $[0, \pi]$. Then the Fourier sine and cosine series converge to $f(x)$ for all $x \in (0, \pi)$ at which f is continuous. For other points, they converge to*

$$\frac{1}{2}\left(f(x_-) + f(x_+)\right).$$

Proof: First, we extend the function either evenly or oddly. Next, we extend it to all of $\mathbb{R}$ to be 2π periodic. Like Riker, we just *make it so.* We're only proving a statement about points in $(0, \pi)$. So, what happens outside of this set of points, well it don't matter. We apply the theorem on pointwise convergence of Fourier series now.

1.4. **How to compute sums using the Integration Theorem for Fourier Series. Example:** Use a Fourier series to compute:

$$\sum_{n\geq 1} \frac{1}{n^4}.$$

Hint: Expand x^2 in a Fourier series. This is an even function, hence no sines in its Fourier series. The other terms

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)dx.$$

We do this integral:

$$\int_0^{\pi} x^2 \cos(nx)dx = \int x^2 \left(\frac{\sin(nx)}{n}\right)' dx = x^2 \frac{\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x \frac{\sin(nx)}{n} dx.$$

Above we did integration by parts. The first part vanishes. The second term we handle with integration by parts again,

$$\int_0^{\pi} x\sin(nx)dx = \int_0^{\pi} x\left(-\cos(nx)/n\right)' dx = -\frac{x\cos(nx)}{n}\Big|_0^{\pi} + \int_0^{\pi} \cos(nx)/n dx.$$

Now this time the second term vanishes because integrating gives us a sine which is 0 at 0 and at π. So, recalling the constant factors, we get

$$\int_0^{\pi} x^2 \cos(nx)dx = \frac{2\pi\cos(\pi n)}{n^2} = \frac{2\pi(-1)^n}{n^2}.$$

Hence our coefficients,

$$a_n = \frac{2 * 2(-1)^n}{n^2}.$$

Moreover, we also compute the term

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}.$$

Hence, the Fourier series expansion of x^2 is

$$\frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n \cos(nx)}{n^2}.$$

Let $x = \pi$. Since our periodically extended function, x^2 is continuous on all of $\mathbb{R}$, the Fourier series converges to its value at $x = \pi$ which means

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n(-1)^n}{n^2} \implies \frac{\pi^2}{6} = \sum_{n\geq 1} \frac{1}{n^2}.$$

To get up to summing n^{-4} we use Theorem 2.4 about integrating Fourier series. We see that

$$c_0 = \frac{\pi^2}{3}.$$

We also see that for $f(t) = t^2$,

$$F(x) := \int_0^x f(t)dt = \frac{x^3}{3}.$$

The series from the theorem is

$$C_0 + 4\sum_{n\geq 1} \frac{(-1)^n \sin(nx)}{n^3}.$$

The term

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx = 0,$$

because $F(x)$ above is odd. Hence, the theorem together with the remark after it says that

$$4\sum_{n\geq 1} \frac{(-1)^n \sin(nx)}{n^3} = \frac{x^3}{3} - \frac{\pi^2 x}{3}, \quad x \in [-\pi, \pi].$$

Exercise: Compute $\sum n^{-3}$.

To proceed, we're going to need to use the theorem once more to get n^4 in the denominator. Before we do this, let's multiply everything by 3 to make it nicer. Then

$$x^3 - \pi^2 x = 12\sum_{n\geq 1} \frac{(-1)^n \sin(nx)}{n^3}, \quad x \in [-\pi, \pi].$$

So, here we have

$$f(t) = t^3 - \pi^2 t \implies F(x) = \int_0^x f(t)dt = \frac{x^4}{4} - \frac{\pi^2 x^2}{2}.$$

We see also that

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = 0.$$

Hence, we apply the theorem directly to F. The theorem says

$$F(x) = C_0 + 12\sum_{n\geq 1} -\frac{(-1)^n \cos(nx)}{n^4}.$$

We compute

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx = \frac{1}{\pi}\int_0^{\pi} \frac{x^4}{4} - \frac{\pi^2 x^2}{2}dx = \frac{\pi^4}{20} - \frac{\pi^4}{6}.$$

Therefore

$$F(x) = \frac{x^4}{4} - \frac{\pi^2 x^2}{2} = \frac{\pi^4}{20} - \frac{\pi^4}{6} - 12\sum_{n\geq 1} \frac{(-1)^n \cos(nx)}{n^4}, \quad x \in [-\pi, \pi].$$

We do the same trick now of choosing

$$x = \pi \implies \cos(nx) = \cos(n\pi) = (-1)^n, \quad (-1)^n(-1)^n = 1 \forall n.$$

Hence,

$$F(\pi) = \frac{\pi^4}{4} - \frac{\pi^4}{2} = \frac{\pi^4}{20} - \frac{\pi^4}{6} - 12\sum_{n\geq 1} \frac{1}{n^4}.$$

Re-arranging things

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{1}{12}\left(\frac{\pi^4}{20} - \frac{\pi^4}{6} + \frac{\pi^4}{2} - \frac{\pi^4}{4}\right).$$

Just for fun, we determine what this is...

$$\frac{\pi^4}{20} - \frac{\pi^4}{6} + \frac{\pi^4}{2} - \frac{\pi^4}{4} = \frac{\pi^4}{2}\left(\frac{1}{10} - \frac{1}{3} + \frac{1}{2}\right) = \frac{\pi^4}{2}\left(\frac{3-10+15}{30}\right)$$
$$= \frac{\pi^4}{2}\left(\frac{8}{30}\right) = \frac{2\pi^4}{15}.$$

So, recalling the factor of $\frac{1}{12}$, we see that

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{2\pi^4}{(12)(15)} = \frac{\pi^4}{6(15)} = \frac{\pi^4}{90}.$$

Wow, who would have guessed that? Not I said the fly!

1.4.1. *Exercises to be done by oneself: Hints.*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$
$$f(x) := x(\pi - |x|).$$
Hint: Use Beta.

(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$
$$f(x) = e^{bx}.$$
Hint: Use Beta.

(3) Use the Fourier series for the function $f(x) = |\sin(x)|$ to compute the sum
$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}.$$
Hint: use Beta to show that the Fourier series of the function defined to be $|\sin(x)|$ for $|x| < \pi$ and extended to be 2π periodic is:
$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2-1}.$$
Use the theorem on the pointwise convergence of Fourier series to compute the value for $x = 0$. Then use algebra to obtain the value for
$$\sum_{n\geq 1} \frac{1}{4n^2-1}.$$
Next, take $x = \frac{\pi}{2}$, and proceed similarly to compute the sum
$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{4n^2-1}.$$

(4) Use the Fourier series for the function $f(x) = x(\pi - |x|)$ to compute the sum
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$
Hint: use Beta to show that the Fourier series of the function $x(\pi - |x|)$ defined on $|x| < \pi$ and extended to be 2π periodic is:
$$\frac{8}{\pi}\sum_{n\geq 1} \frac{\sin(2n-1)x}{(2n-1)^3}.$$

To compute the sum, set $x = \frac{\pi}{2}$ and use the theorem on the pointwise convergence of Fourier series.

(5) Let $f(x)$ be the periodic function such that $f(x) = e^x$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on the rest of $\mathbb{R}$. Let

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

be its Fourier series. Therefore, by Theorem 2.1

$$e^x = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad \forall x \in (-\pi, \pi).$$

If we differentiate this series term-wise then we get $\sum inc_n e^{inx}$. On the other hand, we know that $(e^x)' = e^x$. So, then we should have

$$\sum inc_n e^{inx} = \sum c_n e^{inx} \implies c_n = inc_n \quad \forall n.$$

This is clearly wrong. Where is the mistake?

Hint: What are the hypotheses of the theorem on differentiation of Fourier series (Theorem 2 in today's notes)? Are they all satisfied in this case?

FOURIER ANALYSIS & METHODS

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.01.31

1.1. **Example of the vibrating string.** Assume that at $t = 0$, the ends of the string are fixed, and we have pulled up the middle of it. This makes a shape which mathematically is described by the function

$$v(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

Assume that at $t = 0$ the string is not yet vibrating, so the initial conditions are then

$$\begin{cases} u(x,0) = v(x) \\ u_t(x,0) = 0 \end{cases}$$

We assume the ends of the string are fixed, so we have the boundary conditions

$$u(0) = u(2\pi) = 0.$$

The string is identified with the interval $[0, 2\pi]$. Determine the function $u(x,t)$ which gives the height at the point x on the string at the time $t \ge 0$ which satisfies all these conditions.

1.1.1. *First Step: Separate Variables.* We use our first technique, separation of variables. The wave equation demands that

$$\Box u = 0, \quad \Box u = \partial_{tt} u - \partial_{xx} u.$$

Write

$$u(x,t) = X(x)T(t).$$

Hit it with the wave equation:

$$X(x)T''(t) - X''(x)T(t) = 0.$$

We again *separate the variables* by dividing the whole equation by $X(x)T(t)$. Then we have

$$\frac{T''(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0 \implies \frac{T''}{T} = \frac{X''}{X} = \text{ constant.}$$

The two sides depend on different variables, which makes them both have to be constant. We give that a name, λ. Then, since we have those handy dandy boundary conditions for X (but a much more complicated initial condition for $u(x,0) = v(x)$) we start with X. We solve

$$X'' = \lambda X, \quad X(0) = X(2\pi) = 0.$$

Exercise 1. *Show that the cases $\lambda \geq 0$ won't satisfy the boundary condition.*

We are left with $\lambda < 0$ which by our multivariable calculus theorem tells us that

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To get $X(0) = 0$, we must have $a = 0$. To get $X(2\pi) = 0$ we will need

$$\sqrt{|\lambda|}2\pi = k\pi \quad k \in \mathbb{Z}.$$

Hence

$$\sqrt{|\lambda|} = \frac{k}{2}, \quad k \in \mathbb{Z}.$$

Since $\sin(-x) = -\sin(x)$ are linearly dependent, we only need to take $k \in \mathbb{N}$ (without 0, you know, American $\mathbb{N}$). So, we have X which we index by n, writing

$$X_n(x) = \sin(nx/2) \quad n \in \mathbb{N}.$$

For now, we don't worry about the constant factor. Next, we have the equation for the partner-function (can't forget the partner function!)

$$\frac{T_n''}{T_n} = \lambda_n.$$

Since we know that $\lambda_n < 0$ and $\sqrt{|\lambda_n|} = n/2$ we have

$$\lambda_n = -\frac{n^2}{4}.$$

Hence, our handy dandy multivariable calculus theorem tells us that the solution

$$T_n(t) = a_n\cos(nt/2) + b_n\sin(nt/2).$$

Now, we have

$$u_n(x,t) = X_n(x)T_n(t), \quad \Box u_n = 0 \quad \forall n \in \mathbb{N}.$$

1.1.2. *Supersolution obtained by superposition principle.* Since the PDE is linear and homogeneous, we also have

$$\Box \sum_{n\geq 1} u_n(x,t) = \sum_{n\geq 1} \Box u_n(x,t) = 0.$$

We don't know which of these u_n we need to build our solution according to the initial conditions, so we just take all of them for now.

1.1.3. *Fourier series to find the coefficients using the initial conditions.* We need

$$u(x,t) := \sum_{n\geq 1} u_n(x,t)$$

to satisfy the initial conditions. The first is that

$$u(x,0) = \sum_{n\geq 1} X_n(x)a_n = v(x).$$

We are working on the interval $[0, 2\pi]$. The coefficients are obtained by using X_n as a basis for $\mathcal{L}^2$ on this interval. The coefficients are therefore

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$$a_n = \frac{1}{||X_n||^2}\langle v, X_n\rangle = \frac{\int_0^{2\pi} v(x)\overline{X_n(x)}dx}{\int_0^{2\pi} |X_n(x)|^2 dx}. \tag{1.1}$$

If one wishes to do these integrals, one is welcome to do so. That will not be necessary on the exam, however.

To obtain the b_n coefficients, we use the other initial condition which says that

$$u_t(x,0) = \sum_{n\geq 1} X_n(x)T_n'(0) = \sum_{n\geq 1} X_n(x)\left(-a_n\frac{n}{2}\sin(0) + b_n\frac{n}{2}\cos(0)\right)$$
$$= \sum_{n\geq 1} X_n(x)\frac{n}{2}b_n = 0.$$

These coefficients are calculated in the same way:

$$\frac{n}{2}b_n = \frac{\langle 0, X_n\rangle}{||X_n||^2} = 0\forall n.$$

Hence, our solution is

$$\sum_{n\geq 1} a_n \sin(nx/2)\cos(nt/2),$$

with a_n given in equation (an1 1.1).

1.2. **Summary of methods for solving PDEs on bounded intervals.** Thus far we have collected the following techniques to solve PDEs like the heat and wave equation on *bounded* intervals:

(1) Separation of variables (a means to an end),
(2) Superposition position (smash solutions together to make a supersolution),
(3) Fourier series to find the coefficients obtained using the initial data ($\mathcal{L}^2$ scalar product and divide by the norm).

These methods work well on *bounded* intervals.

1.3. **Another wave equation example.** Solve:

$$u_{tt} = u_{xx}, \quad t > 0, \quad x \in (-1, 1),$$
$$\begin{cases} u(0,x) &= 1 - |x| \\ u_t(0,x) &= 0 \\ u_x(t,-1) &= 0 \\ u_x(t,1) &= 0 \end{cases}$$

We use separation of variables, writing $u(x,t) = X(x)T(t)$. It is just a means to an end. We write the PDE:

$$T''X = X''T.$$

Divide everything by XT to get

$$\frac{T''}{T} = \frac{X''}{X}.$$

Since the two sides depend on different variables, they are both constant. Start with the X side because we have more simple information about it. The boundary conditions that

$$u_x(t,-1) = u_x(t,1) = 0 \implies X'(-1) = X'(1) = 0.$$

So, we have the equation

$$\frac{X''}{X} = \text{ constant, call it } \lambda.$$

Thus we are solving

$$X'' = \lambda X, \quad X'(-1) = X'(1) = 0.$$

Case 1: $\lambda = 0$: In this case, we have solved this equation before. One way to think about it is like the second derivative is like acceleration. If $X'' = 0$, it's like saying X has constant acceleration. Therefore X can only be a linear function. Now, we have the boundary condition which says that $X'(-1) = X'(1) = 0$. So the slope of the linear function must be zero, hence X must be a constant function in this case. So, the only solutions in this case are the constant functions.

Case 2: $\lambda > 0$: In this case, a general solution is of the form:

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

Let us assume that A and B are not both zero. The left boundary condition requires

$$A\sqrt{\lambda}e^{-\sqrt{\lambda}} - \sqrt{\lambda}Be^{\sqrt{\lambda}} = 0.$$

Since $\lambda > 0$ we can divide by $\sqrt{\lambda}$ to say that we must have

$$Ae^{-\sqrt{\lambda}} = Be^{\sqrt{\lambda}} \implies \frac{A}{B} = e^{2\sqrt{\lambda}}.$$

The right boundary condition requires

$$A\sqrt{\lambda}e^{\sqrt{\lambda}} - \sqrt{\lambda}Be^{-\sqrt{\lambda}} = 0.$$

Since $\lambda > 0$, we can divide by $\sqrt{\lambda}$, to make this:

$$Ae^{\sqrt{\lambda}} = Be^{-\sqrt{\lambda}} \implies e^{2\sqrt{\lambda}} = \frac{B}{A}.$$

Hence combining with the other boundary condition we get:

$$\frac{A}{B} = e^{2\sqrt{\lambda}} = \frac{B}{A} \implies A^2 = B^2 \implies A = \pm B \implies \frac{A}{B} = \pm 1.$$

Neither of these are possible because

$$e^{2\sqrt{\lambda}} > 1 \text{ since } 2\sqrt{\lambda} > 0.$$

So, we run amok under the assumption that A and B are not both zero. Hence, the only solution in this case requires $A = B = 0$. This is the waveless wave.

Case 3: $\lambda < 0$: In this case a general solution is of the form:

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To satisfy the left boundary condition we need

$$-a\sqrt{|\lambda|}\sin(-\sqrt{|\lambda|}) + b\sqrt{|\lambda|}\cos(-\sqrt{|\lambda|}) = 0 \iff a\sin(\sqrt{|\lambda|}) = -b\cos(\sqrt{|\lambda|}).$$

To satisfy the right boundary condition we need

$$-a\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}) + b\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}) = 0 \iff a\sin(\sqrt{|\lambda|}) = b\cos(\sqrt{|\lambda|}).$$

Hence we need

eq:bc

$$a\sin(\sqrt{|\lambda|}) = -b\cos(\sqrt{|\lambda|}) = b\cos(\sqrt{|\lambda|}). \tag{1.2}$$

We do not want both a and b to vanish. So, we need to have either

(1) the sine vanishes, so we need $\sin(\sqrt{|\lambda|}) = 0$ which then implies that
$$\sqrt{|\lambda|} = n\pi, \quad n \in \mathbb{Z}$$
(2) or the cosine vanishes so we need $\cos(\sqrt{|\lambda|}) = 0$ which then implies that
$$\sqrt{|\lambda|} = \left(n + \frac{1}{2}\right)\pi, \quad n \in \mathbb{N}.$$

Note that these two cases are *mutually exclusive.* In case (1), by (1.2) this means that $b = 0$. In case (2), by (1.2) this means that $a = 0$. So, we have two types of solutions, which up to constant factor look like:
$$X_m(x) = \begin{cases} \cos(m\pi x/2) & m \text{ is even} \\ \sin(m\pi x/2) & m \text{ is odd} \end{cases}$$
In both cases,
$$\lambda_m = -\frac{m^2\pi^2}{4}.$$

We can now solve for the partner function, $T_m(t)$. The equation is
$$\frac{T_m''}{T_m} = \frac{X_m''}{X_m} = \lambda_m = -\frac{m^2\pi^2}{4}.$$
Therefore, we are in case 3 for the T_m function as well, so we know that
$$T_m(t) = a_m \cos\left(\frac{m\pi t}{2}\right) + b_m \sin\left(\frac{m\pi t}{2}\right).$$
Then we have for
$$u_m(x,t) = X_m(x)T_m(t), \quad \Box u_m = 0 \quad \forall m.$$
(Recall that $\Box = \partial_{tt} - \partial_{xx}$, that is the wave operator). Hence, our functions solve a homogeneous PDE, so we can use the superposition principle to smash them all together to make a super solution:
$$u(x,t) = \sum_{m\in\mathbb{N}} u_m(x,t) = \sum_{n\in\mathbb{N}} X_m(x)\left(a_m \cos\left(\frac{m\pi t}{2}\right) + b_m \sin\left(\frac{m\pi t}{2}\right)\right).$$
How do we determine the coefficients? Using the initial data and a Fourier series for it!!!

The initial data is
$$\begin{cases} u(0,x) & = 1 - |x| \\ u_t(0,x) & = 0 \end{cases}$$
Let us plug $t = 0$ into our solution:
$$u(x,0) = \sum_{m\in\mathbb{N}} X_m(x)a_m.$$
We demand that this is the initial data, so we need
$$1 - |x| = \sum_{m\in\mathbb{N}} X_m(x)a_m.$$
It is a Fourier series on the right side!! We therefore just need to expand the function $1 - |x|$ in a Fourier series. If we think about the basis functions $\{X_m(x)\}_{m\geq 0}$ then
$$a_m = \frac{\langle 1 - |x|, X_m(x)\rangle}{||X_m||^2},$$

where

$$\langle 1-|x|, X_m(x)\rangle = \int_{-1}^{1} (1-|x|)\overline{X_m(x)}dx,$$

$$||X_m||^2 = \int_{-1}^{1} |X_m(x)|^2 dx.$$

On an exam, you are not actually required to compute these integrals!

Now, for the other coefficients (the b_n), we use the condition on the derivative:

$$u_t(x,0) = \sum_{m\in\mathbb{N}} m_n \frac{m\pi}{2} X_m(x) = 0.$$

We know how to Fourier expand the zero function: its coefficients are all just zero. Hence, it suffices to take

$$b_m = 0 \forall m.$$

1.4. **Fourier series on an arbitrary interval.** When we use our tools to solve a PDE on a finite interval, as above, the initial data is *not* a periodic function. Moreover, it was not defined on the interval $(-\pi,\pi)$. The technique still works! It is actually quite beautiful. When we determined the coefficients, we solved for the Fourier coefficients on the interval $(-1,1)$. Here we explain how to do that in general.

For a function f defined on an interval $[a-\ell, a+\ell]$ for some $a\in\mathbb{R}$, and some $\ell>0$, we begin by extending f to be 2ℓ periodic on $\mathbb{R}$. Next, we define

$$g(t) := f\left(\frac{t\ell}{\pi} + a\right) = f(x),$$

that is

$$\frac{t\ell}{\pi} + a = x, \quad t = \frac{(x-a)\pi}{\ell}.$$

Then, the function $g(t)$ is 2π periodic, because

$$g(t+2\pi) = f\left(\frac{(t+2\pi)\ell}{\pi} + a\right) = f\left(\frac{t\ell}{\pi} + a + 2\ell\right) = f\left(\frac{t\ell}{\pi} + a\right).$$

Above, we used the fact that f is 2ℓ periodic. If g is in $\mathcal{L}^2$, then we can expand it into a Fourier series:

$$\sum_{n\in\mathbb{Z}} c_n e^{int},$$

with coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{t\ell}{\pi} + a\right) e^{-int}dt.$$

Substituting in the integral,

$$x = \frac{t\ell}{\pi} + a, \quad dx = \frac{\ell dt}{\pi}$$

the coefficients become:

$$c_n = \frac{1}{2\pi}\frac{\pi}{\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx = \frac{1}{2\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx.$$

Then, we get by substituting for t in terms of x the Fourier series for f,

$$\sum_{n\in\mathbb{Z}} c_n e^{in\left(\frac{(x-a)\pi}{\ell}\right)}.$$

The same relationship holds for the Fourier cosine and sine coefficients:

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \quad n \geq 1,$$

or equivalently

$$a_n = \frac{1}{\ell}\int_{a-\ell}^{a+\ell} f(x)\cos(n(x-a)\pi/\ell)dx, \quad b_n = \frac{1}{\ell}\int_{a-\ell}^{a+\ell} f(x)\sin(n(x-a)\pi/\ell)dx,$$

and the Fourier series has the form

$$\frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(n(x-a)\pi/\ell) + b_n \sin(n(x-a)\pi/\ell).$$

To what does the Fourier series converge?

Theorem 1. *Assume that f is defined on an interval $[a-\ell, a+\ell]$ for some $a \in \mathbb{R}$, and some $\ell > 0$, such that f is piecewise $\mathcal{C}^1$ on this interval. Then the Fourier series for f, defined by*

$$\sum_{n\in\mathbb{Z}} c_n e^{in\left(\frac{(x-a)\pi}{\ell}\right)}, \quad c_n = \frac{1}{2\ell}\int_{a-\ell}^{a+\ell} f(x)e^{-in(x-a)\pi/\ell}dx,$$

or equivalently the series

$$\frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(n(x-a)\pi/\ell) + b_n \sin(n(x-a)\pi/\ell)$$

converges to $f(x)$ for all $x \in (a-\ell, a+\ell)$ at which f is continuous. At a point $x \in (a-\ell, a+\ell)$ where f is not continuous, the series converges to

eq:avg

$$\frac{f(x_+) + f(x_-)}{2}. \tag{1.3}$$

Exercise 2. *Prove the theorem. As a hint: apply the Theorem PCF$\sum$ to the function g above.*

1.5. **Two primary applications of Fourier series.** We now have to main uses for Fourier series.

(1) Solving PDEs on bounded intervals. This proceeds in three steps: (1) separation of variables (a means to an end), (2) smashing all solutions obtained in this way together to create a *super solution* (superposition), and (3) using a Fourier series to express the initial data.
(2) Using Theorem 2.1 to compute nifty sums like:

$$\sum_{n\geq 1}\frac{1}{n^2}.$$

To compute such a sum, you will first compute the Fourier series of a certain function f which is defined on $(-\pi, \pi)$ and extended 2π periodically:

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

Next, substituting a specific value of x you want to recover the desired sum, like $\sum n^{-2}$. You use the theorem to conclude that the series converges to the average of the left and right limit of the function at x. Then re-arrange to obtain your desired sum.

The simplest way to compute the sum

$$\sum_{n\geq 1}\frac{1}{n^4}$$

requires deep theorems about Hilbert spaces, which is our next topic. These theorems will tell us that

$$\sum_{n\in\mathbb{Z}}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx,$$

for

$$f(x) := x^2 \text{ for } |x|\leq\pi, \text{ and extended to be } 2\pi \text{ periodic on } \mathbb{R},$$

with

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx.$$

If we have looked up the Fourier series (or we compute it), we find:

$$\frac{\pi^2}{3}+\sum_{n\geq 1}\frac{4(-1)^n\cos(nx)}{n^2}.$$

This is not given in terms of c_n but we can nonetheless obtain the c_n since:

$$a_n = c_n + c_{-n} = \frac{4(-1)^n}{n^2}, \quad b_n = i(c_n - c_{-n}) = 0 \forall n\geq 1 \implies c_n = c_{-n}$$

and thus

$$a_n = 2c_n \implies c_n = \frac{2(-1)^n}{n^2} = c_{-n} \quad \forall n\geq 1.$$

The magical Hilbert space theory therefore tells us that

$$\sum_{n\in\mathbb{Z}}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x^2|^2dx = \frac{1}{2\pi}\frac{2\pi^5}{5} = \frac{\pi^4}{5}.$$

On the left side,

$$\sum_{n\in\mathbb{Z}}|c_n|^2 = |c_0|^2 + 2\sum_{n\geq 1}|c_n|^2 = \frac{\pi^4}{9}+2\sum_{n\geq 1}\left|\frac{2(-1)^n}{2n^2}\right|^2 = \frac{\pi^4}{9}+2\sum_{n\geq 1}\frac{4}{4n^4}$$
$$= \frac{\pi^4}{9}+8\sum_{n\geq 1}\frac{1}{n^4}.$$

Consequently,

$$\frac{\pi^4}{5} = \frac{\pi^4}{9}+8\sum_{n\geq 1}\frac{1}{n^4} \implies \frac{\pi^4}{5}-\frac{\pi^4}{9} = 8\sum_{n\geq 1}\frac{1}{n^4} \implies \frac{9\pi^4-5\pi^4}{8*45} = \sum_{n\geq 1}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Our main motivation for developing Hilbert space theory (in case we are not simply motivated by the love of the theory itself) are that this theory will:

(1) provide new tools to be able to explicitly evaluate series using Fourier series (as done above);
(2) determine if our solution found by the Fourier series method is indeed *the unique* solution to our PDE on a bounded interval;
(3) provide new tools to be able to solve PDEs in other compact geometric settings (like in a rectangle, disk, annulus, cylinder, box, sphere, and so forth).

1.5.1. *Exercises to be done by oneself: Answers.*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := x(\pi - |x|).$$

Okay, it is

$$\sum_{n \geq 1} \frac{8 \sin((2n-1)x)}{\pi(2n-1)^3}.$$

(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = e^{bx}.$$

Okay, it is

$$\sum_{n \in \mathbb{Z}} \frac{\sinh(b\pi)(-1)^n}{\pi(b - in)} e^{inx}.$$

(3) Use the Fourier series for the function $f(x) = |\sin(x)|$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi - 2}{4}.$$

The Fourier series is

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n \geq 1} \frac{\cos(2nx)}{4n^2-1}.$$

So, to obtain the first sum, one can use $x = 0$. The series will converge to 0, so you get that

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n \geq 1} \frac{1}{4n^2-1} = 0.$$

Then, re-arranging, one obtains the desired sum. To get the sum with the $(-1)^{n+1}$ upstairs, one should use $x = \frac{\pi}{2}$, because then upstairs one has

$$\cos(2n\pi/2) = \cos(n\pi) = (-1)^n.$$

The series will converge to $|\sin(\pi/2)| = 1$. The same idea applies to re-arrange and obtain the desired sup.

(4) Use the Fourier series for the function $f(x) = x(\pi - |x|)$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

We have computed the Fourier series above. The question now is what value of x to use? Well, upstairs we have

$$\sin((2n-1)x).$$

For $x = \pi/2$ this becomes

$$\sin((2n-1)\pi/2).$$

This will alternate between $+1$ like when $n = 1$ and -1 like when $n = 2$. So, we can compute in this way that

$$\sin((2n-1)\pi/2) = (-1)^{n+1}.$$

Consequently, for $x = \pi/2$ the series is

$$\sum_{n\geq 1} \frac{8(-1)^{n+1}}{\pi(2n-1)^3}.$$

It converges to the average of the left and right limits of $f(x)$ at $x = \pi/2$. These are the same and are both equal to

$$\frac{\pi^2}{4}.$$

Hence

$$\frac{\pi^2}{4} = \sum_{n\geq 1} \frac{8(-1)^{n+1}}{\pi(2n-1)^3}.$$

Re-arrange to get the desired sum.

(5) Let $f(x)$ be the periodic function such that $f(x) = e^x$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on the rest of $\mathbb{R}$. Let

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}$$

be its Fourier series. Therefore, by Theorem 2.1

$$e^x = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad \forall x \in (-\pi, \pi).$$

If we differentiate this series term-wise then we get $\sum inc_n e^{inx}$. On the other hand, we know that $(e^x)' = e^x$. So, then we should have

$$\sum inc_n e^{inx} = \sum c_n e^{inx} \implies c_n = inc_n \quad \forall n.$$

This is clearly wrong. Where is the mistake?

DO NOT DIFFERENTIATE THE SERIES TERMWISE!!! That's the mistake. One can only differentiate termwise when the function satisfies the hypotheses of Theorem 2.3. That theorem requires the function to be continuous on $\mathbb{R}$. The function e^x on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$ has discontinuities at $\pi + 2n\pi$ for all $n \in \mathbb{Z}$. So it fails to satisfy the hypotheses of the theorem, thus that theorem does not apply to this function.

(6) Determine the Fourier sine and cosine series of the function

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Okay, they are

$$\frac{\pi}{4} - \frac{2}{\pi} \sum_{n\geq 1} \frac{\cos((4n-2)x)}{(2n-1)^2}, \quad \frac{4}{\pi} \sum_{n\geq 1} (-1)^{n+1} \frac{\sin((2n-1)x)}{(2n-1)^2}.$$

(7) Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 2 \\ -1 & 2 < x < 4 \end{cases}$$

in a cosine series on $[0,4]$. Okay, it is

$$\frac{4}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}}{2n-1}\cos\left(\frac{(2n-1)\pi x}{4}\right).$$

(8) Expand the function e^x in a series of the form

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}, \quad x\in(0,1).$$

Okay, it is

$$(e-1)\sum_{n\in\mathbb{Z}}\frac{e^{2\pi i n x}}{1-2\pi i n}.$$

(9) Define

$$f(t)=\begin{cases} t & 0\leq t\leq 1\\ 1 & 1<t<2\\ 3-t & 2\leq t\leq 3\end{cases}$$

and extend f to be 3-periodic on $\mathbb{R}$. Expand f in a Fourier series. Determine, in the form of a Fourier series, a 3-periodic solution to the equation

$$y''(t)+3y(t)=f(t).$$

This is Extra Exercise 2, and the solution is contained in the extra övningar document on the course homepage.

FOURIER ANALYSIS & METHODS

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Hilbert spaces

Why should we bother to understand Hilbert spaces? Hilbert spaces are important because they are the missing mathematics (which Fourier did not have!) to rigorously justify using Fourier series to solve PDEs. We have learned the following procedure:

(1) Start with a PDE where the x variable is in a finite (bounded) interval.
(2) Separate variables by writing u, (the unsub) as a product like $u(x,t) = X(x)T(t)$. Plug it into the PDE.
(3) Solve for X using the boundary conditions. This will probably give lots of Xs which can be indexed by $\mathbb{N}$.
(4) Each X_n has a partner T_n. Solve for these. Probably, you've got some unknown constants.
(5) Is the PDE homogeneous? If so, $X_1T_1 + X_2T_2 + \ldots$ also solves the PDE so you can smash them together into a big party series. If *not* then you may need to do something else (i.e. steady state solution). In the homogeneous case, you will then use the IC and the collection $\{X_n\}$ to find the coefficients in T_n and end up with a solution of the form
$$\sum_{n\in\mathbb{N}} X_n(x)T_n(t).$$
It's precisely in this last step where the Hilbert space theory is being used to say that you can use the X_n obtain the IC, because the Hilbert space theory tells us when certain functions are basis functions for $\mathcal{L}^2$!

A *Hilbert space* is a complete[1], normed vector space whose norm is defined by a scalar product. The definition of a vector space means that if u and v are elements in your Hilbert space, then for all complex numbers a and b,
$$au + bv \text{ is in your Hilbert space.}$$
So, taking $a = b = 0$, there is always a 0 vector in your Hilbert space. The fact that it is normed means that every element of the Hilbert space has a *length,* which

Date: 2020.02.03.

[1] Every Cauchy sequence converges. Do you remember what a Cauchy sequence is? If not, please look it up or ask!

is equal to its norm. To define this, we describe the scalar product. For a Hilbert space H, the scalar product satisfies:

$$u, v \in H \implies \langle u, v \rangle \in \mathbb{C},$$
$$c \in \mathbb{C} \implies \langle cu, v \rangle = c\langle u, v \rangle,$$
$$u, v, w \in H \implies \langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle,$$
$$\langle u, v \rangle = \overline{\langle v, u \rangle},$$
$$\langle u, u \rangle \geq 0, \quad = 0 \iff u = 0.$$

Therefore, we can define the norm of a vector as

$$||u|| := \sqrt{\langle u, u \rangle}.$$

The norm of a vector is also equal to its distance from the 0 element of the Hilbert space. Similarly,

$$||u - v|| = \sqrt{\langle u - v, u - v \rangle}$$

is the distance between the elements u and v in your Hilbert space. We say that a set of elements

$$\{u_\alpha\} \subset H$$

is an orthonormal basis (ONB) for H if for any $v \in H$ there exist complex numbers (c_α) such that

$$v = \sum c_\alpha u_\alpha, \quad \langle u_\alpha, u_\beta \rangle = \delta_{\alpha,\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta. \end{cases}$$

This is the Kronecker δ. You may be wondering why we haven't written an index for α. Well, that's because à priori, they could be uncountable.

Theorem 1. *A Hilbert space is separable if and only if it has either a finite ONB or a countable ONB.*

There is a cute proof here:
`http://www.polishedproofs.com/relationship-between-a-countable-orthonormal-basis-and-a-cou`

We're only going to be working with Hilbert spaces which have either a finite ONB or a countable ONB. The dimension of a Hilbert space is the number of elements in an ONB. Any finite dimensional Hilbert space is in bijection with the standard one

$$\mathbb{C}^n, \quad u, v \in \mathbb{C}^n \implies \langle u, v \rangle = u \cdot \overline{v}.$$

Thus, writing

$$u = (u_1, \ldots, u_n), \qquad \text{with each component } u_k \in \mathbb{C},\ k = 1, \ldots, n$$

and similarly for v,

$$\langle u, v \rangle = \sum_{k=1}^{n} u_k \overline{v_k}.$$

The bijection between any finite (n) dimensional Hilbert space and $\mathbb{C}^n$ comes from taking an ONB of the Hilbert space and mapping the elements of the ONB to the standard basis vectors of $\mathbb{C}^n$. Here are some useful basic results for Hilbert spaces.

Proposition 2. *Let H be a Hilbert space. For any u and v in H,*

$$||u + v||^2 = ||u||^2 + 2\Re\langle u, v \rangle + ||v||^2.$$

Proof: Compute:

$$||u+v||^2 = \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,v\rangle + \langle v,u\rangle$$
$$= ||u||^2 + \langle u,v\rangle + ||v||^2 + \overline{\langle u,v\rangle}.$$

We all know that for a complex number z,

$$z + \overline{z} = 2\Re(z).$$

So,

$$\langle u,v\rangle + \overline{\langle u,v\rangle} = 2\Re\langle u,v\rangle.$$

1.1. **Cauchy-Schwarz Inequality, Triangle Inequality, and Pythagorean Theorem.**

Proposition 3. *For any Hilbert space, H, for any u and v in H,*

$$|\langle u,v\rangle| \le ||u||||v||.$$

Proof: Assume that at least one of the two is non-zero. Let's assume $v \neq 0$, because otherwise we can just swap their names. We begin by considering the length of the vector u plus v scaled by a factor of t. If $t \to 0$, the length tends to $||u||^2$. What happens for other values of t? We compute it:

$$||u+tv||^2 = ||u||^2 + 2t\Re\langle u,v\rangle + t^2||v||^2, \quad t \in \mathbb{R}.$$

This is a real valued function of t. It's a quadratic function of t in fact. The derivative is

$$2t||v||^2 + 2\Re\langle u,v\rangle.$$

It's an upwards shaped quadratic function, so its unique minimum is when

$$t = -\frac{\Re\langle u,v\rangle}{||v||^2}.$$

If we then check out what happens at this value of t,

$$||u+tv||^2 = ||u||^2 - 2\frac{\Re\langle u,v\rangle}{||v||^2}\Re\langle u,v\rangle + \Re\langle u,v\rangle^2\frac{||v||^2}{||v||^4} = ||u||^2 - \frac{\Re\langle u,v\rangle^2}{||v||^2}.$$

We know that

$$0 \le ||u+tv||^2$$

so we get

$$0 \le ||u||^2 - \frac{\Re\langle u,v\rangle^2}{||v||^2} \implies 0 \le ||u||^2||v||^2 - \Re\langle u,v\rangle^2.$$

This gives us

$$\Re\langle u,v\rangle^2 \le ||u||^2||v||^2.$$

Well, this is annoying because of that silly $\Re$. I wonder how we could make it turn into $|\langle u,v\rangle|$? Also, we don't want to screw up the $||u||^2||v||^2$ part. Well, we know how the scalar product interacts with complex numbers, for $\lambda \in \mathbb{C}$,

$$\langle \lambda u, v\rangle = \lambda\langle u,v\rangle.$$

So, if for example

$$\langle u,v\rangle = re^{i\theta}, r = |\langle u,v\rangle| \text{ and } \theta \in \mathbb{R}.$$

We can modify u, without changing $||u||$,

$$||e^{-i\theta}u|| = ||u||.$$

Moreover

$$\langle e^{-i\theta}u, v\rangle = e^{-i\theta}\langle u, v\rangle = e^{-i\theta}re^{i\theta} = |\langle u, v\rangle|\,.$$

So, if we repeat everything above replacing u with $e^{-i\theta}u$ we get

$$\Re\langle e^{-i\theta}u, v\rangle^2 \leq ||e^{-i\theta}u||^2||v||^2 = ||u||^2||v||^2,$$

and by the above calculation

$$\langle e^{-i\theta}u, v\rangle = |\langle u, v\rangle| \in \mathbb{R} \implies \Re\langle e^{-i\theta}u, v\rangle^2 = |\langle u, v\rangle|^2\,.$$

So, we have

$$|\langle u, v\rangle|^2 \leq ||u||^2||v||^2.$$

Taking the square root of both sides completes the proof of the Cauchy-Schwarz inequality.

We also have a triangle inequality.

Proposition 4. *For any u and v in a Hilbert space H,*

$$||u + v|| \leq ||u|| + ||v||.$$

Proof: We just use the previous two results:

$$||u + v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 \leq ||u||^2 + 2||u||||v|| + ||v||^2 = (||u|| + ||v||)^2.$$

Taking the square root we obtain the triangle inequality.

We have the Pythagorean theorem.

Proposition 5. *If u and v are orthogonal, then*

$$||u + v||^2 = ||u||^2 + ||v||^2.$$

Moreover, if $\{u_n\}_{n=1}^N$ are orthogonal, then

$$||\sum_{n=1}^N u_n||^2 = \sum_{n=1}^N ||u_n||^2.$$

Proof: The first statement follows from

$$||u + v||^2 = ||u||^2 + 2\Re\langle u, v\rangle + ||v||^2 = ||u||^2 + ||v||^2,$$

if u and v are orthogonal, because in that case their scalar product is zero. Moreover, for any collection of orthogonal vectors $\{u_1, \ldots, u_n\}$ we proceed by induction. Assume that

$$||u_1 + \ldots + u_{n-1}||^2 = \sum_{k=1}^{n-1} ||u_k||^2.$$

Then, if u_n is orthogonal to all of $u_1, \ldots, u_{n-1}$ we also have

$$\langle u_n, u_1 + \ldots + u_{n-1}\rangle = \langle u_n, u_1\rangle + \ldots + \langle u_n, u_{n-1}\rangle = 0 + \ldots + 0.$$

Hence u_n is also orthogonal to the sum,
$$\sum_{k=1}^{n-1} u_k.$$
By the Pythagorean theorem,
$$||u_n + \sum_{k=1}^{n-1} u_k||^2 = ||u_n||^2 + ||\sum_{k=1}^{n-1} u_k||^2.$$
By the induction assumption
$$= ||u_n||^2 + \sum_{k=1}^{n-1} ||u_k||^2 = \sum_{k=1}^{n} ||u_k||^2.$$

1.2. **Continuity of the scalar product.**

Proposition 6. *Using only the assumptions that the scalar product satisfies:*
$$\langle u, v\rangle = \overline{\langle v, u\rangle}$$
$$\langle au, v\rangle = a\langle u, v\rangle$$
$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$
$$\langle u, u\rangle \geq 0, \quad \langle u, u\rangle = 0 \iff u = 0,$$
then the scalar product is a continuous function from $H \times H \to \mathbb{C}$.

Proof: It suffices to estimate
$$|\langle u, v\rangle - \langle u', v'\rangle|.$$
I would like to somehow get
$$u - u' \text{ and } v - v'.$$
So, well, just throw them in the first and last
$$\langle u - u', v\rangle = \langle u, v\rangle - \langle u', v\rangle.$$
That shows that
$$\langle u - u', v\rangle + \langle u', v\rangle = \langle u, v\rangle.$$
So, we see that
$$\langle u, v\rangle - \langle u', v'\rangle = \langle u - u', v\rangle + \langle u', v\rangle - \langle u', v'\rangle$$
We can smash the last two terms together because $-1 \in \mathbb{R}$ so
$$-\langle u', v'\rangle = \langle u', -v'\rangle \implies \langle u', v\rangle - \langle u', v'\rangle = \langle u', v - v'\rangle.$$
Hence,
$$|\langle u, v\rangle - \langle u', v'\rangle| = |\langle u - u', v\rangle + \langle u', v - v'\rangle|.$$
By the triangle inequality
$$|\langle u - u', v\rangle + \langle u', v - v'\rangle| \leq |\langle u - u', v\rangle| + |\langle u', v - v'\rangle|.$$
By the Cauchy-Schwarz inequality
$$|\langle u - u', v\rangle| + |\langle u', v - v'\rangle| \leq ||u - u'||||v|| + ||u'||||v - v'||.$$

We therefore see that for any fixed pair $(u,v) \in H \times H$, given $\epsilon > 0$, we can define

$$\delta := \min\left\{ \frac{\varepsilon}{2(||v||+1)}, \frac{\varepsilon}{2(||u||+1)}, 1 \right\}.$$

Then we estimate

$$||u - u'|| < \delta \implies ||u'|| < ||u|| + \delta \leq ||u|| + 1,$$

$$||u - u'|| ||v|| \leq \frac{\varepsilon ||v||}{2(||v||+1)} < \frac{\varepsilon}{2}.$$

and

$$||u'|| ||v - v'|| \leq \frac{(||u||+1)\varepsilon}{2(||u||+1)} \leq \frac{\varepsilon}{2},$$

so we obtain

$$|\langle u, v\rangle - \langle u', v'\rangle| < \varepsilon.$$

Remark 1. *This fact is useful because it allows us to bring limits inside the scalar product. You will see that we do this many times! In particular, if one has two sequences,*

$$\{u_n\}_{n\geq 1}, \quad \{v_n\}_{n \geq 1} \text{ in a Hilbert space, } H,$$

and

$$\lim_{n\to\infty} u_n = u \in H, \quad \lim_{n\to\infty} v_n = v \in H,$$

then the continuity of the scalar product implies that

$$\lim_{n\to\infty} \langle u_n, v_n \rangle = \langle u, v\rangle.$$

This fact allows us to prove an infinite dimensional Pythagorean theorem!

Theorem 7 (Infinite dimensional Pythagorus)**.** *Assume that $\{u_k\}_{k\geq 1}$ are in a Hilbert space, and that*

$$\sum_{k \geq 1} u_k$$

converges to an element u in that Hilbert space. Further, assume that the u_k are pairwise orthogonal. Then we have

$$||u||^2 = \sum_{k\geq 1} ||u_k||^2.$$

Proof: The meaning of

$$\sum_{k\geq 1} u_k = u$$

is that

$$\lim_{n\to\infty} \sum_{k=1}^{n} u_k = u.$$

This is equivalent to

$$\lim_{n\to\infty} ||u_k - u|| = 0.$$

The definition of scalar product says that

$$||u||^2 = \langle u, u\rangle.$$

Let us denote

$$U_n := \sum_{k=1}^{n} u_k.$$

Since it is a finite sum of elements of the Hilbert space, this is an element of the Hilbert space, because Hilbert spaces are vector spaces. The continuity of the scalar product shows that

$$\lim_{n\to\infty} \langle U_n, U_n\rangle = \langle U, U\rangle.$$

For each n, we also have

$$\langle U_n, U_n\rangle = \sum_{k=1}^{n} ||u_k||^2,$$

by the usual (finite) Pythagorean Theorem. Hence, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} ||u_k||^2 = ||U||^2.$$

This shows that the sum on the left converges and is equal to $||U||^2$.

1.3. **Bessel's inequality and the three equivalent conditions to be an ONB.** We prove a very useful inequality.

Theorem 8 (Bessel's Inequality for general Hilbert spaces)**.** *Let* $\{\phi_n\}_{n\in\mathbb{N}}$ *be an orthonormal set in a Hilbert space* H*. Then if* $f \in H$*,*

$$g := \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n \in H,$$

and we have the inequality

$$||g||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \le ||f||^2.$$

Proof: We will prove the inequality above *first,* and then use it to prove that $g \in H$. By the Pythagorean theorem, for each $N \in \mathbb{N}$,

$$\|\sum_{n=1}^{N} \hat{f}_n \phi_n\|^2 = \sum_{n=1}^{N} |\hat{f}_n|^2.$$

Above, we have used the convenient notation

$$\hat{f}_n = \langle f, \phi_n\rangle.$$

We call $\hat{f}_n$ the n^{th} Fourier coefficient of f with respect to the orthonormal set (ONS) $\{\phi_n\}$. We compute that the square of the distance between f and its partial Fourier series

$$0 \le \|f - \sum_{n=1}^{N} \hat{f}_n \phi_n\|^2 = \|f\|^2 - 2\Re\langle f, \sum_{1}^{N} \hat{f}_n \phi_n\rangle + \|\sum_{1}^{N} \hat{f}_n \phi_n\|^2.$$

Let's look at the middle bit:

$$\langle f, \sum_{1}^{N} \hat{f}_n \phi_n\rangle = \sum_{1}^{N} \overline{\hat{f}_n} \langle f, \phi_n\rangle = \sum_{1}^{N} \overline{\hat{f}_n} \hat{f}_n = \sum_{1}^{n} |\hat{f}_n|^2.$$

Hence,

$$0 \leq ||f||^2 - 2\sum_1^N |\hat{f}_n|^2 + \sum_{n=1}^N |\hat{f}_n|^2 = ||f||^2 - \sum_1^N |\hat{f}_n|^2$$

so re-arranging

$$\sum_1^N |\hat{f}_n|^2 \leq ||f||^2.$$

Letting $N \to \infty$, we obtain the inequality

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \leq ||f||^2.$$

To prove that in fact

$$g \in H,$$

we will show that

$$\{F_N\}_{N\geq 1}, \quad F_N := \sum_{n=1}^N \hat{f}_n \phi_n$$

is a Cauchy sequence in H. Since Hilbert spaces are complete, it follows that this Cauchy sequence converges to a limit $F \in H$. So, let $\varepsilon > 0$ be given. Then, by Bessel's inequality, since

$$\sum_1^\infty |\hat{f}_n|^2 < \infty,$$

there exists $N \in \mathbb{N}$ such that

$$\sum_N^\infty |\hat{f}_n|^2 < \varepsilon^2.$$

This is because the tail of any convergent series can be made as small as we like. So, now if we have $N_1 \geq N_2 \geq N$, we estimate

$$||F_{N_1} - F_{N_2}||^2 = ||\sum_{N_2+1}^{N_1} \hat{f}_n \phi_n||^2 = \sum_{N_2+1}^{N_1} |\hat{f}_n|^2$$

$$\leq \sum_{N_2+1}^{\infty} |\hat{f}_n|^2 \leq \sum_{N}^{\infty} |\hat{f}_n|^2 < \varepsilon^2.$$

Consequently we have that for all $N_1 \geq N_2 \geq N$,

$$||F_{N_1} - F_{N_2}|| < \varepsilon.$$

This is the definition of being a Cauchy sequence. Consequently, we obtain that

$$\lim_{N\to\infty} \sum_{n=1}^N \hat{f}_n \phi_n = g \in H.$$

By our infinite Pythagorean theorem, since ϕ_n are orthonormal, we also have that $\hat{f}_n \phi_n \in H$ are orhthogonal. We therefore have

$$||g||^2 = \sum_{n\geq 1} ||\hat{f}_n \phi_n||^2 = \sum_{n\geq 1} |\hat{f}_n|^2 ||\phi_n||^2 = \sum_{n\geq 1} |\hat{f}_n|^2 \leq ||f||^2.$$

♡

1.4. **The 3 equivalent conditions to be an ONB in a Hilbert space.** Perhaps what makes the following theorem so nice is the pleasant setting of a Hilbert space, or translated directly from German, a Hilbert room. Hilbert rooms are cozy. The reason is because there is a notion of orthogonality, so it is very easy to find one's way around, much like the grid-like streets in the USA.

Theorem 9. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be orthonormal in a Hilbert space, H. The following are equivalent:*

$$(1)\quad f \in H \text{ and } \langle f, \phi_n\rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2)\quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n.$$

$$(3)\quad ||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

The last of these is known as Parseval's equation. *If any of these three equivalent conditions hold, then we say that $\{\phi_n\}$ is an orthonormal basis of H.*

Proof: We shall proceed in order prove (1) $\implies$ (2), then (2) $\implies$ (3), and finally (3) $\implies$ (1). Stay calm and carry on.

First we assume statement (1) holds, and then we shall show that (2) must hold as well. Bessel's Inequality Theorem says that

$$g := \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n \in H.$$

So, we would like to prove that in fact $g = f$, somehow using the fact that statement (1) holds true. Idea: let's try to show that $f - g = 0$. This will imply that $f = g$. To use (1) we should compute then

$$\langle f - g, \phi_n\rangle.$$

Let's do this.

$$\langle f - g, \phi_n\rangle = \langle f, \phi_n\rangle - \langle g, \phi_n\rangle.$$

We insert the definition of g as the series,

$$\langle g, \phi_n\rangle = \langle \sum_{m\geq 1} \langle f, \phi_m\rangle \phi_m, \phi_n\rangle = \sum_{m\geq 1} \langle f, \phi_m\rangle \langle \phi_m, \phi_n\rangle = \langle f, \phi_n\rangle.$$

Above, we have used in the second equality the linearity of the inner product and the continuity of the inner product. In the third equality, we have used that $\langle \phi_m, \phi_n\rangle$ is 0 if $m \neq n$, and is 1 if $m = n$. Hence, only the term with $m = n$ survives in the sum. Thus,

$$\langle f - g, \phi_n\rangle = \langle f, \phi_n\rangle - \langle g, \phi_n\rangle = \langle f, \phi_n\rangle - \langle f, \phi_n\rangle = 0, \quad \forall n \in \mathbb{N}.$$

By (1), this shows that $f - g = 0 \implies f = g$.

Next, we shall assume that (2) holds, and we shall use this to demonstrate (3). By (2),

$$f = \sum_{n\in\mathbb{N}} \hat{f}_n \phi_n, \quad \hat{f}_n := \langle f, \phi_n\rangle.$$

To obtain (3), we can simply apply our infinite dimensional Pythagorean theorem, which says that

$$||f||^2 = \sum_{n\in\mathbb{N}} ||\hat{f}_n \phi_n||^2 = \sum_{n\in\mathbb{N}} |\hat{f}_n|^2 ||\phi_n||^2 = \sum_{n\in\mathbb{N}} |\hat{f}_n|^2.$$

Finally, we assume (3) holds and use it to show that (1) must also hold. This is pleasantly straightforward. We assume that for some f in our Hilbert space, $\langle f, \phi_n \rangle = 0$ for all n. Using (3), we compute

$$||f||^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 = \sum_{n \in N} 0 = 0.$$

The only element in a Hilbert space with norm equal to zero is the 0 element. Thus $f = 0$.

1.5. **Exercises for the week: demonstreras.** Those exercises from [1] which shall be demonstrated are:

1. (3.3.9) Suppose $\{\phi_n\}$ is an orthonormal basis for $\mathcal{L}^2(a, b)$. Show that for any $f, g \in \mathcal{L}^2(a, b)$
$$\langle f, g \rangle = \sum \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle}.$$
2. (3.3.10.c) Evaluate the following series by applying Parseval's equation to certain Fourier expansions:
$$\sum_{n \geq 1} \frac{n^2}{(n^2+1)^2}.$$
3. (3.3.10.b) Evaluate the following series by applying Parseval's equation to certain Fourier expansions:
$$\sum_{n \geq 1} \frac{1}{(2n-1)^6}$$
4. (3.4.3) Let D be the unit disk $\{x^2 + y^2 \leq 1\}$ and let $f_n(x, y) = (x + iy)^n$. Show that $\{f_n\}_{n \geq 0}$ is an orthogonal set in $\mathcal{L}^2(D)$, and compute $||f_n||$ for all n.
5. (3.5.4) Find all λ so that there exists a solution $f(x)$ defined on $[0, \ell]$ to the equation
$$f'' + \lambda f = 0, \quad f'(0) = 0, \quad f(\ell) = 0.$$
6. (EO 23) Find all solutions f on $[0, a]$ and corresponding λ to the equation:
$$f'' + \lambda f = 0, \quad f(0) = f'(0), \quad f(a) = -2f'(a).$$
7. (4.2.1) Suppose a rod is mathematicized as the interval $[0, \ell]$, and the end at $x = 0$ is held at temperature zero while the end at $x = \ell$ is insulated. Find a series expansion for the temperature $u(x, t)$ given the initial temperature $f(x)$ and no sinks or sources.

1.6. **Exercises for the week: räkna själv.** Those exercises from [1] which one should solve are:

1. (3.3.1) Show that if $\{f_n\}_{n \geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that
$$\lim_{n \to \infty} f_n = f,$$
then for all $g \in H$ we have
$$\lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle.$$

(2) (3.3.2) Show that for all f, g in a Hilbert space one has

$$||f|| - ||g||| \leq ||f - g||.$$

(3) (3.3.10.d) Use Parseval's equation to compute

$$\sum_{n \geq 1} \frac{\sin^2(na)}{n^4}.$$

(4) (3.4.1) Show that $\{e^{2\pi i(mx+ny)}\}_{n,m\in\mathbb{Z}}$ is an orthogonal set in $\mathcal{L}^2(R)$ where R is any square whose sides have length one and are parallel to the coordinate axes.

(5) (3.4.6) Find an example of a sequence $\{f_n\}$ in $\mathcal{L}^2(0,\infty)$ such that $f_n(x) \to 0$ uniformly for all $x > 0$ but $f_n \not\to 0$ in the $\mathcal{L}^2$ norm.

(6) (3.5.7) Find all solutions f on $[0,1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

(7) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0,\ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell,t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0,t) = 0$. Find a series expansion for the temperature $u(x,t)$ such that the initial temperature is given by $f(x)$.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.04

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The best approximation theorem

The Fourier series of f an element of a Hilbert space, H, with respect to an orthonormal set $\{\phi_n\}$ is

$$\sum_n \hat{f}_n \phi_n,$$

where

$$\hat{f}_n = \langle f, \phi_n \rangle, \text{ and the set } \{\phi_n\} \text{ is orthonormal, meaning } \langle \phi_n, \phi_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

The Fourier series is actually *equal* to f if and only if the orthonormal set is in fact an orthonormal basis. In any case, even though the Fourier series might not be equal to f, it is the *best* approximation to f in the following sense.

Theorem 1 (Best Approximation)**.** *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal set in a Hilbert space, H. If $f \in H$, and*

$$\sum_{n\in\mathbb{N}} c_n \phi_n \in H,$$

then

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

and equality holds $\iff$ $c_n = \langle f, \phi_n \rangle$ is true $\forall n \in \mathbb{N}$.

Proof: We make a few definitions: let

$$g := \sum \widehat{f_n} \phi_n, \quad \widehat{f_n} = \langle f, \phi_n \rangle,$$

and

$$\varphi := \sum c_n \phi_n.$$

Idea: write

$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 + 2\Re\langle f - g, g - \varphi \rangle.$$

Idea: show that

$$\langle f - g, g - \varphi \rangle = 0.$$

Just write it out (stay calm and carry on):

$$\langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$

$$= \sum \overline{\widehat{f_n}}\langle f, \phi_n\rangle - \sum \overline{c_n}\langle f, \phi_n\rangle - \sum \widehat{f_n}\langle \phi_n, \sum \widehat{f_m}\phi_m\rangle + \sum \widehat{f_n}\langle \phi_n, \sum c_m\phi_m\rangle$$

$$= \sum |\widehat{f_n}|^2 - \sum \overline{c_n}\widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n}\overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have

$$||f - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 \geq ||f - g||^2,$$

with equality iff

$$||g - \varphi||^2 = 0.$$

Let us now write out what this norm is, using the definitions of g and φ. By their definitions,

$$g - \varphi = \sum (\widehat{f_n} - c_n)\phi_n.$$

By the Pythagorean theorem, due to the fact that the ϕ_n are an orthonormal set, and hence multiplying them by the scalars, $\widehat{f_n} - c_n$, they remain orthogonal, we have

$$||g - \varphi||^2 = \sum \left|\widehat{f_n} - c_n\right|^2.$$

This is a sum of non-negative terms. Hence, the sum is only zero if all of the terms in the sum are zero. The terms in the sum are all zero iff

$$\left|\widehat{f_n} - c_n\right| = 0 \forall n \iff c_n = \widehat{f_n} \forall n \in \mathbb{N}.$$

Corollary 2. *Assume that $\{\phi_n\}$ is an OS in a Hilbert space H. Then the best approximation to $f \in H$ of the form*

$$\sum_{n=1}^{N} c_n \phi_n$$

is given by taking

$$c_n = \frac{\langle f, \phi_n\rangle}{||\phi_n||^2}.$$

Exercise 1. *Prove this corollary using the best approximation theorem.*

1.1. **Application of the best approximation theorem.** The goal is to find the numbers $\{c_j\}_{j=0}^{3}$ so that

$$\int_{-\pi}^{\pi} |f - \sum_{j=0}^{3} c_j e^{ijx}|^2 dx$$

is minimized. Here,

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

Since the functions e^{ijx} are orthogonal on $\mathcal{L}^2(-\pi, \pi)$ we can apply the best approximation theorem! It says that the best approximation is to set The best approximation theorem's corollary says that

$$c_j = \frac{\hat{f}_j}{||e^{ijx}||^2} = \frac{\langle f, \phi_j \rangle}{||\phi_j||^2}, \quad \phi_j(x) = e^{ijx}.$$

We therefore compute

$$c_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ijx} dx = \begin{cases} \frac{1}{2} & j = 0 \\ \frac{(-1)^j - 1}{-2\pi i j} & j = 1, 2, 3 \end{cases}$$

2. Spectral Theorem Motivation

Partial differential operators act on functions which are elements of certain Hilbert spaces, known as Sobolev spaces. For example, the operator

$$\Delta = -\partial_x^2$$

acts on the Hilbert space H^2. Don't worry about what it is precisely, because all that matters is that it is a Hilbert space. The operator Δ takes elements of the Hilbert space H^2 and sends them to the Hilbert space $\mathcal{L}^2$. It is a linear operator because

$$\partial_x^2(f(x) + g(x)) = f''(x) + g''(x) = \partial_x^2(f(x)) + \partial_x^2(g(x)).$$

Thinking of functions as vectors, then Δ is like a linear map that takes in vectors and spits out vectors. Just like linear maps on finite dimensional vector spaces, which can be represented by a matrix, a linear operator on a Hilbert space can be represented by a matrix. If it is a sufficiently "nice" operator, then there will exist an orthonormal basis of eigenfunctions with corresponding eigenvalues. Here it is useful to recall

Theorem 3 (Spectral Theorem for $\mathbb{C}^n$)**.** *Assume that A is a Hermitian matrix. Then there exists an orthonormal basis of $\mathbb{C}^n$ which consists of eigenvectors of A. Moreover, each of the eigenvalues is real.*

Proof: Remember what Hermitian means. It means that for any $u, v \in \mathbb{C}^n$, we have

$$\langle Au, v \rangle = \langle u, Av \rangle.$$

By the Fundamental Theorem of Algebra, the characteristic polynomial

$$p(x) := \det(A - xI)$$

factors over $\mathbb{C}$. The roots of p are $\{\lambda_k\}_{k=1}^n$. These are by definition the eigenvalues of A. First, we consider the case when A has zero as an eigenvalue. If this is the case, then we define

$$K_0 := \mathrm{Ker}(A) = \{u \in \mathbb{C}^n : Au = 0\}.$$

We note that all nonzero $u \in K_0$ are eigenvectors of A for the eigenvalue 0. Since K_0 is a k-dimensional subspace of $\mathbb{C}^n$, it has an ONB $\{v_1, \ldots, v_k\}$. If $k = n$, we are done. So, assume that $k < n$. Then we consider

$$K_0^\perp = \{u \in \mathbb{C}^n : \langle u, v \rangle = 0 \forall v \in K_0\}.$$

Note that if $u \in K_0^\perp$ then

$$\langle Au, v \rangle = \langle u, Av \rangle = 0 \quad \forall v \in K_0.$$

Hence $A : K_0^\perp \to K_0^\perp$. Moreover, if

$$u \in K_0^\perp, \quad Au = 0 \implies u \in K_0 \cap K_0^\perp \implies u = 0.$$

Hence A is bijective from $K_0^\perp$ to itself. Since A has eigenvalues $\{\lambda_j\}_{j=1}^n$, and 0 appears with multiplicity k, $\lambda_{k+1} \neq 0$. It has some non-zero eigenvector. Let's call it u. Since it is an eigenvector it is not zero, so we define

$$v_{k+1} := \frac{u}{||u||}.$$

Proceeding inductively, we define K_1 to be the span of the vectors $\{v_1, \ldots, v_{k+1}\}$. We look at A restricted to $K_1^\perp$. We note that A maps K_1 to itself because if

$$v = \sum_1^{k+1} c_j v_j \implies Av = \sum_1^{k+1} c_j A v_j = \sum_1^{k+1} c_j \lambda_j v_j \in K_1.$$

Similarly, if $w \in K_1^\perp$,

$$\langle Aw, v\rangle = \langle w, Av\rangle = 0 \forall v \in K_1.$$

So, A maps $K_1^\perp$ into itself. Since the kernel of A is in K_1, A is a surjective and injective map from $K_1^\perp$ into itself. We note that A restricted to $K_1^\perp$ satisfies the same hypotheses as A, in the sense that it is still Hermitian, and it has a characteristic polynomial of degree equal to the dimension of $K_1^\perp$ So, there is an eigenvalue λ_{k+2}, for A as a linear map from $K_1^\perp$ to itself. It has an eigenvector, which we may assume has unit length, contained in $K_1^\perp$. Call it v_{k+2}. Continue inductively until we reach in this way $\{v_1, \ldots, v_n\}$ to span $\mathbb{C}^n$.

Why are the eigenvalues all real? This follows from the fact that if λ is an eigenvalue with eigenvector u then

$$\langle Au, u\rangle = \lambda ||u||^2 = \langle u, Au\rangle = \overline{\lambda} ||u||^2.$$

Since u is an eigenvector it is not zero, so this forces $\lambda = \overline{\lambda}$.

2.1. **An example.** Let us do an example. On $[-\pi, \pi]$, the functions which satisfy

$$\Delta f = \lambda f, \quad f(-\pi) = f(\pi)$$

are

$$f(x) = f_n(x) = e^{inx}.$$

The corresponding

$$\lambda_n = n^2.$$

So, the eigenvalues of Δ with this particular boundary condition are n^2, and the corresponding eigenfunctions are $e^{\pm inx}$. We have proven that these are orthogonal. We note that for all f and g in $\mathcal{L}^2$ which satisfy $f(-\pi) = f(\pi)$, $g(-\pi) = g(\pi)$ and which are also (at least weakly) twice differentiable, we would also get $f'(-\pi) = f'(\pi)$ and similarly for g, so that

$$\langle \Delta f, g\rangle = \int_{-\pi}^{\pi} -f''(x)\overline{g(x)}dx = -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx$$

$$= -f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} + f(x)\overline{g'(x)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx.$$

Due to the boundary conditions, all that survives is

$$-\int_{-\pi}^{\pi} f(x)\overline{g''(x)}dx = \langle f, \Delta g\rangle.$$

So we see that

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle.$$

This is just like the spectral theorem for Hermitian matrices! There is a similar spectral theorem here, a "grown-up linear algebra" theorem, called The Adult Spectral Theorem. This grown-up version of the spectral theorem says that, like a Hermitian matrix, the operator Δ also has an $\mathcal{L}^2$ orthonormal basis of eigenfunctions. Hence, by this Spectral Theorem, we will be able to conclude that the orthonormal *set,*

$$\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}},$$

is an ONB. To state the Adult Spectral Theorem, we need to introduce Regular Sturm-Liouville Problems (SLPs).

2.2. **Regular SLPs.** Let L be a linear, second order ordinary differential operator. So, we can write

$$L(f) = r(x)f''(x) + q(x)f'(x) + p(x)f(x).$$

Above, r, q, and p are specified REAL VALUED functions. As a simple example, take $r(x) = -1$, and $q(x) = p(x) = 0$. Then we have

$$L(f) = \Delta f = -f''(x).$$

We are working with functions defined on an interval $[a, b]$ which is a *finite* interval. So, the Hilbert space in which everything is happening is $\mathcal{L}^2$ on that interval. Like with matrices, we can think about the *adjoint* of the operator L. The adjoint by definition satisfies

$$\langle Lf, g\rangle = \langle f, L^* g\rangle,$$

where we are using L^* to denote the adjoint operator. Whatever it is. On the left side, we know what everything is, so we write it out by definition of the scalar product

$$\langle Lf, g\rangle = \int_a^b L(f)\overline{g(x)}dx = \int_a^b \left(r(x)f''(x) + q(x)f'(x) + p(x)f(x)\right)\overline{g(x)}dx.$$

Integrating by parts, we get

$$\begin{aligned} &= (r\bar{g})f'|_a^b - \int_a^b (r\bar{g})'f' + (qg)f|_a^b - \int_a^b (q\bar{g})'f + \int_a^b pf\bar{g} \\ &= (r\bar{g})f' + (q\bar{g})f|_a^b - \int_a^b \left[(r\bar{g})'f' + (q\bar{g})'f - pf\bar{g}\right]. \end{aligned}$$

We integrate by parts once more on the $(r\bar{g})'f'$ term to get

$$= (r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)|_a^b + \int_a^b (r\bar{g})''f - (q\bar{g})'f + fp\bar{g}.$$

So, if the boundary conditions are chosen to make the stuff evaluated from a to b (these are called the boundary terms in integration by parts) vanish, then we could define

$$L^*g = (rg)'' - (qg)' + pg,$$

since then

$$\langle Lf, g\rangle = \int_a^b (r\bar{g})''f - (q\bar{g})'f + fp\bar{g} = \langle f, L^*g\rangle.$$

Here we use that r, q and p are real valued functions, so $\bar{r} = r$, $\bar{q} = q$, and $\bar{p} = p$. For the spectral theorem to work, we will want to have

$$L = L^*.$$

When this holds, we say that L *is formally self-adjoint.* So, we need

$$Lf = L^*f \iff rf'' + qf' + pf = (rf)'' - (qf)' + pf.$$

We write the things out:

$$rf''+qf'+pf = (rf'+r'f)'-qf'-q'f+pf \iff rf''+qf' = rf''+2r'f'+r''f-qf'-q'f$$

$$\iff qf' = 2r'f' + r''f - qf' - q'f \iff (2q - 2r')f' + (r'' - q')f = 0.$$

To ensure this holds for all f, we set the coefficient functions equal to zero:

$$2q - 2r' = 0 \implies q = r', \quad q' = r''.$$

Well, that just means that $q = r'$. So, we need L to be of the form

$$Lf = rf'' + r'f' + pf = (rf')' + pf.$$

The boundary terms should also vanish, so we want:

$$(r\bar{g})f' - (r\bar{g})'f + (q\bar{g})f)\big|_a^b = (r\bar{g})f' - (r\bar{g})'f + (r'\bar{g})f\big|_a^b = 0,$$

$$\iff r\bar{g}f' - r'\bar{g}f - r\bar{g}'f + r'\bar{g}f\big|_a^b = 0 \iff r\bar{g}f' - r\bar{g}'f\big|_a^b = 0$$

$$\iff r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

So, it suffices to assume that we are working with functions f and g that satisfy

$$(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

Writing this out we get:

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) - (\bar{g}(a)f'(a) - \bar{g}'(a)f(a)) = 0 \iff$$

$$\bar{g}(b)f'(b) - \bar{g}'(b)f(b) = \bar{g}(a)f'(a) - \bar{g}'(a)f(a).$$

This is how we get to the definition of a regular SLP on an interval $[a, b]$. It is specified by

(1) a formally self-adjoint operator

$$L(f) = (rf')' + pf,$$

where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:

$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$

The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions

$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP

$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$

The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition.

We then have a miraculous fact.

Theorem 4 (Adult Spectral Theorem)**.** *For every regular Sturm-Liouville problem as above, there is an orthonormal basis of L^2_w consisting of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. We have*

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Here, L^2_w is the weighted Hilbert space consisting of (the almost everywhere-equivalence classes of measurable) functions on the interval $[a, b]$ which satisfy

$$\int_a^b |f(x)|^2 w(x) dx < \infty,$$

and the scalar product is

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)} w(x) dx.$$

We are not equipped to prove this fact. You can rest assured however that it is done through the techniques of functional analysis and bears similarity to the proof of the spectral theorem for finite dimensional vector spaces. As a corollary to this theorem we obtain

Theorem 5. *The functions*

$$\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$$

are an orthogonal basis for the Hilbert space $\mathcal{L}^2(-\pi, \pi)$.

Proof: These functions satisfy a regular SLP. This SLP is to find all constants λ and functions f such that

$$f'' + \lambda f = 0,$$

and f is 2π periodic. The operator L is just the operator

$$L(f) = f''.$$

The function $r = 1$, $p = 0$, and the weight is just 1. The boundary conditions are thus:

$$f(-\pi) - f(\pi) = 0, \quad f'(-\pi) - f'(\pi) = 0.$$

We can check that this is 'self-adjoint' by plugging it into the required condition. Assume that some totally arbitrary f and g satisfy this condition, so that $g(-\pi) - g(\pi) = 0$ also. Then

$$(\bar{g}f' - \bar{g}'f)|_{-\pi}^{\pi} = \bar{g}(\pi)f'(\pi) - \bar{g}'(\pi)f(\pi) - \bar{g}(-\pi)f'(-\pi) + \bar{g}'(-\pi)f(-\pi) = 0.$$

By our ODE theory, we can already say that all solutions (up to constant factors) to this problem are

$$f_n(x) = e^{inx}, \quad \lambda_n = n^2\pi^2.$$

Now, by the Adult Spectral Theorem, we know that these are an orthogonal basis (they can be normalized if we so desire).

2.3. **Exercises for the week: Hints.** Those exercises from [1] which one should solve are:

(1) (3.3.1) Show that if $\{f_n\}_{n\geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that
$$\lim_{n\to\infty} f_n = f,$$
then for all $g \in H$ we have
$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle.$$
Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f, g\rangle$.

(2) (3.3.2) Show that for all f, g in a Hilbert space one has
$$|||f|| - ||g||| \leq ||f - g||.$$
Hint: First show that for any real numbers a and b,
$$|a - b|^2 = a^2 - 2ab + b^2.$$
Next, apply this fact with $a = ||f||$ and $b = ||g||$ to show that
$$|||f|| - ||g||| = ||f||^2 - 2||f||||g|| + ||g||^2.$$
Compare this to
$$||f - g||^2 = ||f||^2 - 2\Re\langle f, g\rangle + ||g||^2.$$

(3) (3.3.10.d) Use Parseval's equation to compute
$$\sum_{n\geq 1} \frac{\sin^2(na)}{n^4}.$$
Hint: The Fourier series of
$$f(x) := \begin{cases} x & -a < x < a \\ a\frac{\pi - x}{\pi - a} & a < \theta < \pi \\ a\frac{\pi + x}{a - \pi} & -\pi < x < -a \end{cases}$$
where implicitly we are assuming $0 < a < \pi$ is
$$\frac{2}{\pi - a}\sum_{n\geq 1} \frac{\sin(na)}{n^2}\sin(nx)$$

(4) (3.4.1) Show that $\{e^{2\pi i(mx+ny)}\}_{n,m\in\mathbb{Z}}$ is an orthogonal set in $\mathcal{L}^2(R)$ where R is any square whose sides have length one and are parallel to the coordinate axes. Hint: Compute the integral
$$\int_{x=a}^{a+1}\int_{y=b}^{b+1} e^{2\pi i(mx+ny)}e^{-2\pi i(kx+\ell y)}dxdy, \quad m, n, k, \ell \in \mathbb{Z}.$$

(5) (3.4.6) Find an example of a sequence $\{f_n\}$ in $\mathcal{L}^2(0,\infty)$ such that $f_n(x) \to 0$ uniformly for all $x > 0$ but $f_n \not\to 0$ in the $\mathcal{L}^2$ norm. Hint: Oh this is a fun sort of challenge problem... Here is a little bit of idea. The function $\frac{1}{\sqrt{n^2+x}}$ is not in $\mathcal{L}^2(0,\infty)$. How about using this function as an idea, define functions $f_n(x)$ which are say defined in some way for $x \in [0, n]$ and make

them zero for all $x > n$. Get them to decrease uniformly to zero for all x, but get their $\mathcal{L}^2$ norms to be increasing... Play around with it!

(6) (3.5.7) Find all solutions f on $[0,1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

Hint: As we have computed before, consider three cases, $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. Use the boundary conditions to solve for all the possible f.

(7) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0,\ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell,t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0,t) = 0$. Find a series expansion for the temperature $u(x,t)$ such that the initial temperature is given by $f(x)$. Hint: Divide and conquer. First find a so-called steady state solution, that is find a function $g(x)$ which does not depend on t which satisfies

$$(\partial_t - \partial_{xx})g = 0, \quad g(0) = 0, \quad g'(\ell) = A.$$

Now, since g does not depend on t, when you apply the heat operator you just get

$$-g''(x) = 0, \quad g(0) = 0, \quad g'(\ell) = A.$$

Find g which solves this. Now, look for a solution u which satisfies

$$u_t - u_{xx} = 0, \quad u(0,t) = u_x(\ell,t) = 0, \quad u(x,0) = f(x) - g(x).$$

You can use the methods from last week, separation of variables, superposition (since everything including the BCs are homogeneous), and Fourier series (Hilbert spaces!) to solve for u. The full solution will then be

$$u(x,t) + g(x).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.07

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. SLPs

Recall the definition of a regular SLP:

(1) a formally self-adjoint differential operator
$$L(f) = (rf')' + pf,$$
where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:
$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$
The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions
$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP
$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$
The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition. The magical theorem about SLPs says that for such a regular SLP, there exists solutions $\{\phi_n\}_{n\geq 1}$ with corresponding eigenvalues λ_n such that these $\{\phi_n\}_{n\geq 1}$ are an orthogonal basis for the weighted $\mathcal{L}^2$ space, $\mathcal{L}^2_w(a, b)$. Moreover, these eigenvalues are all *real.* Let's see just what makes this theorem so magical...

1.1. SLP example for a PDE.

Here is how the SLP theory can be useful in practice. We are given the problem
$$u_t - u_{xx} = 0, \quad u_x(0, t) = \alpha u(0, t), \quad u_x(l, t) = -\alpha u(l, t), \quad u(x, 0) = f(x).$$
Above, we assume that
$$\alpha > 0, \quad f \in \mathcal{L}^2.$$
These boundary conditions are based on Newton's law of cooling: the temperature gradient across the ends is proportional to the temperature difference between the

ends and the surrounding medium. It is a homogeneous PDE, so we have good chances of being able to solve it using separation of variables. Thus, we write

$$u(x,t) = X(x)T(t) \implies T'(t)X(x) - X''(x)T(t) = 0 \implies \frac{T'}{T} = \frac{X''}{X}.$$

This means both sides are equal to a constant. Call it λ. We start with the x side, because we have more information about that due to the BCs. Are they self-adjoint BCs? Let's check! In the definition of SLP, we are looking for X to satisfy

$$\frac{X''}{X} = \lambda \iff X'' = \lambda X \iff X'' - \lambda X = 0.$$

OBS! The relationship between the constant we have named λ from the PDE has the *opposite* sign as the corresponding term in an SLP. So, the SLP would look like

$$X'' + \Lambda X = 0 \quad \Lambda = -\lambda.$$

The r and w are both 1 in the definition of SLP, and the p is 0. The $a = 0$ and $b = l$. So, we need to check that if f and g satisfy

$$f'(0) = \alpha f(0), \quad g'(l) = -\alpha g(l)$$

then

$$(\bar{g}f' - \bar{g}'f)|_0^l = 0.$$

We plug it in

$$\bar{g}(l)f'(l) - \bar{g}'(l)f(l) - \bar{g}(0)f'(0) + \bar{g}'(0)f(0)$$
$$= -\bar{g}(l)\alpha f(l) + \alpha\bar{g}(l)f(l) - \bar{g}(0)\alpha f(0) + \alpha\bar{g}(0)f(0) = 0.$$

Yes, the BC is a self-adjoint BC. So, the SLP theorem says there exists an $\mathcal{L}^2$ OB of eigenfunctions. What are they? We check the cases.

$$X'' = \lambda X.$$

What if $\lambda = 0$? Then

$$X(x) = ax + b.$$

To get

$$X'(0) = \alpha X(0) \implies a = \alpha b \implies b = \frac{a}{\alpha}.$$

Next,

$$X'(l) = -\alpha X(l) \implies a = -\alpha\left(al + \frac{a}{\alpha}\right) = -a(\alpha l + 1).$$

Presumably $a \neq 0$ because if $a = 0$ the whole solution is just 0. So, we can divide by it and we get

$$\implies 1 = -(\alpha l + 1) \implies \alpha l = -2.$$

Since $l > 0$ and $\alpha > 0$, this is impossible. So, no non-zero solutions for $\lambda = 0$.

Next we try $\lambda > 0$. Then the solution looks like

$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$$

or equivalently, we can use sinh and cosh, to write

$$X(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x).$$

We try out the BCs. They require

$$X'(0) = \alpha X(0) \iff a\sqrt{\lambda}\sinh(0) + b\sqrt{\lambda}\cosh(0) = \alpha\left(a\cosh(0) + b\sinh(0)\right)$$
$$\iff b\sqrt{\lambda} = \alpha a \implies b = \frac{\alpha a}{\sqrt{\lambda}}.$$

We check out the other BC:

$$X'(l) = -\alpha X(l) \iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l)+\alpha a\cosh(\sqrt{\lambda}l) = -\alpha\left(a\cosh(\sqrt{\lambda}l) + \frac{\alpha a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l)\right).$$

$$\iff a\sqrt{\lambda}\sinh(\sqrt{\lambda}l) + \frac{\alpha^2 a}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}l) = -2\alpha a\cosh(\sqrt{\lambda}l)$$

If $a = 0$ the whole solution is zero, so we presume that is not the case and divide by a. Then this requires

$$\frac{\sinh(\sqrt{\lambda}l)}{\cosh(\sqrt{\lambda}l)} = \frac{-2\alpha}{\sqrt{\lambda}+\alpha^2/\sqrt{\lambda}}.$$

The left side is positive, but the right side is negative. $\lightning$

Thus, we finally try $\lambda < 0$. Then the solution looks like

$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$

To get

$$X'(0) = \alpha X(0) \implies b\sqrt{|\lambda|} = \alpha a \implies b = \frac{\alpha a}{\sqrt{|\lambda|}}.$$

Next we need

$$X'(l) = -\alpha X(l)$$

$$\implies -a\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}l)+\frac{\alpha a}{\sqrt{|\lambda|}}\sqrt{|\lambda|}\cos(\sqrt{|\lambda|}l) = -\alpha\left(a\cos(\sqrt{|\lambda|}l) + \frac{\alpha a}{\sqrt{|\lambda|}}\sin(\sqrt{|\lambda|}l)\right).$$

Presumably $a \neq 0$ because if that is the case then the whole solution is 0. So, we may divide by a, and we need

$$2\alpha\cos\sqrt{|\lambda|} = \sin(\sqrt{|\lambda|}l)\left(\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}\right).$$

This is equivalent to

$$\frac{2\alpha}{\sqrt{|\lambda|} - \frac{\alpha^2}{\sqrt{|\lambda|}}} = \tan(\sqrt{|\lambda|}l)$$

$$\iff \frac{2\alpha\sqrt{|\lambda|}}{|\lambda| - \alpha^2} = \tan(\sqrt{|\lambda|}l).$$

Well, that's pretty weird, but according to the SLP theory, the sequence

$$\{\lambda_n\}_{n\geq 1} \text{ and } \{X_n(x)\}_{n\geq 1}, \quad X_n(x) = a_n\left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}}\sin(\sqrt{|\lambda_n|}x)\right)$$

of eigenvalues and corresponding eigenfunctions is an orthogonal basis of $\mathcal{L}^2$. Here since we are solving a PDE, it is most convenient to leave the coefficients simply as a_n and solve for them according to the initial conditions of the PDE.

The partner functions

$$T_n(t) \text{ satisfy } T_n'(t) = \lambda_n T_n(t) \implies T_n(t) = e^{\lambda_n t}.$$

Here it is good to note that the $\lambda_n < 0$ and tend to $-\infty$ as $n \to \infty$ which follows from the Adult Spectral Theorem, because in the SLP terminology,

$$\Lambda_n = -\lambda_n \to \infty \implies \lambda_n \to -\infty.$$

So, for heat, that is realistic. We build the solution using superposition because the PDE is linear and homogeneous, so

$$u(x,t) = \sum_{n\geq 1} T_n(t) X_n(x).$$

Since we wish this to be equal to the initial data at $t = 0$, we demand

$$u(x,0) = \sum_{n\geq 1} a_n \left(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \right) = f(x).$$

By the SLP theory, the functions above form an OB, so we can expand our initial data function in terms of this OB. To do this we compute

$$a_n = \frac{\langle f(x), \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \rangle}{|| \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x)||^2},$$

where

$$\langle f(x), \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x) \rangle = \int_0^l f(x) \overline{(\cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x))} dx,$$

$$|| \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x)||^2 = \int_0^l | \cos(\sqrt{|\lambda_n|}x) + \frac{\alpha}{\sqrt{|\lambda_n|}} \sin(\sqrt{|\lambda_n|}x)|^2 dx.$$

1.2. **SLP example.** SLPs may come from solving a PDE, but to avoid overcomplicating things, sometimes you will just need to solve an SLP by itself. For example:

$$(xf')' + \lambda x^{-1} f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$

In this example the function $r(x) = x$, and the function $p(x) = 0$, whilst the weight function $w(x) = x^{-1}$. Let us consider three cases for λ.

Case $\lambda = 0$**:** If $\lambda = 0$, then the equation becomes

$$xf'' + f' = 0,$$

which we can re-arrange to

$$\frac{f''}{f'} = -\frac{1}{x}.$$

The left side is the derivative of $\log(f')$. So, integrating both sides (saving the constant for later):

$$\log(f') = -\log(x).$$

Exponentiating both sides we get

$$f' = \frac{1}{x} \implies f(x) = A\log(x) + B,$$

for some constants A and B. The boundary conditions demand that

$$f(1) = 0 \implies B = 0.$$

The other boundary condition demands that

$$f(b) = 0 \implies A = 0, \text{ since } b > 1 \text{ so } \log(b) > 0.$$

We are left with the zero function. That is never an eigenfunction. So $\lambda = 0$ is not an eigenvalue for this SLP.

Case $\lambda > 0$**:** If $\lambda > 0$, we observe that the equation we have is something called an Euler equation. (Or we look up the ODE section of Beta and search for this type of ODE, and see that Beta tells us this is an Euler equaiton). Consequently, we look for solutions of the form

$$f(x) = x^\nu.$$

The differential equation we wish to solve is:

$$xf'' + f' + \lambda x^{-1} f = 0 \implies x^2 f'' + xf' + \lambda f = 0,$$

so substituting $f(x) = x^\nu$, this becomes

$$x^2(\nu)(\nu - 1)x^{\nu-2} + x\nu x^{\nu-1} + \lambda x^\nu = 0.$$

This simplifies to:

$$x^\nu \left(\nu^2 - \nu + \nu + \lambda\right) = 0 \implies \nu^2 = -\lambda.$$

Since $\lambda > 0$, this means

$$\nu = \pm i\sqrt{\lambda}.$$

So, a basis of solutions is $x^{i\sqrt{|\lambda|}}$ and $x^{-i\sqrt{\lambda}}$. Note that

$$x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\log(x)}.$$

By Euler's formula, an equivalent basis of solutions is

$$\cos(\sqrt{\lambda}\log(x)), \quad \sin(\sqrt{\lambda}\log(x)).$$

Hence in this case our solution is of the form:

$$f(x) = A\cos(\sqrt{\lambda}\log(x)) + B\sin(\sqrt{\lambda}\log(x)).$$

The boundary conditions demand that

$$f(1) = 0 \implies A = 0.$$

The second boundary condition demands that

$$B\sin(\sqrt{\lambda}\log(b)) = 0.$$

Since we do not seek the zero function, we presume that $B \neq 0$ and thus require

$$\sin(\sqrt{\lambda}\log(b)) = 0 \implies \sqrt{\lambda}\log(b) = n\pi, \quad n \in \mathbb{N}.$$

We therefore have countably many eigenfunctions and eigenvalues, which we may index by the natural numbers, writing

$$\lambda_n = \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = \sin\left(\frac{n\pi\log(x)}{\log(b)}\right).$$

Nice.

The last case to consider is **case** $\lambda < 0$**:** We proceed similarly as above and obtain that a basis of solutions is

$$x^{\pm\sqrt{|\lambda|}}.$$

Write our solution as

$$f(x) = Ax^{\sqrt{|\lambda|}} + Bx^{-\sqrt{|\lambda|}}.$$

The boundary conditions demand that:

$$f(1) = 0 \implies A + B = 0 \implies B = -A.$$

The next boundary condition demands that:

$$f(b) = Ab^{\sqrt{|\lambda|}} - Ab^{-\sqrt{|\lambda|}} = 0 \implies A = 0 \text{ or } b^{\sqrt{|\lambda|}} = b^{-\sqrt{|\lambda|}} \implies b^{2\sqrt{|\lambda|}} = 1 \implies \sqrt{|\lambda|} = 0 \lightning.$$

Thus the only way for the boundary conditions to be satisfied is if the eigenfunction is the zero function, but this is not an eigenfunction! Hence no negative λ solutions.

The magical SLP theorem tells us that these rather peculiar functions

$$\{f_n(x)\}_{n\geq 1}$$

are an orthogonal basis for $\mathcal{L}^2_{1/x}(1,b)$. This means that for any $g \in \mathcal{L}^2_{1/x}(1,b)$, we can expand it as a Fourier series with respect to this basis. The coefficients will be

$$\frac{\langle g, f_n\rangle_{1/x}}{||f_n||^2_{1/x}}, \quad \langle g, f_n\rangle_{1/x} = \int_1^b g(x)\overline{f_n(x)}x^{-1}dx, \quad ||f_n||^2_{1/x} = \int_1^b |f_n(x)|^2 x^{-1}dx.$$

If the function we wish to expand is specified, we could compute these integrals.

1.3. **Another SLP example.** Consider the problem

$$(x^2 f')' + \lambda f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$

Here we have $r(x) = x^2$ and $w(x) = 1$. The equation is:

$$2xf' + x^2 f'' + \lambda f = 0.$$

We shall consider the three cases for λ.

Case $\lambda = 0$**:** In this case the equation simplifies to

$$x^2 f'' + 2xf' = 0 \implies \frac{f''}{f'} = -\frac{2}{x} \implies (\log(f'))' = -\frac{2}{x} \implies \log(f') = -2\log x \implies f' = e^{-2\log x} = x^{-2}.$$

So, this gives us a solution of the form

$$f(x) = -A\frac{1}{x} + B.$$

Let us verify the boundary conditions. We require $f(1) = 0$ so this means

$$-A + B = 0 \implies B = A.$$

We also require $f(b) = 0$ so this means

$$-A\frac{1}{b} + B = 0 = \frac{-A}{b} + A \implies \frac{A}{b} = A \implies b = 1 \text{ or } A = 0.$$

So since $b > 1$ the only solution here is the zero function which is not an eigenfunction.

Case $\lambda > 0$**:** We consider the fact that this is an Euler equation, so we look for solutions of the form $f(x) = x^\nu$. Then the equation looks like:

$$x^2(\nu)(\nu-1)x^{\nu-2} + 2x(\nu)x^{\nu-1} + \lambda x^\nu = 0 \iff x^\nu\left(\nu^2 - \nu + 2\nu + \lambda\right) = 0$$

so we need ν to satisfy:

$$\nu^2 + \nu + \lambda = 0.$$

This is a quadratic equation, so we solve it:

$$\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

So, actually the cases $\lambda > 0$ and $\lambda < 0$ really should split up into whether $\lambda = \frac{1}{4}$ or is larger or smaller. If $\lambda = \frac{1}{4}$, then we are only getting one solution this way, $x^{-1/2}$. To get a second solution we multiply by $\log x$.

Exercise 1. *Plug the function* $x^{-1/2}\log x$ *into the SLP for the value* $\lambda = \frac{1}{4}$. *Verify that it satisfy the equation.*

Now, let's see if such a function will satisfy the boundary conditions. We need

$$Ax^{-1/2} + Bx^{-1/2}\log(x)\Big|_{x=1} = 0 \implies A = 0.$$

We also need

$$Bb^{-1/2}\log(b) = 0, \quad b > 1 \implies B = 0.$$

So we only get the zero solution in this case.

When $\lambda < \frac{1}{4}$, solutions are of the form

$$Ax^{\nu_+} + Bx^{\nu_-}, \quad \nu_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

Exercise 2. *Check the boundary conditions. Verify that they are satisfied if and only if $A = B = 0$.*

Finally we consider $\lambda > \frac{1}{4}$. Then we have

$$\nu_\pm = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}} \implies f(x) = \frac{A}{\sqrt{x}}x^{i\sqrt{\lambda-1/4}} + \frac{B}{\sqrt{x}}x^{-i\sqrt{\lambda-1/4}}.$$

Using Euler's formula, this is equivalently expressed as

$$\frac{\alpha}{\sqrt{x}}\cos(\sqrt{\lambda - 1/4}\log x) + \frac{\beta}{\sqrt{x}}\sin(\sqrt{\lambda - 1/4}\log x).$$

Due to the boundary condition at $x = 1$ we must have $\alpha = 0$. So to obtain the other boundary condition, we need

$$\sin(\sqrt{\lambda - 1/4}\log b) = 0 \implies \sqrt{\lambda - 1/4}\log b = n\pi, \quad n \in \mathbb{N}.$$

Hence

$$\lambda = \lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = x^{-1/2}\sin\left(\frac{n\pi\log x}{\log b}\right).$$

Note that in general we are not bothering to normalize our eigenfunctions because it is rather tedious and not fundamental to our learning experience in this subject.

1.4. **Exercises for the week: Answers.** Those exercises from [1] which one should solve are:

(1) (3.3.1) Show that if $\{f_n\}_{n\geq 1}$ are elements of a Hilbert space, H, and we have for some $f \in H$ that

$$\lim_{n\to\infty} f_n = f,$$

then for all $g \in H$ we have

$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle.$$

Answer: we would like to prove

$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle.$$

This is equivalent to proving

$$\lim_{n\to\infty} \langle f_n, g\rangle - \langle f, g\rangle = 0.$$

So, next we follow the hint and estimate

$$|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle (f_n - f), g\rangle| \leq ||f_n - f||||g||.$$

The meaning of

$$\lim_{n\to\infty} f_n = f$$

in a Hilbert space is that

$$\lim_{n\to\infty} ||f_n - f|| = 0.$$

Hence, by some theorem about the product of limits, as long as they exist (obs! $\lim_{n\to\infty} ||g|| = ||g||$, it's just not changing at all), we have

$$\lim_{n\to\infty} ||f_n - f|| ||g|| = ||g|| \lim_{n\to\infty} ||f_n - f|| = 0.$$

(2) (3.3.2) Show that for all f, g in a Hilbert space one has

$$|||f|| - ||g||| \le ||f - g||.$$

Answer: We follow the hint. For any real numbers a and b,

$$|a - b|^2 = a^2 - 2ab + b^2.$$

Next, we apply this fact with $a = ||f||$ and $b = ||g||$ to obtain that

$$|||f|| - ||g|||^2 = ||f||^2 - 2||f|| ||g|| + ||g||^2.$$

We compare this to

$$||f - g||^2 = ||f||^2 - 2\Re\langle f, g\rangle + ||g||^2,$$

since

$$||f|| ||g|| \ge \Re\langle f, g\rangle \implies ||f||^2 - 2\Re\langle f, g\rangle + ||g||^2 \ge ||f||^2 - 2||f|| ||g|| + ||g||^2.$$

Thus we obtain

$$||f - g||^2 \ge |||f|| - ||g|||^2.$$

Taking the square root of both sides completes the proof.

(3) (3.3.10.d) Use Parseval's equation to compute

$$\sum_{n\ge 1} \frac{\sin^2(na)}{n^4}.$$

Answer:

$$\frac{a^2(\pi - a)^2}{6}.$$

(4) (3.4.1) Show that $\{e^{2\pi i(mx+ny)}\}_{n,m\in\mathbb{Z}}$ is an orthogonal set in $\mathcal{L}^2(R)$ where R is any square whose sides have length one and are parallel to the coordinate axes. Answer:

$$\int_{x=a}^{a+1} \int_{y=b}^{b+1} e^{2\pi i(mx+ny)} e^{-2\pi i(kx+\ell y)} dx dy = \int_{x=a}^{a+1} e^{2\pi i(m-k)x} dx \int_{y=b}^{b+1} e^{2\pi i(n-\ell)y} dy.$$

For $m \ne k$,

$$\int_{x=a}^{a+1} e^{2\pi i(m-k)x} dx = \left. \frac{e^{2\pi i(m-k)x}}{2\pi i(m-k)} \right|_a^{a+1}.$$

The function above is 1 periodic, so this is zero. Same holds for $n \ne \ell$.

(5) (3.4.6) Find an example of a sequence $\{f_n\}$ in $\mathcal{L}^2(0,\infty)$ such that $f_n(x) \to 0$ uniformly for all $x > 0$ but $f_n \not\to 0$ in the $\mathcal{L}^2$ norm. Answer: let

$$f_n(x) := \begin{cases} \frac{1}{\sqrt{x+\sqrt{n}}} & 0 \le x \le n \\ 0 & x > n \end{cases}.$$

Then

$$0 \le \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} \frac{1}{\sqrt{\sqrt{n}}} = 0.$$

So the convergence to zero is uniform on $[0,\infty)$. On the other hand

$$||f_n||^2_{\mathcal{L}^2} = \int_0^\infty |f_n(x)|^2 dx = \int_0^n \frac{1}{x+\sqrt{n}} dx = \ln(x+\sqrt{n})\Big|_{x=0}^n$$

$$= \ln(n+\sqrt{n}) - \ln(\sqrt{n}) = \ln\left(\frac{n+\sqrt{n}}{\sqrt{n}}\right) = \ln(\sqrt{n}+1).$$

This simultaneously shows that $f_n \in \mathcal{L}^2(0,\infty)$ for all n, as well as that the $\mathcal{L}^2$ norm of f_n tends to infinity.

(6) (3.5.7) Find all solutions f on $[0,1]$ and all corresponding λ to the equation:

$$f'' + \lambda f = 0, \quad f(0) = 0, \quad f'(1) = -f(1).$$

Answer: the eigenvalues are $\lambda_n = \nu_n^2$ where ν_n are the positive solutions of $\tan(\nu) = -\nu$, and the eigenfunctions are $\sin(\nu_n x)$.

(7) (4.2.3) Let $f(x)$ be the initial temperature at the point x in a rod of length ℓ, mathematicized as the interval $[0,\ell]$. Assume that heat is supplied at a constant rate at the right end, in particular $u_x(\ell,t) = A$ for a constant value A, and that the left end is held at the constant temperature 0, so that $u(0,t) = 0$. Find a series expansion for the temperature $u(x,t)$ such that the initial temperature is given by $f(x)$. Answer:

$$u(x,t) = Ax + \sum_{n\ge1}\left(b_n + \frac{(-1)^n 8A\ell}{(2n-1)^2\pi^2}\right) e^{-(n-1/2)^2\pi^2 kt/(\ell^2)} \sin((n-1/2)\pi x/\ell).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.10

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. SLPs

Recall the definition of a regular SLP:

(1) a formally self-adjoint differential operator
$$L(f) = (rf')' + pf,$$
where r and p are real valued, r, r', and p are continuous, and $r > 0$ on $[a, b]$.

(2) self-adjoint boundary conditions:
$$B_i(f) = \alpha_i f(a) + \alpha_i' f'(a) + \beta_i f(b) + \beta_i' f'(b) = 0, \quad i = 1, 2.$$
The self adjoint condition further requires that the coefficients $\alpha_i, \alpha_i', \beta_i, \beta_i'$ are such that for all f and g which satisfy these conditions
$$r(\bar{g}f' - \bar{g}'f)\big|_a^b = 0.$$

(3) a positive, continuous function w on $[a, b]$.

The SLP is to find all solutions to the BVP
$$L(f) + \lambda w f = 0, \quad B_i(f) = 0, \quad i = 1, 2.$$
The eigenvalues are all numbers λ for which there exists a corresponding non-zero eigenfunction f so that together they satisfy the above equation, and f satisfies the boundary condition.

1.1. SLP example.

Consider the problem
$$(x^2 f')' + \lambda f = 0, \quad f(1) = f(b) = 0, \quad b > 1.$$
Here we have $r(x) = x^2$ and $w(x) = 1$. The equation is:
$$2xf' + x^2 f'' + \lambda f = 0.$$
We shall consider the three cases for λ.

Case $\lambda = 0$: In this case the equation simplifies to
$$x^2 f'' + 2xf' = 0 \implies \frac{f''}{f'} = -\frac{2}{x} \implies (\log(f'))' = -\frac{2}{x} \implies \log(f') = -2\log x \implies f' = e^{-2\log x} = x^{-2}.$$

So, this gives us a solution of the form

$$f(x) = -A\frac{1}{x} + B.$$

Let us verify the boundary conditions. We require $f(1) = 0$ so this means

$$-A + B = 0 \implies B = A.$$

We also require $f(b) = 0$ so this means

$$-A\frac{1}{b} + B = 0 = \frac{-A}{b} + A \implies \frac{A}{b} = A \implies b = 1 \text{ or } A = 0.$$

So since $b > 1$ the only solution here is the zero function which is not an eigenfunction.

Case $\lambda > 0$: We consider the fact that this is an Euler equation, so we look for solutions of the form $f(x) = x^\nu$. Then the equation looks like:

$$x^2(\nu)(\nu-1)x^{\nu-2} + 2x(\nu)x^{\nu-1} + \lambda x^\nu = 0 \iff x^\nu\left(\nu^2 - \nu + 2\nu + \lambda\right) = 0$$

so we need ν to satisfy:

$$\nu^2 + \nu + \lambda = 0.$$

This is a quadratic equation, so we solve it:

$$\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

So, actually the cases $\lambda > 0$ and $\lambda < 0$ really should split up into whether $\lambda = \frac{1}{4}$ or is larger or smaller. If $\lambda = \frac{1}{4}$, then we are only getting one solution this way, $x^{-1/2}$. To get a second solution we multiply by $\log x$.

Exercise 1. *Plug the function $x^{-1/2}\log x$ into the SLP for the value $\lambda = \frac{1}{4}$. Verify that it satisfy the equation.*

Now, let's see if such a function will satisfy the boundary conditions. We need

$$Ax^{-1/2} + Bx^{-1/2}\log(x)\Big|_{x=1} = 0 \implies A = 0.$$

We also need

$$Bb^{-1/2}\log(b) = 0, \quad b > 1 \implies B = 0.$$

So we only get the zero solution in this case.

When $\lambda < \frac{1}{4}$, solutions are of the form

$$Ax^{\nu_+} + Bx^{\nu_-}, \quad \nu_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

Exercise 2. *Check the boundary conditions. Verify that they are satisfied if and only if $A = B = 0$.*

Finally we consider $\lambda > \frac{1}{4}$. Then we have

$$\nu_\pm = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}} \implies f(x) = \frac{A}{\sqrt{x}}x^{i\sqrt{\lambda-1/4}} + \frac{B}{\sqrt{x}}x^{-i\sqrt{\lambda-1/4}}.$$

Using Euler's formula, this is equivalently expressed as

$$\frac{\alpha}{\sqrt{x}}\cos(\sqrt{\lambda - 1/4}\log x) + \frac{\beta}{\sqrt{x}}\sin(\sqrt{\lambda - 1/4}\log x).$$

Due to the boundary condition at $x = 1$ we must have $\alpha = 0$. So to obtain the other boundary condition, we need

$$\sin(\sqrt{\lambda - 1/4}\log b) = 0 \implies \sqrt{\lambda - 1/4}\log b = n\pi, \quad n \in \mathbb{N}.$$

Hence

$$\lambda = \lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log b)^2}, \quad f_n(x) = x^{-1/2}\sin\left(\frac{n\pi \log x}{\log b}\right).$$

Note that in general we are not bothering to normalize our eigenfunctions because it is rather tedious and not fundamental to our learning experience in this subject.

2. The theory item on SLPs

There is one theory item about SLPs which one *does* need to be able to prove.

Theorem 1 (Cute facts about SLPs)**.** *Let f and g be eigenfunctions for a regular SLP in an interval $[a, b]$ with weight function $w(x) > 0$. Let λ be the eigenvalue for f and μ the eigenvalue for g. Then:*

(1) $\lambda \in \mathbb{R}$ och $\mu \in \mathbb{R}$;
(2) If $\lambda \neq \mu$, then:

$$\int_a^b f(x)\overline{g(x)}w(x)dx = 0.$$

Proof: By definition we have $Lf + \lambda wf = 0$. Moreover, L is self-adjoint, which similar to matrices guarantees that

$$\langle Lf, f\rangle = \langle f, Lf\rangle.$$

By being an eigenfunction,

$$Lf = -\lambda wf.$$

So combining these facts:

$$\langle Lf, f\rangle = \langle -\lambda wf, f\rangle = -\lambda\langle wf, f\rangle$$
$$= \langle f, Lf\rangle = \langle f, -\lambda wf\rangle = -\overline{\lambda}\langle f, wf\rangle.$$

Since w is real valued,

$$\langle wf, f\rangle = \int_a^b w(x)f(x)\overline{f(x)}dx = \int_a^b |f(x)|^2 w(x)dx,$$
$$\langle f, wf\rangle = \int_a^b f(x)\overline{w(x)f(x)}dx = \int_a^b |f(x)|^2 w(x)dx.$$

Since $w > 0$ and f is an eigenfunction,

$$\int_a^b |f(x)|^2 w(x)dx > 0.$$

So, the equation

$$-\lambda\langle wf, f\rangle = -\lambda\int_a^b |f(x)|^2 w(x)dx = -\overline{\lambda}\langle f, wf\rangle = -\overline{\lambda}\int_a^b |f(x)|^2 w(x)dx$$

implies

$$\lambda = \overline{\lambda}.$$

For the second part, we use basically the same argument based on self-adjointness:

$$\langle Lf, g\rangle = \langle f, Lg\rangle.$$

By assumption

$$\langle Lf, g\rangle = -\lambda\langle wf, g\rangle = -\lambda\int_a^b w(x)f(x)\overline{g(x)}dx.$$

Similarly,

$$\langle f, Lg\rangle = \langle f, -\mu wg\rangle = -\overline{\mu}\langle f, wg\rangle = -\mu\langle f, wg\rangle = -\mu\int_a^b f(x)\overline{g(x)}w(x)dx,$$

since $\mu \in \mathbb{R}$ and $w(x)$ is real. So we have

$$-\lambda\int_a^b w(x)f(x)\overline{g(x)}dx = -\mu\int_a^b f(x)\overline{g(x)}w(x)dx.$$

If the integral is non-zero, then it forces $\lambda = \mu$ which is false. Thus the integral must be zero.

3. Solving PDEs with inhomogeneities: turning a ♡ problem into a ♡♡ problem

Let's consider the problem

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

$$u(-\pi, t) = u(\pi, t) = 0$$

$$u_t(x, 0) = 0, \quad x \in [-\pi, \pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 5 \quad x \in [-\pi, \pi], \quad t > 0.$$

We nickname this problem ♡. For the first time, we have an *inhomogeneous PDE.*

Idea: Deal with a time independent inhomogeneity in the PDE by finding a steady state solution.

The idea is that we look for a function $f(x)$ which depends only on x which satisfies the boundary conditions and also satisfies the inhomogeneous PDE. Since f only depends on x, the PDE for f is

$$-f''(x) = 5 \iff f''(x) = -5.$$

This means that

$$f'(x) = -5x + b \implies f(x) = -\frac{5x^2}{2} + bx + c.$$

Now, we want f to satisfy the boundary conditions. So, we want

$$-\frac{5\pi^2}{2} - b\pi + c = 0 = -\frac{5\pi^2}{2} + b\pi + c.$$

If we subtract these equations, then we see that we need to have $b = 0$. If we add these equations then we see that we need

$$-5\pi^2 + 2c = 0 \implies c = \frac{5\pi^2}{2}.$$

Thus, we have found a solution to

$$-f''(x) = 5, \quad f(\pm\pi) = 0,$$

which is

$$f(x) = -\frac{5x^2}{2} + \frac{5\pi^2}{2}.$$

If we then look for a solution to

$$u(x,0) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases} =: v(x)$$

$$u(-\pi,t) = u(\pi,t) = 0$$

$$u_t(x,0) = 0, \quad x \in [-\pi,\pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0,$$

and we add it to f, we will get

$$u(x,0) + f(x) = v(x) + f(x) \neq v(x).$$

The initial condition gets messed up because of f. So, we need to compensate for this. For that reason, we look for a solution to a new problem:

$$u(x,0) = -f(x) + v(x)$$

$$u(-\pi,t) = u(\pi,t) = 0$$

$$u_t(x,0) = 0, \quad x \in [-\pi,\pi]$$

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0.$$

We nickname this new problem ♡♡ because we like it better than ♡. Then, our full solution will be

$$U(x,t) = u(x,t) + f(x).$$

This solution U will then solve ♡. Here it is important to note that when we add u and f, the boundary condition still holds. So, please think about this, because in certain variations on the theme, it could possibly not be true.

Now we can use the techniques we have learned thus far. Separate variables, writing $u(x,t) = X(x)T(t)$. We get the equation

$$T'(t)X(x) - X''(x)T(t) = 0 \iff \frac{T'}{T} = \frac{X''}{X} = \lambda.$$

Since we have super nice BCs for X, we start with the X. We want to solve

$$X''(x) = \lambda X(x), \quad X(-\pi) = X(\pi) = 0.$$

First case: $\lambda = 0$. Then

$$X(x) = ax + b.$$

The BCs say

$$X(-\pi) = -a\pi + b = 0 \implies a\pi = b.$$

Next we need

$$X(\pi) = a\pi + b = 0 \implies b = -a\pi.$$

Combining these,

$$a\pi = -a\pi \implies a = 0 \implies b = 0.$$

So, no solution here because the zero solution doesn't count! Moving right along, let us try

$$\lambda > 0.$$

Then, our solution looks like real exponentials or equivalently sinh and cosh.

HINT: If you interval looks like $[0,l]$, it's probably easiest to work with sinh and cosh because $\sinh(0) = 0$ and $\cosh' = \sinh$. So this will often make things

simpler. On the other hand, if you have an interval like $[a, b]$ with a and be not zero, it may be easier to work with the exponentials. So, that's why I'm choosing to do that here. Hence
$$X(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$
The BCs require
$$X(-\pi) = ae^{-\sqrt{\lambda}\pi} + be^{\sqrt{\lambda}\pi} = 0.$$
Let's multiply by $e^{\sqrt{\lambda}\pi}$, to get
$$a + be^{2\sqrt{\lambda}\pi} = 0 \implies a = -be^{2\sqrt{\lambda}\pi}.$$
We check the other BCs
$$X(\pi) = ae^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0$$
substituting the value of a,
$$-be^{2\sqrt{\lambda}\pi}e^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0.$$
If $b = 0$ the whole solution is 0, so we assume this is not the case and divide by b. Multiplying by $e^{\sqrt{\lambda}\pi}$ we get
$$-e^{4\sqrt{\lambda}\pi} + 1 = 0 \iff e^{4\sqrt{\lambda}\pi} = 1 \iff 4\sqrt{\lambda}\pi = 0 \iff \lambda = 0,$$
which is a contradiction. So, no solutions lurking over here.

Thus, we consider $\lambda < 0$. Then our solution looks like
$$X(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x).$$
We need
$$X(-\pi) = a\cos(-\sqrt{|\lambda|}\pi) + b\sin(-\pi\sqrt{|\lambda|}) = 0 = a\cos(\sqrt{|\lambda|}\pi) - b\sin(\sqrt{|\lambda|}\pi),$$
where we use the evenness of cosine and oddness of sine. We also need
$$X(\pi) = a\cos(\sqrt{|\lambda|}\pi) + b\sin(\sqrt{|\lambda|}\pi) = 0.$$
Adding these equations we see that we need
$$a\cos(\sqrt{|\lambda|}\pi) = 0 \implies a = 0 \text{ or } \sqrt{|\lambda|} = \frac{(2k+1)}{2}, \quad k \in \mathbb{Z}.$$
Subtracting these equations we see that we need
$$b\cos(\sqrt{|\lambda|}\pi) = 0 \implies b = 0 \text{ or } \sqrt{|\lambda|} = \frac{2k}{2}, \quad k \in \mathbb{Z}.$$
I know it looks weird but I wrote it this way to make it looks similar to the one with the cosine. Now, the number $\sqrt{|\lambda|}$ can only have one value. It cannot be two different things at the same time. So, we have two types of solutions
$$X_n(x) = \begin{cases} \cos\left(\frac{nx}{2}\right) & n \text{ is odd} \\ \sin\left(\frac{nx}{2}\right) & n \text{ is even.} \end{cases}$$
Here we have
$$\sqrt{|\lambda_n|} = \frac{n}{2}, \quad \lambda_n = -\frac{n^2}{4}.$$
The partner functions,
$$T_n(t) = \alpha_n \cos(\sqrt{|\lambda_n|}x) + \beta_n \sin(\sqrt{|\lambda_n|}x).$$

We shall determine the coefficients using the IC. First, we write

$$u(x,t) = \sum_{n \geq 1} T_n(t) X_n(x).$$

Next, we use the easier of the two ICs, which is

$$u_t(x,0) = 0.$$

So, we also compute

$$u_t(x,t) = \sum_{n \geq 1} T_n'(t) X_n(x).$$

When we plug in 0, we need to have

$$u_t(x,0) = \sum_{n \geq 1} T_n'(0) X_n(x) = 0.$$

So, to get this, we need

$$T_n'(0) = 0 \forall n.$$

By definition of the T_n,

$$T_n'(0) = \beta_n \sqrt{|\lambda_n|}.$$

So, to make this zero, since $\sqrt{|\lambda_n|} \neq 0$, we need

$$\beta_n = 0 \forall n.$$

Hence, our solution looks like

$$u(x,t) = \sum_{n \geq 1} \alpha_n \cos(\sqrt{|\lambda_n|} t) X_n(x).$$

The other IC says

$$u(x,0) = -f(x) + v(x).$$

Since $\cos(0) = 1$, we see that we need

$$-f(x) + v(x) = \sum_{n \geq 1} \alpha_n X_n(x).$$

This means that we need

$$\alpha_n = \frac{\langle -f + v, X_n \rangle}{||X_n||^2} = \frac{\int_{-\pi}^{\pi} \left(-f(x) + v(x)\right) X_n(x) dx}{\int_{-\pi}^{\pi} |X_n(x)|^2 dx}.$$

It suffices to just leave α_n like this. As we observed before, our full solution is now

$$U(x,t) = u(x,t) + f(x) = -\frac{5x^2}{2} + \frac{5\pi^2}{2} + \sum_{n \geq 1} \alpha_n \cos(\sqrt{|\lambda_n|} t) X_n(x),$$

with X_n defined as above.

folland
3.1. **Exercises from [1] for the week.**

3.1.1. *To be demonstrated.*

(1) (4.2:5) Solve:
$$u_t = k u_{xx} + e^{-2t}\sin(x),$$
with
$$u(x,0) = u(0,t) = u(\pi,t) = 0.$$

(2) (EO 23) Determine the eigenvalues and eigenfunctions of the SLP:
$$f'' + \lambda f = 0, \quad 0 < x < a,$$
$$f(0) - f'(0) = 0, \quad f(a) + 2f'(a) = 0.$$

(3) (EO 24) Determine the eigenvalues and eigenfunctions of the SLP:
$$-e^{-4x}\frac{d}{dx}\left(e^{4x}\frac{du}{dx}\right) = \lambda u, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u'(1) = 0.$$

(4) (EO 1) A function is 2 periodic with $f(x) = (x+1)^2$ for $|x| < 1$. Expand $f(x)$ in a Fourier series. Search for a 2 periodic solution to the equation
$$2y'' - y' - y = f(x).$$

(5) (4.2.6) Solve:
$$u_t = k u_{xx} + Re^{-ct}, \quad R, c > 0,$$
$$u(x,0) = 0 = u(0,t) = u(l,t).$$
Physically this is heat flow in a rod which has a chemical reaction in it such that the reaction produced inside the rod dies out over time.

(6) (4.3.5) Find the general solution of
$$u_{tt} = c^2 u_{xx} - a^2 u,$$
$$u(0,t) = u(l,t) = 0,$$
with arbitrary initial conditions. Physically, this is a model for a string vibrating in an elastic medium where the term $-a^2 u$ represents the force of reaction of the medium on the string.

3.1.2. *To solve oneself.*

(1) (EO 25) Solve the problem:
$$u_{xx} + u_{yy} = y, \quad 0 < x < 2, \quad 0 < y < 1$$
$$u(x,0) = 0, \quad u(x,1) = 0$$
$$u(0,y) = y - y^3, \quad u(2,y) = 0.$$

(2) (EO 27) Solve the problem
$$u_{xx} + u_{yy} + 20u = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(0,y) = u(1,y) = 0$$
$$u(x,0) = 0, \quad u(x,1) = x^2 - x.$$

(3) (4.4:1) Solve the equation
$$u_{xx} + u_{yy} = 0$$
inside the square $0 \le x, y \le l$, subject to the boundary conditions:
$$u(x,0) = u(0,y) = u(l,y) = 0, \quad u(x,l) = x(l-x).$$

(4) (EO 3) Expand the function $\cos(x)$ in a sine series on the interval $(0, \pi/2)$. Use the result to compute

$$\sum_{n\geq 1} \frac{n^2}{(4n^2-1)^2}.$$

(5) (4.2.2) Solve:

$$u_t = ku_{xx}, \quad u(x,0) = f(x),$$
$$u(0,t) = C \neq 0, \quad u_x(l,t) = 0.$$

(6) (4.3.1) Show that the function

$$b_n(t) := \frac{1}{n\pi c}\int_0^t \sin\frac{n\pi c(t-s)}{l}\beta_n(s)ds$$

solves the differential equation:

$$b_n''(t) + \frac{n^2\pi^2c^2}{l^2}b_n(t) = \beta_n(t),$$

as well as the initial conditions $b_n(0) = b_n'(0) = 0$.

(7) (4.4.7) Solve the Dirichlet problem:

$$u_{xx} + u_{yy} = 0 \text{ in } S = \{(r,\theta) : 0 < r_0 \leq r \leq 1, \quad 0 \leq \theta \leq \beta\},$$
$$u(r_0,\theta) = u(1,\theta) = 0, \quad u(r,0) = g(r), \quad u(r,\beta) = h(r).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.12

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Inhomogeneous or non-self adjoint boundary conditions

We wish to solve the homogeneous wave equation inside a rectangle:

$$\Box u = 0 \text{ inside a rectangle, } u(x,y,0) = f(x,y), \quad u_t(x,y,0) = 0,$$

$$u(x,y,t) = g(x,y) \text{ for } (x,y) \text{ on the boundary of the rectangle.}$$

We name this problem $\heartsuit$. Here we have an inhomogeneous boundary condition. So, to solve the problem, we break it into two smaller problems which we tackle *one at a time:* divide and conquer.

> **Idea:** deal with time independent boundary conditions by finding a steady state solution.

So, we begin by looking for

$$\Phi(x,y)$$

to satisfy

$$\Box \Phi = 0 \text{ inside the rectangle,}$$

$$\Phi = g \text{ on the boundary of the rectangle.}$$

Since the physical problem doesn't care where in space the rectangle is sitting, let us put it so that its vertices are at $(0,0)$, $(0,B)$, $(A,0)$, (A,B). Let us call this problem $\heartsuit\heartsuit$.

Once we have found Φ, we will look for a solution w to solve

$$\Box w = 0 \text{ inside the rectangle,}$$

$$w(x,y,t) = 0 \text{ on the boundary of the rectangle,}$$

$$w(x,y,0) = f(x,y) - \Phi(x,y), \quad w_t(x,y,0) = 0.$$

Then, our solution to $\heartsuit$ will be

$$u(x,y,t) = w(x,y,t) + \Phi(x,y).$$

So, we look for Φ to solve $\heartsuit\heartsuit$.

> **Idea:** deal with each inhomogeneous boundary component one at a time.

It is the same principle: divide and conquer. So, first, let us make nice zero boundary conditions on the sides, and just deal with the complicated boundary conditions on the top and bottom. Therefore we look for a function $\phi(x,y)$ which satisfies

$$\Box\phi = 0,$$
$$\phi(0,y) = \phi(A,y) = 0,$$
$$\phi(x,0) = g(x,0), \quad \phi(x,B) = g(x,B).$$

Idea: since the PDE is homogeneous and half of the BCs are good and homogeneous, use separation of variables.

We therefore write the PDE:

$$-X''Y - Y''X = 0 \implies -\frac{Y''}{Y} = \frac{X''}{X} = \lambda.$$

The BCs for X are $X(0) = X(A) = 0$. We have solved this problem. The solutions are, up to constant factors

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right), \quad \lambda_n = -\frac{n^2\pi^2}{A^2}.$$

The equation for the partner function is then:

$$-\frac{Y_n''}{Y_n} = \lambda_n \implies Y_n'' = \frac{n^2\pi^2}{A^2}Y_n.$$

A basis of solutions is given by real exponentials, or equivalently hyperbolic sines and cosines. Since our region contains 0, we have been given a hint that using the hyperbolic sines and cosines may be more simple. So, we follow that hint, with

$$Y_n(y) = a_n \cosh\left(\frac{n\pi y}{A}\right) + b_n \sinh\left(\frac{n\pi y}{A}\right).$$

Next we use superposition to create a super solution, which is legit because the PDE is homogeneous:

$$\phi(x,y) = \sum_{n\geq 1} X_n(x)Y_n(y).$$

To obtain the boundary conditions, we need

$$\phi(x,0) = g(x,0) = \sum_{n\geq 1} a_n X_n(x).$$

Hence, the coefficients

$$a_n = \frac{\langle g(x,0), X_n\rangle}{||X_n||^2} = \frac{\int_0^A g(x,0)\overline{X_n(x)}dx}{\int_0^A |X_n(x)|^2 dx}.$$

For the other BC, we need

$$\phi(x,B) = g(x,B) = \sum_{n\geq 1} X_n(x)\left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right)\right).$$

Therefore we need

$$\left(a_n \cosh\left(\frac{n\pi B}{A}\right) + b_n \sinh\left(\frac{n\pi B}{A}\right)\right) = \frac{\langle g(x,B), X_n\rangle}{||X_n||^2}$$
$$= \frac{\int_0^A g(x,B)X_n(x)dx}{\int_0^A |X_n(x)|^2 dx}.$$

Solving for b_n we get

$$b_n = \frac{1}{\sinh(\frac{n\pi B}{A})}\left(\frac{\langle g(x,B), X_n\rangle}{||X_n||^2} - a_n \cosh\left(\frac{n\pi B}{A}\right)\right).$$

Next, we proceed similarly by searching for a function to fix up the BCs on the left and the right. Having dealt with the inhomogeneous BCs at the top and bottom, we set the BC there equal to zero. In that way, when we sum, we shall not mess up the function ϕ. So, we look for a solution to:

$$\Box\psi(x,y) = 0, \quad \psi(x,0) = \psi(x,B) = 0, \quad \psi(0,y) = g(0,y), \quad \psi(A,y) = g(A,y).$$

By symmetry, the solution will be given by

$$\sum_{n\geq 1} \widetilde{X_n}(y)\widetilde{Y_n(x)},$$

with

$$\widetilde{X_n(y)} = \sin\left(\frac{n\pi y}{B}\right),$$

and

$$\widetilde{Y_n(x)} = \widetilde{a_n}\cosh\left(\frac{n\pi x}{B}\right) + \widetilde{b_n}\sinh\left(\frac{n\pi x}{B}\right).$$

The coefficients come from the boundary conditions:

$$\widetilde{a_n} = \frac{\langle g(0,y), \widetilde{X_n}\rangle}{||\widetilde{X_n}||^2} = \frac{\int_0^B g(0,y)\widetilde{X_n(y)}dy}{\int_0^B |X_n(y)|^2 dy}.$$

The other one

$$\widetilde{b_n} = \frac{1}{\sinh\left(\frac{n\pi A}{B}\right)}\left(\frac{\langle g(A,y), \widetilde{X_n}\rangle}{||\widetilde{X_n}||^2} - \widetilde{a_n}\cosh\left(\frac{n\pi A}{B}\right)\right).$$

So, we have found

$$\psi(x,y) = \sum_{n\geq 1}\widetilde{X_n(y)}\widetilde{Y_n(x)}.$$

The full solution to this part of the problem is

$$\Phi(x,y) = \phi(x,y) + \psi(x,y).$$

Exercise 1. *Verify that this function satisfies both the PDE $\Box\Phi = 0$ as well as all of the boundary conditions.*

To complete the problem, we have only to solve the homogeneous wave equation with the lovely Dirichlet boundary condition and the initial condition with Φ subtracted. So, we are solving:

$$\Box u = 0, \quad u_t(x,y,0) = 0, \quad u(x,y,0) = f(x,y) - \Phi(x,y), \quad u = 0 \text{ on the boundary.}$$

> **Idea:** since we have homogeneous PDE and BC, use separation of variables and superposition.

We use separation of variables for t, x, and y. Write

$$u = TXY.$$

The PDE is

$$T''XY - X''TY - Y''TX = 0 \iff \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda.$$

Since we have nice homogeneous (Dirichlet) boundary conditions, we begin with the functions that depend on the position in the rectangle, that is X and Y.

Their equation is:

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda \implies \frac{X''}{X} = \lambda - \frac{Y''}{Y}.$$

OBS! The left and right sides depend on *different independent variables.* Hence, by the same reasoning that gave us λ, we get that

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu.$$

Let us solve for X first.[1] So, we are looking to solve:

$$X'' = \mu X, \quad X(0) = X(A) = 0.$$

We have solved this before. The solutions are up to constant factors:

$$X_n(x) = \sin\left(\frac{n\pi x}{A}\right) \quad \mu_n = -\frac{n^2\pi^2}{A^2}.$$

This gives the equation for Y,

$$\frac{Y''}{Y} = \lambda - \mu_n, \quad Y(0) = Y(B) = 0.$$

Let us briefly call

$$\nu = \lambda - \mu_n.$$

Then, this is just the same equation but with different names for things:

$$Y'' = \nu Y, \quad Y(0) = Y(B) = 0.$$

Up to constant factors, the solutions are

$$Y_m(y) = \sin\left(\frac{m\pi y}{B}\right) \quad \nu_m = -\frac{m^2\pi^2}{B^2}.$$

Since

$$\nu_m = \lambda - \mu_n \implies \lambda = \lambda_{n,m} = \nu_m + \mu_n = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}.$$

Recalling the equation for the partner function, T, we have

$$T_{n,m}(t) = a_{n,m}\cos(\sqrt{|\lambda_{n,m}|}t) + b_{n,m}\sin(\sqrt{|\lambda_{n,m}|}t).$$

Hence we write

$$u(x,y,t) = \sum_{n,m\geq 1} T_{n,m}(t)X_n(x)Y_m(y).$$

The initial condition

$$u_t(x,y,0) = 0 \implies b_{n,m} = 0 \forall n, m.$$

The other condition is that

$$u(x,y,0) = f(x,y) - \Phi(x,y) = \sum_{n,m\geq 1} a_{n,m}X_n(x)Y_m(y).$$

Hence we require

$$a_{n,m} = \frac{\langle f - \Phi, X_nY_m\rangle}{||X_nY_m||^2} = \frac{\int_{[0,A]\times[0,B]}(f(x,y) - \Phi(x,y))X_n(x)Y_m(y)dxdy}{\int_{[0,A]\times[0,B]}|X_n(x)Y_m(y)|^2dxdy}.$$

[1] In this case, we could solve for either X or Y first, it actually does not matter which you choose.

The full solution is then

$$u(x, y, t) - \Phi(x, y).$$

Remark 1. *The eigenvalues of the two-dimensional SLP we solved above,*

$$\lambda_{n,m} = -\frac{m^2\pi^2}{B^2} - \frac{n^2\pi^2}{A^2}$$

are interesting to compare to the analogous one-dimensional case. In the analogous one dimension case, where we have

$$\mu_n = -\frac{n^2\pi^2}{A^2},$$

you can see that these are all square integer multiples of

$$\mu_1 = -\frac{\pi^2}{A^2}.$$

This is the mathematical reason that vibrating strings sound lovely. On the other hand, as long as the rectangle is not a square, that is $A \neq B$, it is no longer true that the $\lambda_{n,m}$ are all multiples of

$$\lambda_{1,1} = -\frac{\pi^2}{B^2} - \frac{\pi^2}{A^2}.$$

For this reason, vibrating rectangles can sound rather awful. You can listen to something along these lines (okay it's for tori not rectangles, but mathematically basically the same) here: `http://www.toroidalsnark.net/som.html`. Further exploration of the mathematics of music could make for an interesting bachelor's or master's thesis....

2. Heat equation example on an interval with an inhomogeneous boundary condition

We wish to solve the problem:

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x, 0) &= v(x), \\ u_x(4, t) &= 0, \\ u(0, t) &= 20. \end{aligned}$$

Let us call this problem ♡. The boundary conditions are *not zero.* This will mean that the associated SLP does *not* have self-adjoint BCs, which is a big problem. We can use a similar "steady state" trick to deal with this. If the BC $u(0, t) = 20$ were instead $u(0, t) = 0$, then the BCs would be self adjoint BCs. So we want to make it so. Since the PDE is homogeneous, the

> **Idea:** Deal with non-self adjoint BCs which are independent of time by finding a steady state solution.

We want a function $f(x)$ which satisfies the equation

$$-f''(x) = 0,$$

and which gives us the bad BC

$$f(0) = 20.$$

We have a nice homogeneous BC on the other side, so we don't want to mess that up, so we want

$$f'(4) = 0.$$

Then, the function

$$f(x) = ax + b.$$

We use the BCs to compute

$$f(0) = 20 \implies b = 20.$$

$$f'(4) = 0 \implies a = 0.$$

Similar to before, if we add it to the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

it's going to screw up the IC. So, instead we look for the solution of

$$u_t - u_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x) - f(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

This is now a PDE we know how to solve, so we call this problem ♡♡. We use SV to write $u = XT$ (just a means to an end).[2] Next, we get the equation

$$\frac{T'}{T} = \frac{X''}{X} = \lambda.$$

We solve the SLP

$$X'' = \lambda X, \quad X(0) = 0 = X'(4).$$

The reason we know this is an SLP satisfying the hypotheses of the theorem is because we verify that the BC is self-adjoint.

Exercise 2. *Verify that the only solutions for the cases $\lambda \geq 0$ are solutions which are identically zero.*

We only get $\lambda < 0$. Then, the solution is of the form

$$a_n \cos(\sqrt{|\lambda_n|}x) + b_n \sin(\sqrt{|\lambda_n|}x).$$

The BC at 0 tells us that

$$a_n = 0.$$

The BC at 4 tells us that

$$\cos(\sqrt{|\lambda_n|}4) = 0 \implies \sqrt{|\lambda_n|}4 = \frac{2n+1}{2}\pi \implies \sqrt{|\lambda_n|} = \frac{2n+1}{8}\pi.$$

[2] La fin justifie les moyens by M.C. Solaar is recommended listening.

We then also get

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{64}.$$

We shall deal with the coefficients at the very end. So, we set

$$X_n(x) = \sin(\sqrt{|\lambda_n|}x).$$

The partner function

$$\frac{T_n'}{T_n} = \lambda_n \implies T_n(t) = \alpha_n e^{\lambda_n t} = \alpha_n e^{-(2n+1)^2\pi^2 t/64}.$$

We put it all together writing

$$u(x,t) = \sum_{n\geq 1} T_n(t)X_n(x).$$

To make the IC, we need

$$u(x,0) = \sum_{n\geq 1} T_n(0)X_n(x) = v(x) - f(x).$$

Since

$$T_n(0) = \alpha_n,$$

we need

$$\sum_{n\geq 1} \alpha_n X_n(x) = v(x) - f(x).$$

So we want the coefficients to be the Fourier coefficients of $v - f$, thus

$$\alpha_n = \frac{\langle v-f, X_n\rangle}{||X_n||^2} = \frac{\int_0^4 (v(x)-f(x))\overline{X_n(x)}dx}{\int_0^4 |X_n(x)|^2 dx}.$$

Our full solution is

$$U(x,t) = u(x,t) + f(x) = 20 + \sum_{n\geq 1} T_n(t)X_n(x).$$

2.1. **Exercises to solve oneself: hints.**

(1) (EO 25) Solve the problem:

$$u_{xx} + u_{yy} = y, \quad 0 < x < 2, \quad 0 < y < 1$$
$$u(x,0) = 0, \quad u(x,1) = 0$$
$$u(0,y) = y - y^3, \quad u(2,y) = 0.$$

Hint: divide and conquer. First, find a function which is independent of x to solve the inhomogeneous PDE. That is you want $f(y)$ to solve:

$$f''(y) = y, \quad f(0) = 0 = f(1).$$

Next solve the problem

$$v_{xx} + v_{yy} = 0,$$
$$v(x,0) = 0 = v(x,1),$$
$$v(0,y) = y - y^3 - f(y), \quad v(2,y) = -f(y).$$

Show that the solution to the original problem is given by

$$u(x,y) = v(x,y) + f(y).$$

(2) (EO 27) Solve the problem

$$u_{xx} + u_{yy} + 20u = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(0, y) = u(1, y) = 0$$
$$u(x, 0) = 0, \quad u(x, 1) = x^2 - x.$$

Hint: divide and conquer. The PDE is homogeneous. So write $u = XY$. Plug it into the PDE. Solve for X function first because it has nice Dirichlet boundary conditions. Then solve for the partner Y function. Use the condition $u(x, 1) = x^2 - x$ to determine the unknown coefficients.

(3) (4.4:1) Solve the equation

$$u_{xx} + u_{yy} = 0$$

inside the square $0 \le x, y \le l$, subject to the boundary conditions:

$$u(x, 0) = u(0, y) = u(l, y) = 0, \quad u(x, l) = x(l - x).$$

Hint: follow the same procedure as the preceding exercise.

(4) (EO 3) Expand the function $\cos(x)$ in a sine series on the interval $(0, \pi/2)$. Use the result to compute

$$\sum_{n \ge 1} \frac{n^2}{(4n^2 - 1)^2}.$$

Hint: the coefficients in a sine series will be given by

$$\beta_n = \frac{4}{\pi} \int_0^{\pi/2} \cos(x) \sin(2nx) dx.$$

One way to make this integral easier is to expand stuff into complex exponentials from which you can obtain that

$$\cos(ax) \sin(bx) = \frac{1}{2} \left(\sin((a + b)x) - \sin((a - b)x) \right).$$

To compute the big sum at the end, use Parseval's equation.

(5) (4.2.2) Solve:

$$u_t = k u_{xx}, \quad u(x, 0) = f(x),$$
$$u(0, t) = C \neq 0, \quad u_x(l, t) = 0.$$

Hint: First deal with that icky inhomogeneous boundary condition $C \neq 0$ by finding a steady state solution as in lecture. This is $\phi(x)$ which has $\phi''(x) = 0$, $\phi(0) = C$, $\phi'(l) = 0$. Then, look for a solution to solve

$$u_t = k u_{xx}, \quad u(x, 0) = f(x) - \phi(x),$$
$$u(0, t) = 0, \quad u_x(l, t) = 0.$$

For this problem you can now use separation of variables, SLP theory, Hilbert space theory, and finally compute your coefficients using the initial data. It's all coming together!

(6) (4.3.1) Show that the function

$$b_n(t) := \frac{1}{n\pi c} \int_0^t \sin \frac{n\pi c(t - s)}{l} \beta_n(s) ds$$

solves the differential equation:

$$b_n''(t) + \frac{n^2 \pi^2 c^2}{l^2} b_n(t) = \beta_n(t),$$

as well as the initial conditions $b_n(0) = b_n'(0) = 0$. Hint: Use the fundamental theorem of calculus to compute $b_n'(t)$. Check the initial conditions by just substituting 0 for t. Next, compute the derivative of b_n' to get b_n'' and check the equation.

(7) (4.4.7) Solve the Dirichlet problem:

$$u_{xx} + u_{yy} = 0 \text{ in } S = \{(r, \theta) : 0 < r_0 \leq r \leq 1, \quad 0 \leq \theta \leq \beta\},$$
$$u(r_0, \theta) = u(1, \theta) = 0, \quad u(r, 0) = g(r), \quad u(r, \beta) = h(r).$$

Hint: Turn the equation into polar coordinates. An annulus looks like a rectangle in polar coordinates. Next separate variables and write $u = R(r)\Theta(\theta)$. Solve for the R function first because it has the beautiful boundary conditions. This is going to become an Euler equation like we solved for in lecture. So, check your lecture notes to see how we did that (Day 9). Once you have your R functions, they will be like $R_n(r)$, with corresponding λ_n, use this to find the partner Θ_n functions. Finally use g and h to determine the unknown coefficients. This part is a bit like finding the coefficients when solving the Dirichlet problem in a rectangle.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.14

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Inhomogeneous PDE with time dependent inhomogeneity

Solve:

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= v(x), \\ u_x(4,t) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

Non! Sacre bleu! Tabernac![1] This is an inhomogeneous PDE *and the inhomogeneity (tx) depends on time!* A steady-state solution cannot save us. What do we do?

> **Idea:** Use a Fourier Series with non-constant coefficients to deal with time-dependent inhomogeneity.

There's a lovely way to deal with this type of inhomogeneity. We first solve the homogeneous problem.

Exercise 1. *Use separation of variables to solve the homogeneous problem:*

$$w_t - w_{xx} = 0, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} w(x,0) &= v(x) \\ w_x(4,t) &= 0 \\ w(0,t) &= 0. \end{aligned}$$

Having done this, we obtain

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{64}, \quad X_n(x) = \sin(\sqrt{|\lambda_n|}x).$$

$$T_n(t) = \alpha_n e^{\lambda_n t}.$$

[1] This is how they curse in French Canada.

$$\alpha_n = \frac{\langle v, X_n \rangle}{||X_n||^2} = \frac{\int_0^4 v(x)\overline{X_n(x)}dx}{\int_0^4 |X_n(x)|^2 dx},$$

and

$$w(x,t) = \sum_{n \geq 1} T_n(t) X_n(x).$$

Now, we look for a solution to this problem:

$$\phi_t - \phi_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$\phi(x,0) = 0,$$
$$\phi_x(4,t) = 0,$$
$$\phi(0,t) = 0.$$

Idea: look for a solution of the form

$$\sum_{n \geq 1} c_n(t) X_n(x).$$

So, we keep our X_n from the homogeneous problem, and we look for different $c_n(t)$ which will now be *functions of* t. We want the function to satisfy

$$u_t - u_{xx} = tx,$$

so we put the series in the left side into this PDE:

$$\sum_{n \geq 1} c_n'(t) X_n(x) - c_n(t) X_n''(x) = tx.$$

We use the fact the $X_n'' = \lambda_n X_n$, so we want to solve

$$\sum_{n \geq 1} X_n(x) \left(c_n'(t) - c_n(t)\lambda_n \right) = tx.$$

Here is where we do something clever:

Idea: write out tx as a Fourier series in terms of X_n.

The t just goes along for the ride, and

$$tx = t \sum_{n \geq 1} a_n X_n(x),$$

where

$$a_n = \frac{\langle x, X_n \rangle}{||X_n||^2} = \frac{\int_0^4 x X_n(x) dx}{\int_0^4 |X_n|^2 dx}.$$

As usual, we do not need to compute these integrals.

So, we want:

$$\sum_{n \geq 1} X_n(x) \left(c_n'(t) - c_n(t)\lambda_n \right) = tx = \sum_{n \geq 1} t X_n(x) a_n.$$

We equate the coefficients of X_n:

$$(c_n'(t) - \lambda_n c_n(t)) = t a_n.$$

This is an ODE for $c_n(t)$. We also want the IC, $c_n(0) = 0$. The solution to the homogeneous ODE,

$$f' - \lambda_n f = 0 \implies f(t) = e^{\lambda_n t} \text{ times some constant factor.}$$

A particular solution to the inhomogeneous ODE is a linear function of the form:

$$A_n t + B_n \implies A_n - \lambda_n(A_n t + B_n) = a_n t \implies A_n = \frac{-a_n}{\lambda_n}, \quad B_n = \frac{A_n}{\lambda_n} = -\frac{a_n}{\lambda_n^2}.$$

So general solutions are of the form:

$$c_n(t) = C_n e^{\lambda_n t} - \frac{a_n}{\lambda_n} t - \frac{a_n}{\lambda_n^2}, \quad \text{for some constant } C_n.$$

To obtain the initial condition that $c_n'(0) = 0$, we see that we need

$$C_n = \frac{a_n}{\lambda_n^2}.$$

Thus, we have found

$$c_n(t) = \frac{a_n}{\lambda_n^2} e^{\lambda_n t} - \frac{a_n}{\lambda_n} t - \frac{a_n}{\lambda_n^2}.$$

Therefore the solution we seek is

$$u(x,t) = \sum_{n \geq 1} c_n(t) X_n(x),$$

and the full solution to the original problem is

$$U(x,t) = w(x,t) + u(x,t).$$

2. Solving problems where the space variable is in an unbounded region

We will now develop a set of techniques which can be used for solving partial differential equations when the space variable is in an unbounded region. It is straightforward to generalize the definitions of $\mathcal{L}^1$ and $\mathcal{L}^2$ to the real line.

Definition 1 (The real one)**.** *The set*

$$\mathcal{L}^1(\mathbb{R}) = \text{ the set of equivalence classes, of functions which satisfy:}$$

$$f \text{ is measurable, and } \int_{\mathbb{R}} |f(x)| dx < \infty.$$

The function g belongs to the same equivalence class as f if $g = f$ almost everywhere on $\mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 2 (The real one)**.** *The set*

$$\mathcal{L}^2(\mathbb{R}) = \text{ the set of equivalence classes of functions which satisfy:}$$

$$f \text{ is measurable, and } \int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

The function g belongs to the same equivalence class as f if $g = f$ almost everywhere on $\mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 3 (The workable definition of $\mathcal{L}^1(\mathbb{R})$)**.** *It will suffice for the purposes of this humble course to treat $\mathcal{L}^1(\mathbb{R})$ as the set of functions on $\mathbb{R}$ which satisfy*

$$\int_{\mathbb{R}} |f(x)| dx < \infty.$$

The $\mathcal{L}^1(\mathbb{R})$ norm is then defined to be

$$||f||_{\mathcal{L}^1} = \int_{\mathbb{R}} |f(x)| dx.$$

The set of such functions, denoted by $\mathcal{L}^1(\mathbb{R})$, is a complete normed vector space but not *a Hilbert space. A complete normed vector space is also known as a Banach space.*

Definition 4 (The workable definition of $\mathcal{L}^2(\mathbb{R})$)**.** *It will suffice for the purposes of this humble course to treat $\mathcal{L}^2(\mathbb{R})$ as the set of functions on $\mathbb{R}$ which satisfy*

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

This set of functions, denoted by $\mathcal{L}^2(\mathbb{R})$, is a Hilbert space with the scalar product:

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} d\mu.$$

Hence, by definition, the norm on $\mathcal{L}^2(\mathbb{R})$ is

$$||f||_{\mathcal{L}^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 dx}.$$

A lot of things which are true for $\mathcal{L}^2$ on a finite interval are no longer true on $\mathcal{L}^2(\mathbb{R})$. For example, the functions

$$e^{inx}, \sin(x), \cos(x)$$

are all neither in $\mathcal{L}^1(\mathbb{R})$ nor in $\mathcal{L}^2(\mathbb{R})$. Furthermore, there is no relationship between $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$. There are functions which are in $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sqrt{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

is in $\mathcal{L}^1(\mathbb{R})$ but it is *not* in $\mathcal{L}^2(\mathbb{R})$.

Exercise 2. *Verify that this function is in $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$. Compute its $\mathcal{L}^1(\mathbb{R})$ norm.*

On the other hand, the function

$$f(x) = \begin{cases} 0 & x \leq 1 \\ \frac{1}{x} & x > 1 \end{cases}$$

is in $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$.

Exercise 3. *Verify that this function is in $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$. Compute its $\mathcal{L}^2(\mathbb{R})$ norm.*

The function

$$e^{-|x|}$$

is in both $\mathcal{L}^1(\mathbb{R})$ and in $\mathcal{L}^2(\mathbb{R})$.

Exercise 4. *Verify that this function is in both $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$. Compute its $\mathcal{L}^1$ and $\mathcal{L}^2$ norms. Come up with your own examples of functions which are*

(1) *In $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$.*
(2) *In $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$.*
(3) *In both $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$.*

So, all we can say is that

$$\mathcal{L}^1(\mathbb{R}) \not\subset \mathcal{L}^2(\mathbb{R}), \quad \mathcal{L}^2(\mathbb{R}) \not\subset \mathcal{L}^1(\mathbb{R}), \quad \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R}) \neq \emptyset.$$

So, we're in a whole new territory here. To begin we shall define the convolution. This will be super important for solving the heat equation on the real line.

Definition 5. *The convolution of f and g is a function $f * g : \mathbb{R} \to \mathbb{C}$ defined by*

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy,$$

whenever the integral on the right exist.

Proposition 6. *Assume that f and g are both in $\mathcal{L}^2(\mathbb{R})$. Then*

(1) $|f * g(x)| \leq ||f|| ||g||$ *for all* $x \in \mathbb{R}$
(2) $f * (ag + bh) = af * g + bf * h$ *for all* $a, b \in \mathbb{C}$
(3) $f * g = g * f$
(4) $f * (g * h) = (f * g) * h$

Proof: This is useful to do because it helps to familiarize oneself with the convolution. We first estimate

$$|f * g(x)| = \left| \int_{\mathbb{R}} f(x-y)g(y)dy \right| \leq \int_{\mathbb{R}} |f(x-y)||g(y)|dy.$$

The point $x \in \mathbb{R}$ is fixed and arbitrary, so we define a function

$$\phi(y) = f(x-y).$$

Then

$$|f * g(x)| \leq \int_{\mathbb{R}} |\phi(y)||g(y)|dy \leq ||\phi|| ||g||.$$

We compute

$$||\phi||^2 = \int_{\mathbb{R}} |f(x-y)|^2 dy = -\int_{\infty}^{-\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(t)|^2 dt = ||f||^2.$$

Above, we used the substitution $t = x - y$ so $dt = -dy$, and the integral got reversed. The $-$ goes away when we re-reverse the integral. So, in the end we see that

$$|f * g(x)| \leq ||f|| ||g||$$

as desired. The second property follows simply by the linearity of the integral itself. For the third property, we will use substitution again:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

We want to get $g(x-z)$ so we define

$$y = x - z \implies x - y = z, \quad dz = -dy.$$

Hence,

$$f * g(x) = -\int_{\infty}^{-\infty} f(z)g(x-z)dz = \int_{-\infty}^{\infty} g(x-z)f(z)dz = g * f(x).$$

We do something rather similar in the fourth property:

$$f * (g * h)(x) = \int_{\mathbb{R}} f(x-y) \int_{\mathbb{R}} g(y-z)h(z)dzdy.$$

For the other term we have

$$(f*g)*h(x) = \int_{\mathbb{R}} (f*g)(x-y)h(y)dy = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y-z)g(z)h(y)dzdy.$$

So, we define

$$t = y - z \implies x - y = x - t - z, \quad dt = dy.$$

Then

$$f*(g*h)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t-z)g(t)h(z)dzdt.$$

Finally, we call $z = y$ and $t = z$ (sorry if this gives you a headache!) because they are just names, and then we get

$$f*(g*h)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y-z)g(y)h(z)dzdy.$$

If you're worried about the order of integration, don't be. Since everything is in $\mathcal{L}^2$, these integrals converge absolutely, so those Italian magicians, Fubini & Tonelli allow us to do the switch-a-roo with the integrals as much as we like.

One of the useful features of convolution is that we can use it to smooth out non-smooth functions. This is known as *mollification,* which comes from the verb, to mollify, which means to make smooth.[2]

Proposition 7 (Mollification)**.** *If $f \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$, $f' \in \mathcal{L}^2(\mathbb{R})$, and $g \in \mathcal{L}^2(\mathbb{R})$, then $f*g \in \mathcal{C}^1(\mathbb{R})$. Moreover $(f*g)' = f'*g$.*

Proof: Everything converges beautifully so just stick that differentiation right under the integral defining

$$f*g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Hence

$$(f*g)'(x) = \int_{\mathbb{R}} f'(x-y)g(y)dy = f'*g(x).$$

If you are not satisfied with this explanation, a rigorous proof can be obtained using the Dominated Convergence Theorem, but that is a theorem which we cannot prove in the context of this humble course.

2.0.1. *An example.* Let's compute a convolution. Let $f(x) = \frac{1}{1+x^2}$ and

$$g(x) = \begin{cases} 1 & |x| < 3 \\ 0 & |x| > 3 \end{cases}.$$

The function g is not differentiable at the points ± 3. The function f is perfectly smooth on $\mathbb{R}$. Let's convolve them!

$$f*g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} \frac{1}{1+(x-y)^2} g(y)dy = \int_{-3}^{3} \frac{1}{1+(x-y)^2}dy.$$

[2]One can mollify garlic, tahini, chickpeas, soy sauce, olive oil, oregano, black pepper, lemon juice, in suitable proportions, together with a bit of hot sauce like Cholula, Tabasco, or Sriracha, to make hummus.

If we dig deep into our calculus memory, we vaguely recall that

$$(\arctan(t))' = \frac{1}{1+t^2}.$$

So, this integral becomes:

$$-\arctan(x-y)\big|_{-3}^{3} = -\arctan(x-3) + \arctan(x+3).$$

This is indeed a smooth function of x.

3. The Fourier transform

One of the reasons that the convolution is so nice is because it plays well with the Fourier transform. So let us define this Fourier transform.

Proposition 8. *Assume that $f \in \mathcal{L}^1(\mathbb{R})$. Then*

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

is a well-defined complex number for any $\xi \in \mathbb{R}$.

Proof: Simply estimate

$$\left| \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \right| \le \int_{\mathbb{R}} |f(x)| dx < \infty.$$

3.1. Example of computing a Fourier transform.

Example of computing a Fourier transform. Let us get a feel for this by computing a Fourier transform. Consider the function $f(x) = e^{-a|x|}$ where $a > 0$. Then it is certainly in $\mathcal{L}^1(\mathbb{R})$ so we ought to be able to compute its Fourier transform. This is by definition

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} e^{-a|x|} dx = \int_{-\infty}^{0} e^{-ix\xi} e^{ax} dx + \int_{0}^{\infty} e^{-ix\xi} e^{-ax} dx.$$

We compute these integrals by finding a primitive for the integrand:

$$\hat{f}(\xi) = \left. \frac{e^{x(a-i\xi)}}{a-i\xi} \right|_{-\infty}^{0} + \left. \frac{e^{x(-a-i\xi)}}{-a-i\xi} \right|_{0}^{\infty}$$

$$= \frac{1}{a-i\xi} + \frac{1}{a+i\xi} = \frac{a+i\xi+a-i\xi}{a^2+\xi^2} = \frac{2a}{a^2+\xi^2}.$$

3.2. Answers for this week's exercises to be done oneself.

(1) (Eo 25, 27, 3) Please see the end of the Eö document! It has answers!

(2) (4.4:1)

$$u(x,y) = \frac{8l^2}{\pi^3} \sum_{n \ge 1} \frac{1}{(2n-1)^3 \sinh((2n-1)\pi)} \sin\left(\frac{(2n-1)\pi x}{l}\right) \sinh\left(\frac{(2n-1)\pi y}{l}\right).$$

(3) (4.2.2) Here we define first:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{(2n-1)\pi x}{2l}\right) dx.$$

Then the answer to this one is:

$$u(x,t) = C + \sum_{n\geq 1} \left(b_n - \frac{4C}{\pi(2n-1)} \right) \exp\left(-\frac{(n-1/2)^2\pi^2 kt}{l^2} \right) \sin\left(\left(n - \frac{1}{2} \right) \frac{\pi x}{l} \right).$$

(4) (4.3.1) And the answer is... Geez Folland where is the answer? Oh, right this one is to "verify" etc. Well, the way I find this easiest to do is to re-write using the angle addition formula for the sine:

$$b_n(t) = \frac{l}{n\pi c} \int_0^t \sin(n\pi ct/l)\cos(-n\pi cs/l)\beta_n(s)ds + \frac{l}{n\pi c}\int_0^t \cos(n\pi ct/l)\sin(-n\pi cs/l)\beta_n(s)ds.$$

Then we can take out the s-independent terms to the front of the integral, so that

$$b_n(t) = \sin(n\pi ct/l)\frac{l}{n\pi c}\int_0^t \cos(-n\pi cs/l)\beta_n(s)ds + \cos(n\pi ct/l)\frac{l}{n\pi c}\int_0^t \sin(-n\pi cs/l)\beta_n(s)ds.$$

Now we can compute the derivatives and verify the formulas using the product rule together with the fundamental theorem of calculus. Please just ask if you have questions about how this works. Also, if you solved in a different way but ended up correct, that's just peachy too!

(5) (4.4.7) Wow, this answer is long. Let

$$g(r) = \sum c_n \sin\left(\frac{n\pi \log r}{\log r_0} \right)$$

and

$$h(r) = \sum d_n \sin\left(\frac{n\pi \log r}{\log r_0} \right),$$

then

$$u(r,\theta) = \sum_{n\geq 1} (a_n e^{n\pi\theta/\log r_0} + b_n e^{-n\pi\theta/\log r_0}) \sin\left(\frac{n\pi \log r}{\log r_0} \right),$$

where

$$a_n + b_n = c_n, \quad a_n e^{n\pi\beta/\log r_0} + b_n e^{-n\pi\beta/\log r_0} = d_n.$$

Happy Weekend! ♡

FOURIER ANALYSIS & METHODS 2020.02.17

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The illustrious Fourier transform

The following is a useful and fundamental collection of facts about the Fourier transform. It may be useful to introduce the notations

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \hat{f}(\xi).$$

Sometimes we feel like a wide hat, sometimes a narrow hat, and sometimes we need that big $\mathcal{F}$. It is useful to be fluent with all three equivalent notations.

Theorem 1 (Properties of the Fourier transform)**.** *Assume that everything below is well defined. Then, the Fourier transform,*

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

satisfies

(1) $\mathcal{F}(f(x-a))(\xi) = e^{-ia}\hat{f}(\xi)$.
(2) $\mathcal{F}(f')(\xi) = i\xi\hat{f}(\xi)$
(3) $\mathcal{F}(xf(x))(\xi) = i\mathcal{F}(f)'(\xi)$
(4) $\mathcal{F}(f*g)(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

Proof: We just compute (we are being a bit naughty, not bothering with issues of convergence, but all such issues are indeed rigorously verifiable, so not to worry). First

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(x-a)e^{-ix\xi}dx.$$

Change variables. Let $t = x - a$, then $dt = dx$, and $x = t + a$ so

$$\mathcal{F}(f(x-a))(\xi) = \int_{\mathbb{R}} f(t)e^{-i(t+a)\xi}dt = e^{-ia\xi}\hat{f}(\xi).$$

The next one will come from integrating by parts:

$$\int_{\mathbb{R}} f'(x)e^{-ix\xi}dx = f(x)e^{-ix\xi}\big|_{-\infty}^{\infty} - \int_{\mathbb{R}} -i\xi f(x)e^{-ix\xi}dx = i\xi\hat{f}(\xi).$$

The boundary terms vanish because of reasons (again it is $\mathcal{L}^1$ and $\mathcal{L}^2$ theory stuff). Similarly we compute

$$\int_{\mathbb{R}} xf(x)e^{-ix\xi}dx = -\frac{1}{i}\int_{\mathbb{R}} f(x)\frac{d}{d\xi}e^{-ix\xi}dx = i\frac{d}{d\xi}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx = i\mathcal{F}(f)'(\xi).$$

Finally,

$$\mathcal{F}(f*g)(\xi) = \int_{\mathbb{R}} f*g(x)e^{-ix\xi}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-ix\xi}dydx.$$

We do a little sneaky trick

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)e^{-ix\xi}e^{-iy\xi}e^{iy\xi}dydx$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)e^{-i(x-y)\xi}g(y)e^{-iy\xi}dydx.$$

Let $z = x - y$. Then $dz = -dy$ so

$$= \int_{\mathbb{R}}\int_{\infty}^{-\infty} f(z)e^{-iz\xi}(-dz)g(y)e^{-iy\xi}dy = \int_{\mathbb{R}}\int_{\mathbb{R}} f(z)e^{-iz\xi}dzg(y)e^{-iy\xi}dy$$

$$= \hat{f}(\xi)\hat{g}(\xi).$$

It shall be quite useful to know how to "undo" the Fourier transform.

Theorem 2 (Extension of Fourier transform to $\mathcal{L}^2$)**.** *There is a well defined unique extension of the Fourier transform to $\mathcal{L}^2(\mathbb{R})$. The Fourier transform of an element of $\mathcal{L}^2(\mathbb{R})$ is again an element of $\mathcal{L}^2(\mathbb{R})$. Moreover, for any $f \in \mathcal{L}^2(\mathbb{R})$ we have the FIT (Fourier Inversion Theorem):*

eq:fit

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi. \tag{1.1}$$

The theory item FIT is a *Julklapp.* All you need to know is the equation (1.1). The next theorem is also a theory item, with a short proof. The key is to start on the right side and use the FIT.

Theorem 3 (Plancharel)**.** *For any $f \in \mathcal{L}^2(\mathbb{R})$, $\hat{f} \in \mathcal{L}^2(\mathbb{R})$. Moreover,*

$$\langle \hat{f}, \hat{g}\rangle = 2\pi\langle f, g\rangle,$$

and thus

$$||\hat{f}||^2_{\mathcal{L}^2} = 2\pi||f||^2,$$

for all f and g in $\mathcal{L}^2(\mathbb{R})$.

Proof: Start with the right side and use the FIT on f, to write

$$2\pi\langle f, g\rangle = 2\pi\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi}e^{ix\xi}\hat{f}(\xi)\overline{g(x)}d\xi dx = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi}\hat{f}(\xi)\overline{g(x)}d\xi dx.$$

Move the complex conjugate to engulf the $e^{ix\xi}$,

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\overline{g(x)e^{-ix\xi}}d\xi dx.$$

Swap the order of integration and integrate x first:

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\overline{g(x)e^{-ix\xi}}dxd\xi = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \langle \hat{f}, \hat{g}\rangle.$$

We may from time to time use the following cute fact as well.

Lemma 4 (Riemann & Lebesgue)**.** *Assume $f \in \mathcal{L}^1(\mathbb{R})$. Then,*

$$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0.$$

We shall indeed need to actually prove the next one, because it's going to be quite important for solving the heat equation on the real line.

1.1. **The big bad convolution approximation theorem.** This theory item is Theorem 7.3, regarding approximation of a function by convoluting it with a so-called "approximate identity." This theorem and its proof are both rather long. The proof relies very heavily on knowing the definition of limits and how to work with those definitions, so if you're not comfortable with ϵ and δ style arguments, it would be advisable to brush up on these.

Theorem 5. *Let $g \in L^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} g(x)dx = 1.$$

Define

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\varepsilon > 0$,

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof. **Idea 1:** Do manipulations to get a "left side" statement and a "right side" statement.

We would like to show that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy = \alpha f(x+) + \beta f(x-)$$

which is equivalent to showing that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \alpha f(x+) - \beta f(x-) = 0.$$

We now insert the definitions of α and β, so we want to show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$

We can prove this if we show that

$$\heartsuit : \lim_{\varepsilon \to 0} \int_{-\infty} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy = 0$$

and also

$$\star : \lim_{\varepsilon \to 0} \int_0^\infty f(x-y) g_\varepsilon(y) dy - \int_0^\infty f(x-) g(y) dy = 0.$$

In the textbook, Folland proves that $\star$ holds. So, for the sake of diversity, we prove that $\heartsuit$ holds. The argument is the same for both, so proving one of them is sufficient.

Hence, we would like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^0 f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 f(x+) g(y) dy$$

as small as we like. To make this precise, let us assume that "as small as we like" is quantified by a very small $\delta > 0$. Then we show that for sufficiently small ε we obtain

$$\left| \int_{-\infty}^0 f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 f(x+) g(y) dy \right| < \delta.$$

Idea 2: Smash the two integrals together:

$$\int_{-\infty}^0 \left(f(x-y) g_\varepsilon(y) - f(x+) g(y) \right) dy.$$

Well, this is a bit inconvenient, because in the first part we have g_ε, but in the second part it's just g.

Idea 3: Sneak g_ε into the second term. We make a small observation,

$$\int_{-\infty}^0 g(y) dy = \int_{-\infty}^0 g(z/\varepsilon) \frac{dz}{\varepsilon} = \int_{-\infty}^0 g_\varepsilon(z) dz$$

Above, we have made the substitution $z = \varepsilon y$, so $y = z/\varepsilon$, and $dz/\varepsilon = dy$. The limits of integration don't change. By this calculation,

$$\int_{-\infty}^0 f(x+) g(y) dy = \int_{-\infty}^0 f(x+) g_\varepsilon(y) dy.$$

(Above the integration variable was called z, but what's in a name? The name of the integration variable doesn't matter!). Moreover, note that $f(x+)$ is a constant, so it's just sitting there doing nothing. Hence, we have computed that

$$\int_{-\infty}^0 \left(f(x-y) g_\varepsilon(y) - f(x+) g(y) \right) dy = \int_{-\infty}^0 g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy.$$

Remember that $y \leq 0$ where we're integrating. Therefore, $x - y \geq x$.

Idea 4: Use the definition of right hand limit:

$$\lim_{y \uparrow 0} f(x-y) = f(x+) \implies \lim_{y \uparrow 0} f(x-y) - f(x+) = 0.$$

By the definition of limit there exists $y_0 < 0$ such that for all $y \in (y_0, 0)$

$$|f(x-y) - f(x+)| < \widetilde{\delta}.$$

We are using $\widetilde{\delta}$ for now, to indicate that $\widetilde{\delta}$ is going to be something in terms of δ, engineered in such a way that at the end of our argument we get that for ε sufficiently small,

$$\left| \int_{-\infty}^0 g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right| < \delta.$$

To figure out this $\widetilde{\delta}$, we use our estimate on the part of the integral from y_0 to 0,

$$\left|\int_{y_0}^{0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right| \leq \int_{y_0}^{0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy$$
$$\leq \widetilde{\delta}\int_{y_0}^{0}|g_\varepsilon(y)|dy \leq \widetilde{\delta}\int_{\mathbb{R}}|g_\varepsilon(y)|dy = \widetilde{\delta}||g||.$$

Above, we have used the same substitution trick to see that

$$\int_{\mathbb{R}}|g_\varepsilon(y)|dy = \int_{\mathbb{R}}|g(z)|dz = ||g||,$$

where $||g||$ is the $L^1(\mathbb{R})$ norm of g. By assumption, $g \in L^1(\mathbb{R})$, so this L^1 norm is finite. Moreover, because we know that

$$\int_{\mathbb{R}} g(y)dy = 1,$$

we know that

$$||g|| = \int_{\mathbb{R}}|g(y)|dy \geq \left|\int_{\mathbb{R}} g(y)dy\right| = 1.$$

So, let

$$\widetilde{\delta} = \frac{\delta}{2||g||}.$$

Note that we're not dividing by zero, by the above observation that $||g|| \geq 1$. So, this is a perfectly decent number. Then, we have the estimate (repeating the above estimate)

$$\left|\int_{y_0}^{0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right| \leq \int_{y_0}^{0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy$$
$$\leq \widetilde{\delta}\int_{y_0}^{0}|g_\varepsilon(y)|dy \leq \widetilde{\delta}\int_{\mathbb{R}}|g_\varepsilon(y)|dy = \widetilde{\delta}||g|| = \frac{\delta}{2}.$$

Idea 5: To deal with the other part of the integral, from $-\infty$ to y_0, consider the two cases given in the statement of the theorem separately. It is important to remember that

$$y_0 < 0.$$

So, we wish to estimate

$$\left|\int_{-\infty}^{y_0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right|.$$

First, let us assume that f is bounded, which means that there exists $M > 0$ such that $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$. Hence

$$|f(x-y)-f(x+)| \leq |f(x-y)| + |f(x+)| \leq 2M.$$

So, we have the estimate

$$\left|\int_{-\infty}^{y_0}(f(x-y)-f(x+))g_\varepsilon(y)dy\right| \leq \int_{-\infty}^{y_0}|f(x-y)-f(x+)||g_\varepsilon(y)|dy \leq 2M\int_{-\infty}^{y_0}|g_\varepsilon(y)|dy.$$

We shall do a substitution now, letting $z = y/\varepsilon$. Then, as we have computed before,

$$\int_{-\infty}^{y_0}|g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon}|g(z)|dz.$$

Here the limits of integration do change, because $y_0 < 0$. Specifically $y_0 \neq 0$, which is why the top limit changes. We're integrating between $-\infty$ and y_0/ε. We know that $y_0 < 0$. So, when we divide it by a really small, but still positive number, like ε, then $y_0/\varepsilon \to -\infty$ as $\varepsilon \to 0$. Moreover, we know that

$$\int_{-\infty}^{0} |g(y)|dy < \infty.$$

What this really means is that

$$\lim_{R \to -\infty} \int_{R}^{0} |g(y)|dy = \int_{-\infty}^{0} |g(y)|dy < \infty.$$

Hence,

$$\lim_{R \to -\infty} \int_{-\infty}^{0} |g(y)|dy - \int_{R}^{0} |g(y)|dy = 0.$$

Of course, we know what happens when we subtract the integral, which shows that

$$\lim_{R \to -\infty} \int_{-\infty}^{R} |g(y)|dy = 0.$$

Since

$$\lim_{\varepsilon \to 0} y_0/\varepsilon = -\infty,$$

this shows that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{y_0/\varepsilon} |g(y)|dy = 0.$$

Hence, by definition of limit there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{-\infty}^{y_0/\varepsilon} |g(y)|dy < \frac{\delta}{4(M+1)}.$$

Then, combining this with our estimates, above, which we repeat here,

$$\left| \int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy \right| \leq \int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy \leq 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)|dy$$

$$< 2M \frac{\delta}{4(M+1)} < \frac{\delta}{2}.$$

Therefore, we have the estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+) \right) dy \right|$$

$$\leq \int_{-\infty}^{0} |g_\varepsilon(y)||f(x-y) - f(x+)|dy \leq \int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Finally, we consider the other case in the theorem, which is that g vanishes outside a bounded interval. We retain the first part of our estimate, that is

$$\int_{y_0}^{0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy < \frac{\delta}{2}.$$

Next, we again observe that

$$\lim_{\varepsilon \downarrow 0} \frac{y_0}{\varepsilon} = -\infty.$$

By assumption, we know that there exists some $R > 0$ such that

$$g(x) = 0 \forall x \in \mathbb{R} \text{ with } |x| > R.$$

Hence, we may choose ε sufficient small so that

$$\frac{y_0}{\varepsilon} < -R.$$

Specifically, let

$$\varepsilon_0 = \frac{1}{-Ry_0} > 0.$$

Then for all $\varepsilon \in (0, \varepsilon_0)$ we compute that

$$\frac{y_0}{\varepsilon} < -R.$$

Hence for all $y \in (-\infty, y_0/\varepsilon)$ we have $g(y) = 0$. Thus, we compute as before using the substitution $z = y/\varepsilon$,

$$\int_{-\infty}^{y_0} |f(x-y) - f(x+)||g_\varepsilon(y)|dy = \int_{-\infty}^{y_0/\varepsilon} |f(x - \varepsilon z) - f(x+)||g(z)|dz = 0,$$

because $g(z) = 0 \forall z \in (-\infty, y_0/\varepsilon)$. Thus, we have the total estimate that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left| \int_{-\infty}^{0} g_\varepsilon(y) \left(f(x-y) - f(x+)\right) dy \right|$$

$$\leq \int_{-\infty}^{0} |g_\varepsilon(y)||f(x-y)-f(x+)|dy \leq \int_{-\infty}^{y_0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy + \int_{y_0}^{0} |f(x-y)-f(x+)||g_\varepsilon(y)|dy$$

$$< 0 + \frac{\delta}{2} \leq \delta.$$

□

1.2. **Exercises for the week to be demonstrated.** On Monday in the large group we shall have:

(1) (7.2.13.b) Use Plancharel's theorem to compute:

$$\int_{\mathbb{R}} \frac{t^2}{(t^2+a^2)(t^2+b^2)} dt = \frac{\pi}{a+b}.$$

(2) (Eö 12) Let

$$f(t) = \int_0^1 \sqrt{w} e^{w^2} \cos(wt) dw.$$

Compute

$$\int_{\mathbb{R}} |f'(t)|^2 dt.$$

(3) (7.4.1.a,b) Compute the Fourier sine and cosine transforms of e^{-kx}. These are defined, respectively, to be

$$\mathcal{F}_s[f](\xi) = \int_0^\infty f(x) \sin(\xi x) dx, \quad \mathcal{F}_c[f](\xi) = \int_0^\infty f(x) \cos(\xi x) dx.$$

On Wednesday or Friday depending on your group we shall have:

(1) (Eö 6.a, b) Compute the Fourier transforms of:

$$\frac{t}{(t^2+a^2)^2}, \quad \frac{1}{(t^2+a^2)^2}.$$

(2) (Eö 7) A function has Fourier transform

$$\hat{f}(\xi) = \frac{\xi}{1+\xi^4}.$$

Compute

$$\int_{\mathbb{R}} t f(t) dt, \quad f'(0).$$

(3) (7.3.2) Use the Fourier transform to derive the solution of the inhomogeneous heat equation $u_t = k u_{xx} + G(x,t)$ with initial condition $u(x,0) = f(x)$ (assume $f \in \mathcal{L}^2(\mathbb{R})$:

$$u(x,t) = f * K_t(x) + \int_{\mathbb{R}} \int_0^t G(y,s) K_{t-s}(x-y) ds dy.$$

Here

$$K_t(x) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}.$$

1.3. **Exercises for the week to be done oneself.**

(1) (Eö 9) Compute (with help of Fourier transform)

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)} dx.$$

(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a,b) both directly and by using the known case for the interval $(-a,a)$.

(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x} x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1+i\xi)^{-a}$ where Γ is the Gamma function.

(4) (7.2.12) For $a > 0$ let

$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$

Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.

(5) (Eö 6.d,e) Compute the Fourier transform of:

$$e^{-a|t|}\sin(bt), \quad (a,b>0), \quad \frac{t}{t^2+2t+5}.$$

(6) (Eö 15) Find a solution to the equation

$$u(t) + \int_{-\infty}^t e^{\tau - t} u(\tau) d\tau = e^{-2|t|}.$$

(7) (Eö 11) For the function

$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w} e^{iwt} dw,$$

compute

$$\int_{\mathbb{R}} f(t)\cos(t) dt, \quad \int_{\mathbb{R}} |f(t)|^2 dt.$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.19

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Applications of the Fourier transform

We will use the Fourier transform to solve both the homogeneous heat equation as well as the inhomogeneous heat equation. To do this, we briefly recall an important calculation. We would like to compute

$$\int_{\mathbb{R}} e^{-x^2} dx.$$

There is a beautiful trick for doing this calculation. Here is where the idea originates. If this integral were

$$\int_{\mathbb{R}} x e^{-x^2} dx$$

we would know how to compute it. So we would like to be integrating against xdx not just dx. When do we have something like xdx? We have something of this form when we are working in polar coordinates in $\mathbb{R}^2$, because there we have $rdrd\theta$. So, we could compute the integral

$$\int_{\mathbb{R}^2} e^{-r^2} r dr d\theta = 2\pi \int_0^\infty e^{-r^2} r dr = 2\pi \frac{e^{-r^2}}{-2}\Big|_0^\infty = \pi.$$

On the other hand

$$\int_{\mathbb{R}^2} e^{-r^2} r dr d\theta = \int_{\mathbb{R}^2} e^{-x^2-y^2} dx dy = \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \left(\int_{\mathbb{R}} e^{-x^2} dx \right)^2.$$

Thus

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

1.1. **Homogeneous heat equation.** We wish to solve:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = v(x), \end{cases}$$

where our initial data v is assumed to be bounded, continuous, and also in $\mathcal{L}^2(\mathbb{R})$.

> Idea: Fourier transform the PDE with respect to the x variable, because $x \in \mathbb{R}$, whereas $t > 0$, but the Fourier transform integrates over all of $\mathbb{R}$, thus x is the wise choice.

We obtain

$$\hat{u}_t(\xi,t) - \hat{u}_{xx}(\xi,t) = 0.$$

Now, we use the theorem which gave us the properties of the Fourier transform. It says that if we take the Fourier transform of a derivative, $\hat{f'}(\xi) = i\xi\hat{f}(\xi)$. Using this twice,

$$\hat{u}_{xx}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

Now, those of you who are picky about switching limits may not like this, but it is in fact rigorously valid:

$$\partial_t\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t) = 0.$$

Hence

$$\partial_t\hat{u}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

This is a first order homogeneous ODE for u in the t variable. We can solve it!!! We do that and get

$$\hat{u}(\xi,t) = e^{-\xi^2 t}c(\xi).$$

The constant can depend on ξ but not on t. To figure out what the constant should be, we use the IC:

$$\hat{u}(\xi,0) = \hat{v}(\xi) \implies c(\xi) = \hat{v}(\xi).$$

Thus, we have found

$$\hat{u}(\xi,t) = e^{-\xi^2 t}\hat{v}(\xi).$$

Now, we use another property of the Fourier transform which says

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

So, if we can find a function whose Fourier transform is $e^{-\xi^2 t}$, then we can express u as a convolution of that function and v. So, we are looking to find

$$g(x,t) \text{ such that } \hat{g}(x,t) = e^{-\xi^2 t}.$$

We use the FIT:

$$g(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-\xi^2 t}d\xi.$$

We can use some complex analysis to compute this integral. To do this, we shall complete the square in the exponent:

$$-\xi^2 t + ix\xi = -\left(\xi\sqrt{t} - \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}.$$

Therefore we are computing

$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} - \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right)d\xi.$$

Using a contour integral, we can in fact ignore the imaginary part. To see this, first note that we are integrating with respect to ξ, so we can for the moment just consider:

$$\int_{-\infty}^{\infty} \exp\left(-\left(\xi t - \frac{ix}{2\sqrt{t}}\right)^2\right)d\xi.$$

We draw a box. The box has vertices in the complex plane at the points $\pm R$ and $\pm R + \frac{ix}{2\sqrt{t}}$. The integrand above is holomorphic for all ξ inside this box. Therefore the integral around the boundary of the box is zero. When $\xi = \pm R$, the integrand is very small, thus the integrals on the vertical sides of the box tend to zero. Hence the integrals along the two horizontal sides of the box are also adding up to zero, which shows that

$$\int_{-\infty}^{\infty} \exp\left(-\left(\xi t - \frac{ix}{2\sqrt{t}}\right)^2\right) d\xi = \int_{-\infty}^{\infty} \exp(-\xi^2 t^2) d\xi.$$

So, we compute (using a change of variables to $y = \xi\sqrt{t}$ so $t^{-1/2}dy = d\xi$)

$$\int_{\mathbb{R}} e^{-\xi^2 t} d\xi = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-y^2} dy = \frac{\sqrt{\pi}}{\sqrt{t}}.$$

Hence,

$$\int_{\mathbb{R}} \exp\left(-\left(\xi\sqrt{t} - \frac{ix}{2\sqrt{t}}\right)^2 - \frac{x^2}{4t}\right) d\xi = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{x^2}{4t}}.$$

Recalling the factor of $1/(2\pi)$ we have

$$g(x,t) = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{x^2}{4t}} = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

Hence the solution is

$$u(x,t) = g * v(x) = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-(x-y)^2/(4t)} v(y) dy.$$

Exercise 1. *Verify that for all $x \in \mathbb{R}$ and $t > 0$ our solution satisfies the homogeneous heat equation.*

Question 1. *Why is our solution equal to v when $t = 0$?*

If we naively set $t = 0$, we obtain an expression that does not make sense. So, how do we know that this expression indeed gives us our initial data at $t = 0$? We use the big bad convolution approximation theorem! Consider the function

$$\varphi(x) = \frac{e^{-x^2/4}}{2\sqrt{\pi}}.$$

This function satisfies

$$\int_{\mathbb{R}} \varphi(x) dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} dz = 1,$$

using the change of variables $z = \frac{x}{2}$. This function satisfies the hypotheses of the theorem (the so-called g function). We have assumed that v is bounded. Therefore the convolution approximation theorem says that

$$\lim_{\varepsilon \downarrow 0} \varphi_\varepsilon \star v(x) = v(x) \quad \forall x \in \mathbb{R}.$$

Let's re-name ε to $\sqrt{t}$, so that

$$\lim_{\sqrt{t} \downarrow 0} \varphi_{\sqrt{t}} \star v(x) = v(x).$$

Let's write out the

$$\varphi_{\sqrt{t}} \star v(x) = \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{(2\sqrt{t})^2}}}{2\sqrt{\pi}\sqrt{t}} v(y) dy.$$

The theorem says

$$\lim_{\sqrt{t}\downarrow 0} \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{(2\sqrt{t})^2}}}{2\sqrt{\pi}\sqrt{t}} v(y) dy = \lim_{t \downarrow 0} \int_{\mathbb{R}} \frac{e^{-(x-y)^2/(4t)}}{2\sqrt{\pi t}} v(y) dy = \lim_{t \downarrow 0} u(x,t) = v(x) \quad \forall x \in \mathbb{R}.$$

We therefore understand

$$u(x,0) := \lim_{t \downarrow 0} u(x,t) = v(x) \forall x \in \mathbb{R}.$$

With some abstract uniqueness theory, beyond the scope of this humble course, we could also prove that our solution $u(x,t)$ is the unique solution to the heat equation which has initial data equal to $v(x)$ and which is in $\mathcal{L}^2$ for all $t > 0$.

1.2. **Inhomogeneous heat equation.** If you have an inhomogeneous IVP for the heat equation, here are two ways to deal with that:

(1) If the inhomogeneity is *time independent,* look for a steady state solution to solve the inhomogeneous equation. Then, solve the homogeneous equation, but change your initial data. If f is your steady state solution and v was your initial data (before f came along), solve the IVP for the homogeneous heat equation with IC $v - f$ rather than just v.
(2) If the inhomogeneity is *time dependent,* you can try to solve by Fourier transforming the whole PDE.

Since we know how to do the first type of example, let us consider the second type of example. We want to solve an inhomogeneous heat equation on $\mathbb{R}$:

$$u_t - u_{xx} = G(x,t), \quad u(x,0) = v(x) \text{ is continuous, bounded, and in } \mathcal{L}^2.$$

Let's try the Fourier transform method:

$$\partial_t \hat{u}(\xi,t) + \xi^2 \hat{u}(\xi,t) = \hat{G}(\xi,t).$$

This is a first order ODE. If you are a CHEMIST, then you did the special extra part of the course and actually learned how to solve this ODE in t. Pretty cool. To see how this works, treat ξ like a constant, and write

$$f'(t) + \xi^2 f(t) = \hat{G}(\xi,t).$$

The mμthod says to first compute

$$e^{\int \xi^2 dt} = e^{\xi^2 t}.$$

Next compute

$$\int e^{\xi^2 t} \hat{G}(\xi,t) dt.$$

Then, the solution is

$$\frac{\int e^{\xi^2 t} \hat{G}(\xi,t)dt + C(\xi)}{e^{\xi^2 t}} = e^{-\xi^2 t} \int e^{\xi^2 s} \hat{G}(\xi,s) ds + C(\xi) e^{-\xi^2 t}.$$

We would like the initial condition to be satisfied, so when $t = 0$ we should obtain that this is equal to the Fourier transform of the initial data,

$$\hat{v}(\xi).$$

We are free to choose any primitive function of $e^{-\xi^2 s} \hat{G}(\xi,s)$. It is very convenient to choose the one which vanishes when $t = 0$, namely

$$\int_0^t e^{-\xi^2 s} \hat{G}(\xi,s) ds.$$

Then to obtain the initial condition, we just let $C(\xi) = \hat{v}(\xi)$. Thus, our Fourier transformed solution is

$$e^{-\xi^2 t} \int_0^t e^{-\xi^2 s} \hat{G}(\xi, s) ds + \hat{v}(\xi) e^{-\xi^2 t}.$$

We need to figure out from whence this Fourier transform came (equivalently, invert the Fourier transform). This is a linear process, so we can deal with each piece separately and then add them. Well, the second part we did last time. We saw that the Fourier transform of

$$\frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y) dy$$

is

$$\hat{v}(\xi) e^{-\xi^2 t}.$$

Similarly, let's look at the first part. It is

$$e^{-\xi^2 t} \int_0^t e^{\xi^2 s} \hat{G}(\xi, s) ds = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi, s) ds.$$

By the same calculations, the Fourier transform of

$$\frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy = e^{-\xi^2 (t-s)} \hat{G}(\xi, s).$$

Yet again playing switch-a-roo with limits[1],

$$\mathcal{F} \left(\int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy ds \right) (\xi) = \int_0^t e^{-\xi^2 (t-s)} \hat{G}(\xi, s) ds.$$

Therefore, our full solution is

$$\int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy ds + \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} v(y) dy.$$

This solution satisfies our initial data because

$$\lim_{t \downarrow 0} \int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} G(y, s) dy ds = 0,$$

and just as in the homogeneous heat equation, we have by the convolution approximation theorem that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} v(y) dy = v(x) \quad \forall x \in \mathbb{R}.$$

1.3. **Computing tricky integrals (sometimes you can compute integrals that computers cannot!)** The following is a very useful observation:

$$\hat{f}(0) = \int_{\mathbb{R}} f(x) dx.$$

So, if you have the integral of a function, this is equal to the value of its Fourier transform at $\xi = 0$. So, if you can look up the Fourier transform of the function, like in Beta or Folland, then to compute the integral, no need for fancy contour integrals, simply pop $\xi = 0$ into the Fourier transform.

[1]Trust me!

Here is an example:

$$\text{compute: } \int_{\mathbb{R}} \frac{1}{x^2+9} dx.$$

We see this is # 10 in Folland's TABLE 2. On the right side, we get the Fourier transform (with $a = 3$) is given by

$$\frac{\pi}{3} e^{-3|\xi|}.$$

So, this integral is the Fourier transform with $\xi = 0$, hence the value of the integral is

$$\frac{\pi}{3}.$$

That was pretty easy right? For something more complicated, you could have say

$$\int_{\mathbb{R}} f(x)g(x)dx,$$

with some icky functions f and g (see extra övning # 9). Now, you can use that the Fourier transform of a product is

$$(2\pi)^{-1}(\hat{f} * \hat{g})(\xi).$$

Hence, what you have above is

$$\int_{\mathbb{R}} f(x)g(x)dx = \int_{\mathbb{R}} e^{-i(0)x} f(x)g(x)dx = (2\pi)^{-1}(\hat{f} * \hat{g})(0).$$

So, if the Fourier transforms of these functions are somewhat better than the functions f and g, then the stuff on the right could be nicely computable and give you the integral on the left. Try # 9 to see how this works. (If you get stuck, Team Fourier is here to help! Just ask us!)

As another example, there is extra exercise number 10. It says you know the Fourier transform of $f(t)$ is $\frac{1}{|w|^3+1}$. We are then asked to compute

$$\int_{\mathbb{R}} |f * f'|^2 dt.$$

By the Plancharel theorem,

$$\int_{\mathbb{R}} |f * f'|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f * f'}|^2 dt.$$

Now we use the theorem on the properties of the Fourier transform which says

$$\widehat{f * f'}(\xi) = \hat{f}(\xi)\widehat{f'}(\xi).$$

Now we use that same theorem to say that

$$\widehat{f'}(\xi) = i\xi \hat{f}(\xi).$$

So, the stuff on the right is

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi) i\xi \hat{f}(\xi)|^2 d\xi.$$

We are given what the Fourier transform is, so we put it in there:

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\xi^2}{(|\xi|^3+1)^4} d\xi.$$

Now this isn't so terrible. It's an even function so this is

$$\frac{1}{\pi}\int_0^\infty \frac{\xi^2}{(\xi^3+1)^4}d\xi.$$

It just so happens that the derivative of

$$\frac{1}{(\xi^3+1)^3} \text{ is } \frac{-9\xi^2}{(\xi^3+1)^4},$$

so

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi^2}{(\xi^3+1)^4}d\xi = \frac{-1}{9\pi}\left.\frac{1}{(\xi^3+1)^3}\right|_0^\infty = \frac{1}{9\pi}.$$

1.4. **Exercises for the week to be done oneself: hints.**

(1) (Eö 9) Compute (with help of Fourier transform)

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)}dx.$$

Hint: There are disguised zeros and ones hiding all over the place in mathematics. The above is equal to

$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)}e^{-i(0)x}dx = \mathcal{F}\left(\frac{\sin x}{x}\frac{1}{x^2+1}\right)(0).$$

So, we now look at Table 2 in Folland, especially item number 8. It says that the Fourier transform of a product is a convolution of the Fourier transforms. So, we apply this to say

$$\mathcal{F}\left(\frac{\sin x}{x}\frac{1}{x^2+1}\right)(0) = \frac{1}{2\pi}\mathcal{F}\left(\frac{\sin x}{x}\right) * \mathcal{F}\left(\frac{1}{x^2+1}\right)(0).$$

Now we use items 10 and 13 from the same table, together with the definition of the convolution, to substitute for the Fourier transforms on the right side:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \pi\chi_1(0-y)\pi e^{-|y|}dy.$$

Recalling what χ_1 means:

$$= \frac{\pi}{2}\int_{-1}^1 e^{-|y|}dy.$$

I leave it to you do compute the integral!

(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a,b) both directly and by using the known case for the interval $(-a,a)$.

Hint: Well, doing it directly we are computing

$$\int_a^b e^{-ix\xi}dx = \begin{cases} b-a & \xi = 0 \\ \frac{i}{\xi}\left(e^{-bi\xi} - e^{-ai\xi}\right) & \xi \neq 0 \end{cases}$$

To do it the other way, it's convenient to introduce some notations:

$$m := \frac{a+b}{2}, \ell := \frac{b-a}{2}.$$

Then our interval is $[m-\ell, m+\ell]$. So we are computing

$$\int_{m-\ell}^{m+\ell} e^{-ix\xi}dx.$$

To make this more familiar let's do a change of variables so that the integral goes from $-\ell$ to ℓ, so we let $t = x - m$, then $dt = dx$, so we are computing

$$\int_{-\ell}^{\ell} e^{-i(t+m)\xi}dt = e^{-im\xi}\int_{-\ell}^{\ell} e^{-it\xi}dt = e^{-im\xi}\hat{\chi}_{[-\ell,\ell]}(\xi).$$

So now for the Fourier transform of the characteristic function of the interval, that is the function $\chi_{[-\ell,\ell]}$ we can use the item 12 in Table 2 of Folland. With a little algebraic manipulations, one can show that these both roads lead to the same answer.

(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1+i\xi)^{-a}$ where Γ is the Gamma function.

Hint: one is computing

$$\int_0^\infty e^{-x}e^{-ix\xi}x^{a-1}dx = \int_0^\infty e^{-x(1+i\xi)}x^{a-1}dx.$$

On the other hand,

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt.$$

Try doing a substitution to relate these integrals...

(4) (7.2.12) For $a > 0$ let

$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$

Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.

Hint: The idea is basically repeated use of the items in Folland's Table 2, and using the FIT. First, compute the Fourier transform of $f_a * f_b$ which is $\hat{f}_a(\xi)\hat{f}_b(\xi)$, so you can write this stuff down. You will get something like $e^{-|x|\cdots}$. Next, use the FIT to return to $f_a * f_b$. Note that one way to write the FIT is

$$f(x) = \frac{1}{2\pi}\hat{\hat{f}}(-x).$$

Do something similar for the second one...

(5) (Eö 6.d,e) Compute the Fourier transform of:

$$e^{-a|t|}\sin(bt), \quad (a,b>0), \quad \frac{t}{t^2+2t+5}.$$

Hint: I might deal with the first one by splitting up the sine into its complex exponentials, using definition of Fourier transform, and just directly integrating. As for the second one, note that $t^2+2t+5 = (t+1)^2+4$. Do a substitution in the definition of the Fourier transform, let $x = t+1$. Then use item 10 on Folland's Table 2.

(6) (Eö 15) Find a solution to the equation

$$u(t) + \int_{-\infty}^{t} e^{\tau - t}u(\tau)d\tau = e^{-2|t|}.$$

Hint: This is a tricky one! First turn the integral into a convolution. How to do that? Try using $\Theta(\tau)e^{-|\tau|}$. Write out the convolution of that function together with $u(\tau)$. Next, Fourier transform both sides of the equation. So you will get

$$\hat{u}(\xi) + (\widehat{\Theta(\tau)e^{-|\tau|}})(\xi)\hat{u}(\xi) = \widehat{e^{-2|t|}}(\xi).$$

Compute the Fourier transforms of everything except u. Solve the equation for $\hat{u}(\xi)$. Then use the FIT. When you use the FIT, if you do it using contour integrals and the residue, you will need to think about the cases $x > 0$ and $x < 0$ separately. For $x > 0$ the up-rainbow will work. For $x < 0$ the down-rainbow will work.

(7) (Eö 11) For the function

$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w} e^{iwt} dw,$$

compute

$$\int_{\mathbb{R}} f(t)\cos(t)dt, \quad \int_{\mathbb{R}} |f(t)|^2 dt.$$

Hint: This is tricky also. Let me define a new function for us:

$$\phi(w) := \chi_{[0,2]}(w) \frac{\sqrt{2}}{1+w}.$$

Then

$$f(t) = \widehat{\phi}(-t).$$

Oh no she didn't. Yeah. So, for the first one, note that this integral is, expanding the cosine as a sum of complex exponentials

$$\int_{\mathbb{R}} f(t)\cos(t)dt = \frac{1}{2}\left(\hat{f}(1) + \hat{f}(-1)\right).$$

Play around with the FIT and the fact that $f(t) = \widehat{\phi}(-t)$ to figure out the right side. Next, note that

$$\int_{\mathbb{R}} |f(t)|^2 dt = \int_{\mathbb{R}} |\widehat{\phi}(-t)|^2 dt = 2\pi \int_{\mathbb{R}} |\phi(t)|^2 dt.$$

The integral of $|\phi|^2$ is hopefully not that terrible...

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.21

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The Sampling Theorem

This theorem is all about the interaction between Fourier series and Fourier coefficients and how to work with both simultaneously. It is one of the theory items, so its proof is important.

Theorem 1. *Let $f \in L^2(\mathbb{R})$. We take the definition of the Fourier transform of f to be*

$$\int_{\mathbb{R}} e^{-ix\xi} f(x)dx,$$

and we then assume that there is $L > 0$ so that $\hat{f}(\xi) = 0$ $\forall \xi \in \mathbb{R}$ with $|\xi| > L$. Then:

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

Proof:

Idea: Since the Fourier transform $\hat{f}$ has compact support, we can expand it as a Fourier series.

We therefore have

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x)dx.$$

Idea: Use the FIT to express f in terms of its Fourier transform.

We therefore have

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixt} \hat{f}(x)dx = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \hat{f}(x)dx.$$

On the left we have used the fact that $\hat{f}$ is supported in the interval $[-L, L]$, thus the integrand is zero outside of this interval, so we can throw that part of the integral away.

Idea: Substitute the Fourier expansion of $\hat{f}$ into the integral.

So, we have

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} c_n e^{in\pi x/L} dx.$$

From here until the end of the proof, we will essentially just be computing. The coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx = \frac{1}{2L} \int_{\mathbb{R}} e^{ix(-n\pi/L)} \hat{f}(x) dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

In the second equality we have used the fact that $\hat{f}(x) = 0$ for $|x| > L$, so by including that part we don't change the integral. In the third equality we have used the FIT!!! So, we now substitute this into our formula above for

$$f(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{in\pi x/L} dx$$

This is approaching the form we wish to have in the theorem, but the argument of the function f has a pesky negative sign. That can be remedied by switching the order of summation, which does not change the sum, so

$$f(t) = \frac{1}{2L} \int_{-L}^{L} e^{ixt} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{-in\pi x/L} dx.$$

We may also interchange the summation with the integral[1]

$$f(t) = \frac{1}{2L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \int_{-L}^{L} e^{x(it - in\pi/L)} dx.$$

We then compute

$$\int_{-L}^{L} e^{x(it-in\pi/L)} dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)} \sin(Lt - n\pi).$$

Substituting,

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \frac{\sin(Lt - n\pi)}{Lt - n\pi}.$$

Of course my dyslexia has ended up with things being backwards, but it is not a problem because sine is odd so

$$\sin(Lt - n\pi) = -\sin(n\pi - Lt),$$

so

$$\frac{\sin(Lt - n\pi)}{Lt - n\pi} = \frac{-\sin(n\pi - Lt)}{Lt - n\pi} = \frac{\sin(n\pi - Lt}{n\pi - Lt}.$$

[1]None of this makes sense pointwise; we are working over L^2. The key property which allows interchange of limits, integrals, sums, derivatives, etc is *absolute convergence.* This is the case here because elements of L^2 have $\int |f|^2 < \infty$. That is precisely the type of absolute convergence required.

2. Fourier sine and cosine transforms and applications to PDEs on half-spaces

Today we shall investigate some transforms related to the Fourier transform. The first two can be used to solve PDEs on half lines, *if the boundary condition* is suitable.

2.1. Motivation: heat equation on a semi-infinite rod with an insulated end.

We have found ourselves in possession of a giant rod which is insulated at the one end and goes out to infinity at the other. It has an initial temperature distribution given by a function $f(x)$ which is bounded, continuous and an element of $\mathcal{L}^2$. We therefore wish to solve the problem:

$$u_t - u_{xx} = 0, \quad u_x(0,t) = 0, \quad u(x,0) = f(x), \quad x \in [0,\infty).$$

To solve such a problem we will use a Fourier cosine transform together with the Fourier cosine transform inverse theorem.

2.2. Fourier sine and cosine transforms and their inverse formulas.

Definition 2. *Let f be in $\mathcal{L}^1$ or $\mathcal{L}^2$ on $(0,\infty)$. The Fourier cosine transform,*

$$\mathcal{F}_c(f)(\xi) := \int_0^\infty f(x)\cos(\xi x)dx.$$

The Fourier sine transform,

$$\mathcal{F}_s(f)(\xi) := \int_0^\infty f(x)\sin(\xi x)dx.$$

As with the Fourier transform, the Fourier sine and cosine transforms also have inversion formula.

Theorem 3. *Assume that $f \in \mathcal{L}^2[0,\infty)$. Then we have the Fourier cosine inversion formula*

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(f)(\xi)\cos(x\xi)d\xi.$$

We also have the Fourier sine inversion formula

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi.$$

Proof: First, let us extend f evenly to $\mathbb{R}$, denoting this extension by f_e, so that $f_e(-x) = f_e(x)$. We compute the standard Fourier transform:

$$\hat{f}_e(\xi) = \int_{\mathbb{R}} f_e(x)e^{-ix\xi}dx = \int_{\mathbb{R}} f_e(x)(\cos(x\xi) - i\sin(x\xi))dx = 2\int_0^\infty f(x)\cos(x\xi)dx.$$

The term with the sine has dropped out because $f_e(x)\sin(x\xi)$ is an odd function of x. The term with the cosine gets doubled because $f_e(x)\cos(x\xi)$ is an even function. So, all together we have computed:

$$\hat{f}_e(\xi) = 2\int_0^\infty f(x)\cos(x\xi)dx = 2\mathcal{F}_c(f)(\xi).$$

Since the cosine is an even function,

$$\hat{f}_e(\xi) = \hat{f}_e(-\xi).$$

So, we also have that $\mathcal{F}_c(f)$ is an even function. The inversion formula (FIT) says that

$$f_e(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}_e(\xi)d\xi = \frac{1}{\pi}\int_{\mathbb{R}} e^{ix\xi}\mathcal{F}_c(f)(\xi)d\xi$$
$$= \frac{1}{\pi}\int_{\mathbb{R}}(\cos(x\xi) + i\sin(x\xi))\mathcal{F}_c(f)(\xi)d\xi = \frac{2}{\pi}\int_0^\infty e^{ix\xi}\mathcal{F}_c(f)(\xi)d\xi.$$

This is the cosine-FIT! Above we have used the fact that $\mathcal{F}_c(f)$ is an even function. Hence its product with the cosine is also an even function, so that part of the integral gets a factor of two when we integrate only over the positive real line. The product of an even function like $\mathcal{F}_c(f)$ with an odd function, like the sine, is odd, so that integral vanishes.

On the other hand, we may also define the odd extension, f_o which satisfies $f_o(-x) = -f_o(x)$ (for $x \neq 0$). The value of f at zero is not really important at this moment.[2] We compute the standard Fourier transform of the odd extension:

$$\hat{f}_o(\xi) = \int_{\mathbb{R}} f_o(x)e^{-ix\xi}dx = \int_{\mathbb{R}} f_o(x)(\cos(x\xi) - i\sin(x\xi))dx = -2i\int_0^\infty f(x)\sin(x\xi)dx$$
$$= -2i\mathcal{F}_s(f)(\xi).$$

Above, we have used the fact that f_o is odd, and therefore so is its product with the cosine. On the other hand, the product with the sine is an even function, which explains the factor of 2. Since the sine itself is odd, we have that $\hat{f}_o$ is an odd function and similarly $\mathcal{F}_s(f)(\xi)$ is also an odd function. We apply the FIT:

$$f_o(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}_o(\xi)d\xi = -\frac{i}{\pi}\int_{\mathbb{R}}(\cos(x\xi) + i\sin(x\xi))\mathcal{F}_s(f)(\xi)d\xi$$
$$= \frac{1}{\pi}\int_{\mathbb{R}}\sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi = \frac{2}{\pi}\int_0^\infty \sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi).$$

This is the sine-FIT! Above we have used the fact that $\mathcal{F}_s(f)$ is an odd function, and therefore so is its product with the cosine. On the other hand the product of two odd functions is an even function, so that is the reason for the factor of 2.

2.3. **Solving the heat equation on a semi-infinite rod with insulated end.** We wish to solve the problem:

$$u_t - u_{xx} = 0, \quad u_x(0,t) = 0, \quad u(x,0) = f(x), \quad x \in [0,\infty).$$

Assume that by some method, we have obtained a solution $u(x,t)$ defined on $[0,\infty)_x \times [0,\infty)_t$. To see if we may use a Fourier sine or cosine transform method, let us see what happens when we extend our solution evenly or oddly. The even extension would satisfy, by the cosine-FIT:

$$u_e(x,t) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c(u)(\xi)\cos(x\xi)d\xi.$$

The odd extension would satisfy, by the sine-FIT

$$u_o(x,t) = \frac{2}{\pi}\int_0^\infty \sin(x\xi)\mathcal{F}_s(f)(\xi)d\xi.$$

[2]This is because we are working in $\mathcal{L}^2$ which ignores sets of measure zero, and a single point is a set of measure zero.

OBS! The extension matches up with our original function on the positive real line (that is how an extension works!) We need the derivative with respect to x to vanish at $x = 0$. Let's just differentiate these expressions. Note that the x dependence is only in the sine or cosine term so we have:

$$\partial_x u_e(x,t) = -\frac{2}{\pi}\int_0^\infty \mathcal{F}_c(u)(\xi)\xi \sin(x\xi)d\xi \implies \partial_x u_e(0,t) = 0.$$

On the other hand

$$\partial_x u_o(x,t) = \frac{2}{\pi}\int_0^\infty \xi\cos(x\xi)\mathcal{F}_s(u)(\xi)d\xi \implies \partial_x u_o(0,t) = \frac{2}{\pi}\int_0^\infty \xi\mathcal{F}_s(u)(\xi)d\xi = ???$$

The even extension automatically gives us the desired boundary condition whereas the odd extension leads to something complicated looking, which we have no reason to know is zero.

This indicates that we have a decent chance of solving the problem by:

(1) Extending the initial data *evenly* to the real line.
(2) Solving the problem using the Fourier transform on the real line.
(3) Verifying that the solution satisfies all the conditions: the PDE, the IC, and the BC.

We do this. Extend f evenly, and write the extension as f_e. Then the solution to the homogeneous heat equation on the real line with initial data f_e is

$$u_e(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} f_e(y)e^{-\frac{(x-y)^2}{4t}}dy.$$

We split up the integral:

$$\int_{-\infty}^0 f_e(y)e^{-(x-y)^2/(4t)}dy + \int_0^\infty f_e(y)e^{-(x-y)^2/(4t)}dy$$
$$= -\int_\infty^0 f_e(z)e^{-(z+x)^2/(4t)}dz + \int_0^\infty f_e(y)e^{-(x-y)^2/(4t)}dy.$$

Above we made the substitution that $z = -y$ in the first integral. Due to the evenness of f_e, nothing happens when we change $y = -z$. Reversing the limits of integration we obtain

$$-\int_\infty^0 f_e(z)e^{-(z+x)^2/(4t)}dz = \int_0^\infty f_e(z)e^{-(z+x)^2/(4t)}dz = \int_0^\infty f_e(y)e^{-(x+y)^2/(4t)}dy.$$

So, all together we have

$$u_e(x,t) = \frac{1}{2\sqrt{\pi t}}\int_0^\infty f(y)\left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\right)dy.$$

Is this an even function? Let us verify:

$$e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} = e^{-\frac{(-x-y)^2}{4t}} + e^{-\frac{(-x+y)^2}{4t}}.$$

AWESOME! Our solution to the heat equation in this way is EVEN. Therefore, it is the same on the left and right sides. So, we can simply let

$$u(x,t) = u_e(x,t) = \frac{1}{2\sqrt{\pi t}}\int_0^\infty f(y)\left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\right)dy.$$

The way we have built it, it satisfies the IC, BC, and the PDE!

Exercise 1. *Solve:*

$$u_t - u_{xx} = 0, \quad u(0,t) = 0, \quad u(x,0) = \phi(x), \quad x \in [0, \infty).$$

Above, we assume that ϕ is bounded, continuous, and in $\mathcal{L}^2$. Hint: extend ϕ oddly *this time, and use the Fourier sine inverse theorem.*

2.4. **Discrete and fast Fourier transform.** We have seen that computing the Fourier transform is not the easiest thing in the world. The example with the Gaussian involving all those tricks: completing the square, complex analysis and contour integral is a nice and easy case. However, in the *real world* you may come across functions and not know how to compute the Fourier transform by hand, nor be able to find it in BETA. It could be lurking in one of our giant handbooks of calculations (Abramowitz & Stegun, Gradshteyn & Rhizik, to name a few). Or it could simply never have been computed analytically. In this case you may compute something called the *discrete Fourier transform.*

We start with a function, $f(t)$, and think of analyzing $f(t)$ as *time analysis,* whereas analyzing $\hat{f}(\xi)$ as *frequency analysis.* We shall consider a finite dimensional Hilbert space:

$$\mathbb{C}^N = \left\{ (s_n)_{n=0}^{N-1}, \quad s_n \in \mathbb{C}, \quad \langle (s_n), (t_n) \rangle := \sum_{n=0}^{N-1} s_n \overline{t_n} \right\}.$$

Now let

$$e_k(n) := \frac{e^{2\pi i k n/N}}{\sqrt{N}}.$$

Proposition 4. *Let*

$$e_k := (e_k(n))_{n=0}^{N-1}.$$

Then

$$\{e_k\}_{k=0}^{N-1}$$

are an ONB of $\mathbb{C}^N$.

Proof: We simply compute. It is so cute and discrete!

$$\langle e_k, e_j \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k n/N} e^{-2\pi i j n/N} = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i (k-j) n/N}.$$

If $j = k$ the terms are all 1, and so the total is N which divided by N gives 1. Otherwise, we may without loss of generality assume that $k > j$ (swap names if not the case). Then we are staring at a geometric series! We know how to sum it

$$\sum_{n=0}^{N-1} e^{2\pi i (k-j) n/N} = \frac{1 - e^{2\pi i (k-j) N/N}}{1 - e^{2\pi i (k-j)/N}} = 0.$$

Here it is super important that $k - j$ is a number between 1 and $N - 1$. We know this because $0 \leq j < k \leq N - 1$. Hence when we subtract j from k, we get something between 1 and $N - 1$. So we are not dividing by zero!

Now we shall fix T small and N large and look at $f(t)$ just on the interval $[0,(N-1)T]$. Let
$$f(t_n) := f(nT), \quad t_n = nT.$$
Basically, we're going to identify f with an element of $\mathbb{C}^N$, namely
$$(f(t_n))_{n=0}^{N-1}.$$

Definition 5. *Let*
$$w_k := \frac{2\pi k}{NT}.$$
The discrete Fourier transform of f at w_k is defined to be
$$F(w_k) := \langle (f(t_n)), e_k \rangle = \sum_{n=0}^{N-1} \frac{f(t_n)e^{-2\pi ikn/N}}{\sqrt{N}}.$$
This can also be written as
$$\sum_{n=0}^{N-1} \frac{f(t_n)e^{-iw_k t_n}}{\sqrt{N}}.$$

Example 1. *One of the fun facts about the discrete Fourier transform is that we can Fourier transform functions which are neither in $\mathcal{L}^2$ nor in $\mathcal{L}^1$. For example, let's compute the discrete Fourier transform of*
$$f(x) = x, \quad T = \frac{1}{10}, \quad N = 5.$$
So, we identify f with the vector
$$(0, 0.1, 0.2, 0.3, 0.4).$$
Then,
$$F(w_k) := \sum_{n=0}^{4} \frac{ne^{-2\pi ikn/5}}{10\sqrt{5}}.$$
So, we identify the Fourier transform of f with the vector
$$\left(\sum_{n=0}^{4} \frac{n}{10\sqrt{5}}, \sum_{n=0}^{4} \frac{ne^{-2\pi in/5}}{10\sqrt{5}}, \sum_{n=0}^{4} \frac{ne^{-4\pi in/5}}{10\sqrt{5}}, \sum_{n=0}^{4} \frac{ne^{-6\pi in/5}}{10\sqrt{5}}, \sum_{n=0}^{4} \frac{ne^{-8\pi in/5}}{10\sqrt{5}} \right).$$

Proposition 6. *We have the inversion formula*
$$f(t_n) = \sum_{k=0}^{N-1} F(w_k)e_n(k) = \langle (F(w_k)), \overline{e}_n \rangle.$$

Proof: We simply compute. By definition
$$\langle (F(w_k)), \overline{e}_n \rangle = \sum_{k=0}^{N-1} F(w_k)e_n(k).$$
Now, we insert the definition of $F(w_k)$ which gives us another sum, so we use a different index there. Hence we have
$$\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \frac{f(t_m)e^{-iw_k t_m}}{\sqrt{N}} \frac{e^{2\pi ikn/N}}{\sqrt{N}} = \frac{1}{N} \sum \sum f(t_m)e^{-2\pi ikm/N}e^{2\pi ikn/N}$$
$$= \frac{1}{N} \sum \sum f(t_m)e^{2\pi ik(n-m)/N} = \frac{1}{N} \sum_{m=0}^{N-1} f(t_m) \sum_{k=0}^{N-1} e^{2\pi ik(n-m)/N}$$

$$= \sum_{m=0}^{N-1} f(t_m) \sum_{k=0}^{N-1} \frac{e^{-2\pi ikm/N}}{\sqrt{N}} \frac{\overline{e^{-2\pi ikn/N}}}{\sqrt{N}} = \sum_{m=0}^{N-1} f(t_m)\langle e_m, e_n\rangle.$$

By the proposition we just proved before,

$$\langle e_m, e_n\rangle = \delta_{n,m} = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases}$$

So, the only term which survives is when $m = n$, and so we get

$$f(t_n).$$

Example 2. *Now, let's see if the inversion formula actually works for our example... First, we should have*

$$\sum_{k=0}^{4} F(w_k)e_0(k) = \sum_{k=0}^{4}\sum_{n=0}^{4} \frac{ne^{-2\pi ikn/5}}{10\sqrt{5}} \frac{1}{\sqrt{5}}$$

$$= \frac{1}{50}\sum_{n=0}^{4} n \sum_{k=0}^{4} e^{-2\pi ikn/5} = \frac{1}{50}\sum_{n=1}^{4} \frac{1-e^{-2\pi in}}{1-e^{-2\pi in/5}} = 0 = f(t_0).$$

Let's try another value:

$$\sum_{k=0}^{4} F(w_k)e_1(k) = \sum_{k=0}^{4}\sum_{m=0}^{4} \frac{me^{-2\pi ikm/5}}{10\sqrt{5}} \frac{e^{2\pi ik/5}}{\sqrt{5}}$$

$$= \frac{1}{50}\sum_{n=1}^{4} n \sum_{k=0}^{4} e^{-2\pi ik(n-1)/5}.$$

For $n = 2, 3, 4$, the sum over k gives

$$\frac{1-e^{-2\pi i(n-1)}}{1-e^{-2\pi i(n-1)/5}} = 0.$$

For $n = 1$, the sum over k gives 5. Thus, the only term that survives is the term with $n = 1$, for which we obtain

$$\frac{1}{50}(1)(5) = \frac{1}{10} = f(t_1).$$

So, it is indeed working as it should. This is rather tedious, however.

Now, we can see this as matrix multiplication. In the discrete Fourier transform, we sampled f at the finitely many points $t_0, \ldots, t_{N-1}$. We therefore identify f with a vector

$$\begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_{N-1}) \end{bmatrix}.$$

Similarly, the Fourier transform can be identified with the vector:

$$\begin{bmatrix} F(w_0) \\ F(w_1) \\ \cdots \\ F(w_{N-1}) \end{bmatrix}.$$

This vector is the product of the matrix

$$\begin{bmatrix}\bar{e}_0 & \bar{e}_1 & \dots \bar{e}_{N-1}\end{bmatrix}$$

whose columns are

$$\bar{e}_n = \frac{1}{\sqrt{N}}\begin{bmatrix} e^0 \\ e^{-2\pi i n/N} \\ e^{-2\pi i(2)n/N} \\ \dots e^{-2\pi i k n/N} \\ \dots \\ e^{-2\pi i n(N-1)/N}\end{bmatrix}$$

together with the vector

$$\begin{bmatrix} f(t_0) \\ f(t_1) \\ \dots \\ f(t_{N-1})\end{bmatrix}$$

That is

$$\begin{bmatrix} F(w_0) \\ F(w_1) \\ \dots \\ F(w_{N-1})\end{bmatrix} = \begin{bmatrix}\bar{e}_0 & \bar{e}_1 & \dots \bar{e}_{N-1}\end{bmatrix}\begin{bmatrix} f(t_0) \\ f(t_1) \\ \dots \\ f(t_{N-1})\end{bmatrix}$$

This entails a LOT of calculations. We can speed it up by being clever. Many calculations are repeated in fact. Assume that $N = 2^X$ for some giant power X. The idea is to split up into even and odd terms. We do this:

$$F(w_k) = \frac{1}{\sqrt{N}}\left[\sum_{j=0}^{\frac{N}{2}-1} f(t_{2j})e^{-2\pi i k(2j)/N} + \sum_{j=0}^{\frac{N}{2}-1} f(t_{2j+1})e^{-2\pi i k(2j+1)/N}\right].$$

We introduce the slightly cumbersome notation:

$$e_N^k(n) = e^{-2\pi i k n/N}.$$

Then,

$$e_N^k(2j) = e^{-2\pi i k(2j)/N} = e^{-2\pi i k j/(N/2)} = e_{N/2}^k(j).$$

Now we only need an $\frac{N}{2} \times \frac{N}{2}$ matrix! You see, writing this way,

$$F(w_k) = \frac{1}{\sqrt{N}}\left[\sum_{j=0}^{\frac{N}{2}-1} f(t_{2j})e_{N/2}^k(j) + e_N^k(1)\sum_{j=0}^{\frac{N}{2}-1} f(t_{2j+1})e_{N/2}^k(j)\right].$$

We can repeat this many times because N is a power of 2. We just keep chopping in half. If we do this as many times as possible, we will need to do on the order of

$$\frac{N}{2}\log_2(N)$$

computations. This is in comparison to the original method which had an $N \times N$ matrix and was thus on the order of N^2 computations. For example, if $N = 2^{10}$, then comparing $N^2 = 2^{20}$ to $\frac{N}{2}\log_2 N = 2^9 * 10$, we see that

$$\frac{2^{10} * 5}{2^{20}} = \frac{x}{100} \implies 100 * 2^{10} * 5 = 2^{20}x \implies 2^2 * 5^3 * 2^{10}2^{-20} = x,$$

so

$$5^3 2^{-8} = x \approx 0.488.$$

This means the amount of work we are doing by using the FFT is less than 0.5% of the work done using the standard DFT. In other words, we save over 99.5% by doing the FFT. That's why it's called FAST.

2.5. **Answers to the exercises to be done oneself.**

(1) (Eö 9) Compute (with help of Fourier transform)
$$\int_{\mathbb{R}} \frac{\sin(x)}{x(x^2+1)} dx.$$
(This is in the back of the EÖ document!)

(2) (Eö 67) Compute the Fourier transform of the characteristic function for the interval (a, b) both directly and by using the known case for the interval $(-a, a)$. (This is in the back of the EÖ document!)

(3) (7.2.8) Given $a > 0$ let $f(x) = e^{-x}x^{a-1}$ for $x > 0$, $f(x) = 0$ for $x \le 0$. Show that $\hat{f}(\xi) = \Gamma(a)(1 + i\xi)^{-a}$ where Γ is the Gamma function.

Well, there are not really answers to make sense of here. My hint was to do a substitution of variables:
$$\hat{f}(\xi) = \int_0^\infty e^{-ix\xi - x} x^{a-1} dx.$$
On the other hand
$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt.$$
So let's try making
$$x(1 + i\xi) = t \implies dx(1 + i\xi) = dt \implies \frac{dt}{1 + i\xi} = dx.$$
Our integral becomes
$$\hat{f}(\xi) = \int_0^{(1+i\xi)\infty} e^{-t} \left(\frac{t}{1+i\xi}\right)^{a-1} \frac{dt}{1+i\xi}$$
$$= (1 + i\xi)^{-a} \int_0^{(1+i\xi)\infty} e^{-t} t^{a-1} dt.$$

We integrate along the line from 0 to $(1 + i\xi)R = R + iR\xi$. For $\xi > 0$ that is the first diagonal bit. Next, integrate from $R + iR\xi$ to R. The integrate back along the real axis from R to zero. Our integrand is $e^{-z}z^{a-1}$. Inside the triangle it's holomorphic. So by complex analysis the integral around the triangle is zero. Since $|e^{-z}| = e^{-x}$ if $z = x + iy$ for $x, y \in \mathbb{R}$, along the right side of the triangle the integral is super small, tending to zero. That says the the integral along this funny diagonal line and the integral going from R to 0 are tending to be equal. More precisely $\lim_{R\to\infty} \int_0^{R(1+i\xi)} f(z)dz + \int_R^0 f(z)dz = 0$. Hence since flipping the integral changes its sign $\lim_{R\to\infty} \int_0^{R(1+i\xi)} f(z)dz = \int_0^\infty f(z)dz$. So
$$\hat{f}(\xi) = (1 + i\xi)^{-a} \int_0^{(1+i\xi)\infty} e^{-t}t^{a-1}dt = (1 + i\xi)^{-a} \int_0^\infty e^{-t}t^{a-1}dt.$$
This is $(1 + i\xi)^{-a}\Gamma(a)$.

(4) (7.2.12) For $a > 0$ let
$$f_a(x) = \frac{a}{\pi(x^2+a^2)}, \quad g_a(x) = \frac{\sin(ax)}{\pi x}.$$
Use the Fourier transform to show that: $f_a * f_b = f_{a+b}$ and $g_a * g_b = g_{\min(a,b)}$.
So we transform:
$$\widehat{f_a * f_b}(\xi) = \widehat{f_a}(\xi)\widehat{f_b}(\xi) = e^{-a|\xi|-b|\xi|} = e^{-(a+b)|\xi}.$$
Now we use the FIT to say:
$$f_a * f_b(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(a+b)|\xi|}e^{ix\xi}d\xi.$$
OBS! The integral on the right side this is the Fourier transform of $e^{-(a+b)|\xi|}$ at the point $-x$ rather than x. So we use our beloved Table 2 (item 11) to say that the Fourier transform of this function at the point $-x$ is
$$2(a+b)(x^2+(a+b)^2)^{-1},$$
so substituting
$$f_a * f_b(x) = \frac{1}{2\pi}2(a+b)(x^2+(a+b)^2)^{-1} = \frac{(a+b)}{\pi(x^2+(a+b)^2)} = f_{a+b}(x).$$
We do the same trick to solve the g exercise, yo.
$$\widehat{g_a * g_b}(\xi) = \widehat{g_a}(\xi)\widehat{g_b}(\xi) = \chi_a(\xi)\chi_b(\xi) = \chi_{\min(a,b)}(\xi).$$
The last step follows from the the definition of the characteristic function. So, we use the FIT again to say:
$$g_a * g_b(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\chi_{\min(a,b)}(\xi)d\xi.$$
Same trick: integral on the right is the Fourier transform of $\chi_{\min(a,b)}$ at the point $-x$ (rather than x). So we use our favorite Table 2 to say that
$$g_a * g_b(x) = \frac{1}{2\pi}x^{-1}2\sin(\min(a,b)x) = \frac{\sin(\min(a,b)x)}{\pi x} = g_{\min(a,b)}(x).$$

(5) (Eö 6.d,e) Compute the Fourier transform of:
$$e^{-a|t|}\sin(bt), \quad (a,b>0), \quad \frac{t}{t^2+2t+5}.$$
(This is in the back of the EÖ document!)

(6) (Eö 15) Find a solution to the equation
$$u(t) + \int_{-\infty}^{t} e^{\tau - t}u(\tau)d\tau = e^{-2|t|}.$$
(This is in the back of the EÖ document!)

(7) (Eö 11) For the function
$$f(t) = \int_0^2 \frac{\sqrt{w}}{1+w}e^{iwt}dw,$$
compute
$$\int_{\mathbb{R}} f(t)\cos(t)dt, \quad \int_{\mathbb{R}} |f(t)|^2dt.$$
(This is in the back of the EÖ document!)

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.24

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Dirichlet problem in a quadrant

Consider the problem

$$u_{xx} + u_{yy} = 0, \quad x, y > 0, \quad u(x, 0) = f(x), \quad u(0, y) = g(y).$$

To deal with these inhomogeneities let us instead solve two nicer problems:

(1) $w_{xx} + w_{yy} = 0, \quad x, y > 0, \quad w(x, 0) = f(x), \quad w(0, y) = 0.$
(2) $v_{xx} + v_{yy} = 0, \quad x, y > 0, \quad v(x, 0) = 0, \quad v(0, y) = g(y).$

The full solution will then be obtained by setting

$$u(x, y) = w(x, y) + v(x, y).$$

Exercise 1. *Verify that if w and v solve the problems above, then indeed u solves the original problem.*

We would like to use Fourier methods, but the problems we have above $x, y > 0$. The Fourier transform is defined on the whole plane. So, we may wish to use an even or odd extension.

> Idea: To solve a problem like $w_{xx}+w_{yy} = 0, \quad x, y > 0, \quad w(x, 0) = f(x), \quad w(0, y) = 0$, look at the boundary condition. The solution should vanish at $x = 0$. Now think about sine and cosine. Which of these vanishes at $x = 0$? The sine. That is an odd function. So this gives us the clue to extend oddly.

We define therefore

$$f_o(x) := \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}.$$

Now, we take the Fourier transform of the PDE in the x variable. We obtain:

$$-\xi^2 \hat{w}(\xi, y) + \partial_{yy}\hat{w}(\xi, y) = 0 \implies \hat{w}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y}.$$

The functions A and B can depend on ξ but not on y. We would like to use Fourier methods which requires staying within $\mathcal{L}^2$. Hence we do not want the second solution because $y > 0$ so it is very much not in $\mathcal{L}^2$. Thus, we keep the first solution. The boundary condition at $y = 0$ acts like an initial condition, at least from x's perspective:

$$\hat{w}(\xi, 0) = A(\xi) = \hat{f}_o(\xi) \implies \hat{w}(\xi, y) = \hat{f}_o(\xi)e^{-|\xi|y}.$$

We look at table 2 of Folland to find a function whose Fourier transform is $e^{-|\xi|y}$. OBS! The transform is occurring in the x variable, from whose perspective y is a constant. Thus, the item on the table is a slight modification of 10, in particular the function

$$\frac{y}{\pi}(x^2 + y^2)^{-1} \text{ has Fourier transform in the } x \text{ variable } e^{-y|\xi|}.$$

Thus, we have found

$$\hat{w}(\xi, y) = \hat{f}_o(\xi)\widehat{\frac{y}{\pi}(x^2 + y^2)^{-1}}(\xi).$$

The Fourier transform sends convolutions to products, which tells us that

$$w(x, y) = \int_{-\infty}^{\infty} f_o(z)\frac{y}{\pi((x - z)^2 + y^2)}dz = \int_{-\infty}^{0} f_o(z)\frac{y}{\pi((x - z)^2 + y^2)}dz + \int_0^{\infty} f(z)\frac{y}{\pi((x - z)^2 + y^2)}dz$$

We do a substitution in the first integral, with $t = -z$

$$= \int_{-\infty}^{0} f_o(z)\frac{y}{\pi((x - z)^2 + y^2)}dz = -\int_{\infty}^{0} f_o(-t)\frac{y}{\pi((x + t)^2 + y^2)}dt$$

$$= \int_{\infty}^{0} f(t)\frac{y}{\pi((x + t)^2 + y^2)}dt = -\int_0^{\infty} f(t)\frac{y}{\pi((x + t)^2 + y^2}dt.$$

Re-naming the variable of integration z, we get

$$w(x, y) = \int_0^{\infty} f(z)\left[\frac{y}{\pi((x - z)^2 + y^2)} - \frac{y}{\pi((x + z)^2 + y^2)}\right]dz.$$

The other problem is basically identical, we simply Fourier transform in the y variable. Thus the solution to the second problem is

$$v(x, y) = \int_0^{\infty} g(z)\left[\frac{x}{\pi((y - z)^2 + x^2)} - \frac{x}{\pi((y + z)^2 + x^2)}\right]dz.$$

We obtain the full solution by adding:

$$u(x, y) = w(x, y) + v(x, y).$$

2. The Laplace transform

We shall now enter Chapter 8, and learn about another useful transform, known as the Laplace transform.

Definition 1. *Assume that*

lap0 (2.1) $$f(t) = 0 \quad \forall t < 0,$$

and that there exists $a, C > 0$ *such that*

lapa (2.2) $$|f(t)| \leq Ce^{at} \quad \forall t \geq 0.$$

Then for we define for $z \in \mathbb{C}$ *with* $\Re(z) > a$ *the Laplace transform of* f *at the point* z *to be*

$$\mathfrak{L}f(z) = \hat{f}(-iz) = \int_0^{\infty} f(t)e^{-zt}dt.$$

We may also use the notation

$$\widetilde{f}(z) = \mathfrak{L}f(z).$$

Let us verify that the Laplace transform is well defined. To do so, we estimate

$$|\mathfrak{L}f(z)| \le \int_0^\infty |f(t)e^{-zt}|dt \le \int_0^\infty Ce^{at}|e^{-zt}|dt = \int_0^\infty e^{at}e^{-\Re(z)t}dt$$
$$= \left.\frac{e^{t(a-\Re(z))}}{a-\Re(z)}\right|_0^\infty = \frac{1}{\Re(z)-a}.$$

Above we have used the fact that

$$\left|e^{\text{complex number}}\right| = e^{\text{real part}}.$$

Due to this beautiful convergence, $\mathfrak{L}f(z)$ is holomorphic in the half plane $\Re(z) > a$. This is because we may differentiate under the integral sign due to the absolute convergence of the integral. The assumption that $f(t) = 0$ for all negative t is not actually necessary, we could just make it so. For this purpose we define the *heavyside* function, commonly denoted by

$$\Theta(t) := \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}.$$

If we have some f defined on $\mathbb{R}$ which satisfies (2.2) but is not (2.1), we can apply the Laplace transform to Θf. Another thing which can happen is that we have a function which is only defined on $[0, \infty)$. In that case, we can just extend it to be identically zero on $(-\infty, 0)$.

Let's familiarize ourselves with the Laplace transform by demonstrating some of its fundamental properties.

Proposition 2 (Properties of $\mathfrak{L}$)**.** *Assume f and g satisfy (2.2) and (2.1), then*

(1) $\mathfrak{L}f(x+iy) \to 0$ as $|y| \to \infty$ for all $x > a$.
(2) $\mathfrak{L}f(x+iy) \to 0$ as $x \to \infty$ for all y.
(3) $\mathfrak{L}(\Theta(t-a)f(t-a))(z) = e^{-az}\mathfrak{L}f(z)$.
(4) $\mathfrak{L}(e^{ct}f(t))(z) = \mathfrak{L}f(z-c)$.
(5) $\mathfrak{L}(f(at)) = a^{-1}\mathfrak{L}f(a^{-1}z)$.
*(6) *** If f is continuous and piecewise $\mathcal{C}^1$ on $[0,\infty)$, and f' satisfies (2.2) and (2.1), then*
$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$
(7) $\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}f(z)$.
(8) $\mathfrak{L}(tf(t))(z) = -(\mathfrak{L}f)'(z)$.
(9) $\mathfrak{L}(f * g)(z) = \mathfrak{L}f(z)\mathfrak{L}g(z)$.
(10) If $t^{-1}f(t)$ satisfies (2.1) and (2.2), then
$$\mathfrak{L}(t^{-1}f(t))(z) = \int_z^\infty \mathfrak{L}f(w)dw.$$
The integral is any contour in the w-plane which starts at z along which $\Im w$ stays bounded and $\Re w \to \infty$.

Proof: There's a bunch of stars next to #6 because it's the reason the Laplace transform is useful for solving PDEs and ODEs. It's quite similar to how the Fourier transform takes in derivatives and spits out multiplication. Intuitively, this fact about $\mathfrak{L}$ should jive with the similar fact about $\mathcal{F}$ because well, the Laplace transform is just the Fourier transform taken at a complex point.

(1) The first statement

$$\mathfrak{L}f(z) = \int_0^\infty e^{-(x+iy)t} f(t)dt = \int_0^\infty e^{-xt} f(t) e^{-iyt} dt = \hat{g}(y),$$

for the function

$$g(t) = e^{-xt} f(t).$$

The Riemann-Lebesgue Lemma says that $\hat{g}(y) \to 0$ when $|y| \to \infty$.

(2) The second statement is more satisfying because we just compute and estimate directly. We did this estimate above already, where we got

$$|\mathfrak{L}f(z)| \leq \frac{1}{\Re(z) - a} \to \infty \text{ when } \Re(z) = x \to \infty.$$

(3) The third statement is also a direct computation:

$$\mathfrak{L}(\Theta(t-a)f(t-a))(z) = \int_0^\infty \Theta(t-a) f(t-a) e^{-zt} dt = \int_{-a}^\infty \Theta(s) f(s) e^{-z(s+a)} ds.$$

Above we did the substitution $s = t - a$ so $ds = dt$. Since f and the Heavyside function are zero for negative s, and the Heavyside function is 1 for positive s, this is

$$e^{-za} \int_0^\infty f(s) e^{-zs} ds = e^{-za} \mathfrak{L}f(z).$$

(4) Similarly, we directly compute

$$\mathfrak{L}(e^{ct} f)(z) = \int_0^\infty e^{ct} e^{-zt} f(t) dt = \int_0^\infty e^{-(z-c)t} f(t) dt = \mathfrak{L}f(z-c).$$

(5) Again no surprise, we compute

$$\mathfrak{L}(f(at))(z) = \int_0^\infty e^{-zt} f(at) dt = \int_0^\infty e^{-zs/a} f(s) \frac{ds}{a} = a^{-1} \mathfrak{L}f(z/a).$$

Here we used the substitution $s = at$ so $a^{-1} ds = dt$.

(6) Now we are finally getting to the important one:

$$\mathfrak{L}(f')(z) = \int_0^\infty e^{-zt} f'(t) dt = e^{-zt} f(t)\Big|_0^\infty + \int_0^\infty z e^{-zt} f(t) dt.$$

We have used integration by parts above. By (lapa 2.2) and since $\Re(z) > a$, the limit as $t \to \infty$ is zero, and so we get

$$\mathfrak{L}(f')(z) = -f(0) + z\mathfrak{L}f(z).$$

Awesome.

(7) Next we define

$$F(t) = \int_0^t f(s) ds.$$

Then, we use the preceding fact:

$$\mathfrak{L}(F')(z) = z\mathfrak{L}F(z) - F(0) = z\mathfrak{L}F(z).$$

Since $F' = f$ we get

$$z^{-1}\mathfrak{L}(f)(z) = \mathfrak{L}(\int_0^t f(s) ds)(z).$$

(8) Next, we compute:

$$\mathfrak{L}(tf(t))(z) = \int_0^\infty te^{-zt}f(t)dt = \int_0^\infty \frac{d}{dz}\left(-e^{-zt}\right)f(t)dt$$
$$= \frac{d}{dz}\left(-\int_0^\infty e^{-zt}f(t)dt\right) = -(\mathfrak{L}f)'(z).$$

Yes, we have used the absolute convergence of the integral to swap limits. It's legit yo.

(9) Nearing the finish line, we compute

$$\mathfrak{L}(f*g)(z) = \mathcal{F}(f*g)(-iz) = \hat{f}(-iz)\hat{g}(-iz) = \mathfrak{L}f(z)\mathfrak{L}g(z).$$

(10) Finally, note that by (2.2), if $t^{-1}f(t)$ satisfies this, then at the point $t=0$ apparently the function f vanishes, so that the function $t^{-1}f(t)$ is well defined. So, don't panic about this point!!! We next define the holomorphic function

$$F(z) = \int_z^\infty \widetilde{f}(w)dw.$$

Since $\widetilde{f}(w) \to 0$ when $\Re(w) \to \infty$ and $\Im(w)$ stays bounded, the fundamental theorem of calculus says that

$$F'(z) = -\widetilde{f}(z).$$

On the other hand,

$$\frac{d}{dz}\int_0^\infty t^{-1}f(t)e^{-zt}dt = \int_0^\infty -f(t)e^{-zt}dt = -\widetilde{f}(z).$$

Hence,

$$F(z) = \int_0^\infty t^{-1}f(t)e^{-zt}dt + c,$$

for some constant c. Since

$$\lim_{\Re z\to\infty} F(z) = 0 = \lim_{\Re(z)\to\infty}\int_0^\infty t^{-1}f(t)e^{-zt}dt \implies c=0.$$

2.1. **Application to solving ODEs.** We see that

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Let's do it again:

$$\mathfrak{L}(f'')(z) = z\mathfrak{L}(f')(z) - f'(0) = z\left(z\mathfrak{L}f(z) - f(0)\right) - f'(0) = z^2\mathfrak{L}f(z) - zf(0) - f'(0).$$

In general:

Proposition 3. *Assume that everything is defined, then*

$$\mathfrak{L}(f^{(k)})(z) = z^k\mathfrak{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0)z^{j-1}.$$

Proof: Well clearly we should do a proof by induction! Check the base case first:

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Here $k = 1$ and the sum has only one term with $j = k = 1$. It works. Now we assume the above formula holds and we show it for $k+1$. We compute

$$\mathfrak{L}(f^{(k+1)})(z) = \mathfrak{L}((f^{(k)})')(z) = z\mathfrak{L}(f^{(k)})(z) - f^{(k)}(0).$$

By induction this is

$$z\left(z^k\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}\right) - f^{(k)}(0).$$

This is

$$z^{k+1}\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0).$$

Let us change our sum: let $j+1 = l$. Then our sum is

$$\sum_{l=2}^{k+1} f^{k-(l-1)}(0)z^{l-1} = \sum_{l=2}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

Observe that

$$f^{(k)}(0) = f^{k+1-1}(0)z^{1-1}.$$

Hence

$$-\sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0) = -\sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

So, we have computed

$$\mathfrak{L}(f^{(k+1)})(z) = z^{k+1}\mathfrak{L}f(z) - \sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

That is the formula for $k+1$, which is what we needed to obtain.

For this reason one can use $\mathfrak{L}$ to solve linear constant coefficient ODEs which can be *non-homogeneous!* Let us see how this works... A linear, constant coefficient ODE of order n looks like:

$$\sum_{k=0}^{n} c_k u^{(k)}(t) = f(t).$$

In order for the solution to be unique, there must be specified initial conditions on u, that is

$$u(0), u'(0), \ldots u^{(n-1)}(0).$$

We are not requiring $f(t)$ to be the zero function, so the ODE could be *inhomogeneous.* Notoriously difficult to solve right? NOT ANYMORE! We hit both sides of the ODE with $\mathfrak{L}$:

$$\sum_{k=0}^{n} c_k\mathfrak{L}(u^{(k)})(z) = \widetilde{f}(z).$$

Let's write out the left side using our proposition. First we have

$$c_0\widetilde{u}(z).$$

Then we have

$$c_1\left(z\widetilde{u}(z) - u(0)\right).$$

By our proposition, we computed that for $k \geq 1$,

$$\mathfrak{L}(c_k u^{(k)})(z) = c_k\left(z^k\widetilde{u}(z) - \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}\right).$$

Therefore the left side of the ODE becomes

$$c_0\widetilde{u}(z) + \sum_{k=1}^{n} c_k\left(z^k\widetilde{u}(z) - \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}\right)$$

$$= \sum_{k=0}^{n} c_k z^k\widetilde{u}(z) - \sum_{k=1}^{n} c_k \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}.$$

We therefore define two polynomials

$$P(z) := \sum_{k=0}^{n} c_k z^k,$$

$$Q(z) := -\sum_{k=1}^{n} c_k \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}.$$

Our ODE has been LAPLACE-TRANSFORMED into

$$P(z)\widetilde{u}(z) + Q(z) = \widetilde{f}(z).$$

We can solve this for $\widetilde{u}(z)$:

$$\widetilde{u}(z) = \frac{\widetilde{f}(z) - Q(z)}{P(z)}.$$

Hence to get our solution $u(t)$ we just need to invert the Laplace transform of the right side, that is our solution will be

$$u(t) = \mathfrak{L}^{-1}\left(\frac{\widetilde{f}(z) - Q(z)}{P(z)}\right).$$

2.2. **Exercises for the week.** These exercises will be demonstrated for you.

(1) (Eö 55)
(2) (7.4.4) Solve the heat equation $u_t = ku_{xx}$ on the half line $x > 0$ with boundary conditions $u(x,0) = f(x)$ and initial condition $u(0,t) = 0$. Do the same for the inhomogeneous heat equation $u_t = ku_{xx} + G(x,t)$ with the same initial and boundary conditions.
(3) (7.4.6) Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $x > 0$, $0 < y < 1$ with the boundary conditions $u_x(0,y) = 0$, $u_y(x,0) = 0$, $u(x,1) = e^{-x}$. Express the answer as a Fourier integral.
(4) (8.4.2) Find the temperature in a semi-infinite rod (the half-line $x > 0$) if its initial temperature is zero, and the end $x = 0$ is held at temperature 1 for $0 < t < 1$ and temperature 0 thereafter.
(5) (Eö 14)
(6) (Eö 45)

2.2.1. *Exercises for the week to be done oneself.*

(1) (7.3.1) Use the Fourier transform to find a solution of the ordinary differential equation $u'' - u + 2g(x) = 0$ where $g \in \mathcal{L}^1(\mathbb{R})$. The solution obtained in this way is the one that vanishes at $\pm\infty$.
(2) (7.4.7) Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $x > 0$, $0 < y < 1$ with the boundary conditions $u(0, y) = 0$, $u(x, 0) = 0$, $u(x, 1) = e^{-x}$. Express the answer as a Fourier integral.
(3) (Eö 47)
(4) (8.4.1) Solve:
$$u_t = ku_{xx} - au, \quad x > 0, \quad u(x,0) = 0, \quad u(0,t) = f(t).$$
(5) (8.4.3) Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c:
$$u_t = ku_{xx} \text{ for } x > 0, \quad u(x,0) = 0, \quad u_x(0,t) = -c.$$
With the aid of the computation:
$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}} e^{-a^2/(4t)}\right)(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}},$$
show that
$$u(x,t) = c\sqrt{\frac{k}{\pi}} \int_0^t s^{-1/2} e^{-x^2/(4ks)} ds.$$
By substituting
$$\sigma = \frac{x}{\sqrt{4ks}}$$
and then integrating by parts, show that
$$u(x,t) = c\sqrt{\frac{4kt}{\pi}} e^{-x^2/(4kt)} - cx\,\mathrm{erfc}\left(\frac{x}{\sqrt{4kt}}\right).$$
(6) (Eö 12)

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.26

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Application of the Laplace transform to solving PDEs

Let us consider the *telegraph equation,*

$$u_{xx} = \alpha u_{tt} + \beta u_t + \gamma u.$$

This is homogeneous, and generalizes both the heat equation ($\alpha = \gamma = 0$, and $\beta = 1$) as well as the wave equation ($\beta = \gamma = 0$, and $\alpha = 1$). According to those who know more physics than I, this corresponds to an electromagnetic signal on a cable.

We wish to solve the problem on a half line with the following boundary and initial conditions:

$$u(0,t) = f(t), \quad u(x,0) = u_t(x,0) = 0.$$

Tip 1. *If we have a half-line problem with boundary condition at $x = 0$ that is* a function of t *try using the Laplace transform in the t variable.*

We follow the tip and hit the whole PDE with the Laplace transform in the t variable. This gives

$$\widetilde{u}_{xx}(x,z) = \alpha \mathfrak{L}(u_{tt})(x,z) + \beta \mathfrak{L}(u_t)(x,z) + \gamma \widetilde{u}(x,z).$$

We use the properties of the Laplace transform and the initial conditions which say

$$u(x,0) = 0, \quad u_t(x,0) = 0,$$

so

$$\widetilde{u}_{xx}(x,z) = \alpha z^2 \widetilde{u}(x,z) + \beta z \widetilde{u}(x,z) + \gamma \widetilde{u}(x,z).$$

This is simply

$$\widetilde{u}_{xx}(x,z) = \left(\alpha z^2 + \beta z + \gamma\right) \widetilde{u}(x,z).$$

It's a second order, linear, constant coefficient, homogeneous ODE for the x variable. Let

$$q = \sqrt{\alpha z^2 + \beta z + \gamma}.$$

Our solution to the ODE is of the form

$$\widetilde{u}(x,z) = a(z) e^{qx} + b(z) e^{-qx}.$$

We have that lovely BC at $x = 0$: $u(0,t) = f(t)$. Hence,

$$\widetilde{u}(0,z) = \widetilde{f}(z) \implies a(z) + b(z) = \widetilde{f}(z).$$

Note that here we are extending f to $(-\infty, 0)$ to be identically equal to zero so that we may Laplace transform it. Assume that $\Re(q) > 0$. (If this weren't the case, just swap q and $-q$). To be able to invert the Laplace transform and get the solution to our PDE, we will not want $\widetilde{u}(x,z) \to \infty$ when $x \to \infty$. Hence, we throw out the e^{qx} solution and just use

$$\widetilde{u}(x,z) = b(z)e^{-qx}.$$

Therefore, $b(z) = \widetilde{f}(z)$. So, our Laplace-transformed solution is

$$\widetilde{u}(x,z) = \widetilde{f}(z)e^{-qx}.$$

By the properties of the Laplace transform, if we can find $g(x,t)$ such that

$$\widetilde{g}(x,z) = e^{-qx},$$

then the solution to this PDE will be

soln

$$u(x,t) = f * g(x,t) = \int_{\mathbb{R}} f(t-s)\Theta(t-s)g(x,s)\Theta(s)ds = \int_0^t f(t-s)g(x,s)ds. \tag{1.1}$$

The reason for those heavyside functions is that $f(*) = 0$ for $* < 0$ and $g(x,*) = 0$ for $* < 0$. To guarantee that this holds, we multiply $f(t-s)$ by $\Theta(t-s)$ and multiply $g(x,s)$ by $\Theta(s)$.

Now, recalling the definition of q, we are looking for

$$g(x,t) \text{ with } \widetilde{g(x,z)} = e^{-x\sqrt{\alpha z^2 + \beta z + \gamma}}.$$

To find such a g, we would like to invert the Laplace transform.

1.1. **Inverting the Laplace transform.** The Laplace transform is closely related to the Fourier transform, and it is this fact, together with the FIT, that will guide our way to the LIT (Laplace Inverse Theorem).

$$\widetilde{f}(z) = \int_0^\infty f(t)e^{-zt}dt = \int_0^\infty f(t)e^{-\Re(z)t - i\Im(z)t}dt.$$

For this reason, let's define

$$g(t) = e^{-\Re(z)t}f(t),$$

so we also have

$$f(t) = e^{\Re(z)t}g(t).$$

Then

$$\mathfrak{L}f(z) = \hat{g}(\Im(z)) = \int_{\mathbb{R}} f(t)e^{-\Re(z)}e^{-i\Im(z)t}dt,$$

because $f(t) = 0$ for all $t < 0$. Let's apply the FIT to the function, g:

$$g(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)e^{i\xi t}d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \mathfrak{L}f(\Re(z) + i\xi)e^{i\xi t}d\xi.$$

To make this look less imposing, let us write $y = \xi$. So, we have

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\Re(z) + iy)e^{iyt}dy.$$

Since $f(t) = e^{\Re(z)t}g(t)$, we have

$$f(t) = e^{\Re(z)t}\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\Re(z) + iy)e^{iyt}dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\Re(z) + iy)e^{\Re(z)t + iyt}dy.$$

We would like to understand this as a complex integral. If we parametrize the vertical path for $w \in \mathbb{C}$ with $\Re(w) = \Re(z)$ by $w = \Re(z) + iy$, then $dw = idy$. We do not have an i. Hence, what we are computing is

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma_z} \widetilde{f}(w) e^{wt} dw,$$

where Γ_z is the upward vertical path along the line $\Re(w) = \Re(z)$. This is the LIT: Laplace inversion formula:

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma_z} \widetilde{f}(w) e^{wt} dw.$$

By definition of the Laplace transform, this should hold for $z \in \mathbb{C}$ with $\Re(z) > a$ where a comes from the growth estimate on f, that is $|f(t)| \leq Ce^{at}$ for all $t \geq 0$ for constants a and C. If we naively look at this equation, we see that the left side is *independent of* z. So, the right side ought to be as well. It is.

Theorem 1 (LIT)**.** *Assume that f is Laplace-transformable. Denote by $\widetilde{f}$ its Laplace transform. Then for $b > a$,*

$$f(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \widetilde{f}(z) e^{zt} dz.$$

Conversely, assume that $F(z)$ is analytic in $\Re(z) > a$. For $b > a$, $R > 0$, and $t \in \mathbb{R}$, let

$$f_{R,b}(t) = \frac{1}{2\pi i} \int_{b-iR}^{b+iR} F(z) e^{zt} dz.$$

Assume that for some $\alpha > 1/2$ and $C > 0$ we have

$$|F(z)| \leq C(1 + |z|)^{-\alpha}, \quad \forall z \in \mathbb{C} \text{ with } \Re(z) > a,$$

and assume that for some $b > a$, $f_{R,b}(t)$ converges pointwise as $R \to \infty$ to $f(t)$ for a Laplace transformable f. Then

$$\lim_{R \to \infty} f_{R,b}(t) = f(t) \quad \forall b > a,$$

and

$$F(z) = \mathfrak{L} f(z).$$

Proof: Let us draw and define a contour, with our amazing tikz skillz yo.

By assumption the function F is analytic inside the box, and e^{zt} is an entire function. Therefore

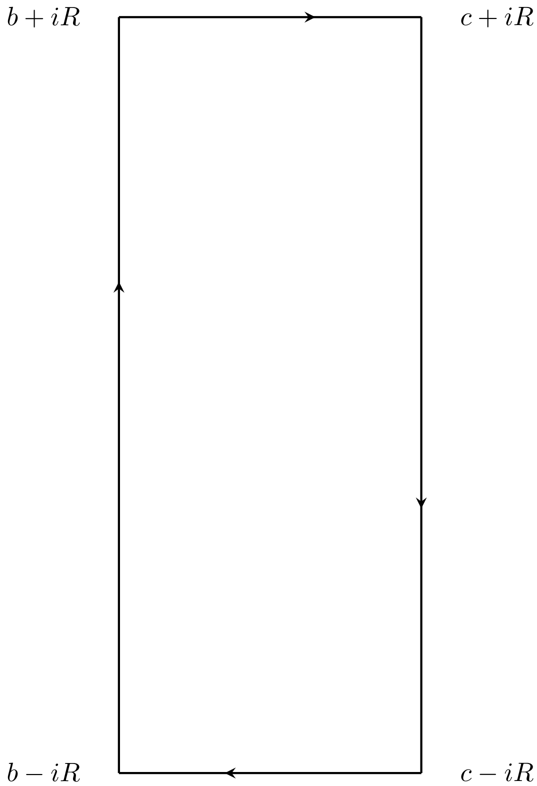
$$\int_{\Gamma_R} F(z) e^{zt} dz = 0.$$

So, we wish to show that the limit as $R \to \infty$ of the top and bottom integrals is zero. To obtain this, we either wave our hands like Folland or actually estimate:

$$\int_{b \pm iR}^{c \pm iR} |F(z)||e^{zt}| dz \leq |c - b| e^{ct} \max_{b \leq x \leq c} \frac{C}{(1 + |x \pm iR|)^{\alpha}}.$$

Above we used the fact that between $b \pm iR$ and $c \pm iR$, $|e^{zt}| \leq e^{ct}$ together with the estimate assumed on F. Next, we note that

$$|x \pm iR| = \sqrt{x^2 + R^2} \geq R.$$



box

FIGURE 1. The contour over which we integral. Call the contour Γ_R. As one can see, we assume that $c > b$.

Therefore we estimate from above by

$$|c-b|e^{ct}\frac{C}{(1+R)^{\alpha}} \to 0 \text{ as } R \to \infty.$$

Therefore, if for some $b > a$,

$$\lim_{R\to\infty} f_{R,b}(t) = f(t),$$

this means that

$$\lim_{R\to\infty} \int_{b-iR}^{b+iR} F(z)e^{zt}dz - \int_{c-iR}^{c+iR} F(z)e^{zt}dz = 0.$$

To see this, observe that

$$\int_{\Gamma_R} F(z)e^{zt}dz = 0 \quad \forall R.$$

Moreover, the top and bottom integrals go to zero as $R \to \infty$. Hence the sum of the left and right integrals also tends to zero as $R \to \infty$. So,

$$\lim_{R\to\infty} \int_{b-iR}^{b+iR} F(z)e^{zt}dz = \lim_{R\to\infty} \int_{c-iR}^{c+iR} F(z)e^{zt}dz \implies \lim_{R\to\infty} f_{R,b}(t) = f(t) = \lim_{R\to\infty} f_{R,c}(t).$$

Now, let us parametrize the complex integral. We use $\gamma(s) = b + is$ so $\dot\gamma(s) = ids$. Hence

$$\int_{b-iR}^{b+iR} F(z)e^{zt}dz = \int_{-R}^{R} F(b+is)e^{(b+is)t}ids = ie^{bt}\int_{-R}^{R} F(b+is)e^{ist}ds.$$

Moreover, we have assumed that

$$\lim_{R\to\infty} f_{R,b}(t) = \lim_{R\to\infty} \frac{ie^{bt}}{2\pi i}\int_{-R}^{R} F(b+is)e^{ist}ds = f(t)$$

so

$$\lim_{R\to\infty}\int_{-R}^{R} F(b+is)e^{ist}ds = 2\pi e^{-bt}f(t).$$

Let us define here

$$g_{R,b}(s) = \begin{cases} F(b+is) & |s| \le R \\ 0 & |s| > R \end{cases}.$$

Then

$$\int_{-R}^{R} F(b+is)e^{ist}ds = \int_{\mathbb{R}} g_{R,b}(s)e^{ist}ds = \widehat{g_{R,b}}(-t).$$

Moreover,

$$\lim_{R\to\infty} \widehat{g_{R,b}(-t)} = 2\pi e^{-bt}f(t).$$

Similarly

$$\lim_{R\to\infty} \widehat{g_{R,b}(t)} = 2\pi e^{bt}f(-t).$$

On the other hand,

$$\lim_{R\to\infty} g_{R,b}(s) = F(b+is).$$

By the FIT,

$$F(b+it) = \frac{1}{2\pi}\int_{\mathbb{R}} 2\pi e^{bs}f(-s)e^{its}ds.$$

It is more natural to do a change of variables, letting $\sigma = -s$, so $d\sigma = -ds$, and we get

$$F(b+it) = \int_{\sigma=\infty}^{\sigma=-\infty} e^{-b\sigma}f(\sigma)e^{-it\sigma}(-d\sigma) = \int_{-\infty}^{\infty} e^{-\sigma(b+it)}f(\sigma)d\sigma$$

$$= \int_{0}^{\infty} e^{-\sigma(b+it)}f(\sigma)d\sigma = \mathfrak{L}f(b+it).$$

Here we use the fact that f satisfies the growth estimate needed to be Laplace transformable.

1.2. **Computing an inverse Laplace transform to solve the heat equation.** For the case in which our telegraph equation is the heat equation, we have $\alpha = \gamma = 0$, and $\beta = 1$. Consequently, the square rooted polynomial in z we had named q is of the simple form:

$$q = \sqrt{z}.$$

Our Laplace-transformed solution is:

$$\widetilde{f}(z) e^{-\sqrt{z}x}.$$

Since the Laplace transform turns convolutions into multiplication, we would like to find $g(x,t)$ so that

$$\widetilde{g}(x,z) = e^{-\sqrt{z}x}.$$

Then, the solution will be given as in (1.1).

We are therefore looking for $g(x,t)$ so that

$$\widetilde{g}(x,z) = e^{-\sqrt{z}x}.$$

If we try to apply the LIT directly, we should compute

$$\int_{b-i\infty}^{b+i\infty} e^{-x\sqrt{z}} e^{zt} dz.$$

Do you know how to integrate that? I do not. It is pretty scary looking. For starters, there is the $\sqrt{z}$. This really needs to be understood using the complex logarithm which is, as the name suggests, complex.

Tip 2. *Always be careful with* $\log(z)$ *in* $\mathbb{C}$. *It is* not *entire. It is a log. Logs come from trees which have branches. Complex logs always have branches and branch cuts. You have been warned.*

So, since trying to compute the inverse Laplace transform directly seems impossible, let us try to make a reasonable guess at a function whose Laplace transform might be what we need to solve the heat equation. To solve the heat equation on $\mathbb{R}$ we used

$$e^{-x^2/(4t)} (4\pi t)^{-1/2}.$$

So, since the Laplace and Fourier transforms are closely related, and we are solving the heat equation on $[0,\infty)$, which is an unbounded interval, this is a good candidate. We shall compute its Laplace transform and see what we get. If we are super lucky, it will just give us the function we want. If we are less lucky, but still pretty lucky, the process of computing the Laplace transform together with the properties of the Laplace transform will show us how to get $g(x,t)$ whose Laplace transform is $\widetilde{g}(x,z) = e^{-\sqrt{z}x}$.

Let us therefore define:

$$\star = \int_0^\infty e^{-tz} e^{-x^2/(4t)} (4\pi t)^{-1/2} dt.$$

We are computing the Laplace transform of $\Theta(t) h(x,t)$ where

$$h(x,t) = e^{-x^2/(4t)} (4\pi t)^{-1/2}.$$

Now, we see that

$$\star = \int_0^\infty (4\pi t)^{-1/2} \exp\left(-(\sqrt{tz})^2 - \left(\frac{x}{2\sqrt{t}} \right)^2 \right) dt.$$

We do the completing the square trick in the exponent:

$$\star = \int_0^\infty (4\pi t)^{-1/2} \exp\left(-\left(\sqrt{tz} - \frac{x}{2\sqrt{t}}\right)^2 - x\sqrt{z}\right) dt$$
$$= e^{-x\sqrt{z}} \int_0^\infty \frac{1}{2\sqrt{\pi t}} \exp\left(-\left(\sqrt{tz} - \frac{x}{2\sqrt{t}}\right)^2 \right).$$

To compute this we need to use a very very clever trick. First, let us clean up our integral just a little bit to remove that pesky $\sqrt{t}$ which is getting divided. We let $s = \sqrt{t}$. Then

$$ds = \frac{dt}{2\sqrt{t}}$$

So,

$$\star = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi}} \int_0^\infty e^{-(s\sqrt{z} - x/(2s))^2} ds.$$

Theorem 2 (Cauchy & Schlömilch transform)**.**

$$\int_0^\infty af((as - b/s)^2) ds = \int_0^\infty f(y^2) dy.$$

Proof: The proof is so clever.[1]

We do a substitution in the integral. Let $t = \frac{b}{as}$. Then

$$dt = -\frac{b}{as^2} ds \implies -\frac{as^2}{b} dt = ds.$$

We see that

$$t^2 = \frac{b^2}{a^2 s^2} \implies \frac{a^2 s^2}{b^2} = t^{-2} \implies \frac{as^2}{b} = \frac{b}{at^2}.$$

Next, when $s \to 0$ and $s > 0$ we see that $t \to \infty$. On the other hand, when $s \to \infty$, $t \to 0$. We also see that

$$as = \frac{t}{b}, \quad -\frac{b}{s} = -ta.$$

So, let us call

$$\heartsuit = \int_0^\infty af((as - b/s)^2) ds = \int_\infty^0 af((t/b - ta)^2) \left(-\frac{b}{at^2}\right) dt$$
$$= \int_0^\infty f((t/b - at)^2) \frac{b}{t^2} dt.$$

Note that

$$(t/b - at)^2 = (-(at - t/b))^2 = (at - t/b)^2.$$

Hence we have computed

$$\heartsuit = \int_0^\infty f((at - t/b)^2) \frac{b}{t^2} dt.$$

[1] I don't know if Cauchy and Schlömilch actually had anything to do with this formula. Oscar Schlömilch was elected a foreign member of the Royal Swedish Academy of Sciences in 1862. He was a German mathematician who lived from April 13, 1823 until February 7, 1901. Augustin-Louis Cauchy was a French mathematician who lived August 21, 1789 until May 23, 1857. Did they ever meet? Why is this named after them?

Therefore

$$2\heartsuit = \int_0^\infty af((as-b/s)^2)ds + \int_0^\infty f((at-t/b)^2)\frac{b}{t^2}dt$$
$$= a\int_0^\infty f((as-b/s)^2)ds + b\int_0^\infty f((as-b/s)^2)\frac{ds}{s^2}.$$

As a consequence,

$$\heartsuit = \frac{1}{2}\int_0^\infty f((as-b/s)^2)\left(a+\frac{b}{s^2}\right)ds.$$

Now we let

$$y = as - \frac{b}{s} \implies dy = \left(a + \frac{b}{s^2}\right)ds.$$

We note that when $s \to 0$, $y \to -\infty$, and on the flip side, when $s \to \infty$, $y \to \infty$. Therefore

$$\heartsuit = \frac{1}{2}\int_{-\infty}^\infty f(y^2)dy = \int_0^\infty f(y^2)dy,$$

since $f(y^2)$ is an even function.

We will use the Cauchy & Schlömilch transform with

$$a = \sqrt{z}, \quad b = \frac{x}{2}, \quad f(s) = e^{-s^2}.$$

Then, it says that

$$\int_0^\infty \sqrt{z}\exp(-(as-b/s)^2)ds = \int_0^\infty \sqrt{z}\exp\left(-\left(s\sqrt{z}-\frac{x}{2s}\right)^2\right)ds$$
$$= \int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}.$$

Now we were computing

$$\star = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi}}\int_0^\infty e^{-(s\sqrt{z}-x/(2s))^2}ds = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi z}}\int_0^\infty \sqrt{z}e^{-(s\sqrt{z}-x/(2s))^2}ds$$
$$= \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

So, we have computed

$$\mathfrak{L}\left(\Theta(t)h(x,t)\right)(z) = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

This is almost what we wanted, except for the $2\sqrt{z}$ in the denominator. Here we use the properties of the Laplace transform. Consider the function:

$$\int_z^\infty \frac{e^{-x\sqrt{w}}}{2\sqrt{w}}dw = -\frac{e^{-x\sqrt{w}}}{x}\Bigg|_{w=z}^\infty = \frac{e^{-x\sqrt{z}}}{x}.$$

By the properties of the Laplace transform

$$\mathfrak{L}(t^{-1}f(t))(z) = \int_z^\infty \widetilde{f}(w)dw.$$

So,

$$\mathfrak{L}(t^{-1}\Theta(t)h(x,t))(z) = \int_z^\infty \frac{e^{-x\sqrt{w}}}{2\sqrt{w}}dw = \frac{e^{-x\sqrt{z}}}{x}.$$

because we computed

$$\mathfrak{L}\left(\Theta(t)h(x,t)\right)(z) = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}.$$

We can simply multiply both sides by x to get

$$\mathfrak{L}(t^{-1}x\Theta(t)h(x,t))(z) = e^{-x\sqrt{z}}$$

as desired. Let us summarize this phenomenal calculation as a theorem for future reference.

Theorem 3. *The Laplace transform of*

$$g(x,s) := \frac{x}{s}\Theta(s)h(x,s), \quad h(x,s) = \frac{1}{\sqrt{4\pi s}}e^{-\frac{x^2}{4s}}, \quad \Theta(s) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

in the variable s is

$$\mathfrak{L}(g)(x,z) = e^{-x\sqrt{z}}.$$

Therefore going back to our problem, the solution

$$u(x,t) = (f(s) * (s^{-1}x\Theta(s)h(x,s))(t) = \int_{\mathbb{R}} f(t-s)g(x,s)ds$$
$$= \int_0^t \frac{f(t-s)}{2\sqrt{\pi}s^{3/2}}xe^{-\frac{x^2}{4s}}ds.$$

This is because f is zero for negative times.

Remark 1. *One of the things I love about this class is that you begin to approach actual research mathematics. I think that must be exciting for you, because calculus (envariabelanalys) is like 300 years old. Cauchy's complex analysis is also a few hundred years old. That's not very close to actual current year 2019 math! Here is an example of how the Cauchy-Schlömilch transform is super awesome (and look, this paper is only 9 years old which is super young by research terms):*
https://arxiv.org/abs/1004.2445

1.2.1. *Hints to: exercises for the week to be done oneself.*

(1) (7.3.1) Use the Fourier transform to find a solution of the ordinary differential equation

$$u'' - u + 2g(x) = 0, \quad g \in \mathcal{L}^1(\mathbb{R}).$$

Hint: Hit the whole equation with the Fourier transform in the x variable. So you are getting

$$-\xi^2\hat{u}(\xi) - \hat{u}(\xi) = -2\hat{g}(\xi).$$

Solving for $\hat{u}(\xi)$ we get

$$\hat{u}(\xi) = 2\frac{\hat{g}(\xi)}{1+\xi^2}.$$

From here, we see we got a product. The Fourier transform of a convolution results in a product. So, find a function whose Fourier transform is $\frac{1}{1+\xi^2}$. Then, you can express the solution as the convolution of $2g$ with this!

(2) (7.4.7) We are tasked with solving the following problem:

$$u_{xx} + u_{yy} = 0, \quad x > 0, 0 < y < 1, \quad u(0, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = e^{-x}.$$

Hint: Extend the boundary data $u(x, 1) = e^{-x}$, since you want to have 0 at $x = 0$, oddly. Then hit the PDE with the Fourier transform in the x variable. This will result in an ODE for y:

$$-\xi^2 \hat{u}(\xi, y) + \hat{u}_{yy}(\xi, y) = 0 \implies \hat{u}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi e^{|\xi|y}.$$

Discard the solution which grows exponentially. Use the boundary condition at $y = 1$ to determine the coefficient function. Invert the Fourier transform. Come to consultation time if you are still stuck!

(3) (Eö 47) We wish to find a solution to

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad 0 < y < a,$$

with

$$u(x, 0) = 0, \quad u(x, a) = f(x).$$

Hint: Fourier transform the PDE in the x variable. This will result in an ODE for y. The solution will turn out to involve hyperbolic trig functions. To obtain the inequality use Plancharel's theorem. Also, you can be relieved that it is just fine to leave the solution as a Fourier integral!

(4) (8.4.1) Solve:

$$u_t = ku_{xx} - au, \quad x > 0, \quad u(x, 0) = 0, \quad u(0, t) = f(t).$$

Hint: Let's hit the PDE with the Laplace transform in the t variable and see what happens. It is a little bit different this time:

$$z\mathfrak{L}u(x, z) = k\mathfrak{L}u(x, z)_{xx} - a\mathfrak{L}u(x, z).$$

So we re-arrange and have

$$(z + a)\mathfrak{L}u(x, z) = k\mathfrak{L}u(x, z)_{xx} \implies \frac{z + a}{k}\mathfrak{L}u(x, z) = \mathfrak{L}u(x, z)_{xx}.$$

This is similar, and our solution is of the form

$$A(z)e^{-x\sqrt{(z+a)/k}} + B(z)e^{x\sqrt{(z+a)/k}}.$$

We want this to be bounded for z large, so we strike the second solution. The initial condition says we want

$$A(z) = \mathfrak{L}f(z).$$

So our Laplace-transformed solution is:

$$\mathfrak{L}f(z)e^{-x\sqrt{(z+a)/k}}.$$

This is a product. We can express our solution as a convolution if we find something whose Laplace transform is that exponential term. Let's write the exponential a little differently:

$$e^{-\frac{x}{\sqrt{k}}\sqrt{z--a}}.$$

We see that item 3 on table 3 with $c = -a$ shows that

$$\mathfrak{L}(e^{-at}f(t))(z) = \mathfrak{L}f(z - -a).$$

So if we find a function whose Laplace transform is $e^{-\frac{x}{\sqrt{k}}\sqrt{z}}$ then we will be done. We see that item 27 on table 3 gives us just that:

$$\mathfrak{L}(t^{-3/2}e^{-b^2/(4t)})(z) = 2b^{-1}\sqrt{\pi}e^{-b\sqrt{z}}.$$

(We already have one thing called a running around, so I changed the name here to b). Consequently

$$\mathfrak{L}(e^{-at}t^{-3/2}e^{-b^2/(4t)})(z) = 2b^{-1}\sqrt{\pi}e^{-b\sqrt{z+a}}.$$

Now just figure out what you need b to equal to make this work. Your solution will be a convolution of f and the correct thing to make the right side equal to $e^{-\frac{x}{\sqrt{k}}\sqrt{z--a}}$.

(5) (8.4.3) Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c:

$$u_t = ku_{xx} \text{ for } x > 0, \quad u(x,0) = 0, \quad u_x(0,t) = -c.$$

With the aid of the computation:

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}}e^{-a^2/(4t)}\right)(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}},$$

show that

$$u(x,t) = c\sqrt{\frac{k}{\pi}}\int_0^t s^{-1/2}e^{-x^2/(4ks)}ds.$$

Hint: Let's hit the PDE with the Laplace transform in the t variable. We get

$$\mathfrak{L}(u_t)(x,z) = k\mathfrak{L}(u_{xx})(x,z).$$

By the properties of the Laplace transform, and the IC,

$$\mathfrak{L}(u_t)(x,z) = z\mathfrak{L}(u)(x,z) - u(x,0) = z\mathfrak{L}(u)(x,z).$$

So we have the equation:

$$\frac{z}{k}\mathfrak{L}u(x,z) = \mathfrak{L}u(x,z)_{xx}.$$

This is an ODE now for the Laplace transform of our solution. The solution is of the form:

$$\mathfrak{L}u(x,z) = A(z)e^{-x\sqrt{z/k}} + B(z)e^{x\sqrt{z/k}}.$$

We want this to be bounded for large z so we strike the second solution. The boundary condition we have is that $u_x(0,t) = -c$, so when we transform this, we want

$$\mathfrak{L}u_x(0,z) = -\mathfrak{L}(c)(z).$$

We can Laplace transform the constant function:

$$\int_0^\infty ce^{-tz}dt = \frac{c}{z}.$$

On the other hand, taking the derivative of $A(z)e^{-\sqrt{z/k}x}$ with respect to x and then setting $x = 0$ we get:

$$-\sqrt{\frac{z}{k}}A(z) \implies -\sqrt{\frac{z}{k}}A(z) = -\frac{c}{z}.$$

So, we want

$$A(z) = \frac{c\sqrt{k}}{z^{3/2}}.$$

Thus our Laplace transformed solution is:

$$\mathfrak{L}u(x,z) = \frac{c\sqrt{k}}{z^{3/2}} e^{-x\sqrt{z/k}} = c\sqrt{k}\frac{1}{z}\left(\frac{e^{-x\sqrt{z/k}}}{\sqrt{z}}\right).$$

From here on out we can follow Folland's hint and use Table 3 which says that the Laplace transform of

$$\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}(f)(z).$$

So, we have

$$\mathfrak{L}\left(\int_0^t \frac{1}{\sqrt{\pi s}} e^{-a^2/(4s)} ds\right)(z) = \frac{e^{-a\sqrt{z}}}{z\sqrt{z}}.$$

Now just deal with the constant factors and choose a correctly...

(6) (Eö 12) We define

$$f(t) = \int_0^1 \sqrt{w} e^{w^2} \cos(wt) dw.$$

We are supposed to then somehow compute

$$\int_{\mathbb{R}} |f'(t)|^2 dt.$$

Hint: This definition of f looks remarkably like a Fourier transform of something... The right side is an $\mathcal{L}^2$ norm, so we have the Parseval (is that the right name?) formula which says that

$$\int_{\mathbb{R}} |f'(t)|^2 dt = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f'(t)}|^2 dt.$$

Then we look to Table 2 of Folland which says that

$$\widehat{f'(\xi)} = i\xi \hat{f}(\xi).$$

So we just need to compute

$$\frac{1}{2\pi}\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi.$$

To solve this, the function f requires further inspection... it is very close to being a Fourier transform. Let us make it so. Begin by extending evenly (the presence of cosine hints at this...)

$$f(t) = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|} e^{w^2}\cos(wt)dw = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}e^{-iwt}dw.$$

The reason for the last step is that the function (without the cosine) is even. So if we throw in $e^{-iwt} = \cos(-wt) + i\sin(-wt) = \cos(wt) - i\sin(wt)$ the integral with the sine will be zero since sine is odd and the rest of the integrand is zero. So we recognize

$$f(t) = \mathcal{F}\left(\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)(t).$$

By the FIT
$$\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)e^{iwt}dt = \frac{1}{2\pi}\hat{f}(-w) = \frac{1}{2\pi}\hat{f}(w).$$
This is because f is even and so it's Fourier transform is also even. So, we see that
$$\pi\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \hat{f}(w).$$
Hence, we just need to compute
$$\frac{1}{2}\int_{\mathbb{R}} w^2\left(\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)^2 dw = \frac{1}{2}\int_{-1}^{1}|w|w^2e^{2w^2}dw$$
$$= \int_0^1 w^3e^{2w^2}dw.$$
Write the integrand as $(w^2)(we^{2w^2})$. Integrate by parts. It should end nicely.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.02.26

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Geometric settings in which we can solve PDEs

We are on the home stretch! So far, the geometric settings we can handle are:

(1) finite intervals and rectangles, using Fourier series and SLP techniques;
(2) the entire real line, using Fourier transform;
(3) with nice boundary conditions, a half line using even or odd extensions;
(4) with a time-dependent boundary condition, a half line using Laplace transform;
(5) combining techniques to deal with half-spaces and quadrants.

2. Fun with drums and Bessel functions

Why do drums sound the way they do? This is actually a question that even today we do not completely understand. You'll soon understand why...

We shall solve the initial value problem for a vibrating drum. We begin by mathematicizing the drumhead as a circular membrane. Since it is a drumhead, the boundary is attached to the rest of the drum, so the boundary does not vibrate, it remains fixed. We think of the drumhead as being instantaneously still at the moment when we hit it. Consequently, the height on the drum at a point $z = (x, y)$ and time t satisfies:

$$u_{tt} - u_{xx} - u_{yy} = 0, \quad x^2+y^2 \le L^2, \quad \begin{cases} u(x,y,t) = 0 & (x,y) \text{ on the boundary} \\ u_t(x,y,0) = 0 & \\ u(x,y,0) = f(x,y) & \end{cases}.$$

To solve this problem, we see that it is pretty decent and homogeneous, and it is also occurring in a bounded region of the plane. So we see if we can use separation of variables. For this we first separate the time and space variables. So our equation is

$$T''(t)S(x,y) - S_{xx}(x,y)T - S_{yy}(x,y)T = 0.$$

We divide everything by TS, move things around, and get

$$\frac{T''}{T} = \frac{S_{xx} + S_{yy}}{S}.$$

Since each side depends on a different variable, we have the equation

$$\frac{S_{xx}+S_{yy}}{S}=\lambda=\frac{T''}{T}.$$

Which side to solve first? We have the nice homogeneous boundary condition for the space variables, so we should solve for the space variables first. Consequently we seek a solution to:

$$S_{xx}+S_{yy}=\lambda S.$$

Expressing the boundary using x and y it is:

$$x^2+y^2=L^2.$$

This is not something of the form "variable equals value." It is more complicated. The reason is because the natural coordinate system for a disk is *not* the square Cartesian coordinates. The natural coordinate system is the polar coordinate system.

Exercise 1. *Show that the differential operator*

$$\partial_{xx}+\partial_{yy}$$

in polar coordinates (r,θ) *becomes*

$$\partial_{rr}+r^{-1}\partial_r+r^{-2}\partial_{\theta\theta}.$$

Hint: use the chain rule!

In terms of polar coordinates the boundary is at $r=L$. This is the type of expression we usually have for a boundary. The function S should vanish at $r=L$. Moreover, we are on a disk. So, the function S at θ and $\theta+2k\pi$ should be the same for all $k\in\mathbb{Z}$. Let us separate variables, writing $S=R(r)\Theta(\theta)$. Then our equation becomes

$$R''\Theta+r^{-1}R'\Theta+r^{-2}\Theta''=\lambda R\Theta,\quad R(L)=0,\quad \Theta(\theta+2k\pi)=\Theta(\theta).$$

Let's get the different variables cordoned off to different sides of the equation. So, we first divide by $R\Theta$:

$$\frac{R''}{R}+r^{-1}\frac{R'}{R}+r^{-2}\frac{\Theta''}{\Theta}=\lambda.$$

Multiply everything by r^2 to liberate the term with Θ from any r dependence:

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}=r^2\lambda \iff r^2\frac{R''}{R}+r\frac{R'}{R}-r^2\lambda=-\frac{\Theta''}{\Theta}.$$

Each side depends on a different variable, so they are both constant. Since we have the lovely periodicity condition for Θ, and its equation is more simple, let us look for its solution first. We have

$$-\frac{\Theta''}{\Theta}=\text{ constant }=\mu,\quad \Theta(\theta+2k\pi)=\Theta(\theta).$$

So, we are looking for a 2π periodic function which has Θ'' equal to a constant times Θ. The only functions which have this are sines and cosines! Equivalently, we may use complex exponentials. So, we may choose to use

$$\{\sin(nx),\cos(nx)\}_{n\in\mathbb{N}_0},\text{ or }\{e^{inx}\}_{n\in\mathbb{Z}}.$$

Either of these will do the job. The numbers

$$\mu=\mu_n=-n^2.$$

So, now let us take the value of μ_n and use it to find the partner function R_n. It satisfies

$$r^2\frac{R_n''}{R_n} + r\frac{R_n'}{R_n} - r^2\lambda = -\frac{\Theta_n''}{\Theta_n} = - - n^2 = n^2.$$

Re-arranging the equation, we get

eq:almostb (2.1)
$$r^2R_n'' + rR_n' - r^2\lambda - n^2R_n = 0. \tag{2.1}$$

This is quite close to Bessel's equation.

Definition 1. *The differential equation*

$$x^2u''(x) + xu'(x) + (x^2 - \alpha^2)u(x) = 0, \quad \alpha \in \mathbb{C}$$

is Bessel's equation. The differential equation

$$u^2u''(x) + xu'(x) - (x^2 + \alpha^2)u(x) = 0,$$

is the modified Bessel equation.

So, let's try to relate our equation (eq:almostb 2.1). The main differences are: λ factor attached to r^2 term and different signs. Let us consider first the case in which $\lambda < 0$. Then $-\lambda > 0$. So, let us write

$$R_n(r) = F_n(x), \quad x = r\sqrt{|\lambda|} \implies R_n'(r) = F_n'(x)\sqrt{|\lambda|}.$$

So we also have

$$rR_n'(r) = \frac{x}{\sqrt{|\lambda|}}R_n'(r) = \frac{x}{\sqrt{|\lambda|}}F_n'(x)\sqrt{|\lambda|} = xF_n'(x).$$

Similarly we get

$$r^2R_n''(r) = x^2F_n''(x).$$

Moreover, since $\lambda < 0$,

$$-r^2\lambda = x^2.$$

So for the function F_n the differential equation (eq:almostb 2.1) is

$$x^2F_n''(x) + xF_n'(x) + x^2F_n(x) - n^2F_n(x).$$

This is

$$x^2F_n''(x) + xF_n'(x) + (x^2 - n^2)F_n(x) = 0.$$

This is Bessel's equation! The solution in this case is given by the function

$$F_n(x) = J_n(x) \implies R_n(r) = J_n(r\sqrt{|\lambda|}).$$

What should $\sqrt{|\lambda|}$ be? This comes from the boundary condition. We need

$$R_n(L) = 0 \implies J_n(L\sqrt{|\lambda|}) = 0 \implies L\sqrt{|\lambda|} \text{ is a number where } J_n \text{ vanishes.}$$

Theorem 2. *The Bessel function J_n has infinitely many zeros along the real axis. We may therefore write $\{z_{n,m}\}_{m\geq 1}$ to indicate the m^{th} positive zero of the Bessel function J_n.*

Consequently, we require

$$L\sqrt{|\lambda|} = z_{n,m} \qquad \text{for some } m \geq 1.$$

This shows that (recalling $\lambda < 0$ in this case)

$$\lambda = \lambda_{n,m} = -\frac{z_{n,m}^2}{L^2}.$$

eq:almostb

Exercise 2. *Consider the case $\lambda > 0$. Do a similar change of variables to (2.1) to show that in this case we obtain the modified Bessel equation:*

$$x^2 F_n''(x) + xF_n'(x) - (x^2 + n^2)F_n(x) = 0.$$

Check the literature to see that the solutions are the modified Bessel functions, I_n and K_n. Verify in the literature that the functions $K_n(x) \to \infty$ when $x \to 0$. So, these do not yield physically viable solutions to the wave equation because there is no reason for our drum to go off to infinity at the center point. Verify that the functions $I_n(x)$ do not have any positive real zeros, so there is no way to obtain the boundary condition $R_n(L) = 0$. Hence, these too can be discarded.

So, with the exercise, we are able to conclude that only the case $\lambda < 0$ yields physically viable solutions. Equipped with this knowledge, we may return to our equation for the time dependent function.

$$\frac{T_{n,m}''}{T_{n,m}} = \lambda_{n,m} = -\frac{z_{n,m}^2}{L^2} \implies T_{n,m}(t) = a_{n,m}\cos(z_{n,m}t/L) + b_{n,m}\sin(z_{n,m}t/L).$$

The coefficients shall be determined by our initial conditions. Using superposition to create a super solution we have

$$u(t, r, \theta) = \sum_{n,m \geq 1} \left(a_{n,m}\cos(z_{n,m}t/L) + b_{n,m}\sin(z_{n,m}t/L)\right) J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta)).$$

The time derivative should vanish when $t = 0$, which means that the coefficients

$$b_{n,m} = 0 \quad \forall n, m.$$

The other condition is

$$u(0, r, \theta) = \sum_{n,m \geq 1} a_{n,m} J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta)) = f(r, \theta).$$

So, we would like to have a sort of Fourier expansion in terms of these Bessel functions and sines and cosines. We will have a theorem which says that indeed this is true. Thus

$$a_{n,m} = \frac{\langle f, J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))\rangle}{||J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))||^2}.$$

Here since we are doing things on a disk and using polar coordinates, our scalar products are:

$$\langle f, J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))\rangle = \int_0^L \int_0^{2\pi} f(r, \theta)\overline{J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))} r dr d\theta,$$

and

$$||J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))||^2 = \int_0^L \int_0^{2\pi} |J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))|^2 r dr d\theta.$$

2.1. **What are Bessel functions?** So, what exactly are these Bessel functions? We shall see that they are a bit like the redneck cousins of the sine and cosine functions. Let us write Bessel's equation in this way:

$$x^2 f'' + xf' + (x^2 - \nu^2)f = 0.$$

Assume that f has a series expansion (we will later see that this assumption luckily works out - if it didn't - we'd just have to keep trying other methods). Then we write

$$f(x) = \sum_{j \geq 0} a_j x^{j+b}.$$

Stick it into the ODE:

$$x^2 \sum_{j \geq 0} a_j (j+b)(j+b-1) x^{j+b-2} + x \sum_{j \geq 0} a_j (j+b) x^{j+b-1} + (x^2 - \nu^2) \sum_{j \geq 0} a_j x^{j+b} = 0.$$

Pull the factors of x inside the sum:

$$\sum_{j \geq 0} a_j (j+b)(j+b-1) x^{j+b} + \sum_{j \geq 0} a_j (j+b) x^{j+b} + \sum_{j \geq 0} a_j x^{j+b+2} - \nu^2 a_j x^{j+b} = 0.$$

Begin with $j = 0$. To make the sum vanish, it will certainly suffice to make all the individual terms in the sum vanish. So we would like to have

$$a_0 \left(b(b-1) + b - \nu^2 \right) x^b = 0.$$

This will be true if

$$a_0 = 0 \text{ or } b^2 - \nu^2 = 0 \implies b = \pm \nu.$$

Next look at $j = 1$. We need

$$a_1 \left((1+b)(1+b-1) + (1+b) - \nu^2 \right) x^{b+1} = 0.$$

Let's simplify what's in the parentheses, so we need

$$a_1 \left((1+b)^2 - \nu^2 \right) = 0.$$

So, here are our options:

(1) Let $b = \nu$, set $a_1 = 0$, and be free to choose a_0 OR
(2) Let $(1+b) = \nu$, set $a_0 = 0$, and be free to choose a_1.

If we think about it, the second option is rather like doing the first one for $\nu - 1$ instead of ν. So, the two options are basically equivalent, but the first one is a bit more simple, so that is what we choose to do. We set $b = \nu$, $a_1 = 0$, and we shall choose $a_0 \neq 0$ later.

What happens with the higher terms? Once $j \geq 2$ the term with $a_j x^{j+b+2}$ gets involved. Let's group the terms in the series in a nice way:

$$\sum_{j \geq 0} x^{j+b} a_j \left((j+b)(j+b-1) + (j+b) - \nu^2 \right) + a_j x^{j+b+2} = 0.$$

This is

$$\sum_{j \geq 0} x^{j+b} a_j \left((j+b)^2 - \nu^2 \right) + a_j x^{j+b+2} = 0.$$

We figured out how to make the terms with the powers x^b and x^{b+1} vanish. For the higher powers, the coefficient of

$$x^{j+b+2} \text{ is } a_{j+2} \left((j+2+b)^2 - \nu^2 \right) + a_j.$$

Therefore, we need

$$a_{j+2} \left((j+2+b)^2 - \nu^2 \right) = -a_j \implies a_{j+2} = -\frac{a_j}{(j+2+b)^2 - \nu^2)}.$$

Recalling that we picked $b = \nu$, this means

$$a_{j+2} = -\frac{a_j}{(j+2+\nu)^2 - \nu^2},$$

so we are not dividing by zero which is a good thing. Equivalently, for $j \geq 2$, we have

$$a_j = -\frac{a_{j-2}}{(j+\nu)^2 - \nu^2} = -\frac{a_{j-2}}{j^2 + 2\nu j} = -\frac{a_{j-2}}{j(j+2\nu)}.$$

We therefore see that since we picked $a_1 = 0$, all of the odd terms are zero. On the other hand, for the even terms, we can figure out what these are using induction. I claim that

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1+\nu)(2+\nu)\ldots(k+\nu)}.$$

To begin we check the base case which has $k = 1$:

$$a_2 = -\frac{a_0}{2(2+2\nu)} = -\frac{a_0}{4(1+\nu)} = \frac{(-1)^1 a_0}{2^{2(1)} 1!(1+\nu)}.$$

So the formula is correct. We next assume that it holds for k and verify using what we computed above that it works for $k+1$. We have for $j = 2k+2$,

$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+2+2\nu)}.$$

We insert the expression for a_{2k} by the induction assumption that the formula holds for k:

$$a_{2k+2} = -\frac{(-1)^k a_0}{(2k+2)(2k+2+2\nu)2^{2k} k!(1+\nu)(2+\nu)\ldots(k+\nu)}.$$

We note that

$$(2k+2)(2k+2+2\nu) = 4(k+1)(k+1+\nu) = 2^2(k+1)(k+1+\nu).$$

So

$$a_{2k+2} = -\frac{(-1)^k a_0}{2^{2(k+1)}(k+1)k!(1+\nu)(2+\nu)\ldots(k+\nu)(k+1+\nu)}.$$

Finally we note that

$$(k+1)k! = (k+1)!.$$

So,

$$a_{2k+2} = -\frac{(-1)^k a_0}{2^{2(k+1)}(k+1)!(1+\nu)(2+\nu)\ldots(k+\nu)(k+1+\nu)}.$$

This is the formula for $k+1$, so it is indeed correct. Before we proceed, we recall one of the many special functions,

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C}, \quad \Re(s) > 1.$$

Exercise 3. *Use integration by parts to show that*

$$s\Gamma(s) = \Gamma(s+1).$$

Next, show that $\Gamma(1) = 1$*. Use induction to show that* $\Gamma(n+1) = n!$ *for* $n \geq 1$*.*

Since $\Gamma(1) = 1$*, this is the reason we define*

$$0! := 1.$$

Moreover, viewing Γ *as an extension of the factorial function to real numbers, we can compute silly expressions like*

$$\pi! = \Gamma(\pi+1), \quad e! = \Gamma(e+1), \quad i! = \Gamma(i+1).$$

Use the so-called functional equation $s\Gamma(s) = \Gamma(s+1)$ to show that Γ extends to a meromorphic function whose only poles occur at the points 0 and the negative integers.

So, motivated by the form of the coefficients, the tradition is to choose

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

Therefore coefficient

$$a_{2k} = \frac{(-1)^k}{2^{2k+\nu} k!(1+\nu)(2+\nu)\dots(k+\nu)\Gamma(\nu+1)} = \frac{(-1)^k}{2^{2k+\nu} k!\Gamma(k+\nu+1)}.$$

This is because

$$(\nu+1)\Gamma(\nu+1) = \Gamma(\nu+2).$$

Next

$$(\nu+2)\Gamma(\nu+2) = \Gamma(\nu+3).$$

We continue all the way to

$$(\nu+k)\Gamma(\nu+k) = \Gamma(\nu+k+1).$$

We have therefore arrived at the definition of the Bessel function of order ν,

$$J_\nu(x) := \sum_{k\geq 0} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

For the special case $\nu = n \in \mathbb{N}$, the Bessel function is defined for good reason via

$$J_{-n}(x) = (-1)^n J_n(x).$$

The Weber Bessel function is defined for $\nu \notin \mathbb{N}$ to be

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

The second linearly independent solution to Bessel's equation is then defined for $n \in \mathbb{N}$ to be

$$Y_n(x) := \lim_{\nu\to n} Y_\nu(x),$$

and this is well defined. If you are curious about Bessel functions, there are books by Olver, Watson, and Lebedev to name a few. What is most important about Y_n is that it blows up when $x \to 0$. That's okay. Since $J_n(x) \to 0$ as $x \to 0$, for $n \geq 1$, this shows that Y_n and J_n are certainly linearly independent! Hence they indeed form a basis of solutions to the Bessel equation.

2.1.1. *Solutions to: exercises for the week to be done oneself.*

(1) (7.3.1) Use the Fourier transform to find a solution of the ordinary differential equation

$$u'' - u + 2g(x) = 0, \quad g \in \mathcal{L}^1(\mathbb{R}).$$

Answer: $u(x) = g * e^{-|x|} = e^{-x}\int_{-\infty}^x e^y g(y)dy + e^x \int_x^\infty e^{-y} g(y)dy$.

(2) (7.4.7) We are tasked with solving the following problem:

$$u_{xx} + u_{yy} = 0, \quad x > 0, 0 < y < 1, \quad u(0,y) = 0, \quad u(x,0) = 0, \quad u(x,1) = e^{-x}.$$

Answer: $u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\xi \sin(\xi x)\sinh(\xi y)}{(1+\xi^2)\sinh(\xi)} d\xi$.

(3) (Eö 47) We wish to find a solution to

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad 0 < y < a,$$

with

$$u(x, 0) = 0, \quad u(x, a) = f(x).$$

Answer: The solution is

$$\frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \frac{\sinh(\xi y)}{\sinh(\xi a)} e^{ix\xi} dx.$$

To obtain the inequality, one can use Plancharel's theorem which says that

$$\int_{\mathbb{R}} |u(x,y)|^2 dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{u}(\xi, y)|^2 d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left| \frac{\sinh(\xi y)}{\sinh(\xi a)} \right|^2 d\xi \le \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |f(x)|^2 dx.$$

We used that $y \le a$ to obtain that the ratio of hyperbolic sines is ≤ 1, and in the last step we used Plancharel again.

(4) (8.4.1) $u(x,t) = \frac{x}{\sqrt{4\pi k}} \int_0^t f(t-s) e^{-as} s^{-3/2} e^{-s^2/(4ks)} ds.$

(5) (8.4.3) Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c:

$$u_t = k u_{xx} \text{ for } x > 0, \quad u(x, 0) = 0, \quad u_x(0, t) = -c.$$

With the aid of the computation:

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}} e^{-a^2/(4t)} \right)(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}},$$

show that

$$u(x,t) = c\sqrt{\frac{k}{\pi}} \int_0^t s^{-1/2} e^{-x^2/(4ks)} ds.$$

Answer: We hit the PDE with the Laplace transform in the t variable. We get

$$\mathfrak{L}(u_t)(x,z) = k\mathfrak{L}(u_{xx})(x,z).$$

By the properties of the Laplace transform, and the IC,

$$\mathfrak{L}(u_t)(x,z) = z\mathfrak{L}(u)(x,z) - u(x,0) = z\mathfrak{L}(u)(x,z).$$

So we have the equation:

$$\frac{z}{k} \mathfrak{L}u(x,z) = \mathfrak{L}u(x,z)_{xx}.$$

This is an ODE now for the Laplace transform of our solution. The solution is of the form:

$$\mathfrak{L}u(x,z) = A(z) e^{-x\sqrt{z/k}} + B(z) e^{x\sqrt{z/k}}.$$

We want this to be bounded for large z so we strike the second solution. The boundary condition we have is that $u_x(0,t) = -c$, so when we transform this, we want

$$\mathfrak{L}u_x(0,z) = -\mathfrak{L}(c)(z).$$

We can Laplace transform the constant function:

$$\int_0^\infty c e^{-tz} dt = \frac{c}{z}.$$

On the other hand, taking the derivative of $A(z)e^{-\sqrt{z/k}x}$ with respect to x and then setting $x=0$ we get:

$$-\sqrt{\frac{z}{k}}A(z) \implies -\sqrt{\frac{z}{k}}A(z) = -\frac{c}{z}.$$

So, we want

$$A(z) = \frac{c\sqrt{k}}{z^{3/2}}.$$

Thus our Laplace transformed solution is:

$$\mathfrak{L}u(x,z) = \frac{c\sqrt{k}}{z^{3/2}}e^{-x\sqrt{z/k}} = c\sqrt{k}\frac{1}{z}\left(\frac{e^{-x\sqrt{z/k}}}{\sqrt{z}}\right).$$

From here on out we can follow Folland's hint and use Table 3 which says that the Laplace transform of

$$\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}(f)(z).$$

So, we have

$$\mathfrak{L}\left(\int_0^t \frac{1}{\sqrt{\pi s}}e^{-a^2/(4s)}ds\right)(z) = \frac{e^{-a\sqrt{z}}}{z\sqrt{z}}.$$

To get the correct right side, we choose

$$a = \frac{x}{\sqrt{k}}.$$

To get the constant factor of $c\sqrt{k}$ as well, we multiply both sides of the equation by $c\sqrt{k}$. So, we have

$$\mathfrak{L}\left(c\sqrt{k}\int_0^t \frac{1}{\sqrt{\pi s}}e^{-x^2/(4\sqrt{k}s)}ds\right)(z) = c\sqrt{k}e^{-x\sqrt{z/k}}.$$

Hence, the solution to the problem before it was hit with the Laplace transform is

$$c\sqrt{k}\int_0^t \frac{1}{\sqrt{\pi s}}e^{-x^2/(4\sqrt{k}s)}ds.$$

(6) (Eö 12) We define

$$f(t) = \int_0^1 \sqrt{w}e^{w^2}\cos(wt)dw.$$

We are supposed to then somehow compute

$$\int_{\mathbb{R}} |f'(t)|^2 dt.$$

Hint: This definition of f looks remarkably like a Fourier transform of something... The right side is an $\mathcal{L}^2$ norm, so we have the Parseval (is that the right name?) formula which says that

$$\int_{\mathbb{R}} |f'(t)|^2 dt = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f'(t)}|^2 dt.$$

Then we look to Table 2 of Folland which says that

$$\widehat{f'(\xi)} = i\xi\hat{f}(\xi).$$

So we just need to compute

$$\frac{1}{2\pi}\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi.$$

To solve this, the function f requires further inspection... it is very close to being a Fourier transform. Let us make it so. Begin by extending evenly (the presence of cosine hints at this...)

$$f(t) = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\cos(wt)dw = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}e^{-iwt}dw.$$

The reason for the last step is that the function (without the cosine) is even. So if we throw in $e^{-iwt} = \cos(-wt) + i\sin(-wt) = \cos(wt) - i\sin(wt)$ the integral with the sine will be zero since sine is odd and the rest of the integrand is zero. So we recognize

$$f(t) = \mathcal{F}\left(\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)(t).$$

By the FIT

$$\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)e^{iwt}dt = \frac{1}{2\pi}\hat{f}(-w) = \frac{1}{2\pi}\hat{f}(w).$$

This is because f is even and so it's Fourier transform is also even. So, we see that

$$\pi\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \hat{f}(w).$$

Hence, we just need to compute

$$\frac{1}{2}\int_{\mathbb{R}} w^2\left(\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)^2 dw = \frac{1}{2}\int_{-1}^{1}|w|w^2e^{2w^2}dw$$

$$= \int_0^1 w^3e^{2w^2}dw.$$

Write the integrand as $(w^2)(we^{2w^2})$. Integrate by parts. It should end nicely.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.03.02

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. 2019.03.04

There are several interesting facts about Bessel functions. Entire books have been written on these special functions.

1.1. Fun facts about Bessel functions.

Theorem 1 (Recurrence Formulas)**.** *For all x and ν*

$$(x^{-\nu} J_\nu(x))' = -x^{-\nu} J_{\nu+1}(x)$$
$$(x^{\nu} J_\nu(x))' = x^{\nu} J_{\nu-1}(x)$$
$$xJ_\nu'(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x)$$
$$xJ_\nu'(x) + \nu J_\nu(x) = xJ_{\nu-1}(x)$$
$$xJ_{\nu-1}(x) + xJ_{\nu+1}(x) = 2\nu J_\nu(x)$$
$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x)$$

Proof: Can you guess what we do? That's right - use the definition!!!! First,

$$x^{-\nu} J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n \frac{x^{2n}}{2^{2n+\nu}}}{n!\Gamma(n+\nu+1)}.$$

Take the derivative of the sum termwise. This is totally legitimate because this series converges locally uniformly in $\mathbb{C}$. So, we compute

$$\sum_{n\geq 1} \frac{(-1)^n 2n \frac{x^{2n-1}}{2^{2n+\nu}}}{n!\Gamma(n+\nu+1)} = \sum_{m\geq 0} \frac{(-1)^{m+1} 2(m+1) \frac{x^{2m+1}}{2^{2m+2+\nu}}}{(m+1)!\Gamma(m+2+\nu)}.$$

Above we re-indexed the sum by defining $n = m+1$. Next we do some simplifying around

$$= -\sum_{m\geq 0} \frac{(-1)^m \frac{x^{2m+1}}{2^{2m+1+\nu}}}{m!\Gamma(m+2+\nu)} = -x^{-\nu} \sum_{m\geq 0} \frac{(-1)^m \frac{x^{2m+1+\nu}}{2^{2m+1+\nu}}}{m!\Gamma(m+2+\nu)} = -x^{-\nu} J_{\nu+1}(x).$$

Next we compute similarly the derivative of $x^\nu J_\nu$ is

$$\sum_{n\geq 0} \frac{(-1)^n (2n+2\nu) \frac{x^{2n+2\nu-1}}{2^{2n+\nu}}}{n!\Gamma(n+\nu+1)}.$$

We factor out a 2 to get

$$\sum_{n\geq 0} \frac{(-1)^n (n+\nu) \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}}{n!\Gamma(n+\nu+1)}.$$

Note that

$$\Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu) \implies \frac{(n+\nu)}{\Gamma(n+\nu+1)} = \frac{1}{\Gamma(n+\nu)}.$$

So, above we have

$$\sum_{n\geq 0} \frac{(-1)^n \frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}}{n!\Gamma(n+\nu)} = x^\nu J_{\nu-1}(x).$$

To do the third one it is basically expanding out the first one:

$$(x^{-\nu} J_\nu(x))' = -\nu x^{-\nu-1} J_\nu + x^{-\nu} J_\nu' = -x^{-\nu} J_{\nu+1}.$$

Multiply through by $x^{\nu+1}$ to get

$$-\nu J_\nu + x J_\nu' = -x J_{\nu+1}.$$

We do similarly in the second formula:

$$\nu x^{\nu-1} J_\nu + x^\nu J_\nu' = x^\nu J_{\nu-1}.$$

Multiply by $x^{-\nu+1}$ to get

$$\nu J_\nu + x J_\nu' = x J_{\nu-1}.$$

Next, to get the fifth formula, subtract the third formula from the fourth. Finally, to get the sixth formula, add the third formula to the fourth.

We shall prove two lovely facts about the Bessel functions. The following fact is a theory item!

1.2. The generating function for the Bessel functions.

This is a lovely, follow your nose and use the definitions type of proof.

Theorem 2. *For all x and for all $z \neq 0$, the Bessel functions, J_n satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}.$$

Proof. We begin by writing out the familiar Taylor series expansion for the exponential functions

$$e^{xz/2} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

These converge beautifully, absolutely and uniformly for z in compact subsets of $\mathbb{C} \setminus \{0\}$. So, since we presume that $z \neq 0$, we can multiply these series and fool around with them to try to make the Bessel functions pop out... Thus, we write

bessel1

$$e^{xz/2}e^{-x/(2z)} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k \geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}. \tag{1.1}$$

Here is where the one and only clever idea enters into this proof, but it's rather straightforward to come up with it. We would like a sum with $n = -\infty$ to ∞. So we look around into the above expression on the right, hunting for something which ranges from $-\infty$ to ∞. The only part which does this is $j - k$, because each of j and k range over 0 to ∞. Thus, we keep k as it is, and we let $n = j - k$. Then $j + k = n + 2k$, and $j = n + k$. However, now, we have $j! = (n + k)!$, but this is problematic if $n + k < 0$. There were no negative factorials in our original expression! So, to remedy this, we use the equivalent definition via the Gamma function,

$$j! = \Gamma(j + 1), \quad k! = \Gamma(k + 1).$$

Moreover, we observe that in (1.1), $j!$ and $k!$ are for j and k non-negative. We also observe that

$$\frac{1}{\Gamma(m)} = 0, \quad m \in \mathbb{Z}, \quad m \leq 0.$$

Hence, we can write

$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n + k + 1)k!}.$$

This is because for all the terms with $n + k + 1 \leq 0$, which would correspond to $(n+k)!$ with $n+k < 0$, those terms ought not to be there, but indeed, the $\frac{1}{\Gamma(n+k+1)}$ causes those terms to vanish!

Now, by definition,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(k + n + 1)}.$$

Hence, we have indeed see that

$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} J_n(x)z^n.$$

$\square$

1.3. **Integral representation of the Bessel functions.** Let $z = e^{i\theta}$ for $\theta \in \mathbb{R}$. Then the theorem on the generating function for the Bessel functions says

$$\sum_{n \in \mathbb{Z}} J_n(x)z^n = e^{\frac{xz}{2} - \frac{x}{2z}}.$$

So, we use the fact that

$$\frac{1}{e^{i\theta}} = e^{-i\theta},$$

together with this formula to see that

$$\sum_{n \in \mathbb{Z}} J_n(x)e^{in\theta} = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}.$$

By Euler's formula,

$$\sum_{n\in\mathbb{Z}} J_n(x)e^{in\theta} = e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta).$$

Therefore, the left side is the Fourier expansion of the function on the right. OMG!!! Hence, the Bessel functions are actually *Fourier coefficients* of this function! So,

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta}e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)d\theta.$$

Note that

$$\sin(x\sin(-\theta) - n(-\theta)) = \sin(-x\sin\theta - n(-\theta)) = -\sin(x\sin\theta - n\theta).$$

So the sine part is odd and integrates to zero. We therefore have

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta - n\theta)d\theta.$$

This formula can be super useful. For example, we see that the Bessel functions have yet another property similar to their straight-laced Swedish ancestors, the sine and cosine. They satisfy $|J_n(\theta)| \le 1 \forall x$.

1.4. **Applications to solving PDEs in circular type regions.** We shall now see how to generalize our Bessel function techniques to solve problems on pieces of circular sectors. Consider a circular sector of radius ρ and opening angle α. In the eyes of polar coordinates, this is a rectangle, $[0,\rho]\times[0,\alpha]$. That is, this set in $\mathbb{R}^2$ is in polar coordinates

$$\{(r,\theta)\in\mathbb{R}^2 : 0\le r\le\rho, \text{ and } 0\le\theta\le\alpha\}.$$

This is much the same as how we describe a rectangle using *rectangular* coordinates, (x,y).

To solve both the heat equation and the wave equation in a circular sector, we can use the same SLP and Fourier series style techniques we used on rectangles. The homogeneous heat equation is:

$$\partial_t u + \Delta u = 0, \quad \Delta = -\partial_{xx} - \partial_{yy}.$$

The homogeneous wave equation is:

$$u_{tt} + \Delta u = 0.$$

If we have neat and tidy (self-adjoint) boundary conditions, we can use separation of variables. Writing our function as $T(t)S(x,y)$, we obtain the equations:

heat $T'S + T\Delta S = 0$ which, dividing by the product TS becomes

$$\frac{\Delta S}{S} = -\frac{T'}{T} = \text{ constant.}$$

wave $T''S + T\Delta S = 0$ which, dividing by the product TS becomes

$$\frac{\Delta S}{S} = -\frac{T''}{T} = \text{ constant.}$$

So we see that in both cases we need to solve an equation of the form

$$\Delta S = \lambda S, \quad \lambda \text{ is a constant.}$$

After we solve this, we can then continue with solving both the heat equation and the wave equation.

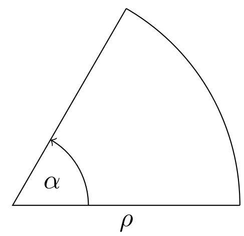


FIGURE 1. A circular sector of opening angle α and radius ρ.

1.5. **Dirichlet boundary condition on a circular sector.** Let's assume that we have the Dirichlet boundary condition on the boundary of the circular sector. So, we are looking for a function S which is zero on the boundary.

The boundary condition in polar coordinates is:

$$r = \rho, \quad \theta = 0, \quad \theta = \alpha.$$

So, it makes a lot more sense to use these coordinates. To proceed, we need to *write the operator using polar coordinates also!* We have previously computed in an exercise that in polar coordinates, the operator is:

$$\Delta = -\partial_{rr} - r^{-1}\partial_r - r^{-2}\partial_{\theta\theta}.$$

Let us try to solve $\Delta S = \lambda S$ in the circular sector using separation of variables. So, we have

$$R(r) \text{ and } \Theta(\theta).$$

The first one only depends on the r coordinate, whereas the second one only depends on the θ coordinate. Now, our PDE is:

$$-R''(r)\Theta(\theta) - r^{-1}R'(r)\Theta(\theta) - r^{-2}\Theta''(\theta)R(r) = \lambda R(r)\Theta(\theta).$$

First, we multiply everything by r^2, then we divide it all by ΘR to get

$$\frac{-r^2R'' - rR'}{R} - \frac{\Theta''}{\Theta} = \lambda \implies \frac{-r^2R'' - rR'}{R} - \lambda r^2 = \frac{\Theta''}{\Theta}.$$

Since the two sides depend on different variables, they are both constant. It turns out that the Θ side is much easier to deal with, so we look at solving it:

$$\frac{\Theta''}{\Theta} = \mu, \quad \Theta(0) = \Theta(\alpha) = 0.$$

We have solved such an equation a few times before. There are no non-zero solutions for $\mu > 0$. For $\mu < 0$ solutions are, up to constant factors,

$$\Theta_m(\theta) = \sin\left(\frac{m\pi\theta}{\alpha}\right), \quad \mu_m = -\frac{m^2\pi^2}{\alpha^2}.$$

As a consequence, we get the equation for R,

$$\frac{-r^2R'' - rR'}{R} - \lambda r^2 = \mu_m.$$

We multiply this equation by R, obtaining

$$-r^2R'' - rR' - \lambda r^2 R = \mu_m R.$$

This is equivalent to

$$r^2R'' + rR' + (\lambda r^2 + \mu_m)R = 0.$$

We make a small clever change of variables. Let

$$x = \sqrt{\lambda} r, \quad f(x) := R(r), \quad r = \frac{x}{\sqrt{\lambda}}.$$

Then by the chain rule

$$R'(r) = \sqrt{\lambda} f'(x), \quad R''(r) = \lambda f''(x).$$

So, the equation becomes

$$\left(\frac{x^2}{\lambda}\right) \lambda f''(x) + \frac{x}{\sqrt{\lambda}} \sqrt{\lambda} f'(x) + (x^2 + \mu_m) f(x) = 0.$$

This simplifies, recalling that $\mu_m = -m^2\pi^2/\alpha^2$,

besseleq

$$x^2 f''(x) + x f'(x) + (x^2 - m^2\pi^2/\alpha^2) f(x) = 0. \tag{1.2}$$

This is the definition of Bessel's equation of order $\frac{m\pi}{\alpha}$. Consequently, a solution to this equation is

$$J_{m\pi/\alpha}(x) = J_{m\pi/\alpha}(\sqrt{\lambda} r).$$

To satisfy the boundary condition, we would like

$$J_{m\pi/\alpha}(\sqrt{\lambda}\rho) = 0.$$

So, $\sqrt{\lambda}\rho$ should be a point at which this Bessel function vanishes. We have a useful fact about these zeros.

Theorem 3. *The Bessel function $J_{m\pi/\alpha}$ has infinitely many positive zeros which can be indexed as*

$$\{z_{m,k}\}_{k \geq 1},$$

where $z_{m,k}$ is the k^{th} positive zero.

Consequently, we shall have

$$J_{m\pi/\alpha}(z_{m,k} r/\rho), \quad \lambda_{m,k} = \frac{z_{m,k}^2}{\rho^2}.$$

We therefore have the collection of functions

$$S_{m,k}(\theta, r) = \sin(m\pi\theta/\alpha) J_{m\pi/\alpha}\left(\frac{z_{m,k} r}{\rho}\right).$$

Now we may obtain the time part of the solution.

heat Let us look for a solution to the homogeneous heat equation which satisfies

$$u(r, \theta, 0) = f(r, \theta).$$

Then, the partner functions T shall be given by:

$$\frac{\Delta S}{S} = -\frac{T'}{T} = \lambda_{m,k} \implies T_{m,k}(t) = A_{m,k} e^{-\lambda_{m,k} t}.$$

By superposition our full solution is therefore

$$u(r, \theta, t) = \sum_{m,k} A_{m,k} e^{-\lambda_{m,k} t} S_{m,k}(r, \theta).$$

wave Let us look for a solution to the homogeneous wave equation which satisfies

$$w(r,\theta,0) = g(r,\theta), \quad w_t(r,\theta,0) = 0.$$

$$\frac{\Delta S}{S} = -\frac{T''}{T} = \lambda_{m,k} \implies T_{m,k}(t) = a_{m,k}\cos(z_{m,k}t/\rho) + b_{m,k}\sin(z_{m,k}t/\rho).$$

By superposition our full solution is therefore

$$w(r,\theta,t) = \sum_{m,k}(a_{m,k}\cos(z_{m,k}t/\rho) + b_{m,k}\sin(z_{m,k}t/\rho))S_{m,k}(r,\theta).$$

To determine the coefficients, we shall use the following theorem.

Theorem 4. *The set of functions*

$$\sin(m\pi\theta/\alpha)J_{m\pi/\alpha}\left(\frac{z_{m,k}r}{\rho}\right), \quad k \geq 0, \quad m \geq 1$$

are an orthogonal basis for $\mathcal{L}^2$ on the sector of radius ρ and opening angle α. Above, $z_{m,k}$ is the k^{th} positive zero of $J_{m\pi/\alpha}$.

Consequently, for the heat equation we demand

$$u(r,\theta,0) = \sum_{m,k} A_{m,k}S_{m,k}(r,\theta) = f(r,\theta),$$

which shows us that the coefficients should be

$$A_{m,k} = \frac{\langle f, S_{m,k}\rangle}{||S_{m,k}||^2},$$

where

$$\langle f, S_{m,k}\rangle = \int_0^\alpha \int_0^\rho f(r,\theta)\overline{S_{m,k}(r,\theta)}rdrd\theta,$$

and

$$||S_{m,k}||^2 = \int_0^\alpha \int_0^\rho |S_{m,k}(r,\theta)|^2 rdrd\theta.$$

For the wave equation we demand

$$w(r,\theta,0) = \sum_{m,k} a_{m,k}S_{m,k}(r,\theta) = g(r,\theta) \implies a_{m,k} = \frac{\langle g, S_{m,k}\rangle}{||S_{m,k}||^2}.$$

The second condition tells us what the other coefficients should be:

$$w_t(r,\theta,0) = \sum_{m,k} z_{m,k}/\rho b_{m,k}S_{m,k}(r,\theta) = 0 \implies b_{m,k} = 0 \forall m,k.$$

1.6. **Bessel functions for Neumann boundary condition.** This theorem is another type of "adult spectral theorem."

Theorem 5. *Assume that $\nu \geq 0$ and $\rho > 0$. Assume that $c \geq -\nu$. Let*

$$\{z_k\}_{k\geq 1}$$

be the positive zeros of $cJ_\nu(x) + xJ_\nu'(x)$, and let

$$\psi_k(x) = J_\nu(z_k x/\rho).$$

If $c > -\nu$ then $\{\psi_k\}_{k\geq 1}$ is an orthogonal basis for $\mathcal{L}^2_w$ on the interval $(0,b)$ for the weight function $w(x) = x$. If $c > -\nu$, then $\{\psi_k\}_{k\geq 0}$ is an orthogonal basis for $\mathcal{L}^2_w$ on the interval $(0,b)$ for the weight function $w(x) = x$, with $\psi_0(x) = x^\nu$.

Let us see how to apply this theorem when we are solving the heat (and wave) equations with the Neumann boundary condition. We follow the same procedure as for the heat equation. Let us name the sector

$$\Sigma.$$

$$u_t + \Delta u = 0, \quad \text{inside } \Sigma,$$
$$u(r, \theta, 0) = v(r, \theta) \text{ inside } \Sigma$$
$$\text{the outward pointing normal derivative of } u = 0 \text{ on the boundary of } \Sigma.$$

We do the same procedure as before. We arrive at the equation for the Θ part:

$$\Theta'' = \mu\Theta, \quad \Theta'(0) = \Theta'(\alpha) = 0.$$

You can do the exercise to show that the only solutions are for $\mu < 0$, and to satisfy the boundary conditions, up to constant multiples

$$\Theta_m(\theta) = \cos(m\pi/\alpha), \quad \mu_m = -\frac{m^2\pi^2}{\alpha^2}, \quad m \geq 0.$$

Then, we again arrive at the Bessel equation of order $m\pi/\alpha$ for the function R. So, we get that

$$R_m(r) = J_{\nu_m}(\sqrt{\lambda}r), \quad \nu_m = m\pi/\alpha.$$

The boundary condition for R_m is that

$$R_m'(\rho) = 0.$$

So, this means we need

$$\sqrt{\lambda}J_{\nu_m}'(\sqrt{\lambda}\rho) = 0.$$

In other words, $\sqrt{\lambda}$ needs to be a solution of the equation

$$xJ_{\nu_m}'(\rho x) = 0.$$

If z_k is a solution to

$$xJ_{\nu_m}'(x) = 0,$$

then

$$z_k J_{\nu_m}'(z_k) = 0 \implies \frac{z_k}{\rho}J_{\nu_m}'(z_k\rho/\rho) = 0.$$

So, to satisfy the boundary condition, we need

$$\sqrt{\lambda} = \frac{z_k}{\rho} \implies \sqrt{\lambda}J_{\nu_m}'(\sqrt{\lambda}\rho) = 0.$$

Really, z_k also depends on m, so that is why we write $z_{m,k}$ to mean the k^{th} positive solution of the equation

$$xJ_{\nu_m}'(x) = 0.$$

Our function

$$R_{m,k}(r) = J_{\nu_m}(z_{m,k}r/\rho).$$

This also shows that

$$\lambda_{m,k} = \frac{z_{m,k}^2}{\rho^2}.$$

Now, we recall the equation for the partner function, T,

$$T_{m,k}'(t) = -\lambda_{m,k}T_{m,k}(t).$$

So, up to constant factors,

$$T_{m,k}(t) = e^{-\lambda_{m,k}t}.$$

To apply the theorem, we note that

$$\nu_m = m\pi/\alpha > 0 \forall m \in \mathbb{N}.$$

Therefore taking $c = 0$ in the theorem, $c \geq -\nu_m$ for all m. The theorem then tells us that the set

$$\{R_{m,k}(r)\}_{k\geq 1} = \{J_{\nu_m}(z_{m,k}r/\rho)\}_{k\geq 1}$$

is an orthogonal basis for $\mathcal{L}^2(0,\rho)$ with respect to integrating against rdr. We also know that the $\Theta_m(\theta)$ functions are an orthogonal basis for $\mathcal{L}^2(0,\alpha)$ with respect to integrating against $d\theta$. Consequently, the entire collection

$$S_{m,k}(r,\theta) = \Theta_m(\theta)R_{m,k}(r)$$

is an orthogonal basis for $\mathcal{L}^2(\Sigma)$. This is because integrating on $\mathcal{L}^2(\Sigma)$ in polar coordinates is integrating

$$\int_\Sigma v(r,\theta)rdrd\theta = \int_0^\rho \int_0^\alpha v(r,\theta)rdrd\theta.$$

So, the theorem says that we can expand the initial data in a Fourier series with respect to the orthogonal basis functions $S_{m,k}$. We therefore write the solution

$$u(r,\theta,t) = \sum_{m,k} \widehat{v_{m,k}} T_{m,k}(t) S_{m,k}(r,\theta),$$

where

$$\widehat{v_{m,k}} = \frac{\int_\Sigma v(r,\theta)S_{m,k}(r)rdrd\theta}{||S_{m,k}||^2}$$

$$= \frac{\int_0^r \int_0^\theta \sin(m\pi\theta/\alpha)J_{m\pi/\alpha}(z_{m,k}r/\rho)v(r,\theta)rdrd\theta}{\int_0^r \int_0^\theta \sin(m\pi\theta/\alpha)^2 J_{m\pi/\alpha}(z_{m,k}r/\rho)^2 rdrd\theta}.$$

1.7. **Exercises to be demonstrated.**

(1) Eö 28
(2) (5.5.2) A circular cylinder of radius ρ is at the constant temperature A. At time $t = 0$ it is tightly wrapped in a sheath of the same material of thickness δ, thus forming a cylinder of radius $\rho + \delta$. The sheath is initially at temperature B, and its outside surface is maintained at temperature B. If the ends of the new, enlarged cylinder are insulated, find the temperature inside at subsequent times.
(3) Eö 30
(4) Eö 52
(5) Eö 53
(6) (5.5.4) A cylindrical uranium rod of radius 1 generates heat within itself at a constant rate a (think radioactive material). Its ends are insulated and its circular surface is immersed in a cooling bath at temperature zero. Thus

$$u_t = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + a, \quad u(1,t) = 0.$$

First find the steady state temperature $v(r)$ in the rod. Then find the temperature in the rod if its initial temperature is zero.

1.8. **Exercises to be done oneself.**

(1) (5.2.4) Demonstrate the identity:
$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$
(2) (5.5.1) A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$, $(c > 0)$.
(3) (5.5.5) Solve the problem
$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r, \theta, z) : 0 \leq r \leq b, 0 \leq z \leq l\}$$
$$u(r, \theta, 0) = 0, \quad u(r, \theta, l) = g(r, \theta), \quad u(b, \theta, z) = 0.$$
(4) (5.5.6) Find the steady-state temperature in the cylinder $0 \leq r \leq 1$, $0 \leq z \leq 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$.
(5) Eö 29

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.03.03

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Solving PDEs with the help of SLPs

We have seen how the process of solving PDEs like the heat and wave equation often leads to a set of functions which comprise an orthogonal basis for $\mathcal{L}^2$ or a weighted $\mathcal{L}^2$ space. These basis functions generally come from separation of variables. When we solve the "space" part of the PDE, we very often end up solving a type of SLP. The easiest examples are:

$$f'' = \lambda f, \quad f(a) = 0 = f(b), \text{ for f defined on the interval, } [a, b]$$
$$f'' = \lambda f, \quad f'(a) = 0 = f'(b), \text{ for f defined on the interval, } [a, b]$$
$$f'' = \lambda f, \quad f(a) = 0 = f'(b), \text{ for f defined on the interval, } [a, b]$$
$$f'' = \lambda f, \quad f'(a) = 0 = f(b), \text{ for f defined on the interval, } [a, b].$$

A more challenging example comes from solving the heat and wave equations on a circular sector. There, when we did separation of variables, we got the nice type of SLP above for the angular variable (θ), and we got a more complicated SLP for the radial variable. This turned into a Bessel equation. We used the initial data to determine the coefficients in our series expansion, by writing the initial data as a Fourier-Bessel type series.

2. The French polynomials

In *other* geometric settings, this same process will lead to *other* special functions. In the last part of this course, based on chapter 6 in Folland, we will look at *the French polynomials,*

(1) Legendre polynomials
(2) Hermite polynomials
(3) Laguerre polynomials

We can imagine that now we may be solving PDEs in more exotic geometric settings, like French Polynesia. Hence, more exotic functions will play the role of the SLP part of the problem. Three such types of functions are the aforementioned French polynomials.

2.1. Legendre polynomials.

These French polynomials arise from using spherical coordinates to solve the wave and heat equations on a three-dimensional sphere.

2.2. **Hermite polynomials.** These French polynomials arise from using parabolic coordinates to solve the wave and heat equations in a parabolic shaped region.

2.3. **Laguerre polynomials.** These French polynomials arise from the quantum mechanics of the hydrogem atom.

2.4. **Orthogonal polynomials general theory.** For the purpose of this course, it is most important that you learn how to *use* the French polynomials. Depending on how much time we have, we may go into the details of the origins of the French polynomials, but these details are rather complicated and will not be examined. So, we prioritize that which shall be examined. The following proposition is therefore useful.

Proposition 1. *Assume that $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of polynomials such that p_n is of degree n for each n. Assume that $p_0 \neq 0$. Then for each $k \in \mathbb{N}$, any polynomial of degree k is a linear combination of $\{p_j\}_{j=0}^k$.*

Proof: The proof is by induction. If q_0 is a polynomial of degree 0, then we may simply write

$$q_0 = \frac{q_0}{p_0} p_0.$$

This is okay because p_0 is degree zero, so it is a constant, and $p_0 \neq 0$, so the coefficient q_0/p_0 is also a constant. Assume that we have verified the proposition for all $0, 1, \ldots k$. We wish to show that it holds for $k+1$. So, let q be a polynomial of degree $k+1$. This means that

$$q(x) = ax^{k+1} + l.o.t. \quad l.o.t. \text{ means lower order terms}$$

has

$$a \neq 0.$$

Moreover, since p_{k+1} is of degree $k+1$ (not of a lower degree), it is of the form

$$p_{k+1} = bx^{k+1} + l.o.t., \quad b \neq 0.$$

So, let us consider

$$q(x) - \frac{a}{b} p_{k+1}(x) = p(x) \text{ which is degree } k.$$

By induction, p is a linear combination of $p_0, \ldots, p_k$. Therefore

$$q(x) = \frac{a}{b} p_{k+1} + \sum_{j=0}^{k} c_j p_j,$$

for some constants $\{c_j\}_{j=0}^k$.

Proposition 2. *Let $\{p_k\}_{k=0}^\infty$ be a set of polynomials such that each p_k is of degree k, and $p_0 \neq 0$. Moreover, assume that they are $\mathcal{L}^2$ orthogonal on a finite bounded interval $[a,b]$. Then these polynomials comprise an orthogonal basis of $\mathcal{L}^2$ on the interval $[a,b]$.*

Proof: Assume that some $f \in \mathcal{L}^2$ on the interval is orthogonal to all of these polynomials. Therefore by the preceding proposition, f is orthogonal to *all* polynomials. To see this, note that if p is a polynomial of degree n, then there exist numbers $c_0, \ldots, c_n$ such that

$$p = \sum_{j=0}^{n} c_j p_j \implies \langle f, p \rangle = \sum_{j=0}^{n} \overline{c_j} \langle f, p_j \rangle = 0.$$

We shall use the fact that continuous functions are dense in $\mathcal{L}^2$. Therefore given $\varepsilon > 0$, there exists a continuous function, g, such that

$$||f - g|| < \frac{\varepsilon}{2(||f|| + 1)}.$$

Next, we use the Stone-Weierstrass Theorem which says that all continuous functions on bounded intervals can be approximated by polynomials. Therefore, there exists a polynomial p such that

$$||g - p|| < \frac{\varepsilon}{2(||f|| + 1)}.$$

Finally, we compute

$$||f||^2 = \langle f, f \rangle = \langle f - g + g - p + p, f \rangle = \langle f - g, f \rangle + \langle g - p, f \rangle + \langle p, f \rangle$$
$$= \langle f - g, f \rangle + \langle g - p, f \rangle.$$

By the Cauchy-Schwarz inequality,

$$||f||^2 \le ||f - g|| ||f|| + ||g - p|| ||f|| < \frac{||f||\varepsilon}{2(||f|| + 1)} + \frac{||f||\varepsilon}{2(||f|| + 1)} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $||f|| = 0$. Hence by the three equivalent conditions to be an orthogonal basis, we have that the polynomials are an orthogonal basis of $\mathcal{L}^2$ on the interval.

2.5. **Best approximations.** We recall a slight variation of the best approximation theorem:

Theorem 3. *Let $\{\phi_n\}_{n \in \mathbb{N}}$ be an orthonormal set set in a Hilbert space, H. If $f \in H$,*

$$||f - \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n|| \le ||f - \sum_{n \in \mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n \in \mathbb{N}} \in \ell^2,$$

and $=$ holds $\iff$ $c_n = \langle f, \phi_n \rangle$ holds $\forall n \in \mathbb{N}$. More generally, let $\{\phi_n\}_{n=0}^N$ be an orthogonal, non-zero set in a Hilbert space H. Then,

$$||f - \sum_{n=0}^{N} \frac{\langle f, \phi_n \rangle}{||\phi_n||^2} \phi_n|| \le ||f - \sum_{n=0}^{N} c_n \phi_n||, \quad \forall \{c_n\}_{n=0}^N \in \mathbb{C}^{N+1}.$$

Equality holds if and only if

$$c_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2}, \quad n = 0, \ldots, N.$$

How to prove it? The only difference is the last part, but we can use the proof of the first part. Define $\psi_n = 0$ for $n > N$. Next define

$$\psi_n = \frac{\phi_n}{||\phi_n||}, \quad n = 0, \ldots, N.$$

Repeat the argument in the proof of the best approximation theorem using $\{\psi_n\}_{n\in\mathbb{N}}$ instead of ϕ_n.

$$||f - \sum_{n\in\mathbb{N}} c_n\psi_n||^2 = ||f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n + \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n||^2$$

$$= ||f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n||^2 + ||\sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n||^2 + 2\Re\langle f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n, \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n\rangle.$$

The scalar product

$$\langle f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n, \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n\rangle = \langle f, \sum_{n\in\mathbb{N}} (\hat{f}_n - c_n)\Psi_n\rangle - \sum_{n\in\mathbb{N}} \hat{f}_n\langle\psi_n, \sum_{m\in\mathbb{N}} (\hat{f}_m - c_m)\Psi_n\rangle.$$

By the orthogonality and definition of Ψ_n, and the definition of $\hat{f}_n$,

$$= \sum_{n\in\mathbb{N}} \hat{f}_n\overline{(\hat{f}_n - c_n)} - \sum_{n\in\mathbb{N}} \hat{f}_n \sum_{m\in\mathbb{N}} \overline{(\hat{f}_m - c_m)}\langle\psi_n, \psi_m\rangle$$

$$= \sum_{n\in\mathbb{N}} \hat{f}_n\overline{(\hat{f}_n - c_n)} - \sum_{n\in\mathbb{N}} \hat{f}_n\overline{(\hat{f}_n - c_n)} = 0.$$

Therefore

$$||f - \sum_{n\in\mathbb{N}} c_n\psi_n||^2 = ||f - \sum_{n\in\mathbb{N}} \hat{f}_n\psi_n||^2 + ||\sum_{n\in\mathbb{N}} \hat{f}_n\psi_n - \sum_{n\in\mathbb{N}} c_n\psi_n||^2$$

$$= ||f - \sum_{n=0}^{N} \hat{f}_n\psi_n||^2 + \sum_{n=0}^{N} |\hat{f}_n - c_n|^2 \le ||f - \sum_{n=0}^{N} \hat{f}_n\psi_n||^2,$$

with equality if and only if $c_n = \hat{f}_n$ for all n. Since

$$\sum_{n=0}^{N} \hat{f}_n\psi_n = \sum_{n=0}^{N} \frac{\langle f, \phi_n\rangle}{||\phi_n||^2}\phi_n,$$

this completes the proof.

2.5.1. *Applications: best approximation problems.* This shows us that if we have a finite orthogonal set of non-zero vectors in a Hilbert space, then for any element of that Hilbert space, the best approximation of f in terms of those vectors is given by

$$\sum_{n=0}^{N} \frac{\langle f, \phi_n\rangle}{||\phi_n||^2}\phi_n.$$

Here is the setup of questions which can be solved using this theory. Either:

(1) You are given functions defined on an interval which are $\mathcal{L}^2$ orthogonal on that interval (possibly with respect to a weight function which is also specified). Either you recognize that they orthogonal because you've seen them before (like sines, cosines, from problems you have solved previously) *or* you *compute* that they are $\mathcal{L}^2$ orthogonal on the interval. Then, you are asked to find the numbers $c_0, c_1, \dots c_N$ so that the $\mathcal{L}^2$ norm, or the weighted $\mathcal{L}^2$ norm of $f - \sum_{k=0}^N c_k \phi_k$ is minimized, where the function f is also specified.
(2) You are asked to find the polyonomial of at most degree N such that the $\mathcal{L}^2$ norm (or weighted $\mathcal{L}^2$ norm) of $f - p$ where p is a polynomial is minimized.

In the first case, you compute

$$c_k = \frac{\langle f, \phi_k \rangle}{||\phi_k||^2}.$$

In the second case you need to *build up a set of orthogonal or orthonormal polynomials.* Then, you let ϕ_k be defined to be the polynomial of degree k you have built. Proceed the same as in the first case, and your answer shall be

$$\sum_{k=0}^N c_k \phi_k.$$

If you don't like the thought of building up a set of orthogonal polynomials, if you are lucky, then it may be possible to suitably modify some of the French polynomials to be orthogonal on the interval under investigation, with respect to the (possibly weighted) $\mathcal{L}^2$ norm. So, we shall proceed to study the French polynomials. Depending on how much time we have, we may also be able to get into their "origin stories."

2.6. **The Legendre polynomials.** The *Legendre polynomials,* are defined to be

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2-1)^n \right).$$

OMG like why on earth are they defined in such a bizarre way, right? What did you expect, they are French polynomials! Of course they are not defined in some simple way, mais non, they must be all fancy and shrouded in mystery and intrigue. Actually though, the reason comes from the PDE in which they arise as solving one part of the separation of variables for the heat and wave equations in three dimensions using spherical coordinates. First, let us do a small calculation involving these polynomials:

$$(x^2-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (x^2)^k = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}.$$

Therefore, if we differentiate n times, only the terms with $k \geq n/2$ survive. Differentiating a term x^{2k} once we get $2kx^{2k-1}$. Differentiating n times gives

$$\frac{d^n}{dx^n}(x^{2k}) = x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

If we want to be really persnickety, we prove this by induction. For $n=1$, we get that

$$(x^{2k})' = 2kx^{2k-1}.$$

Which is correct. If we assume the formula is true for n, then differentiating $n+1$ times using the formula for n we get

$$(2k-n)x^{2k-(n+1)}\prod_{j=0}^{n-1}(2k-j) = x^{2k-(n+1)}\prod_{j=0}^{n}(2k-j).$$

See, it is correct. As a result,

$$P_n(x) = \frac{1}{2^n n!}\sum_{k\geq n/2}^{n}(-1)^{n-k}\binom{n}{k}x^{2k-n}\prod_{j=0}^{n-1}(2k-j).$$

So, we see that this is indeed a polynomial of degree n.

Next time we will prove the following theorem about the Legendre polynomials.

Theorem 4. *The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$ and*

$$||P_n||^2 = \frac{2}{2n+1}.$$

Here, we shall simply use this theorem to do an example.

Exercise 1. *Find the polynomial $p(x)$ of at most degree four which minimizes the following integral*

$$\int_{-1}^{1}|p(x)-e^x|^2 dx.$$

Based on our theoretical knowledge, the 'best approximation' can be created using the Legendre polynomials. Let

$$f(x) := e^x.$$

Then, the 'best approximation' in terms of the Legendre polynomials is

$$p(x) = \sum_{n=0}^{4} c_n P_n(x),$$

where $P_n(x)$ is the Legendre polynomial of degree n, and

$$c_n := \frac{\langle f, P_n\rangle}{||P_n||^2} = \frac{\int_{-1}^{1} e^x \overline{P_n(x)}dx}{\frac{2}{2n+1}}.$$

The beautiful fact is that we do not need to compute these integrals.

2.7. **Hints for the exercises to be done oneself.**

(1) (5.5.1) A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$, $(c > 0)$. Hint: Since the ends are insulated the problem is reduced to a disk. Moreover, since the initial condition is radially symmetric, the solution will also continue to be radially symmetric for all later times. Thus, you just need $u(r,t)$ a function depending on the radius and the time. Write $u(r,t) = R(r)T(t)$ and put into the heat equation remembering to use polar coordinates for the PDE. Solve for R first. Use the boundary condition. There will be J_0s and the λ_ks will come from an equation that you need $J_0(\lambda_k r)$ to satisfy (BC!). Then solve for the time part, and finally get the coefficients using the initial condition.

(2) (5.5.5) Solve the problem

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r, \theta, z) : 0 \le r \le b, 0 \le z \le l\}$$

$$u(r, \theta, 0) = 0, \quad u(r, \theta, l) = g(r, \theta), \quad u(b, \theta, z) = 0.$$

Hint: There is only one inhomogeneous part of the equation, and that is the boundary condition when $z = l$. Otherwise, observe that since we are in a cylinder, the function must be 2π periodic in the theta variable. So, let us separate variables writing $u = R(r)\Theta(\theta)Z(z)$. Put this into the PDE. First solve for the theta dependent function. I am guessing you will get either $e^{in\theta}$ for $n \in \mathbb{Z}$ or $\sin(n\theta)$ and $\cos(n\theta)$, and these are equivalent to each other... Next, I would solve for the R function. This has the zero boundary condition: $R(b) = 0$. So, I am guessing you will get $J_n(z_{n,k}r/b)$ where $z_{n,k}$ is the k^{th} positive zero of the Bessel function J_n for $n \in \mathbb{N}$. Last but not least, use these to solve for your Z function. Since the PDE is homogeneous, smash them all together into a super-solution using superposition. Use the condition $u(r, \theta, l) = g(r, \theta)$ to specify what the constants in your solution need to be.

(3) (5.2.4) Demonstrate the identity:

$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$

Hint: Use the recurrence formulas. Integrating by parts is a reasonable idea as well.

(4) (5.5.6) Find the steady-state temperature in the cylinder $0 \le r \le 1$, $0 \le z \le 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$. Hint: This is a radially symmetric problem, so you'll have the variables r, z. No thetas. No t because you're asked to find the 'steady-state temperature' so, this is the temperature that is independent of time. Use separation of variables, writing $u(r, z) = R(r)Z(z)$. The boundary condition for R will be that $R'(1) = 0$, because no heat is lost outside the circular surface. The boundary condition for Z is weird. So, solve for R first. The operator $\partial_{xx} + \partial_{yy} + \partial_{zz}$ in these coordinates is

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz}.$$

Since it is steady state, you're solving $\Delta RZ = 0$. Solve for R first. Then use it to solve for Z. This will involve expanding $f(r)$ in a series...

(5) Eö 29 Hint: Oh geez. Look at that boundary condition. It depends on time. Well, let's not panic. This is a new trick. Look at the function $(t+1)$. You want that sitting at $x = 0$, but you want to kill it at $x = 1$. How to achieve this using t and x?

$$(t + 1)(1 - x).$$

This takes care of the boundary condition at $x = 0$, the boundary condition at $x = 1$, and the initial condition at $t = 0$. Does it screw up the PDE? Well,

$$(\partial_t - 2\partial_{xx})(t + 1)(1 - x) = 1 - x.$$

So now you've got an inhomogeneous PDE. Use the series technique. First, find the basis

$$X_n \text{ with } X_n(0) = X_n(1) = 0, \quad X_n'' = \lambda_n X_n.$$

Find the lambdas. Next write

$$v(x,t) = \sum_{n\geq 1} T_n(t) X_n(x).$$

Expand $-(1-x)$ in an X_n Fourier series, like

$$-(1-x) = \sum_{n\geq 1} b_n X_n(x).$$

Stick v into the PDE. Set it equal to the series for $-(1-x)$. Use the differential equation satisfied by X_n. Equate the coefficients of X_n on the left and right. This will give an ODE for T_n. Use as initial condition $T_n(0) = 0$. Your full solution will be

$$(t+1)(1-x) + v(x,t).$$

Check that it satisfies everything required. If you're stuck, go back to the first exercise demonstrated on Monday's big group session for inspiration! Also, it might make you feel better to know that I first tried doing some Laplace transform business with this, and it became horrible. Realized that it was so complicated, there must be a better way. Indeed there is.

(6) Eö 35 (sorry forgot this one before) Hint: Since you're in a cylinder, use polar coordinates for x and y, but keep z just as it is. The PDE is therefore

$$(\partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + \partial_{zz})u = 0.$$

The function should vanish at $z = 0$ and $z = L$. It's got a strange boundary condition at $r = R$. It might be good to change this R into a ρ in case you'd like to use separation of variables. Try to solve the problem using separation of variables. Solve for Z first. Since the boundary data doesn't depend on θ but only depends on z, the solution is independent of θ. So you're just going to have Z and R. You'll get the coefficients from the boundary data, which might look weird, but should read

$$u = \sin\left(\frac{\pi z}{L}\right) - \sin\left(\frac{\pi z}{L}\right)\cos\left(\frac{\pi z}{L}\right).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).

FOURIER ANALYSIS & METHODS 2020.03.06

JULIE ROWLETT

ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The Legendre polynomials and applications

Theorem 1. *The Legendre polynomials are orthogonal in* $\mathcal{L}^2(-1,1)$*, and*

$$||P_n||^2 = \frac{2}{2n+1}.$$

Proof: We first prove the orthogonality. Assume that $n > m$. Then, since they have this constant stuff out front, we compute

$$2^n n! 2^m m! \langle P_n, P_m \rangle = \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m dx.$$

Let us integrate by parts once:

$$= \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m \Big|_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

Consider the boundary term:

$$\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}}(x-1)^n(x+1)^n.$$

This vanishes at $x = \pm 1$, because the polynomial vanishes to order n whereas we only differentiate $n-1$ times. So, we have shown that

$$2^n n! 2^m m! \langle P_n, P_m \rangle = -\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m.$$

We repeat this $n-1$ more times. We note that for all $j < n$,

$$\frac{d^j}{dx^j}(x^2-1)^n \text{ vanishes at } x = \pm 1.$$

For this reason, all of the boundary terms from integrating by parts vanish. So, we just get

$$(-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^m dx = (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^n}{dx^n}\frac{d^m}{dx^m}(x^2-1)^m dx$$

Remember that $n > m$. We computed that $\frac{d^m}{dx^m}(x^2-1)^m$ is a polynomial of degree m. So, if we differentiate it more than m times we get zero. So, we're integrating zero! Hence it is zero.

For the second part, we need to compute:

$$(x^2-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (x^2)^k = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}.$$

Therefore, if we differentiate n times, only the terms with $k \geq n/2$ survive. Differentiating a term x^{2k} once we get $2kx^{2k-1}$. Differentiating n times gives

$$\frac{d^n}{dx^n}(x^{2k}) = x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

If we want to be really persnickety, we prove this by induction. For $n=1$, we get that

$$(x^{2k})' = 2kx^{2k-1}.$$

Which is correct. If we assume the formula is true for n, then differentiating $n+1$ times using the formula for n we get

$$(2k-n)x^{2k-(n+1)} \prod_{j=0}^{n-1} (2k-j) = x^{2k-(n+1)} \prod_{j=0}^{n} (2k-j).$$

See, it is correct. As a result,

$$P_n(x) = \frac{1}{2^n n!} \sum_{k \geq n/2}^n (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

So, we see that this is indeed a polynomial of degree n. With this formula, we can write

$$P_n(x) = \frac{1}{2^n n!} \sum_{k \geq n/2}^n (-1)^{n-k} \binom{n}{k} x^{2k-n} \prod_{j=0}^{n-1} (2k-j).$$

Differentiating n times gives us just the term with the highest power of x, so we have

$$\frac{d^n}{dx^n} P_n(x) = \frac{1}{2^n n!} n! \prod_{j=0}^{n-1} (2n-j) = \frac{(2n)!}{2^n n!}.$$

Consequently,

$$\begin{aligned}
\langle P_n, P_n \rangle = (-1)^n \frac{1}{2^n n!} \frac{(2n)!}{2^n n!} \int_{-1}^1 (x^2-1)^n dx &= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx \\
&= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^{2k} dx \\
&= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^n (-1)^{n-k} \frac{x^{2k+1}}{2k+1} \binom{n}{k} \Bigg|_0^1 \\
&= (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{1}{2k+1} \\
&= \frac{2(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{2k+1}.
\end{aligned}$$

This looks super complicated. Apparently by some miracle of life

$$\int_0^1 (1-x^2)^n dx = \frac{\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+3/2)}.$$

Since

$$\langle P_n, P_n \rangle = (-1)^n \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (x^2-1)^n dx = \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (1-x^2)^n dx,$$

we get

$$\frac{\Gamma(n+1)\Gamma(1/2)2(2n)!}{2^{2n}(n!)^2\Gamma(n+3/2)}.$$

We use the properties of the Γ function together with the fact that $\Gamma(1/2) = \sqrt{\pi}$ to obtain

$$\frac{\sqrt{\pi}2(2n)!}{2^{2n}n!(n+1/2)\Gamma(n+1/2)}.$$

Let us consider

$$2(n+1/2)\Gamma(n+1/2) = (2n+1)\Gamma(n+1/2).$$

Next consider

$$2(n-1/2)\Gamma(n-1/2) = (2n-1)\Gamma(n-1/2).$$

Proceeding this way, the denominator becomes

$$2^n n!(2n+1)(2n-1)\dots 1\sqrt{\pi}.$$

However, now looking at the first part

$$2^n n! = 2n(2n-2)(2n-4)\dots 2.$$

So together we get

$$(2n+1)!\sqrt{\pi}.$$

Hence putting this in the denominator of the expression we had above, we have

$$\frac{\sqrt{\pi}2(2n)!}{(2n+1)!\sqrt{\pi}} = \frac{2}{2n+1}.$$

Corollary 2. *The Legendre polynomials are an orthogonal basis for $\mathcal{L}^2$ on the interval* $[-1,1]$.

Theorem 3. *The even degree Legendre polynomials $\{P_{2n}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$. The odd degree Legendre polynomials $\{P_{2n+1}\}_{n\in\mathbb{N}}$ are an orthogonal basis for $\mathcal{L}^2(0,1)$.*

Proof: Let f be defined on $[0,1]$. We can extend f to $[-1,1]$ either evenly or oddly. First, assume we have extended f evenly. Then, since $f \in \mathcal{L}^2$ on $[0,1]$,

$$\int_{-1}^1 |f_e(x)|^2 dx = 2\int_0^1 |f(x)|^2 dx < \infty.$$

Therefore f_e is in $\mathcal{L}^2$ on the interval $[-1,1]$. We have proven that the Legendre polynomials are an orthogonal basis. So, we can expand f_e in a Legendre polynomial series, as

$$\sum_{n\geq 0} \hat{f}_e(n) P_n,$$

where

$$\hat{f}_e(n) = \frac{\langle f_e, P_n \rangle}{||P_n||^2.}$$

By definition,

$$\langle f_e, P_n \rangle = \int_{-1}^{1} f_e(x) P_n(x) dx.$$

Since f_e is even, the product $f_e(x)P_n(x)$ is an *odd* function whenever n is odd. Hence all of the odd coefficients vanish. Moreover,

$$\langle f_e, P_{2n} \rangle = 2 \int_0^1 f(x) P_{2n}(x)) dx.$$

We also have

$$||P_{2n}||^2 = 2 \int_0^1 |P_{2n}(x)|^2 dx.$$

Consequently

$$f = \sum_{n \in \mathbb{N}} \left(\frac{\int_0^1 f(x) P_{2n}(x) dx}{\int_0^1 |P_{2n}(x)|^2 dx} \right) P_{2n}.$$

We can also extend f oddly. This odd extension satisfies

$$\int_{-1}^{1} |f_o(x)|^2 dx = \int_{-1}^{0} |f_o(x)|^2 dx + \int_0^1 |f_o(x)|^2 dx = 2 \int_0^1 |f_o(x)|^2 dx < \infty.$$

So, the odd extension is also in $\mathcal{L}^2$ on the interval $[-1, 1]$. We can expand f_o in a Legendre polynomial series, as

$$\sum_{n \geq 0} \hat{f}_o(n) P_n,$$

where

$$\hat{f}_o(n) = \frac{\langle f_o, P_n \rangle}{||P_n||^2.}$$

By definition,

$$\langle f_o, P_n \rangle = \int_{-1}^{1} f_o(x) P_n(x) dx.$$

Since f_o is odd, the product $f_o(x)P_n(x)$ is an *odd* function whenever n is *even.* Hence all of the even coefficients vanish. Moreover,

$$\langle f_o, P_{2n+1} \rangle = 2 \int_0^1 f(x) P_{2n+1}(x)) dx,$$

because the product of two odd functions is an even function. We also have

$$||P_{2n+1}||^2 = \int_{-1}^{0} |P_{2n+1}(x)|^2 dx + \int_0^1 |P_{2n+1}(x)|^2 dx = 2 \int_0^1 |P_{2n+1}(x)|^2 dx.$$

Consequently

$$f = \sum_{n \in \mathbb{N}} \left(\frac{\int_0^1 f(x) P_{2n+1}(x) dx}{\int_0^1 |P_{2n+1}(x)|^2 dx} \right) P_{2n+1}.$$

1.1. **Applications of Legendre polynomials to best approximations on bounded integrals.**

Exercise 1. *Find the polynomial $q(x)$ of at most degree 10 which minimizes the following integral*

$$\int_{-\pi}^{\pi} |q(x) - \sin(x)|^2 dx.$$

To do this exercise, we need different polynomials... If Legendre polynomials are orthogonal on $(-1, 1)$, can we somehow use them to create orthogonal polynomials on $(-\pi, \pi)$? Let's think about changing variables. How about setting

$$t = \frac{x}{\pi}.$$

Then,

$$\int_{-\pi}^{\pi} P_n(x/\pi)\overline{P_m(x/\pi)}dx = \int_{-1}^{1} P_n(t)\overline{P_m(t)}\pi dt = \begin{cases} 0 & n \neq m \\ \frac{2\pi}{2n+1} & n = m \end{cases}.$$

Therefore the polynomials

$$P_n(x/\pi)$$

are orthogonal on $x \in (-\pi, \pi)$, and their norms squared on that interval are

$$\frac{2\pi}{2n+1}.$$

The best approximation is therefore the polynomial

$$q(x) = \sum_{n=0}^{10} a_n P_n(x/\pi), \quad a_n := \frac{\int_{-\pi}^{\pi} \sin(x)\overline{P_n(x/\pi)}dx}{\frac{2\pi}{2n+1}}.$$

Exercise 2. *Find the polynomial $p(x)$ of degree at most 100 which minimizes the following integral*

$$\int_0^{10} |e^{x^2} - p(x)|^2 dx.$$

Yikes! Well, let's not panic just yet. The number 100 is even. Hence, we know that the even degree Legendre polynomials are an orthogonal basis for $\mathcal{L}^2(0, 1)$. So, we can use the even degree Legendre polynomials if we can just deal with this interval not being $(0, 1)$ but being $(0, 10)$. To figure this out, let's think about changing variables... As before, think about changing variables,

$$t = x/10,$$

so that

$$\int_0^{10} P_{2n}(x/10)P_{2m}(x/10)dx = \int_0^1 P_{2n}(t)P_{2m}(t)10dt = \begin{cases} 0 & n \neq m \\ \frac{10}{4n+1} & n = m \end{cases}$$

The last calculation we obtained by recalling our calculation

$$\int_{-1}^{1} |P_n(x)|^2 dx = (-1)^n \frac{(2n)!}{(2^n n!)^2} \int_{-1}^{1} (x^2-1)^n dx = \frac{2}{2n+1} \implies \int_0^1 |P_{2n}(x)|^2 dx = \frac{1}{4n+1}.$$

So, the functions $P_{2n}(x/10)$ are an orthogonal basis for $\mathcal{L}^2(0,10)$. Consequently the Best Approximation Theorem says that the best approximation is given by the polynomial

$$p(x) = \sum_{n=0}^{50} c_n P_{2n}(x/10), \quad c_n = \frac{\int_0^{10} e^{x^2} \overline{P_{2n}(x/10)} dx}{\frac{10}{4n+1}}.$$

Exercise 3. *Find the polynomial $p(x)$ of degree at most 99 which minimizes the following integral*

$$\int_0^{10} |e^{x^2} - p(x)|^2 dx.$$

Here, we can recycle our previous solution since 99 is odd, so we can use the odd degree Legendre polynomials in this case to form an orthogonal basis for $\mathcal{L}^2(0,10)$. Our polynomial shall be

$$p(x) = \sum_{n=0}^{49} c_n P_{2n+1}(x/10), \quad c_n = \frac{\int_0^{10} e^{x^2} \overline{P_{2n+1}(x/10)} dx}{\frac{10}{2(2n+1)+1}}.$$

1.2. **Legendre polynomials for best approximations on arbitrary intervals.** Let's consider a best approximation problem on an interval (a,b). First, we find its midpoint,

$$m = \frac{a+b}{2}.$$

Next, we find its length

$$\ell = \frac{b-a}{2}.$$

Then the interval

$$(a,b) = (m-\ell, m+\ell).$$

Since we know about the Legendre polynomials, P_n, on $(-1,1)$ since $x \mapsto \frac{x-m}{\ell} = t$ sends (a,b) to $(-1,1)$,

$$P_n\left(\frac{x-m}{\ell}\right) \qquad \text{are orthogonal on } (a,b).$$

In case this is not super obvious, let us compute using the substitution $t = \frac{x-m}{\ell}$,

$$\int_a^b P_n\left(\frac{x-m}{\ell}\right) P_k\left(\frac{x-m}{\ell}\right) dx = \int_{-1}^1 \ell P_n(t) P_k(t) dt = 0 \text{ if } n \neq k.$$

We have simply used substitution in the integral with $t = \frac{x-m}{\ell}$. So, these modified Legendre polynomials are orthogonal on (a,b). Moreover

$$\int_a^b P_n^2\left(\frac{x-m}{\ell}\right) dx = \int_{-1}^1 \ell P_n^2(t) dt = \ell ||P_n||^2 = \frac{2\ell}{2n+1}.$$

So, we simply expand the function f using this version of the Legendre polynomials. Let

$$c_n = \frac{\int_a^b f(x) P_n\left(\frac{x-m}{\ell}\right) dx}{\int_a^b [P_n((x-m)/\ell)]^2 dx}.$$

The best approximation amongst all polynomials of degree at most N is therefore

$$P(x) = \sum_{n=0}^{N} c_n P_n\left(\frac{x-m}{\ell}\right).$$

2. Les polynomes d'hermite

These polynomials shall be a basis for $\mathcal{L}^2(\mathbb{R})$ with respect to the weight function e^{-x^2}.

Definition 4. *The Hermite polynomials are defined to be*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Proposition 5. *The Hermite polynomials are polynomials with the degree of H_n equal to n.*

Proof: The proof is by induction. For $n = 0$, this is certainly true, as $H_0 = 1$. Next, let us assume that

$$\frac{d^n}{dx^n} e^{-x^2} = p_n(x) e^{-x^2},$$

is true for a polynomial, p_n which is of degree n. Then,

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = \frac{d}{dx}\left(p_n(x)e^{-x^2}\right) = p_n'(x)e^{-x^2} - 2xp_n(x)e^{-x^2} = \left(p_n'(x) - 2xp_n(x)\right) e^{-x^2}.$$

Let

$$p_{n+1} = p_n'(x) - 2xp_n(x).$$

Then we see that since p_n is of degree n, p_{n+1} is of degree $n+1$. Moreover

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = p_{n+1}(x) e^{-x^2}.$$

So, in fact, the Hermite polynomials satisfy:

$$H_0 = 1, \quad H_{n+1} = -\left(H_n'(x) - 2xH_n(x)\right).$$

Proposition 6. *The Hermite polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function e^{-x^2}. Moreover, with respect to this weight function $||H_n||^2 = 2^n n! \sqrt{\pi}$.*

Proof: Assume $n > m \geq 0$. We compute

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = \int_{\mathbb{R}} (-1)^n \frac{d^n}{dx^n} e^{-x^2} H_m(x) dx.$$

We use integration by parts n times, noting that the rapid decay of e^{-x^2} kills all boundary terms. We therefore get

$$\int_{\mathbb{R}} e^{-x^2} \frac{d^n}{dx^n} H_m(x) dx = 0.$$

This is because the polyhomial, H_m, is of degree $m < n$. Therefore differentiating it n times results in zero. Finally, for $n = m$, we have by the same integration by parts,

$$\int_{\mathbb{R}} H_n^2(x) e^{-x^2} dx = \int_{\mathbb{R}} e^{-x^2} \frac{d^n}{dx^n} H_n(x) dx.$$

The n^{th} derivative of H_n is just the n^{th} derivative of the highest order term. By our preceding calculation, the highest order term in H_n is

$$(2x)^n.$$

Differentiating n times gives
$$2^n n!.$$
Thus
$$\int_{\mathbb{R}} H_n^2(x)e^{-x^2}dx = 2^n n! \int_{\mathbb{R}} e^{-x^2}dx = 2^n n!\sqrt{\pi}.$$

We may wish to use the following lovely fact, but we shall not prove it.

Theorem 7. *The Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-x^2}.*

2.1. **Answers to the exercises to be done oneself.**

(1) (5.2.4) Demonstrate the identity:
$$\int_0^x sJ_0(s)ds = xJ_1(x), \quad \int_0^x J_1(s)ds = 1 - J_0(x).$$
Well, one of the recurrence formulas says
$$\frac{d}{dx}(xJ_1(x)) = xJ_0(x).$$
Thus a function whose derivative is equal to $sJ_0(s)$ is the function $xJ_1(x)$. Hence we can evaluate
$$\int_0^x sJ_0(s)ds = \left. sJ_1(s)\right|_{s=0}^{s=x} = xJ_1(x).$$
Another of the recurrence formulas says
$$\frac{d}{dx}J_0(x) = -J_1(x).$$
So,
$$\int_0^x J_1(s)ds = -\left. J_0(s)\right|_{s=0}^{s=x} = J_0(0) - J_0(x) = 1 - J_0(x).$$

(2) (5.5.1) A cylinder of radius b is initially at the constant temperature A. Find the temperatures in it at subsequent times if its ends are insulated and its circular surface obeys Newton's law of cooling, $u_r + cu = 0$, $(c > 0)$. Answer:
$$u(r,t) = 2A\sum_{k\geq 1} \frac{\lambda_k J_1(\lambda_k)}{(\lambda_k^2 + b^2c^2)J_0(\lambda_k)^2} J_0\left(\frac{\lambda_k r}{b}\right) e^{-\lambda_k^2 t/b^2},$$
where λ_k is the $k^t h$ positive solution to
$$\lambda_k J_0'(\lambda_k) + bcJ_0(\lambda_k) = 0.$$

(3) (5.5.5) Solve the problem
$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = 0 \text{ in } D = \{(r,\theta,z) : 0 \leq r \leq b, 0 \leq z \leq l\}$$
$$u(r,\theta,0) = 0, \quad u(r,\theta,l) = g(r,\theta), \quad u(b,\theta,z) = 0.$$
Answer:
$$u(r,\theta,z) = \sum_{n\geq 0}\sum_{k\geq 1}(a_{kn}\cos n\theta + b_{kn}\sin n\theta)J_n\left(\frac{\lambda_{k,n} r}{b}\right)\sinh\left(\frac{\lambda_{k,n} z}{b}\right),$$

where

$$b_{k,n} = \frac{2}{b^2 \pi \sinh \lambda_{k,n}} \int_{-\pi}^{\pi} \int_0^b g(r\theta) \frac{J_n(\lambda_{k,n} r)}{J_{n+1}(\lambda_{k,n})^2} \sin n\theta r dr d\theta,$$

and similarly for $a_{k,n}$ where $\lambda_{k,n}$ is the k^{th} positive zero of J_n.

(4) (5.5.6) Find the steady-state temperature in the cylinder $0 \le r \le 1$, $0 \le z \le 1$ when the circular surface is insulated, the bottom is kept at temperature 0, and the top is kept at temperature $f(r)$. Answer:

$$u(r, z) = a_0 z + \sum_{k \ge 1} a_k J_0(\lambda_k r) \sinh(\lambda_k z),$$

where λ_k is the k^{th} positive zero of J_0,

$$a_0 = 2 \int_0^1 r f(r) dr,$$

and

$$a_k = \frac{2}{J_0(\lambda_k)^2 \sinh \lambda_k} \int_0^1 r f(r) J_0(\lambda_k r) dr, \quad k > 0.$$

(5) Eö 29 (answer is in there!)

(6) Eö 35 (answer is in there!)

FOURIER ANALYSIS & METHODS 2020.03.09

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. The generating function for the Hermite polynomials

This theory item is similar to the analogous result for the Bessel functions, but with a bit of a twist.

Theorem 1. *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

satisfy

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Proof: The key idea with which to begin is to consider instead

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2}.$$

We consider the Taylor series expansion of this guy, with respect to z, viewing x as a parameter. By definition, the Taylor series expansion for

$$e^{-(x-z)^2} = \sum_{n \geq 0} a_n z^n,$$

where

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} e^{-(x-z)^2}, \quad \text{evaluated at } z = 0.$$

To compute these coefficients, we use the chain rule, introducing a new variable $u = x - z$. Then,

$$\frac{d}{dz} e^{-(x-z)^2} = -\frac{d}{du} e^{-u^2},$$

and more generally, each time we differentiate, we get a -1 popping out, so

$$\frac{d^n}{dz^n} e^{-(x-z)^2} = (-1)^n \frac{d^n}{du^n} e^{-u^2},$$

Hence, evaluating with $z = 0$, we have

$$a_n = \frac{1}{n!} (-1)^n \frac{d^n}{du^n} e^{-u^2}, \quad \text{evaluated at } u = x.$$

The reason it's evaluated at $u = x$ is because in our original expression we're expanding in a Taylor series around $z = 0$ and $z = 0 \iff u = x$ since $u = x - z$. Now, of course, we have

$$\frac{d^n}{du^n} e^{-u^2}, \quad \text{evaluated at } u = x = \frac{d^n}{dx^n} e^{-x^2}.$$

Hence, we have the Taylor series expansion

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2} = \sum_{n \geq 0} \frac{z^n}{n!} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Now, we multiply both sides by e^{x^2} to obtain

$$e^{2xz-z^2} = e^{x^2} \sum_{n \geq 0} \frac{z^n}{n!} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

We can bring e^{x^2} inside because everything converges beautifully. Then, we have

$$e^{2xz-z^2} = \sum_{n \geq 0} \frac{z^n}{n!} e^{x^2} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Voilà! The definition of the Hermite polynomials is staring us straight in the face! Hence, we have computed

$$e^{2xz-z^2} = \sum_{n \geq 0} \frac{z^n}{n!} H_n(x).$$

1.1. **Applications to best approximations.**

Exercise 1. *Find the polynomial of at most degree* 40 *which minimizes*

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-x^2} dx,$$

where f is some function in the weighted $\mathcal{L}^2$ space on $\mathbb{R}$ with weight e^{-x^2}.

We know that the Hermite polynomials are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with the weight function e^{-x^2}. We see that same weight function in the integral. Therefore, we can rely on the theory of the Hermite polynomials! Consequently, we define

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(x) e^{-x^2} dx}{||H_n||^2},$$

where

$$||H_n||^2 = \int_{\mathbb{R}} H_n^2(x) e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{40} c_n H_n(x).$$

Some variations on this theme are created by changing the weight function.

Exercise 2. *Find the polynomial of at most degree* 60 *which minimizes*

$$\int_{\mathbb{R}} |f(x) - P(x)|^2 e^{-2x^2} dx.$$

This is not the correct weight function for H_n. However, we can make it so. The correct weight function for $H_n(x)$ is e^{-x^2}. So, if the exponential has $2x^2 = (\sqrt{2}x)^2$, then we should change the variable in H_n as well. We will then have, via the substitution $t = \sqrt{2}x$,

$$\int_{\mathbb{R}} H_n(\sqrt{2}x) H_m(\sqrt{2}x) e^{-2x^2} dx = \int_{\mathbb{R}} H_n(t) H_m(t) e^{-t^2} \frac{dt}{\sqrt{2}} = 0, \quad n \neq m.$$

Moreover, the norm squared is now

$$\int_{\mathbb{R}} H_n^2(t) e^{-t^2} \frac{dt}{\sqrt{2}} = \frac{||H_n||^2}{\sqrt{2}} = \frac{2^n n! \sqrt{\pi}}{\sqrt{2}}.$$

Consequently, the functions $H_n(\sqrt{2}x)$ are an orthogonal basis for $\mathcal{L}^2$ on $\mathbb{R}$ with respect to the weight function e^{-2x^2}. We have computed the norms squared above. The coefficients are therefore

$$c_n = \frac{\int_{\mathbb{R}} f(x) H_n(\sqrt{2}x) e^{-2x^2} dx}{2^n n! \sqrt{\pi}/\sqrt{2}}.$$

The polynomial is

$$P(x) = \sum_{n=0}^{60} c_n H_n(\sqrt{2}x).$$

2. The Laguerre polynomials

The Laguerre polynomials come from understanding the quantum mechanics of the hydrogen atom. We shall not get into this[1]

Definition 2. *The Laguerre polynomials,*

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x}).$$

We summarize their properties in the following

Theorem 3 (Properties of Laguerre polynomials)**.** *The Laguerre polynomials are an orthogonal basis for $\mathcal{L}^2$ on $(0, \infty)$ with the weight function $x^\alpha e^{-x}$. Their norms squared,*

$$||L_n^\alpha||^2 = \frac{\Gamma(n + \alpha + 1)}{n!}.$$

They satisfy the Laguerre equation

$$[x^{\alpha+1} e^{-x} (L_n^\alpha)']' + n x^\alpha e^{-x} L_n^\alpha = 0.$$

For $x > 0$ and $|z| < 1$,

$$\sum_{n=0}^{\infty} L_n^\alpha(x) z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}}.$$

[1] Alex Jones does get into it: `https://www.youtube.com/watch?v=i91XV07Vsc0`. Check out the Alex Jones Prison Planet `https://www.youtube.com/watch?v=kn_dHspHd8M`. Turns out that Alex Jones's crazy ranting makes for decent death metal vocals. The gay frogs and America first remix are pretty decent too.

Exercise 3. *Find the polynomial of at most degree* 7 *which minimizes*

$$\int_0^\infty |f(x) - P(x)|^2 x^\alpha e^{-x} dx.$$

Since the Laguerre polynomials are an orthogonal basis for $\mathcal{L}^2(0,\infty)$ with weight function $x^\alpha e^{-x}$, we define

$$c_n = \frac{\int_0^\infty f(x) L_n^\alpha(x) x^\alpha e^{-x} dx}{||L_n^\alpha||^2}.$$

The polynomial we seek is:

$$P(x) = \sum_{n=0}^{7} c_n L_n^\alpha(x).$$

3. Best approximation summary

Assume that based on theoretical considerations we know that a certain collection of functions

$$e^{inx}, \quad \cos, \quad \sin, \quad \text{orthogonal polynomials}, \quad \text{Bessel functions}, \quad \text{weird SLP functions},$$

are an orthogonal basis on a bounded interval. In the case of SLP functions, *do not forget the weight function* in case the weight function is not simply 1. Let us call such function ϕ_n. Then the best approximation to any f in $\mathcal{L}^2$ of the bounded interval under consideration is its Fourier-ϕ_n expansion, which is

$$\sum \frac{\langle f, \phi_n \rangle}{||\phi_n||^2} \phi_n(x).$$

Recall

$$\langle f, \phi_n \rangle = \int f(y) \overline{\phi_n(y)} w(y) dy, \quad \text{if the weight function is } w(y),$$

and

$$||\phi_n||^2 = \langle \phi_n, \phi_n \rangle.$$

One can also do best approximations using Hermite and Laguerre polynomials on $\mathbb{R}$ and $(0,\infty)$, respectively, with the weight functions e^{-x^2} and $x^\alpha e^{-x}$, respectively. It works in very much the same way in all these cases.

4. Distributions done the right way

The mathematical concept of a *distribution,* or, as they are sometimes called, *generalized function,* has been badly abused not only by physicists but also by mathematicians. You may have already heard about the so-called "delta function." It's not really a function. It's not a 'generalized function.' It has its very own terminology, and that is that it is a distribution. Now, distributions are not as mysterious and weird as the mystique in which they are often shrouded.

Distributions are functions which themselves take as input a function. A particularly nice class of distributions are the *tempered distributions.* These distributions take in a Schwarz class function and spit out a number.

Definition 4. *Assume that f is a smooth function on $\mathbb{R}$. Then, we say that $f \in \mathcal{S}$ if for all k and for all n,*

$$\lim_{|x|\to\infty} x^n f^{(k)}(x) = 0.$$

In other words, f and all its derivatives decay rapidly at $\pm\infty$. There are quite a few functions which satisfy this. For example, all smooth functions which live on a bounded interval (compactly supported) satisfy this property.

Exercise 4. *Show that if $f \in \mathcal{S}$ then all of the derivatives of f are in $\mathcal{S}$. Show that if $f \in \mathcal{S}$ then its Fourier transform is also in $\mathcal{S}$.*

Definition 5. *A* tempered distribution *is a function which maps $\mathcal{S}$ to $\mathbb{C}$, which satisfies the following conditions:*

- *It is linear, so for a distribution denoted by L, we have*
$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$
for all f and g in $\mathcal{C}_c^\infty(\mathbb{R})$ and for all complex numbers α and β.
- *There is a non-negative integer N and a constant $C \geq 0$ such that for all $f \in \mathcal{S}$*
$$|L(f)| \leq C \sum_{j+k \leq N} \sup_{x \in \mathbb{R}} |x^j f^{(k)}(x)|.$$

Let's do an example. We define a distribution in the following way. For $f \in \mathcal{C}_c^\infty(\mathbb{R})$,
$$L(f) := f(0).$$
That is, the distribution takes in the function, f, and spits out the value of f at the point $0 \in \mathbb{R}$. This distribution satisfies for any f and g in $\mathcal{C}_c^\infty(\mathbb{R})$ and for any α and $\beta \in \mathbb{C}$,
$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g).$$
Moreover, we have the estimate that
$$|L(f)| \leq |f(0)| \leq \sup_{x \in \mathbb{R}} |f(x)|.$$
So the estimate required is satisfied with $N = 0$ and $C = 1$. This distribution has a name. It is called the *delta* distribution. It is usually written with the letter δ. It is nothing other than a function which takes a function as its input and spits out a number as its output.

Exercise 5. *Assume that $f \in \mathcal{C}_c^\infty(\mathbb{R})$. Show that by defining*
$$L_f(g) = \int_{\mathbb{R}} f(x)g(x)dx, \quad g \in \mathcal{C}_c^\infty(\mathbb{R}),$$
L_f is a tempered distribution.

In fact, the assumption that $f \in \mathcal{C}_c^\infty(\mathbb{R})$ wasn't even necessary. You can show that for $f \in \mathcal{L}^2(\mathbb{R})$ or $f \in \mathcal{L}^1(\mathbb{R})$, the distribution, L_f defined above (it takes in a function $g \in \mathcal{C}_c^\infty(\mathbb{R})$ and integrates the product with f over $\mathbb{R}$), is a distribution. So, here's something which is rather cool. The elements in $\mathcal{L}^2(\mathbb{R})$ and $\mathcal{L}^1(\mathbb{R})$ are in general *not* differentiable at all. However, the *distributions* we can make out of them *are* differentiable. Here's how we do that.

Definition 6. *The derivative of a tempered distribution, L is another tempered distribution, denoted by $L' \in \mathcal{D}(\mathbb{R})$, which is defined by*
$$L'(g) = -L(g'), \quad g \in \mathcal{S}.$$

To see that this definition makes sense, we think about the special case where $L = L_f$, and $f \in \mathcal{S}$. Then, we *can* take the derivative of f, and it is also an element of $\mathcal{S}$. So, we can define $L_{f'}$ in the analogous way. Let's write it down when it takes in $g \in \mathcal{S}$,

$$L_{f'}(g) = \int_{\mathbb{R}} f'(x)g(x)dx.$$

We can do integration by parts. The boundary terms vanish, so we get

$$L_{f'}(g) = \int_{\mathbb{R}} f'(x)g(x)dx = -\int_{\mathbb{R}} f(x)g'(x)dx.$$

So,

$$L_{f'}(g) = -L_f(g') = (L_f)'(g).$$

This is why it makes a lot of sense to define the derivative of a distribution in this way. For the heavyside function, we define

$$L_H, \quad L_H(g) = \int_0^\infty g(x)dx.$$

Then, we compute that

$$L'_H(g) = -L_H(g') = -\int_0^\infty g'(x)dx.$$

Due to the fact that $g \in \mathcal{S}$,

$$\lim_{x\to\infty} g(x) = 0.$$

Hence, we have

$$-\int_0^\infty g'(x)dx = -(0 - g(0)) = g(0) = \delta(g).$$

So, we see that the derivative of L_H is the δ distribution! Pretty neat!

In this way, distributions can solve differential equations! For example, we'd say that a distribution L satisfies the equation

$$L'' + \lambda L = 0$$

if, for every $g \in \mathcal{S}$ we have

$$L''(g) + \lambda L(g) = 0.$$

This turns out to be incredibly useful and important in the theory of partial differential equations. However, the way it usually works is that instead of actually finding a distribution which solves the PDE, one shows by abstract mathematics that there *exists* a distribution which solves the PDE. Then, one can use clever methods to show that the mere existence of a distribution solving the PDE, which is called a *weak solution,* actually implies that there exists a genuinely differentiable solution to the PDE. We don't want to get ahead of ourselves here, so conclude with one last exercise, which proves that you can differentiate distributions as many times as you like!

Exercise 6. *Use induction to show that you can differentiate a distribution as many times as you like, by defining*

$$L^{(k)}(g) := (-1)^k L(g^{(k)}).$$

In a similar way, we can define the Fourier transform of a distribution.

Definition 7. *Assume that L is a tempered distribution. The Fourier transform of L is the distribution, $\hat{L}$ which for $f \in \mathcal{S}$ acts as follows*

$$\hat{L}(f) := L(\hat{f}).$$

In this way, we can compute the Fourier transform of our favorite distribution, δ.

$$\hat{\delta}(f) := \delta(\hat{f}) = \hat{f}(0) = \int_{\mathbb{R}} f(x)dx.$$

So, we could think of the Fourier transform of δ as the distribution which acts by

$$\hat{\delta} : f \in \mathcal{S} \mapsto \int_{\mathbb{R}} f(x)dx.$$

On the other hand, by the FIT,

$$\delta(f) = f(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)d\xi = \frac{1}{2\pi}\hat{\delta}(\hat{f}) = \frac{1}{2\pi}\hat{\hat{\delta}}(f).$$

So that's kind of cute. It says that

$$\delta = \frac{1}{2\pi}\hat{\hat{\delta}}.$$