

FOURIER ANALYSIS - A collection of techniques for solving the most important PDEs of physics (chemistry, engineering)
Heat, waves, energy, quantum physics, mechanics.

- EXAMPLES
- ① $\partial_t - \Delta$ heat operator, Δ Laplace operator
 $\Delta = \operatorname{div} \nabla$, on \mathbb{R}^n $\Delta = \partial_x^2$
 - ② $\partial_t^2 - \Delta$ wave operator, \square
 - ③ $\Delta u + \lambda u = 0$ for some function u , $\lambda \in \mathbb{C}$

PDEs Technique zero Turn PDE into ODE

Separation of variables

Ex. Vibrating string of length = L. Identify with $[0, L] \subset \mathbb{R}'$

Let $u(x, t) =$ height at point $x \in [0, L]$ at time $t \geq 0$

~~All~~ ends are fixed (immobile)

Natural height is zero $\Rightarrow u(0, t) = u(L, t) = 0 \quad \forall t \geq 0$

Vibration \Leftrightarrow wave equation: $u_{xx} = c^2 u_{tt}$

We may assume that constant $c=1$. Why? We may change the units of time, i.e. let $\tau = ct \Rightarrow u_{xx} = u_{\tau\tau}$

Assume (separation of variables) $u(x, t) = f(x)g(t)$

The equation is then $\partial_x^2(f(x)g(t)) = \partial_t^2(f(x)g(t))$

$$\frac{f''(x)g(t)}{f(x)g(t)} = \frac{f(x)g''(t)}{f(x)g(t)} \quad (\text{divide by } fg)$$

$$\Rightarrow \lambda = \frac{f''(x)}{f(x)} = \frac{g''(t)}{g(t)} \Rightarrow \text{both sides must be constant.}$$

Deal with $\frac{f''(x)}{f(x)}$ first because we know more about x.

$$u(0, t) = f(0)g(t) = 0 \quad \forall t \Rightarrow f(0) = 0$$

Similarly $f'(l) = 0$. We have $f''(x) = \lambda f(x)$, $f(0) = 0 = f(l)$

What can λ be?

$\lambda = 0$: $f''(x) = 0 \Rightarrow f(x) = ax + b$, $a, b \in \mathbb{R}$, $b = 0$, $a = 0$
 Thus the only solution for $\lambda = 0$ is $f(x) = 0$. Boring!

$\lambda > 0$: $f''(x) = \lambda f(x) \Rightarrow f(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$ (immobile string)

Exercise: Show that the only values of a, b such that $f(0) = f(l) = 0$ are $a = b = 0$

$\lambda < 0$: $f''(x) = \lambda f(x) \Rightarrow f(x) = a \cos(\sqrt{|\lambda|}x) + b \sin(\sqrt{|\lambda|}x)$

$f(0) = a = 0$, $f(l) = b \sin(\sqrt{|\lambda|}l) = 0$, don't want $b = 0$ Boring!
 $\sin(\sqrt{|\lambda|}l) = 0 \Leftrightarrow \sqrt{|\lambda|}l = k\pi$ for some $k \in \mathbb{Z}$, $|\lambda| = \frac{k^2\pi^2}{l^2}$ $k \geq 1$

$$\lambda = -\frac{k^2\pi^2}{l^2}$$

Thus λ can be $-\frac{k^2\pi^2}{L^2}$ for any $k \in \mathbb{N}$

Let us write $\lambda_k = -\frac{k^2\pi^2}{L^2}$ (higher notes - shorter strings)

$f_k(x) = b_k \sin\left(\frac{k\pi x}{L}\right) \rightarrow g_k$ satisfies $g_k''(t) = -\frac{k^2\pi^2}{L^2} g_k(t)$

Thus $g_k(t) = \alpha_k \cos\left(\frac{k\pi t}{L}\right) + \beta_k \sin\left(\frac{k\pi t}{L}\right)$

(Later): Solution: $u(x, t) = \sum_{k \geq 1} \left(\alpha_k \cos\left(\frac{k\pi t}{L}\right) + \beta_k \sin\left(\frac{k\pi t}{L}\right) \right) \sin\left(\frac{k\pi x}{L}\right)$
determined by $u(x, 0)$ and $u_t(x, 0)$

Linear PDEs of order two

DEF A second order linear PDE for an unknown func. u of n variables is an equation for u and its partial derivatives up to order two, of the form $L(u) = f$, f is known function given in the PDE, and L is a PD operator

$$L(u) = a(x) u(x) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}(x) + \sum_{i,j=1}^n c_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$

for known (given functions) $a, b_i, c_{ij} \quad x \in \mathbb{R}^n$

Why is it called linear? $L(u+v) = L(u) + L(v)$ because $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear. For any constant λ , $L(\lambda u) = \lambda L(u)$. $L(au+ bv) = aL(u) + bL(v)$

"Superposition principle" If $L(u) = f$ and $L(v) = g$ then $u+v$ is a solution to $L(u+v) = f+g$. Especially important is $L(u) = 0$. If $L(v) = 0 \Rightarrow L(u+v) = 0$

Ex. $u_k(x, t) = f_k(x) g_k(t)$ satisfies $L(u_k) = 0$ where $L = \underline{\partial_x^2 - \partial_t^2}$

$$\Rightarrow L\left(\sum_{k \geq 1} u_k\right) = 0$$

this is a linear 2nd order PDE for an unknown func. of $n=2$ (x, t) variables

THM (Recall)

A basis (recall from linear algebra) of solutions to the ODE $au'' + bu' + cu = 0 \quad a, b, c \in \mathbb{R}$ is

(i) $\{e^{rx}, e^{r_2 x}\}$

(ii) $\{e^{rx}, x e^{r_2 x}\}$

(iii) $\{\sin(I(r)x) e^{R(r)x}, \cos(I(r)x) e^{R(r)x}\}$

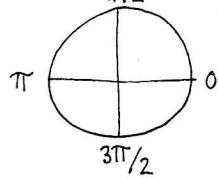
(i) $b^2 > 4ac, r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(ii) $b^2 = 4ac, r = \frac{-b}{2a}$

(iii) $b^2 < 4ac, r = -\frac{b}{2a} + \frac{i}{2a} \sqrt{2ac - b^2},$

$$\boxed{\begin{aligned} I(r) &= \text{Im}(r) \\ R(r) &= \text{Re}(r) \end{aligned}}$$

Heat flow on a rod bent into a circle



use $x \in [0, 2\pi]$ to indicate position on the rod

$u(x, t)$ = temp at point x at time t .

No sources, no sinks \Rightarrow Heat eq: $u_t = u_{xx} \Leftrightarrow u_t - u_{xx} = 0$

Separate variables (technique 0) : $u(x, t) = f(x)g(t)$

$$\text{Eq: } g'(t)f(x) - g(t)f''(x) = 0 \Rightarrow \frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} \Rightarrow \text{both sides constant} = \lambda$$

$$f''(x) = \lambda f(x)$$

Rod $\cong \mathbb{R}$, but identify points which are 2π apart.

$$f(x + 2\pi) = f(x) \quad \forall x$$

Exercise: Show that the only solutions with $\lambda \geq 0$ and $f(x + 2\pi) = f(x)$ is $f(x) \equiv 0$.

$\Rightarrow \lambda < 0$ (if we want non trivial solution)

$$\Rightarrow f(x) = a \cos(\sqrt{|\lambda|}x) + b \sin(\sqrt{|\lambda|}x) \quad 2\pi \text{ periodic.}$$

$$2\pi \text{ periodic} \Rightarrow \sqrt{|\lambda|} \in \mathbb{N}, \quad \lambda = -n^2 \text{ for } n \in \mathbb{N}$$

$$\text{Let } \lambda_n = -n^2, \quad f_n = a_n \cos(nx) + b_n \sin(nx)$$

$$g_n \text{ satisfies } g_n'(t) = \lambda_n g_n(t) = -n^2 g_n(t) \Rightarrow g_n(t) = \alpha_n e^{-n^2 t}$$

All solutions that can be found by this method

$$u_n(x, t) = f_n(x)g_n(t), \text{ satisfies } \partial_t u_n - \partial_{xx} u_n = 0$$

$$\Rightarrow \sum_{n \geq 1} u_n(x, t) \text{ also satisfies } \partial_t - \partial_{xx} = 0$$

$$u(x, t) = \sum_{n \geq 1} e^{-n^2 t} (a_n \cos(nx) + b_n \sin(nx)) \text{ satisfies } \partial_t u - \partial_{xx} u = 0$$

$$\text{If we know } u(x, 0) = \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx) \Rightarrow$$

\Rightarrow we know $u(x, t) \quad \forall t > 0$. ① Given $u(x, 0)$ can we find $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ such that $u(x, 0) = \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$?

② How do we find them? What are they?

F2

2.1 FOURIER SERIES OF PERIODIC FUNCTIONS

DEF $f: \mathbb{R} \rightarrow \mathbb{R}$ is periodic with period p if $\forall x \in \mathbb{R} \quad f(x+p) = f(x)$

t.ex. $f(x) = \sin x \quad p = 2\pi, \quad f(x) = \cos(\frac{x}{2}) \quad p = 4\pi$

Lemma 2.1 Integration of periodic func.

If f is periodic with period p , then $\int_a^{a+p} f(x) dx$ has the same value $\forall a \in \mathbb{R}$.

Proof: Assume that f is continuous

Idea: want to use fund. thm. calculus.

$$F(x) = \int_x^{x+p} f(t) dt = \int_0^{x+p} f(t) dt - \int_0^x f(t) dt$$

Want to show that F is constant

$$\text{Differentiate: } F'(x) = f(x+p) - f(x) = 0$$

This is always 0 thus F is a constant

$\therefore F(x)$ has the same value $\forall x \in \mathbb{R}$ ■

Errata: for the heat eq. we also have the $n=0$ solution

$$\Rightarrow u(x, 0) = \sum_{n=0}^{\infty} e^{-n\omega t} (a_n \cos(nx) + b_n \sin(nx))$$

Motivation for Fourier Series

Solve PDEs like the wave and heat eq. on compact sets (for the x variable)

i.e. $\begin{cases} [0, l] & x \in [0, l] \\ \textcircled{O} & x \in [0, 2\pi], \text{ etc.} \end{cases}$ NOT $x \in \mathbb{R}$

We would like to find $\{a_n\}, \{b_n\}$ given $u_0(x)$ such that

$$u_0(x) = \frac{a_0}{2} + \underbrace{\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)}_{\text{linear combination}}$$

Idea: express u_0 in terms of an ONB of functions

$$\text{Recall: } \cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad x \in \mathbb{R}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\text{Exercise: ① We can write } u_0(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)}$$

$$\Leftrightarrow \text{can write } u_0(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{② } c_n = \frac{1}{2} (a_n - ib_n) \quad \text{for } n \geq 1$$

$$c_n = \frac{1}{2} (a_n + ib_n) \quad \text{for } n \leq -1$$

$$c_0 = \frac{a_0}{2}$$

$$\text{③ Equivalently, } \begin{aligned} a_n &= c_n + c_{-n}, & n \geq 0 \\ b_n &= i(c_n - c_{-n}), & \forall n \geq 0 \end{aligned}$$

Fourier series
of $u_0(x)$

In linear algebra, write $\vec{u}_o = \sum_{k=1}^n \langle \vec{u}_o, \vec{v}_k \rangle \vec{v}_k$ where $\vec{u}_o \in \mathbb{R}^n$
 and $\{\vec{v}_k\}_{k=1}^n$ are an ONB. (orthonormal basis)

We would like to do the same for functions.
 ∵ Need an inner (scalar) product for functions \langle , \rangle
 (Hilbert spaces - more to come in ch 3)

$$\text{DEF } \langle f, g \rangle = \int f \bar{g}$$

For now, 2π periodic functions, use $\int_{-\pi}^{\pi} f(x) \bar{g(x)} dx$

Lemma $\{e^{inx}\}_{n \in \mathbb{Z}}$ are orthogonal i.e. $\langle e^{inx}, e^{imx} \rangle = 0$ if $n \neq m$

$$\begin{aligned} \text{By definition } \langle e^{inx}, e^{imx} \rangle &= \int_{-\pi}^{\pi} e^{inx} \bar{e^{imx}} dx = \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = \\ &= \int_{-\pi}^{\pi} e^{i(n-m)x} dx = \left[\frac{e^{i(n-m)x}}{i(n-m)} \right]_{-\pi}^{\pi} = 0 \quad \text{by } 2\pi \text{ periodicity} \end{aligned}$$

$$\text{DEF } \|f\|_{L^2}^2 = \int_{-\pi}^{\pi} |f|^2 = \langle f, f \rangle$$

$$\begin{aligned} \langle e^{inx}, e^{inx} \rangle &= \int_{-\pi}^{\pi} e^{inx} \bar{e^{inx}} dx = \int_{-\pi}^{\pi} e^{inx} e^{-inx} dx = \int_{-\pi}^{\pi} |e^{inx}|^2 dx = \int_{-\pi}^{\pi} 1 dx = 2\pi \\ \Rightarrow \|e^{inx}\|_{L^2}^2 &= 2\pi, \|e^{inx}\|_{L^2} = \sqrt{2\pi} \end{aligned}$$

Thus $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}$ is an orthonormal set

We will also see that it is an ONB \Rightarrow Any $u_o \in L^2$ can be expressed as a linear combination of $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}$, $u_o(x) = \sum_{n \in \mathbb{Z}} \langle u_o, \frac{e^{inx}}{\sqrt{2\pi}} \rangle \frac{e^{inx}}{\sqrt{2\pi}}$

From this morning found solutions $u_n(x, t) = f_n(x) g_n(t) = e^{-n^2 t} (a_n \cos(nx) + b_n \sin(nx))$

All satisfy $\partial_t u_n - \partial_{xx} u_n = 0$
 Solution depends on $u(x, 0) = u_o(x)$
 $\sum_{n \geq 0} u_n(x, t)$ also solution.

$$\text{Want } \sum_{n \geq 0} u_n(x, 0) = u_o(x) = \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx)$$

The coefficients $\{c_n\}$, $\{a_n\}$, $\{b_n\}$ are "Fourier coefficients"

$$u_o(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \Rightarrow c_n = \frac{\langle u_o, e^{inx} \rangle}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_o(x) e^{-inx} dx$$

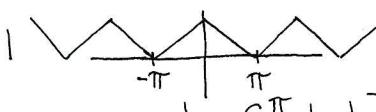
"Fourier series" of u_o

Lemma If f is even, $b_n = 0 \quad \forall n$
 If f is odd, $a_n = 0 \quad \forall n \geq 1$
 $a_0 = 2c_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

Exercise: Proof!

CONVERGENCE As long as $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \Rightarrow$
 $\Rightarrow \sum_{n \in \mathbb{Z}} c_n e^{inx}$ with $c_n = \frac{\langle f, e^{inx} \rangle}{2\pi}$ converges "almost everywhere"

If f is C^1 \Rightarrow converges everywhere, that is $\forall x$

Ex. ① $f(x) = |x|$ 

By definition $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{inx} dx = \frac{1}{2\pi} \left(\int_{-\pi}^0 -xe^{-inx} dx + \int_0^{\pi} xe^{-inx} dx \right)$

Check that we get the same $\int_0^{2\pi}$

$= \frac{1}{2\pi} \left(\left[\frac{-xe^{-inx}}{-in} \right]_{-\pi}^0 - \int_{-\pi}^0 \frac{(-1)e^{-inx}}{-in} dx + \left[\frac{xe^{-inx}}{-in} \right]_0^{\pi} - \int_0^{\pi} \frac{(1)e^{-inx}}{-in} dx \right) =$

$= \frac{1}{2\pi} \left(\left[\frac{-e^{-inx}}{(-in)} \right]_{-\pi}^0 + \left[\frac{e^{-inx}}{(-in)} \right]_0^{\pi} \right) = \begin{cases} 0 & n \text{ even} \\ -\frac{2}{\pi n^2} & n \text{ odd} \end{cases}$

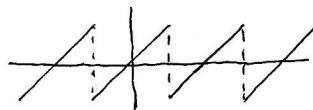
$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{\pi}{2}$$

$$a_n = c_n + c_{-n} = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{\pi n^2}, & n \text{ odd} \end{cases}$$

$$\Rightarrow |x| = \frac{\pi}{2} + \sum_{\substack{n \geq 1 \\ \text{odd}}} \frac{-4}{\pi n^2} \cos(nx) = \frac{\pi}{2} + \sum_{\substack{n \in \mathbb{Z} \\ \text{odd}}} e^{inx} \left(\frac{-2}{\pi n^2} \right) \quad \text{converges!}$$

Exercise: compute $\sum_{\substack{n \geq 1 \\ \text{odd}}} \frac{1}{n^2}$

② $f(x) = x$, $x \in (-\pi, \pi)$. Extend to be 2π periodic but discontinuous at $\pm k\pi$, $k \in \mathbb{N}$



f is odd $\Rightarrow a_n = 0 \quad \forall n$

Exercise: show that $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$

and $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx =$$

$$= \frac{2}{\pi} \left(\left[\frac{-x \cos(nx)}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos(nx)}{n} dx \right) =$$

$$= \frac{2(-1)^{n+1}}{n} \Rightarrow x = \sum_{n \geq 1} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

converges but rather slowly

THM Bessel inequality

Let f be 2π periodic, then the Fourier coefficients

$$\text{satisfy } \sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2$$

If $\int_{-\pi}^{\pi} |f|^2 = \infty$, done

Otherwise, assume finite.

$$\text{Use } \Sigma = \sum_{n=-N}^N$$

$$0 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f - \Sigma c_n e^{inx}|^2$$

$$\begin{aligned} 0 &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (f - \Sigma)(\bar{f} - \bar{\Sigma}) = \frac{1}{2\pi} |f|^2 - (\Sigma) \bar{f} - f \bar{\Sigma} + \Sigma \bar{\Sigma} = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 - \Sigma c_n \int_{-\pi}^{\pi} e^{inx} \bar{f} - \Sigma \bar{c}_n \int_{-\pi}^{\pi} f \bar{e}^{inx} + \int_{-\pi}^{\pi} \Sigma \bar{\Sigma} = (*) \end{aligned}$$

$$2\pi c_n = \int_{-\pi}^{\pi} e^{-inx} f \Rightarrow 2\pi \bar{c}_n = \int_{-\pi}^{\pi} \bar{f} e^{inx}$$

$$\Rightarrow (*) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 - 2\pi \Sigma |c_n|^2$$

$$0 \leq (*) \Rightarrow 2\pi \Sigma |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2$$

Corollary to Bessel's inequality

$$\textcircled{1} \sum_{n \geq 0} |a_n|^2 + |b_n|^2 \leq \sum_{n \in \mathbb{Z}} 2|c_n|^2 \leq C \int_{-\pi}^{\pi} |f|^2$$

\textcircled{2} a_n, b_n and $c_n \rightarrow 0$ as $|n| \rightarrow \infty$, as long as $\int_{-\pi}^{\pi} |f|^2 < \infty$

Proof \textcircled{1} $|a_n|^2 + |b_n|^2 = a_n \bar{a}_n + b_n \bar{b}_n = (c_n + c_{-n})(\bar{c}_n + \bar{c}_{-n}) + i(c_n - c_{-n})(-i)(\bar{c}_n - \bar{c}_{-n})$

$$\text{Expand } \Rightarrow = 2(|c_n|^2 + |c_{-n}|^2)$$

\textcircled{2} The terms in a convergent sum (or series) tend to 0. ■

By Bessel's inequality, $\left\| \sum_{-N}^N c_n e^{inx} - f(x) \right\|_{L^2} \xrightarrow{N \rightarrow \infty} 0$

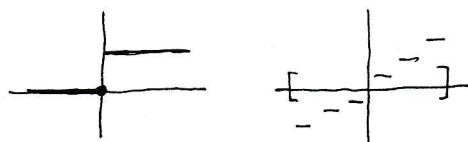
i.e. $\int_{-\pi}^{\pi} \left| \sum_{-N}^N c_n e^{inx} - f(x) \right|^2 dx \rightarrow 0 \text{ as } N \rightarrow \infty$

$\Rightarrow \sum_{-N}^N c_n e^{inx}$ converges to $f(x)$ for almost every x , $\forall x$ not in a set of Lebesgue measure 0.

Pointwise (everywhere) convergence depends on the regularity (differentiability) of f .

DEF f piecewise C^0 on $I = \text{interval}$, if \exists finite set $\{x_k\}_{k=1}^n$ such that f is C^0 on $I \setminus \{x_k\}_{k=1}^n$

Ex.



Analogous def. for pw C^j , $j \geq 0$.

THM 2.1 Assume f is piecewise C^1 , 2π periodic. Then

$$\lim_{N \rightarrow \infty} \sum_{-N}^N c_n e^{inx} = \frac{1}{2} (f(x_-) + f(x_+)) \quad \forall x \in \mathbb{R}$$

$$\left(f(x_-) = \lim_{\substack{t \rightarrow x \\ t < x}} f(t) \quad f(x_+) = \lim_{\substack{t \rightarrow x \\ t > x}} f(t) \right)$$

Thus $\forall x$ at which f is continuous, $\lim_{N \rightarrow \infty} \sum_{-N}^N c_n e^{inx} = f(x)$

Proof $\sum_{-N}^N c_n e^{inx} = \frac{1}{2\pi} \sum_{-N}^N \left(\int_{-\pi}^{\pi} f(t) e^{-int} dt \right) e^{inx} =$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-N}^N f(t) e^{in(x-t)} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-N}^N f(x+y) e^{-iny} dy =$$

want to see
 x inside f

Let $y = t - x$
 $\Rightarrow dy = dt$

by Lemma on integration
of periodic function.

$$= \int_{-\pi}^{\pi} f(x+y) \left(\sum_{n=-N}^{N} \frac{e^{-iny}}{2\pi} \right) dy$$

DEF $D_N(y) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{-iny}$ is the N^{th} Dirichlet Kernel

Folland:
Left &
right
limits
exist

Two equivalent ways to express $D_N(y)$

$$\textcircled{1} \quad \frac{1}{2\pi} \left(\sum_{n=-N}^{N} e^{-iny} \right) = \frac{1}{2\pi} \left(1 + \sum_{n=1}^{N} 2 \cos(ny) \right)$$

$$\textcircled{2} \quad \begin{aligned} \text{Geometric series: } D_N(y) &= \frac{1}{2\pi} e^{-Ny} \left(\sum_{n=0}^{2N} e^{-iny} \right) = \\ &= \frac{e^{-Ny}}{2\pi} \left(\frac{1 - e^{i(2N+1)y}}{1 - e^{iy}} \right) = \frac{e^{-Ny} - e^{i(N+1)y}}{2\pi (1 - e^{iy})} = \\ &= \frac{e^{-Ny} - e^{i(N+1)y}}{2\pi (1 - e^{iy})} \left(\frac{e^{-iy/2}}{e^{-iy/2}} \right) = \\ &= \frac{e^{-i(N+\frac{1}{2})y} - e^{i(N+\frac{1}{2})y}}{2\pi (e^{-iy/2} - e^{iy/2})} = \frac{\sin((N+\frac{1}{2})y)}{2\pi \sin(y/2)} \end{aligned}$$

looks like sines but not quite

Idea: $D_N(y)$  $\approx \delta$ distribution

$$\begin{aligned} \text{Compute: } \int_{-\pi}^{\pi} D_N(y) dy &= \int_0^{\pi} \frac{1}{2\pi} \left(1 + \sum_{n=0}^{N} 2 \cos(ny) \right) dy = \frac{1}{2} \\ &= \int_{-\pi}^{\pi} D_N(y) dy \end{aligned}$$

$$\Rightarrow \frac{1}{2} f(x_-) = \int_{-\pi}^0 D_N(y) f(x_-) dy, \quad \frac{1}{2} f(x_+) = \int_0^{\pi} D_N(y) f(x_+) dy$$

Thus, we need to show that

$$\begin{aligned} \lim_{N \rightarrow \infty} \underbrace{\int_{-\pi}^0 (f(x+y) - f(x_-)) D_N(y) dy}_{E_N(x)} + \int_0^{\pi} (f(x+y) - f(x_+)) D_N(y) dy &= \\ &= \lim_{N \rightarrow \infty} E_N(x) = 0. \end{aligned}$$

Idea: Use corollary to Bessel inequality

$\int_{-\pi}^{\pi} g(y) \sin((N+\frac{1}{2})y) dy \sim \text{Fourier sine coefficient of } g.$

$$\text{Let } g(y) = \begin{cases} \frac{f(x+y) - f(x_-)}{\sin(y/2)} & y < 0 \\ \frac{f(x+y) - f(x_+)}{\sin(y/2)} & y > 0 \end{cases}$$

$$\text{By L'hopital's rule, } \lim_{\substack{y \rightarrow 0^+ \\ y < 0}} g(y) = 2 f'(x_-)$$

$$\lim_{\substack{y \rightarrow 0^+ \\ y > 0}} g(y) = 2 f'(x_+)$$

Thus g is piecewise continuous, actually piecewise C^1

$$\Rightarrow \text{It certainly has } \int_{-\pi}^{\pi} |g(y)|^2 dy < \infty$$

$\Rightarrow g$'s fourier coefficients tend to 0.

$$E_N(y) = \int_{-\pi}^{\pi} \frac{g(y) \sin((N+1/2)y)}{2\pi} dy = b_{N+1/2} \text{ for } g \quad (b_{N+1/2} \rightarrow 0) \text{ finished}$$

$$\sin((N+1/2)y) = \sin(Ny) \cos(y/2) + \cos(Ny) \sin(y/2)$$

Exercise: fix g to make this work....

COR If f and g have the same Fourier coefficients and are pw $C^1 \Rightarrow f = g$.

2.3 DERIVATIVES

THM 2.2 Let f be 2π periodic, continuous, pw C^1 , then f' has Fourier coefficients

$$\begin{cases} a_n' = n b_n \\ b_n' = -n a_n \\ c_n' = i n c_n \end{cases}$$

DO NOT DIFFERENTIATE THE SERIES !!!

USE THE DEF OF a_n, b_n, c_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \underbrace{\cos(nx)}_{(\sin(nx))'} dx = \left[\frac{f(x) \sin(nx)}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{f'(x) \sin(nx)}{n} dx$$

$$\text{By def } b_n' = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx$$

$$\text{Thus } a_n = -\frac{b_n'}{n} \Rightarrow b_n' = -n a_n$$

$$\text{By def } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \underbrace{\sin(nx)}_{(-\cos(nx))'} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \frac{\cos(nx)}{n} dx = \frac{a_n'}{n}$$

$$\Rightarrow a_n' = n b_n$$

$$\text{By def } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \underbrace{e^{-inx}}_{(-e^{-inx})'} dx \stackrel{\text{PI}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f'(x) e^{-inx}}{in} dx = \frac{c_n'}{in}$$

$$\Rightarrow c_n' = i n c_n$$

■

THM 2.6 Let f be 2π periodic, $f \in C^{k-1}$ and $f^{(k-1)}$ pw C^1 , and f is pw C^k . Then the fourier coeffs of f satisfy $\sum |n^k a_n|^2 < \infty$, $\sum |n^k b_n|^2 < \infty$, $\sum |n^k c_n|^2 < \infty$

If $|c_n| \leq |n|^{-k-\alpha} c$ for some $c > 0$ and $\alpha > 1$
 $\Rightarrow f$ is C^k

THM 2.4 f pw C^0 , 2π periodic, $F(x) := \int_0^x f(t) dt$

Then if $C_0 = 0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \Rightarrow F(t) = C_0 + \sum_{n \neq 0} \frac{c_n e^{int}}{in}$

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx$$

$$\text{Moreover, } A_n = -\frac{b_n}{n}, \quad B_n = \frac{a_n}{n}$$

Compute: $\sum_{n \geq 1} \frac{1}{n^4}$

Exercise: Use fourier series and convergence
 to prove $\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\textcircled{1} \quad f(x) = x^2 = \frac{\pi^2}{3} + 4 \sum_{n \geq 1} \frac{(-1)^n \cos(nx)}{n^2}$$

Idea: use thm 2.4 twice Try this, finish next time.

The end of the proof of thm 2.1 (convg. of Fourier series pointwise)

$$E_N(x) = \int_{-\pi}^{\pi} \frac{g(t) \sin((N+1/2)t)}{2\pi} dt \quad \text{with} \quad g(t) = \begin{cases} \frac{f(x+t) - f(x-)}{\sin(t/2)} & t < 0 \\ \frac{f(x+t) - f(x-)}{\sin(t/2)} & t > 0 \end{cases}$$

This is a bounded, pw \mathcal{C}' func. on $[-\pi, \pi]$

" $b_{N+1/2}$ " for g tends to 0 as $N \rightarrow \infty$

$$\sin((N+1/2)t) = \sin(Nt) \cos(t/2) + \cos(Nt) \sin(t/2)$$

Thus $E_N(x) = \frac{1}{2} \beta_N + \frac{1}{2} \alpha_N$ where α_N is the N^{th} fourier cosine coefficient of the func. $g(t) \sin(t/2)$

$$\text{Note that } \int_{-\pi}^{\pi} |g(t) \sin(t/2)|^2 dt \leq \int_{-\pi}^{\pi} |g(t)|^2 dt < \infty$$

β_N is the N^{th} fourier sine coefficient of the func. $g(t) \cos(t/2)$

$$\text{and } \int_{-\pi}^{\pi} |g(t) \cos(t/2)|^2 dt \leq \int_{-\pi}^{\pi} |g(t)|^2 dt < \infty$$

Thus by Bessel inequality Corrolary, both $\alpha_N \rightarrow 0$ and $\beta_N \rightarrow 0$ as $N \rightarrow \infty$ ■

THM 2.5 If f is periodic, continuous, pw \mathcal{C}' , then in fact $\sum_{-N}^N c_n e^{inx}$ ^(2π periodic) $\rightarrow f(x)$ absolutely and uniformly $\forall x \in \mathbb{R}$

Proof By the assumptions on f , f' is piecewise continuous. \Rightarrow bounded, i.e. $\exists M > 0$ with $|f'(x)| \leq M \quad \forall x \in [-\pi, \pi]$

$$\Rightarrow \int_{-\pi}^{\pi} |f'(x)|^2 dx \leq 2\pi M^2$$

$$\Rightarrow \sum_{-\infty}^{\infty} |c_n'|^2 < \infty$$

$$\text{By thm 2.2: } |c_n| = |c_n'(\frac{1}{n})| \quad \forall n \neq 0$$

Note that $\forall \alpha = (\alpha_1, \dots, \alpha_N)$ and $\beta = (\beta_1, \dots, \beta_N)$ with $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N \geq 0$, then we have

$$|\alpha \cdot \beta| \leq \|\alpha\| \|\beta\|$$

$$\sum_{k=1}^N \alpha_k \beta_k \leq \sqrt{\sum_{k=1}^N \alpha_k^2} \sqrt{\sum_{k=1}^N \beta_k^2}. \text{ So if we instead have}$$

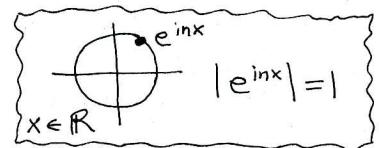
$$\text{two nonnegative sequences } \sum_{k=1}^{\infty} \alpha_k \beta_k \leq \sqrt{\sum_{k=1}^{\infty} \alpha_k^2} \sqrt{\sum_{k=1}^{\infty} \beta_k^2}$$

$$\text{Thus } \sum_{k \in \mathbb{Z}} |c_k| = \sum_{k \in \mathbb{Z}, k \neq 0} |c_k| \left| \frac{1}{k} \right| \leq \sqrt{\sum_{k \in \mathbb{Z}, k \neq 0} |c_k|^2} \sqrt{\underbrace{\sum_{k \in \mathbb{Z}, k \neq 0} \frac{1}{k^2}}_{\pi/\sqrt{3}}}$$

Thus $\sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n \neq 0} c_n e^{inx}$, has

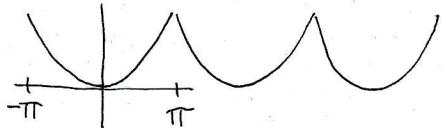
$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} |c_n e^{inx}| = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} |c_n| \leq \sqrt{\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} |c_n|^2} \frac{\pi}{\sqrt{3}} \quad \forall x \in \mathbb{R}$$

\therefore converges abs. unif. $\forall x \in \mathbb{R}$



SOME APPLICATIONS

1. $f(x) = x^2$



pw e^{∞}
continuous everywhere

Compute $b_n = 0 \quad \forall n$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}, \quad \text{for } x \in [-\pi, \pi]$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ compute... Let } x = \pi \Rightarrow \pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2}$$

$$\Rightarrow \left(\pi^2 - \frac{\pi^2}{3}\right) \cdot \frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Next, we consider $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$...

Thm 2.4 If $a_0 = 0$ ($\Leftrightarrow c_0 = 0$) then ...

$$x^2 - \frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2} \quad \text{has } a_0 = 0 \Rightarrow \text{Apply Thm 2.4}$$

$$t^2 - \frac{\pi^2}{3} = f(t)$$

$$F(x) = \int_0^x f(t) dt = C_0 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n^3}, \quad C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx$$

$$\frac{x^3}{3} - \frac{\pi^2 x}{3} = F(x), \quad \text{thus} \quad C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^3}{3} - \frac{\pi^2 x}{3} dx = 0$$

$$x^3 - \pi^2 x = 12 \sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n^3} \quad \text{Exercise: Compute } \sum_{n=1}^{\infty} \frac{1}{n^3}$$

We use thm 2.4 once more:

$$\int_0^x (t^3 - \pi^2 t) dt = F(x) = \frac{x^4}{4} - \frac{\pi^2 x^2}{2} = C_0 + 12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(nx)}{n^4}$$

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{x^4}{4} - \frac{\pi^2 x^2}{2} \right) dx = \frac{1}{\pi} \int_0^{\pi} \left(\frac{x^4}{4} - \frac{\pi^2 x^2}{2} \right) dx = \frac{1}{\pi} \left(\frac{\pi^5}{20} - \frac{\pi^5}{6} \right) =$$

$$= -\frac{7\pi^4}{60}$$

$$\Rightarrow x^4 - \pi^2(2)x^2 = -\frac{(4)\pi^4}{60} + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(nx)}{n^4}$$

$$\text{Let } x=\pi \Rightarrow \pi^4 - 2\pi^4 = -\frac{7\pi^4}{15} + 48 \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n^4}}_{-\sum_{n=1}^{\infty} \frac{1}{n^4}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{48} \left(\pi^4 - \frac{7\pi^4}{15} \right) = \frac{\pi^4}{80}$$

OTHER INTERVALS

$[0, \pi]$ for f here, can extend in two convenient ways to $[-\pi, \pi]$: $\frac{\text{odd}}{f(-x) = -f(x)}$ or $\frac{\text{even}}{f(-x) = f(x)}$, $f_{\text{odd}}, f_{\text{even}}$

Note: If $f(0) \neq 0$ then the odd extension will not be continuous at 0.



Then we know that in the odd case $a_n^{\text{odd}} = 0 \quad \forall n$

$$b_n^{\text{odd}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\text{odd}}(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

because $f_{\text{odd}} = f$ on $[0, \pi]$

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n^{\text{odd}} \sin(nx)$$

For f_{even} , $b_n^{\text{even}} = 0 \quad \forall n$

$$a_n^{\text{even}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\text{even}}(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

because $f_{\text{even}} = f$ on $[0, \pi]$

$$f_{\text{even}}(x) = \sum_{n=0}^{\infty} a_n^{\text{even}} \cos(nx)$$

IHM 2.7 Fourier sine and cosine series for f piecewise C¹ on $[0, \pi]$ converge to $f(x) \quad \forall x \in (0, \pi)$ at which f is continuous.

For discontinuous points, they convg. to $\frac{f(x_-) + f(x_+)}{2}$

For a function on $[a-l, a+l]$ for $a \in \mathbb{R}, l > 0$. Extend f to be $2l$ periodic

$$f(x) = f\left(\frac{\pm l}{\pi} + a\right) = g(t) \quad \text{is on } [-\pi, \pi] \text{ and } 2\pi \text{ periodic}$$

Do all our Fourier Series Stuff with g .

$$g(t) = \sum_{n \in \mathbb{Z}} c_n e^{int}, \quad \text{with } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{tl}{\pi} + a\right) e^{-int} dt = \left\{x = \frac{tl}{\pi} + a\right\} =$$

$$= \frac{1}{2\pi} \int_{a-1}^{a+1} f(x) e^{-in(x-a)\pi/l} \frac{\pi}{l} dx =$$

$$= \frac{1}{2l} \int_{a-1}^{a+1} f(x) e^{-in(x-a)\pi/l} dx$$

Thus $f(x) = g(t) = \sum_{n \in \mathbb{Z}} c_n e^{in(x-a)\pi/l}$, $c_n = \frac{1}{2l} \int_{a-l}^{a+l} f(x) e^{-in(x-a)\pi/l} dx$

PDE APPLICATION



Ends fixed. String at time $t=0$ is still but displaced

let go \Rightarrow Vibrates according to the wave eq.

Solve it, $u(x,t)$

$$u_t(x,0) = 0 \quad \forall x \in [0,\pi]$$

$$u(x,0) = \begin{cases} \frac{x}{\pi} & x \in [0, \pi/2] \\ 1 - \frac{x}{\pi} & x \in [\pi/2, \pi] \end{cases} \quad (\text{IC})$$

$$u(0,t) = 0 \quad \forall t \geq 0, \quad u(\pi,t) = 0 \quad \forall t \geq 0 \quad (\text{BC})$$

Separate variables. Write $f(x)g(t)$, $\square(fg) = 0$, $\square = \partial_t^2 - \partial_x^2$

$$\Rightarrow -f''(x)g(t) + g''(t)f(x) = 0 \Rightarrow g''(t)f(x) = f''(x)g(t)$$

$$\Rightarrow \frac{g''(t)}{g(t)} = \frac{f''(x)}{f(x)} \quad \begin{matrix} \text{(not interested in)} \\ \text{(solution } f=0, g=0) \end{matrix} \Rightarrow \text{Both sides constant}$$

$$\frac{f''(x)}{f(x)} = \lambda \quad \text{constant} \quad f''(x) = \lambda f(x), \quad f(0) = 0, \quad f(\pi) = 0$$

We have seen that the only non zero solutions are with $\lambda = -n^2$, $n \in \mathbb{N}$

$$f_n(x) = \sin(nx) \Rightarrow g_n(t) \text{ satisfies } g_n''(t) = -n^2 g_n(t)$$

$$\Rightarrow g_n(t) = a_n \cos(nt) + b_n \sin(nt)$$

$$\square(f_n g_n) = 0 \quad \forall n \Rightarrow \square\left(\sum_{n=1}^{\infty} f_n g_n\right) = 0$$

$$f(x) = a \cos(nx) + b \sin(nx)$$

Let us write $u(x,t) = \sum_{n \geq 1} \sin(nx) (a_n \cos(nt) + b_n \sin(nt))$

Use the IC to find the a_n and b_n

$$u_t(x,0) = \sum_{n \geq 1} \sin(nx) (-\sin(0) a_n n + n b_n) = \\ = \sum_{n \geq 1} n b_n \sin(nx) . \text{ should be } 0.$$

The Fourier coefficients of the 0 function are all 0.

$$\Rightarrow n b_n = 0 \quad \forall n \Rightarrow b_n = 0 \quad \forall n \geq 1$$

$$u(x,t) = \sum_{n \geq 1} \sin(nx) a_n \cos(nt)$$

$$u(x,0) = \sum_{n \geq 1} a_n \sin(nx)$$

$$\text{let } u_0(x) = \begin{cases} \frac{x}{\pi} & [0, \pi/2] \ni x \\ 1 - \frac{x}{\pi} & [\pi/2, \pi] \ni x \end{cases}$$

The a_n are the Fourier sine coeffs of u_0

$$a_n = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin(nx) dx \quad (\text{plug in in } u(x,t))$$

ÖVN
S1

$$2.1.4 \quad f(\theta) = \begin{cases} 0 & -\pi < \theta \leq 0 \\ \theta & 0 < \theta \leq \pi \end{cases}$$

$$\text{Solution: } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \theta d\theta = \frac{\pi}{2}$$

$$\begin{aligned} n \neq 0, \quad a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \theta \cos(n\theta) d\theta = \\ &= \frac{1}{\pi n} \int_0^{\pi} \theta (\sin(n\theta))' d\theta = \\ &= \frac{1}{\pi n} \left([\theta \sin(n\theta)]_0^{\pi} - \int_0^{\pi} \sin(n\theta) d\theta \right) = \frac{1}{n\pi} \frac{1}{n} [\cosh(n\theta)]_0^{\pi} \\ &= \frac{1}{n^2\pi} (\cosh(n\pi) - 1) = \begin{cases} 0 & n \text{ even} \\ -\frac{2}{n^2\pi} & n \text{ odd} \end{cases} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \theta \sin(n\theta) d\theta = \\ &= -\frac{1}{n\pi} \int_0^{\pi} \theta (\cos(n\theta))' = -\frac{1}{n\pi} \left([\theta \cos(n\theta)]_0^{\pi} - \int_0^{\pi} \cos(n\theta) d\theta \right) = \\ &= -\frac{1}{n\pi} \left(\pi \cos(n\pi) - \left[\frac{1}{n} \sin(n\theta) \right]_0^{\pi} \right) = -\frac{1}{n\pi} \pi \cos(n\pi) = \\ &= \begin{cases} -\frac{1}{n} & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases} \end{aligned}$$

$$\begin{array}{l} \text{sin odd} \\ \text{cos even} \end{array} \quad f(\theta) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)\theta)}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n\theta)}{n}$$

$$2.1.8. \quad f(\theta) = |\sin \theta|$$

$$\begin{aligned} \text{Solution: } a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin \theta| d\theta = \frac{2}{\pi} \int_0^{\pi} \sin \theta d\theta = \left[-\frac{2}{\pi} \cos \theta \right]_0^{\pi} = \\ &= 4/\pi \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin \theta| \cos(n\theta) d\theta = \frac{2}{\pi} \int_0^{\pi} \sin \theta \cos(n\theta) d\theta =$$

$$\left\{ \sin \alpha \cos \beta = \frac{1}{2} (\sin(\alpha+\beta) + \sin(\alpha-\beta)) \right\}$$

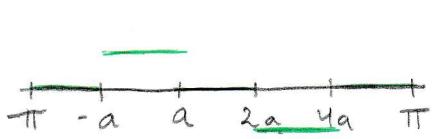
$$= \frac{2}{2\pi} \int_0^{\pi} (\sin(\theta+n\theta) + \sin(\theta-n\theta)) d\theta =$$

$$= \frac{1}{\pi} \left[\frac{-\cos((n+1)\pi)+1}{n+1} + \frac{\cos((n-1)\pi)-1}{n-1} \right] = \begin{cases} 0 & n \text{ odd} \\ \frac{4}{\pi(n^2-1)} & n \text{ even} \end{cases}$$

$f(\theta)$ is even $\Rightarrow b_n = 0$

$$f(\theta) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{4n^2 - 1}$$

2.1.14. $f(\theta) = \begin{cases} 1 & -a < \theta < a \\ -1 & 2a < \theta < 4a \\ 0 & \text{elsewhere in } (-\pi, \pi] \end{cases}$



Solution: $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = \frac{1}{\pi} \int_{-a}^a d\theta - \frac{1}{\pi} \int_{2a}^{4a} d\theta = 0$

$$\begin{aligned} n \neq 0, \quad a_n &= \frac{1}{\pi} \int_{-a}^a \cos(n\theta) d\theta - \frac{1}{\pi} \int_{2a}^{4a} \cos(n\theta) d\theta = \\ &= \frac{2}{n\pi} \sin(an) - \frac{1}{n\pi} \sin(4an) - \frac{1}{n\pi} \sin(2an) = \left\{ \begin{array}{l} \sin 2x = \\ = 2 \sin x \cos x \end{array} \right\} \\ &= \frac{2 \sin(an)}{n\pi} (1 - 2 \cos(2an) \cos(an) + \cos(an)) = \left\{ \begin{array}{l} \cos x \cos y = \\ = \frac{1}{2} (\cos(x-y) + \cos(x+y)) \end{array} \right\} \\ &= \frac{2 \sin(an)}{n\pi} (1 - \cos(an) + \cos(3an) + \cos(an)) \end{aligned}$$

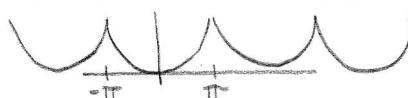
$$b_n = \frac{1}{\pi} \int_{-a}^a \sin(n\theta) d\theta - \frac{1}{\pi} \int_{2a}^{4a} \sin(n\theta) d\theta = \frac{1}{n\pi} (\cos(4an) - \cos(2an)) =$$

$$= \left\{ \cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right) \right\} =$$

$$= -\frac{2}{n\pi} \sin(na) \sin(3na)$$

$$\Rightarrow f(\theta) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(na) \left((1 - \cos(3na)) \cos(n\theta) - \sin(3na) \sin(n\theta) \right)$$

2.1.16. $f(\theta) = \theta^2$ on $[-\pi, \pi]$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 d\theta = \left[\frac{1}{3\pi} \theta^3 \right]_{-\pi}^{\pi} = \frac{2}{3} \pi^3$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \cos(n\theta) d\theta = \frac{1}{n\pi} \int_{-\pi}^{\pi} \theta^2 (\sin(n\theta))' d\theta = \\ &= \underbrace{\left[\frac{1}{n\pi} \theta^2 \sin(n\theta) \right]_{-\pi}^{\pi}}_{=0} - \frac{2}{\pi n} \int_{-\pi}^{\pi} \theta \sin(n\theta) d\theta = \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\pi n^2} \int_{-\pi}^{\pi} \theta (\cos(n\theta))^2 d\theta = \\
 &= \frac{2}{\pi n^2} \left[\theta \cos(n\theta) \right]_{-\pi}^{\pi} - \frac{2}{\pi n^2} \int_{-\pi}^{\pi} \cos(n\theta) d\theta = \\
 &= \frac{4 \cos(n\pi)}{n^2}, \quad f(\theta) \text{ is even} \Rightarrow b_n = 0
 \end{aligned}$$

$$f(\theta) = \frac{2}{3} \pi^3 + \sum_{n=1}^{\infty} \frac{4 \cos(n\pi)}{n^2} \cos(n\theta)$$

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F5

Recap • Any bounded (measurable - but everything is) function f can be expressed as a Fourier Series $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$. Converges almost everywhere

on an interval, $[a-l, a+l]$, extend f to be $2l$ periodic on \mathbb{R}

For the interval $[-\pi, \pi]$, $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$, $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

Also written as $c_n = \hat{f}(n)$, or \hat{f}_n is the n th Fourier coeff. of f .

THM 2.1 If f is pw C^1 and continuous everywhere $\Rightarrow \sum_{n \in \mathbb{Z}} c_n e^{inx} = f(x) \quad \forall x$ (i.e. converges everywhere)

Applications ① Compute $\sum_{n \geq 1} \frac{1}{n^2}$ etc. Compute \sum

② Solve PDEs (like heat & wave eq) on bounded regions of space ($\mathbb{R}^1, \mathbb{R}^2, \text{etc}$)

What if we want to solve on all of \mathbb{R}^n ? For example

$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ on \mathbb{R}^n , $\square := \partial_t - \Delta$ heat operator

Solve: $u(x, 0) = u_0(x)$, $x \in \mathbb{R}^n$, $\square u = 0 \quad \forall t > 0$

Heat kernel on \mathbb{R}^n is $\frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}}$ and func. on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$

$$u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} u_0(y) e^{-|x-y|^2/4t} dy$$

How do we find this?

$f(x) = \frac{1}{2l} \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}$, f on $[-l, l]$ extended to \mathbb{R} to be $2l$ periodic.

Fix x . As $l \rightarrow \infty$, summing $c_n e^{in\pi x/l}$ is like a Riemann integral. Let $(d\xi = \frac{l}{l})$, $\xi = \frac{n}{l}$, $d\xi = \frac{1}{l}$

$$f(x) \approx \frac{1}{2} \int_{-\infty}^{\infty} c_n e^{i\pi x \xi} d\xi = \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_l^l f(y) e^{-in\pi y/l} dy \right) d\xi$$

$$f(x) \approx \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{i\pi x \xi} e^{-i\pi x \xi} dy d\xi$$

$$\text{let } \eta = \xi/2 \Rightarrow f(x) \approx \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(y) e^{-2\pi i y \eta} dy}_{\hat{f}(\eta)} e^{2\pi i n x} d\eta$$

DEF The Fourier transform

$$\hat{f}(\eta) = \int_{\mathbb{R}} f(y) e^{-2\pi i y \eta} dy$$

$F(f)(\eta)$, an equivalent notation

DEF $L^1(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{C}, \text{ measurable} - \text{everything is} \}$

with $\left\{ \int_{\mathbb{R}} |f(y)| dy < \infty \right\} / \sim$

need to care only about

$f \sim g$ if \exists set $N \subset \mathbb{R}$ which is negligible, null this when solving prob.
set, set of Lebesgue measure zero,
 $f(x) = g(x) \quad \forall x \in \mathbb{R} \setminus N$

What are the null sets in $\mathbb{R}, \mathbb{R}_2, \dots$?

Lebesgue measure on \mathbb{R}^n is n-dimensional volume.

In \mathbb{R}^1 it is length. A point has length=0.
Thus every point is a null set.

Any countable union of null sets is also a null set.
 $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all null sets.

$[0,1]$ has measure = 1. $(0,1)$ has measure 1
 $(0,1) \setminus \mathbb{Q}$ still has measure = 1

What is the 2 dimensional Lebesgue measure of a line?
Zero!

Everything of lower dimension has Lebesgue measure zero.

DEF $L^2(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{C}, \text{ measurable} - \text{everything is} \}$

with $\left\{ \int_{\mathbb{R}} |f(y)|^2 dy < \infty \right\} / \sim$

PROP Assume $f \in L^1(\mathbb{R})$. Then $\hat{f}(\eta) \in \mathbb{C} \quad \forall \eta \in \mathbb{R}$.

$$\left| \int_{\mathbb{R}} f(y) e^{-2\pi i \eta y} dy \right| \leq \int_{\mathbb{R}} |f(y)| e^{-2\pi i \eta y} dy \leq$$

$$\leq \int_{\mathbb{R}} |f(y)| dy < \infty$$

$$|f(y) e^{-2\pi i \eta y}| = |f(y)| |\underbrace{e^{-2\pi i \eta y}}_{}| = |f(y)|$$

$$\Rightarrow \text{since } f \in L^1. \text{ Thus } \int_{\mathbb{R}} f(y) e^{-2\pi i \eta y} dy = \hat{f}(\eta) \in \mathbb{C} \text{ is}$$

Thus, $\forall f \in L^1(\mathbb{R})$, $\hat{f}(\eta)$ is well defined $\forall \eta \in \mathbb{R}$

Almost everywhere means everywhere except a set of measure 0.

f and g are the same as elements of L^1 if they are
the same almost everywhere

$f \sim g \Leftrightarrow f = g$ almost everywhere

$\{ \} / \sim$ (can change it at points, still same func.)

#WAM

\mathbb{Z}/\sim , where $x \sim y$ iff $x-y$ is an integer multiple of 5. Thus $\mathbb{Z}/\sim = \mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$

DEF The convolution of f and g , $f * g(x)$ is

$$\int_{\mathbb{R}} f(x-y) g(y) dy$$

THM 7.1 ① If f and $g \in L^2 \Rightarrow |f * g(x)| \leq \sqrt{\int_R |f|^2} \sqrt{\int_R |g|^2}$

- ② $f * (ag + bh) = af * g + bf * h$, \forall functions f, g, h
such that $f * g$ is defined (finite) and $f * h$ is
defined. $a, b \in \mathbb{C}$

$$\textcircled{3} \quad f * g = g * f$$

$$④ f * (g * h) = (f * g) * h$$

$$\text{Proof: } \textcircled{1} \quad \left| \int_{\mathbb{R}} f(x-y) g(y) dy \right| \leq \int_{\mathbb{R}} |f(x-y)| |g(y)| dy \leq \sqrt{\int_{\mathbb{R}} |f(x-y)|^2 dy} \sqrt{\int_{\mathbb{R}} |g(y)|^2 dy} =$$

Schwarz inequality
for L^2

$$= \sqrt{\int_{\mathbb{R}} |f(z)|^2 dz} \sqrt{\int_{\mathbb{R}} |g(y)|^2 dy} = \sqrt{\int_{\mathbb{R}} |f|^2} \sqrt{\int_{\mathbb{R}} |g|^2}$$

② Linearity of the integral and def.

$$\textcircled{3} \quad \int_{\mathbb{R}} f(x-y) g(y) dy = \left\{ y = x-z \right\} = \int_{\mathbb{R}} g(x-z) f(z) dz = g * f(x)$$

④ Exercise! Hint: use the definition and substitution / change of variables. ■

THM 7.2 Mollification If $f \in C^1$ and $f * g$ and $f' * g$

are defined, then $f * g \in \mathcal{C}$, and $(f * g)' = f' * g$

Convolution with a smooth function and a not smooth function creates a smooth function

$$\begin{matrix} f & * & g \\ \uparrow & & \nwarrow \\ \text{smooth} & & \text{terrible} \end{matrix} = \begin{matrix} (f * g) \\ \uparrow \\ \text{SMOOTH!} \end{matrix}$$

"Mollify your peas"

THM 7.5 Properties of the Fourier transform

Don't have
to memorize

$$\textcircled{1} \quad \forall a \in \mathbb{R}, \quad \mathcal{F}((f(x-a)))(\xi) = e^{-2\pi i \xi a} \hat{f}(\xi)$$

$$\mathcal{F}(e^{2\pi i ax} f(x))(\xi) = \hat{f}(\xi - a)$$

$$\textcircled{2} \text{ If } f' \in \mathcal{L}' \text{ then } \widetilde{\mathcal{F}}(f')(\xi) = 2\pi i \xi \hat{f}(\xi)$$

$$\textcircled{3} \quad \text{If } x f(x) \in L^1, \quad \mathcal{F}(x f(x))(\xi) = -\frac{\mathcal{F}(f)(\xi)}{2\pi i}$$

↑ multi. no derivative ↑ derivative

$$\textcircled{4} \quad \mathcal{F}(f * g)(\xi) = \hat{f}(\xi) \hat{g}(\xi)$$

Proof: ① $\int_{\mathbb{R}} f(x-a) e^{-2\pi i \xi x} dx = \begin{cases} y = x-a \\ dy = dx \end{cases} = \int_{\mathbb{R}} f(y) e^{-2\pi i (\xi+a)y} dy$

$$= e^{-2\pi i a \xi} \hat{f}(\xi)$$

$$\int_{\mathbb{R}} e^{2\pi i ax} f(x) e^{-2\pi i x \xi} dx = \int_{\mathbb{R}} f(x) e^{-2\pi i (\xi-a)x} dx =$$

$$= \hat{f}(\xi - a)$$

② $\int_{\mathbb{R}} f'(x) e^{-2\pi i x \xi} dx = - \int f(x) (-2\pi i \xi) e^{-2\pi i x \xi} dx =$

$$= (2\pi i \xi) \hat{f}(\xi)$$

Integration
by parts

what happened
to "boundary terms"? Because we are assuming
the Fourier transform is
well defined \Rightarrow the boundary terms vanish

③ $\int x f(x) e^{-2\pi i x \xi} dx$

$$(\hat{f}(\xi))' = \left(\int f(x) e^{-2\pi i x \xi} dx \right)' = \int (-2\pi i x) f(x) e^{-2\pi i x \xi} dx$$

④ $\iint f(x-y) g(y) e^{-2\pi i x \xi} dy dx \quad \text{Exercise, finish next time!}$

$$\underline{2.2.4} \text{ Show: } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Solution: a) $\theta^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta)$
 $\theta = \pi \Rightarrow \cos(n\pi) = (-1)^n$

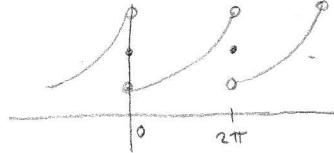
$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3} \frac{\pi^2}{4} = \frac{\pi^2}{6}$$

b) $\theta = 0 \Rightarrow 0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$

$$\underline{2.2.7} \text{ Show: } \sum_{n=1}^{\infty} \frac{1}{n^2+b^2} = \frac{\pi}{2b} \coth(b\pi) - \frac{1}{\pi b^2}, \quad -\pi < \theta < \pi$$

Solution: $f(\theta) = e^{b\theta} = \frac{e^{2\pi b} - 1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{b-in}, \quad 0 < \theta < 2\pi$
(2.1.17 / 2.1.18)



$$f(0_+) = 1$$

$$f(0_-) = e^{2\pi b}$$

By thm 2.1, $\lim_{N \rightarrow \infty} S_N f(\theta) = \frac{1}{2}(f(\theta) + f(\theta_+))$

$$\frac{1+e^{2\pi b}}{2} = \frac{e^{2\pi b}-1}{2\pi} \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{b-in} = \frac{e^{2\pi b}-1}{2\pi} \left(2b \sum_{n=1}^{\infty} \frac{1}{b^2+n^2} + \frac{1}{b} \right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{b^2+n^2} = \frac{\pi}{2b} \coth(\pi b) + \frac{1}{2b^2}$$

$$\underline{2.3.2} \text{ Show } \theta^3 - \pi^2 \theta = 12 \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n^3} \quad (-\pi < \theta < \pi)$$

Solution: Thm 2.4: Case $a_0 \neq 0$

$$F(\theta) = \int_0^\theta \theta'^2 d\theta' = \frac{\theta^3}{3}$$

$$\text{by } \Rightarrow C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) d\theta = 0$$

Recall: $\theta^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta)$

then $\overset{\text{small}}{\downarrow} C_0 = \frac{\pi^2}{3}$

$$\Rightarrow \frac{\theta^3}{3} - \frac{\theta \pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(n\theta)$$

b) $\theta^4 - 2\pi^2 \theta^2 = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(n\theta)}{n^4}$

Solution: use (a). $f(\theta) = \theta^3 - \pi^2 \theta$, then $C_0 = 0$

$$F(\theta) = \int_0^\theta \theta^3 - \pi^2 \theta d\theta = \frac{\theta^4}{4} - \frac{\pi^2 \theta^2}{2}$$

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\theta^4}{4} - \frac{\pi^2 \theta^2}{2} \right) d\theta = \frac{1}{2\pi} \left[\frac{\theta^5}{20} - \frac{\pi^2 \theta^3}{6} \right]_{-\pi}^{\pi} = -\frac{7}{60} \pi^4$$

$$\Rightarrow \frac{\theta^4}{4} - \frac{\pi^2 \theta^2}{2} = -\frac{7}{60} \pi^4 + \frac{1}{12} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(n\theta)}{n^4}$$

c) Show $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Solution: set $\theta = \pi$ in (b)

$$\begin{aligned} \pi^4 - 2\pi^4 &= 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n}{n^4} - \frac{7}{15} \pi^4 \\ \Rightarrow \pi^4 - \frac{7}{15} \pi^4 &= 48 \sum_{n=1}^{\infty} \frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \end{aligned}$$

2.3.3 Evaluate $\sum_{n=1}^{\infty} (2n-1)^{-4} \cdot \cos((2n-1)\theta)$ on $-\pi < \theta < \pi$

by using $\theta(\pi - |\theta|) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{(2n-1)^3}$, $-\pi < \theta < \pi$

Solution: Thm 2.4 $f(\theta) = \theta(\pi - |\theta|) \Rightarrow C_0 = 0$

$$\begin{aligned} F(\theta) &= \int_0^\theta \theta(\pi - |\theta|) d\theta = \frac{\pi \theta^2}{2} - \frac{|\theta^3|}{3} \\ C_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\pi \theta^2}{2} - \frac{|\theta^3|}{3} \right) d\theta = \frac{1}{\pi} \int_0^\pi \left(\frac{\pi \theta^2}{2} - \frac{\theta^3}{3} \right) d\theta = \\ &= \frac{1}{\pi} \left(\frac{\pi^4}{6} - \frac{\pi^4}{12} \right) = \frac{\pi^3}{12} \end{aligned}$$

$$\text{Thm 2.4} \Rightarrow \frac{\pi \theta^2}{2} - \frac{|\theta^3|}{3} = -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\theta)}{(2n-1)^4} + \frac{\pi^3}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos((2n-1)\theta)}{(2n-1)^4} = -\frac{\theta^2 \pi^2}{16} + \frac{\theta^2 |\theta|}{24} \pi^2 + \frac{\pi^4}{96}$$

2.3.4 for $0 < \theta < \pi$, $\sin \theta = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{4n^2-1}$ (*)

and $\cos \theta = \frac{d}{d\theta} \sin \theta = - \int_{\pi/2}^{\theta} \sin \phi d\phi$

Show that (*) can be differentiated and integrated termwise to yield two different expressions for $\cos \theta$, $0 < \theta < \pi$, and reconcile these two expressions.

Solution: $\frac{d}{d\theta} (\sin \theta) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{2n \sin(2n\theta)}{4n^2-1} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(2n\theta)}{4n^2-1}$

$$\begin{aligned} - \int_{\pi/2}^{\theta} \sin \phi d\phi &= - \int_{\pi/4}^{\theta} \left(\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\phi)}{4n^2-1} \right) d\phi = \\ &= \left[-\frac{2\phi}{\pi} \right]_{\pi/2}^{\theta} + \frac{4}{\pi} \sum_{n=1}^{\infty} \int_{\pi/2}^{\theta} \frac{\cos(2n\phi)}{4n^2-1} d\phi = \\ &= -\frac{2\theta}{\pi} + 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2-1)} = \left\{ \text{Note: } \frac{1}{2n(4n^2-1)} = \frac{4n}{4n^2-1} - \frac{1}{n} \right\} = \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2} \left(\frac{4n}{4n^2-1} - \frac{1}{n} \right) - \frac{2\theta}{\pi} + 1 = \end{aligned}$$

$$= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(2n\theta)}{4n^2-1} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n} - \frac{2\theta}{\pi} + 1$$

Need to check: $-\frac{2\theta}{\pi} + 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n} = 0$

eq 2.17: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n\theta)}{n} = \frac{\theta}{2}, \quad -\pi < \theta < \pi$

$$-\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n} = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n} =$$

$$= -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{2n-1} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{2n-1} =$$

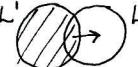
$$= \underbrace{\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\theta)(-1)^{n+1}}{n}}_{= \frac{4}{\pi} \frac{\theta}{2} \text{ by 2.17}} - \underbrace{\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{2n-1}}_{= -1 \text{ by 2.16}}$$

$$\Rightarrow -\frac{2\theta}{\pi} + 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2-1)} = -\frac{2\theta}{\pi} + 1 + \frac{4}{\pi} \frac{\theta}{2} - 1 = 0$$

~~~~~

25/1  
F6

$$\begin{aligned} \text{Proof of (4): } & \iint f(x-y) g(y) e^{-2\pi i x \xi} dx dy = \\ &= \iint f(x-y) e^{-2\pi i \xi(x-y)} g(y) e^{-2\pi i \xi y} dx dy = \{z = x-y\} = \\ &= \iint f(z) e^{-2\pi i \xi z} dz g(y) e^{-2\pi i \xi y} dy = \hat{f}(\xi) \hat{g}(\xi) \blacksquare \end{aligned}$$

IHM  $\mathcal{F}$  is defined on  $L^1 \cap L^2$  and has a canonical, well defined extension to all of  $L^2$ . 

Moreover,  $\forall f \in L^2, \hat{f} \in L^2$

DEF  $\mathcal{F}^{-1}(f)(\xi) = \mathcal{F}(f)(-\xi) = \underbrace{\int e^{2\pi i x \xi} f(x) dx}_{\hat{f}(\xi)} \quad (\text{Inverse})$

$$\forall f \in L^2, \mathcal{F}^{-1}(\mathcal{F}(f)) = f$$

DEF On  $L^2(\mathbb{R})$ ,  $\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx$

$$\|f\|_{L^2} = \sqrt{\langle f, f \rangle} = \sqrt{\int_{\mathbb{R}} |f|^2}$$

As with vectors,  $|\langle f, g \rangle| \leq \|f\|_{L^2} \|g\|_{L^2}$

APPLICATION Heat equation on  $\mathbb{R}$   $\begin{cases} u(x, 0) = u_0(x) \\ \frac{\partial}{\partial t} u(x, t) = 0 \end{cases} \quad \forall t > 0$

Idea:  $\mathcal{F}$  turns derivs into multiplication

$$\mathcal{F}(\frac{\partial}{\partial t} u(x, t))(\xi) = 0 \Rightarrow \hat{u}_t(\xi, t) - (2\pi i \xi)^2 \hat{u}(\xi, t) \quad (\text{apply 7.5(2) twice})$$

This is an ODE in  $t$  variable for  $\hat{u}(\xi, t)$

$$\hat{u}_t(\xi, t) = -4\pi^2 \xi^2 \hat{u}(\xi, t) \quad (\text{i.e. } f' = cf)$$

$$\Rightarrow \hat{u}(\xi, t) = e^{-4\pi^2 \xi^2 t} c(\xi) \quad \text{Recall } u(x, 0) = u_0(x)$$

$$\Rightarrow \hat{u}(\xi, 0) = c(\xi) = \hat{u}_0(\xi) \Rightarrow \hat{u}(\xi, t) = \hat{u}_0(\xi) e^{-4\pi^2 \xi^2 t} \quad \text{Find } h(x) \text{ such that } \hat{h}(\xi) =$$

$$\text{Then } \hat{u}(\xi, t) = \hat{u}_0(\xi) \hat{h}(\xi) = \mathcal{F}(u_0 * h(x))(\xi)$$

$$\Rightarrow u(x, t) = u_0 * h(x) = \int_{\mathbb{R}} u_0(x-y) h(y) dy = \int_{\mathbb{R}} u_0(y) h(x-y) dy$$

By the inversion formula,  $h(x, t) = \mathcal{F}^{-1}(\hat{h}(\xi, t))(x) =$

$$\stackrel{\text{def}}{=} \int_{\mathbb{R}} \hat{h}(\xi, t) e^{2\pi i \xi x} d\xi = \int_{\mathbb{R}} e^{-4\pi^2 \xi^2 t + 2\pi i \xi x} d\xi \quad \text{turn into } e^{-(y^2)} dy$$

$$\int_{\mathbb{R}} e^{-(4\pi^2 \xi^2 t - 2\pi i \xi x)} d\xi = \int_{\mathbb{R}} e^{-((2\pi \xi \sqrt{t})^2 + 2(2\pi \xi \sqrt{t})(\frac{-ix}{2\pi t})) - \frac{x^2}{4t}} d\xi =$$

complete the square

$$= e^{-x^2/4t} \int_{\mathbb{R}} e^{-((2\pi \xi \sqrt{t})^2 - \frac{ix}{2\pi t})} d\xi = \left\{ y = 2\pi \xi \sqrt{t} - \frac{ix}{2\pi t}, dy = 2\pi \sqrt{t} d\xi \right\} =$$

$$= e^{-x^2/4t} \int_{\mathbb{R}} e^{-y^2} \frac{dy}{2\pi\sqrt{t}} = \frac{e^{-x^2/4t}}{\sqrt{\pi} 2\sqrt{t}}$$

Thus  $h(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$

The solution is  $u(x,t) = \underbrace{\int_{\mathbb{R}} u_0(y) \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy}$

In fact, on  $\mathbb{R}^n$ ,  $u(x,t) = \int_{\mathbb{R}^n} \frac{u_0(y) e^{-\|x-y\|^2/4t}}{(4\pi t)^{n/2}} dy$

$$\left( \begin{aligned} \int_{\mathbb{R}} e^{-x^2} dx &= \sqrt{\int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-x^2} dx} = \sqrt{\int_{\mathbb{R}^2} e^{-x^2-y^2} dx dy} = \{ \text{polar coord.} \} = \\ &= \sqrt{\int_{\mathbb{R}^2} e^{-r^2} r dr d\theta} = \sqrt{\int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta} = \\ &= \sqrt{\int_0^{2\pi} \left[ -\frac{e^{-r^2}}{2} \right]_0^\infty d\theta} = \sqrt{\int_0^{2\pi} \frac{1}{2} d\theta} = \sqrt{\frac{2\pi}{2}} = \sqrt{\pi} \end{aligned} \right)$$

THM 7.3 Assume  $g \in L^1(\mathbb{R})$  with  $\int_{\mathbb{R}} g(y) dy = 1$

let  $\alpha = \int_{-\infty}^0 g(y) dy$ ,  $\beta = \int_0^\infty g(y) dy$

Assume  $f$  is piecewise continuous and either  
①  $f$  is bounded on  $\mathbb{R}$  or ②  $f$  has compact support

Then  $\lim_{\varepsilon \rightarrow 0} (f * \underbrace{\frac{1}{\varepsilon} g(x/\varepsilon)}_{g_\varepsilon(x)}) = \alpha f(x_+) + \beta f(x_-) \quad \forall x \in \mathbb{R}$

so if  $f$  is continuous at  $x$   $\lim_{\varepsilon \rightarrow 0} (f * \frac{1}{\varepsilon} g(x/\varepsilon)) = f(x)$

Compact support =  $\exists$  compact set  $K$  such that  $g \equiv 0$  off  $K$ .

Proof  $\lim_{\varepsilon \rightarrow 0} \left( f * \underbrace{\left( \frac{1}{\varepsilon} g(x/\varepsilon) \right)}_{g_\varepsilon(x)} - \alpha f(x_+) - \beta f(x_-) \right) = 0$

$$\int f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 g(y) f(x_+) dy - \int_0^\infty g(y) f(x_-) dy$$

$$\underbrace{\int_{-\infty}^0 (f(x-y) g_\varepsilon(y) - f(x_+) g(y)) dy}_{(*)} + \underbrace{\int_0^\infty (f(x-y) g_\varepsilon(y) - g(y) f(x_-)) dy}$$

Show each of these  $\rightarrow 0$  as  $\varepsilon \rightarrow 0$

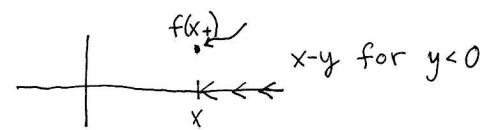
Folland proves this part  $\rightarrow 0$  as  $\varepsilon \rightarrow 0$

Want to get  $\varepsilon$  in both parts (both  $g$ 's)

Note that  $f(x_+) \int_{-\infty}^0 g(y) dy = \left\{ \begin{array}{l} z = \varepsilon y \\ \frac{dz}{\varepsilon} = dy \end{array} \right\} = f(x_+) \int_{-\infty}^0 \frac{g(\frac{z}{\varepsilon})}{\varepsilon} dz, \forall \varepsilon > 0$

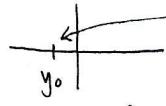
$$\Rightarrow (*) = \int_{-\infty}^0 (f(x-y) - f(x_+)) g_\varepsilon(y) dy$$

By definition  $\lim_{\substack{y \rightarrow 0 \\ y < 0}} (f(x-y) - f(x_+)) = 0$



Let  $\delta > 0$ . Show that  $|(*)| < \delta$  for  $\varepsilon$  small

Then  $\exists y_0 < 0$  such that  $|f(x-y) - f(x_+)| < \frac{\delta}{2(G+1)} \forall y \in [y_0, 0]$



Thus  $\underbrace{\int_{y_0}^0 (f(x-y) - f(x_+)) g_\varepsilon(y) dy}_{\text{estimate}}$ , let  $G = \int_{-\infty}^{\infty} |g(y)| dy < \infty$

$$\begin{aligned} \left| \int_{y_0}^0 (f(x-y) - f(x_+)) g_\varepsilon(y) dy \right| &\leq \int_{y_0}^0 |f(x-y) - f(x_+)| |g_\varepsilon(y)| dy \leq \\ &\leq \frac{\delta}{2(G+1)} \int_{y_0}^0 |g_\varepsilon(y)| dy \leq \frac{\delta}{2(G+1)} \int_{-\infty}^{\infty} |g_\varepsilon(y)| dy = \frac{\delta G}{2(G+1)} < \frac{\delta}{2} \end{aligned}$$

Next, show that for  $\varepsilon$  small,  $\left| \int_{-\infty}^{y_0/\varepsilon} (f(x-\varepsilon z) - f(x_+)) g(z) dz \right| < \frac{\delta}{2}$

let  $z = \frac{y}{\varepsilon} \Rightarrow \int_{-\infty}^{y_0/\varepsilon} (f(x-\varepsilon z) - f(x_+)) g(z) dz$

- If  $g$  has compact support  $[-K, K]$  for some  $K > 0$

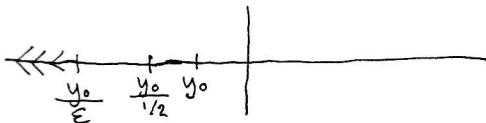
Then we can let  $\varepsilon$  be small enough so that  $\frac{y_0}{\varepsilon} < -K$

$$\Rightarrow \int_{-\infty}^{y_0/\varepsilon} (f(x-\varepsilon z) - f(x_+)) g(z) dz = 0$$

$$\therefore |*| < \frac{\delta}{2} \quad \forall \varepsilon \text{ with } \frac{y_0}{\varepsilon} < -K \Leftrightarrow \frac{(-y_0)}{\varepsilon} > K \Leftrightarrow \frac{-y_0}{K} > \varepsilon$$

- If instead  $f$  is bounded,  $\exists M > 0$  such that  $|f(y)| \leq M \quad \forall y \in \mathbb{R}$

$$\left| \int_{-\infty}^{y_0/\varepsilon} \underbrace{(f(x-\varepsilon z) - f(x_+)) g(z) dz}_{|\cdot| \leq 2M} \right| \leq \int_{-\infty}^{y_0/\varepsilon} 2M |g(z)| dz$$



Note that  $\int_{-\infty}^{\infty} |g(z)| dz = G < \infty$ . Also  $\int_{-\infty}^0 |g(z)| dz \leq G < \infty$

$$\int_{-\infty}^0 |g(z)| dz = \int_{-\infty}^{y_0/\varepsilon} |g(z)| dz + \int_{y_0/\varepsilon}^0 |g(z)| dz$$

By defn.  $\int_{-\infty}^0 |g(z)| dz = \lim_{R \rightarrow -\infty} \int_R^0 |g(z)| dz$

$$\Rightarrow \int_{y_0/\varepsilon}^0 |g(z)| dz \text{ as } \varepsilon \rightarrow 0$$

Hence  $\exists \varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0 \Rightarrow \int_{-\infty}^{y_0/\varepsilon} |g(z)| dz < \frac{\delta}{(M+1)4}$

$$\Rightarrow \int_{-\infty}^{y_0/\varepsilon} 2M |g(z)| dz < \frac{\delta(2M)}{4(M+1)} < \frac{\delta}{2} \Rightarrow |(*)| < \delta \blacksquare$$

Riemann Lebesgue Lemma If  $f \in L^1(\mathbb{R})$ , then

$$\hat{f}(\xi) \rightarrow 0 \quad \text{as } |\xi| \rightarrow 0 \quad (\xi \in \mathbb{R})$$

Fourier inversion - pointwise -  $\mathcal{F}^{-1}(\mathcal{F}(f)) = f \quad \forall f \in L^2$

Let  $f \in L^1(\mathbb{R}) \cap C^0(\mathbb{R})$ , then

$$f(x) = \lim_{\epsilon \rightarrow 0} \int e^{2\pi i \xi x} \frac{e^{-\xi^2 \epsilon^2}}{\sqrt{\pi}} \hat{f}(\xi) d\xi$$

The proof uses thm 7.3 (Exercise: try to figure the proof out!)

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### QUIZ

- Definitions:
1. Fourier series on  $[-\pi, \pi]$  coeff's etc.
  2. Fourier transform  $\hat{f}(\xi)$  when this defined
  3. Convolution
  4. Understand - thm 2.1 pointwise convergence of fourier series.
  5. Idea - solving PDE (not messy) like in exercises.

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DEF Let  $f$  be integ. over  $[0, \pi]$

Fourier cosine series:  $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta)$ ,  $a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos(n\theta) d\theta$

Fourier sine series:  $\sum_{n=1}^{\infty} b_n \sin(n\theta)$ ,  $b_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin(n\theta) d\theta$

THM Let  $f$  be piecewise smooth  $[0, \pi]$

Fourier cosine series and Fourier sine series converge to  $\frac{1}{2}(f(\theta+) + f(\theta-))$  when  $\theta \in (0, \pi)$

Fourier cosine series converges to  $f(\theta+)$  at  $\theta=0$ ,

to  $f(\pi-)$  at  $\theta=\pi$

Fourier sine series converges to 0 at  $\theta=0, \theta=\pi$

2.4.3  $f(\theta) = \sin \theta$   $[0, \pi]$

$$\text{cosine series: } a_n = \frac{2}{\pi} \int_0^{\pi} \sin \theta \cos(n\theta) d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin \theta| \cos(n\theta) d\theta =$$

$$= \begin{cases} -\frac{4}{\pi(n^2-1)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\frac{2}{\pi} + \sum_{k=1}^{\infty} -\frac{4}{\pi((2k)^2-1)} \cos(2k\theta)$$

sine series:

$(\sin(n\theta))$  form basis  $L_2[0, \pi]$ )

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin \theta \sin(n\theta) d\theta = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} (\cos(\theta+n\theta) - \cos(\theta-n\theta)) d\theta =$$

$$= \frac{2}{\pi} \left( \left[ \frac{\sin((n+1)\theta)}{n+1} \right]_0^{\pi} - \left[ \frac{\sin((1-n)\theta)}{1-n} \right]_0^{\pi} \right) = 0$$

$$b_1 = \frac{2}{\pi} \int_0^{\pi} \sin \theta \sin \theta d\theta = 1$$

so sine series:  $\sin \theta$

Cosine ser. conv. to 0 at  $\theta=0, f(\pi-) = 0$   $\theta=\pi$

Sine ser. conv. to 0 at  $\theta=0, \theta=\pi$

2.4.4  $f(\theta) = \cos \theta$   $[0, \pi]$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos(n\theta) d\theta = \frac{2}{\pi} \int_0^{\pi} \cos \theta \cos(n\theta) d\theta =$$

$$= \frac{1}{\pi} \int_0^{\pi} (\cos(\theta+n\theta) + \cos(\theta-n\theta)) d\theta = \frac{1}{\pi} \left( \left[ \frac{\sin(\theta+n\theta)}{1+n} \right]_0^{\pi} + \left[ \frac{\sin(\theta-n\theta)}{1-n} \right]_0^{\pi} \right) = 0$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} \cos \theta \cos \theta d\theta = 1.$$

$\cos \theta$  is cosine series.

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos \theta \sin(n\theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \sin(\theta+n\theta) + \sin(n\theta-\theta) d\theta = \{ n \neq 1 \}$$

$$= \frac{1}{\pi} \left( - \left[ \frac{\cos(\theta+n\theta)}{1+n} \right]_0^{\pi} - \left[ \frac{\cos(\theta-n\theta)}{n-1} \right]_0^{\pi} \right) = \begin{cases} 0, & n \text{ odd} \\ \frac{1}{\pi} \left( \frac{-1+1}{1+n} + \frac{1+1}{n-1} \right) = \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

$$b_1 = \frac{2}{\pi} \int_0^{\pi} \cos \theta \sin \theta d\theta = 0 , \quad \sum_{k=1}^{\infty} \frac{4(2k)}{\pi((2k)^2-1)} \sin(2k\theta) \text{ sine series}$$

Cosine series convg. to 1 at  $\theta=0$ , to -1 at  $\theta=\pi$   
 Sine series convg. to 0 at  $\theta=0, \theta=\pi$

2.4.8  $f(x) = 1-x$  on  $[0, l]$ ,  $l=1$ . Find Fourier cosine series

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx = 2 \int_0^1 (1-x) \cos(n\pi x) dx =$$

$$= \frac{2}{n\pi} \underbrace{\left[ (1-x) \sin(n\pi x) \right]_0^1}_{=0} + \frac{2}{n\pi} \int_0^1 \sin(n\pi x) dx = \frac{-2}{(n\pi)^2} \left[ \cos(n\pi x) \right]_0^1 =$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{4}{(n\pi)^2}, & n \text{ odd} \end{cases}$$

so cosine series:

$$a_0 = 2 \int_0^1 (1-x) dx = 1 , \quad \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{((2k-1)\pi)^2} \cos((2k-1)x)$$

E01  $f(x) = (1+x)^2$ ,  $[-l, l]$ ,  $l=1$ ,  $2l$ -periodic

a) Find complex Fourier series. b) Find  $2l$ -periodic solution of  
 $2y'' - y' - y = f(x)$

Solution: a)  $c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-inx/l} dx = \frac{1}{2} \int_{-1}^1 (1+x)^2 e^{-inx} dx =$

$$= -\frac{1}{2in\pi} \left[ (1+x)^2 e^{-inx} \right]_{-1}^1 + \frac{1 \cdot 2}{2in\pi} \int_{-1}^1 (1+x) e^{-inx} dx =$$

$$= -\frac{1 \cdot 4}{2in\pi} e^{-in\pi} - \frac{1}{(in\pi)^2} \left[ (1+x) e^{-inx} \right]_{-1}^1 + \underbrace{\frac{1}{(in\pi)^2} \int_{-1}^1 e^{-inx} dx}_{=0} =$$

$$= -\frac{2}{in\pi} e^{-in\pi} - \frac{2}{(in\pi)^2} e^{-in\pi} = \left\{ e^{-in\pi} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases} \right\} =$$

$$= -\frac{2(-1)^n (in\pi + 1)}{(in\pi)^2}$$

$$c_0 = \frac{1}{2} \int_{-1}^1 (1+x)^2 dx = \frac{1}{6} \left[ (1+x)^3 \right]_{-1}^1 = \frac{4}{3}$$

$$f(x) = \frac{4}{3} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} -\frac{2(-1)^n (in\pi + 1)}{(in\pi)^2} e^{inx}$$

b) Let  $\sum_{n=-\infty}^{\infty} \tilde{c}_n e^{inx}$  be Fourier series of  $y$ ,  $y(x) = \sum \tilde{c}_n e^{inx}$

$$2y'' - y' - y = 2 \sum_{n=-\infty}^{\infty} \tilde{c}_n (in\pi)^2 e^{inx} - \sum_{n=-\infty}^{\infty} \tilde{c}_n (in\pi) e^{inx} - \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{inx} =$$

$$= \sum_{n=-\infty}^{\infty} \tilde{c}_n (2(in\pi)^2 - in\pi - 1) e^{inx} = f(x) = \frac{4}{3} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} -\frac{2(-1)^n (in\pi + 1)}{(in\pi)^2} e^{inx}$$

$$\tilde{c}_n = -\frac{2(-1)^n (in\pi + 1)}{(in\pi)^2 (2(in\pi)^2 - in\pi - 1)}$$

$$\text{for } n \neq 0, -\tilde{c}_0 = \frac{4}{3}, \tilde{c}_0 = -\frac{4}{3}$$

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F7

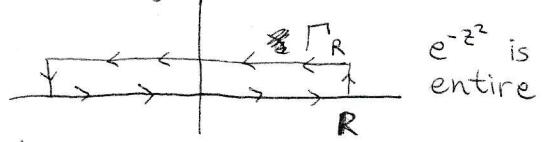
$$\int_{\mathbb{R}} e^{-((2\pi i \xi)^2 + 2(2\pi i \xi) \left(\frac{ix}{2\sqrt{t}}\right) + (\frac{ix}{2\sqrt{t}})^2)} d\xi = \left\{ y = 2\pi i \xi \sqrt{t} + \frac{ix}{2\sqrt{t}} \right\}$$

$$= \int_{\mathbb{R}} e^{-y^2} \frac{dy}{2\pi \sqrt{t}}$$

$$\left\{ dy = 2\pi \sqrt{t} d\xi \right\}$$

Why not  $\int_{-\infty + \frac{ix}{2\sqrt{t}}}^{\infty + \frac{ix}{2\sqrt{t}}} ?$  Claim that these integrals are the same!

$$\int_{-\infty + \frac{ix}{2\sqrt{t}}}^{\infty + \frac{ix}{2\sqrt{t}}} e^{-z^2} dz = \int_{-\infty}^{\infty} e^{-z^2} dz$$



Thus ∀ closed curve like  $\Gamma_R$ ,  $\int_{\Gamma_R} f(z) dz = 0$

Write  $z = a+ib$ ,  $z^2 = a^2 + b^2 + i\ldots$

$$\Rightarrow |e^{-z^2}| = |e^{-(a^2+b^2)+i\ldots}| = |e^{-(a^2+b^2)}||e^{i\ldots}| = |e^{-a^2+b^2}|$$

Thus on  $\psi \uparrow |e^{-z^2}|$  is small. Hence  $\lim_{R \rightarrow \infty} \int_{\Gamma_R} e^{-z^2} dz =$

$$= \int_{-\infty}^{\infty} e^{-z^2} dz + \int_{-\infty + \frac{ix}{2\sqrt{t}}}^{-\infty + \frac{ix}{2\sqrt{t}}} e^{-z^2} dz = 0 \Rightarrow \int_{-\infty}^{\infty} e^{-z^2} dz = - \int_{\infty + \frac{ix}{2\sqrt{t}}}^{-\infty + \frac{ix}{2\sqrt{t}}} e^{-z^2} dz = \int_{-\infty + \frac{ix}{2\sqrt{t}}}^{\infty + \frac{ix}{2\sqrt{t}}} e^{-z^2} dz$$

Pointwise Fourier Inversion: Let  $f \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$  then

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{2\pi i \xi x} e^{-\varepsilon^2 \xi^2} \hat{f}(\xi) d\xi = f(x) \quad \forall x \in \mathbb{R}$$

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}^{-1}(e^{-\varepsilon^2 \xi^2} \hat{f}(\xi))(x) = f(x)$$

Idea: Use thm 7.2:  $\lim_{\varepsilon \rightarrow 0} f * g_\varepsilon(x) = f(x)$  if hyp. satisfied

$$\begin{aligned} (\mathcal{F}^{-1}(e^{-\varepsilon^2 \xi^2} \hat{f}(\xi)))(x) &= \mathcal{F}(e^{-\varepsilon^2 \xi^2} \hat{f}(\xi))(-x) = (e^{-\varepsilon^2 \xi^2} * \hat{f}(\xi))(-x) \\ &= \underbrace{\mathcal{F}^{-1}(e^{-\varepsilon^2 \xi^2})}_{} * f(x) \\ \int_{\mathbb{R}} e^{-\varepsilon^2 \xi^2 + 2\pi i \xi x} d\xi &= \mathcal{F}^{-1}(e^{-\varepsilon^2 \xi^2})(x) \end{aligned}$$

Same idea  $\Rightarrow$  complete the square  $-(\varepsilon^2 \xi^2 - 2\pi i \xi x) =$

$$= -((\varepsilon \xi)^2 + 2(\frac{-\pi i x}{\varepsilon})(\varepsilon \xi) - \frac{\pi^2 x^2}{\varepsilon^2}) - \frac{\pi^2 x^2}{\varepsilon^2}$$

$$\Rightarrow \int_{\mathbb{R}} e^{-\pi^2 x^2/\varepsilon^2} e^{-(\varepsilon \xi + \frac{\pi i x}{\varepsilon})^2} d\xi = \left\{ y = \varepsilon \xi + \frac{i\pi x}{\varepsilon}, dy = \varepsilon d\xi \right\} =$$

$$= e^{-\pi^2 x^2/\varepsilon^2} \int_{\mathbb{R}} e^{-y^2} \frac{dy}{\varepsilon} = \frac{e^{-\pi^2 x^2/\varepsilon^2} \sqrt{\pi}}{\varepsilon}$$

↑  
same calculation

Thus we have  $\lim_{\varepsilon \rightarrow 0} g_\varepsilon * f(x)$ , where  $g(x) = e^{-\pi^2 x^2/\varepsilon^2} \sqrt{\pi} \Rightarrow g_\varepsilon(x) = \frac{e^{-\pi^2 x^2/\varepsilon^2} \sqrt{\pi}}{\varepsilon}$   
 $\int_{\mathbb{R}} g(x) dx = 1$

$$\alpha = \int_{-\infty}^0 g(x) dx = \frac{1}{2} = \int_0^\infty g(x) dx = \beta$$

Thm  $\Rightarrow \lim_{\varepsilon \rightarrow 0} g_\varepsilon * f(x) = \alpha f(x_+) + \beta f(x_-)$  where

$$f(x_+) = \lim_{\substack{y \rightarrow x \\ y > x}} f(y) \quad f(x_-) = \lim_{\substack{y \rightarrow x \\ y < x}} f(y), \quad f \text{ is continuous}$$

$$\text{so } f(x_+) = f(x_-) = f(x), \quad \alpha f(x_+) + \beta f(x_-) = f(x) \blacksquare$$

Plancharel theorem  $\forall f, g \in L^2(\mathbb{R})$ ,  $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$

$$\text{Proof } \langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx = \left\{ \begin{array}{l} \text{use } F^{-1}(F(f))(x) = f(x) \\ L^2 \text{ equality} \end{array} \right\} =$$

$$= \int e^{2\pi i x \xi} \underbrace{\int e^{-2\pi iy\xi} f(y) dy}_{\hat{f}(\xi)} \overline{\int g(x) dx} d\xi =$$

$$= \iint_{\mathbb{R} \times \mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} \overline{g(x)} dx d\xi =$$

$$= \int e^{2\pi i x \xi} \overline{\int g(x) dx} \int \hat{f}(\xi) d\xi =$$

$$= \int \overline{e^{-2\pi i x \xi} g(x)} dx \int \hat{f}(\xi) d\xi = \int e^{-2\pi i x \xi} g(x) \int \hat{f}(\xi) d\xi.$$

$$\text{Thus } \langle f, g \rangle = \int \hat{f}(\xi) \overline{\hat{g}(\xi)} = \langle \hat{f}, \hat{g} \rangle \blacksquare$$

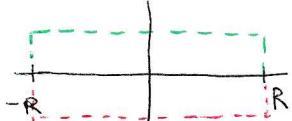
Is it in general easy to compute Fourier transforms of func's?

NO!

$$\text{Ex. } F\left(\frac{1}{1+x^2}\right)(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} \left(\frac{1}{1+x^2}\right) dx \rightarrow \text{Residue thm!}$$

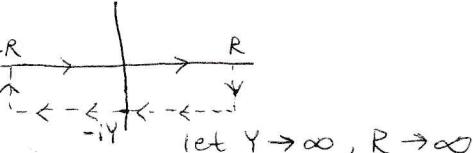
$$f(z) = \frac{e^{-2\pi iz\xi}}{1+z^2}, \quad \xi \in \mathbb{R}, \quad \text{has two simple poles at } \pm i$$

$$\int_{\text{Box}} = 2\pi i \sum_{\text{in box}} \text{Res}$$



$$|f(z)| = \frac{|e^{-2\pi i (\alpha+ib)\xi}|}{|1+z^2|} = \frac{e^{2\pi b\xi}}{|1+z^2|}$$

If  $\xi > 0$ , use box below  $\mathbb{R}$ -axis



$$\int_{\text{box}} f(z) dz = -2\pi i \text{Res}_{z=i} f(z) =$$

$$= -2\pi i \left. \frac{e^{-2\pi iz\xi}}{(z-i)} \right|_{z=i} = -2\pi i \frac{e^{+2\pi i^2 \xi}}{-2i} = +\pi e^{-2\pi \xi}$$

OBS!  $\Gamma_R$  is negatively oriented

If  $\xi < 0$ , use above axis box in the same way  $\Rightarrow \pi e^{2\pi \xi}$

$$\text{Thus } \mathcal{F}\left(\frac{1}{1+x^2}\right)(\xi) = \begin{cases} \pi e^{-2\pi|\xi|} & \xi \geq 0 \\ \pi e^{2\pi|\xi|} & \xi < 0 \end{cases} = \pi e^{-2\pi|\xi|}$$

APPLICATION OF F: Compute say  $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

How? This is  $\hat{f}(0)$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i \xi x}}{1+x^2} dx \quad \text{with } \xi = 0$$

### SEPARATION OF VARIABLES AND "CONTINUOUS SUPERPOSITION"

We want to solve the heat eq. on  $\mathbb{R}$  ( $[0, \infty)$  later...)  
Write solution as  $T(t)X(x) \Rightarrow T'(t)X(x) + X''(x)T(t) = 0$

$$\Rightarrow \frac{T'}{T} = \frac{-X''}{X} \Rightarrow \text{both sides are constant} = \lambda$$

$T(t) = e^{\lambda t}$ , no sources,  $\lambda$  can not be positive!  
(positive  $\lambda$  will increase heat over time, not possible without sources).  $\lambda \leq 0$

$$T_\xi(t) = e^{-\xi^2 t} \Rightarrow X(x) = e^{\pm i \xi x} \quad (\xi \text{ can be any real nbr})$$

Initial condition:  $u_0(x)$  temp at  $x \in \mathbb{R}$ , at time  $t=0$

$$u(x,t) = \int_{-\infty}^{\infty} C(\xi) e^{-\xi^2 t} e^{i \xi x} d\xi = \{\text{change variables}\} =$$

$$= \int_{-\infty}^{\infty} e^{2\pi i y x} \cancel{\widetilde{C}(y)} e^{-4\pi^2 y^2 t} dy$$

$$u(x,0) = u_0(x) = \int_{\mathbb{R}} e^{2\pi i y x} \widetilde{C}(y) dy \Rightarrow \widetilde{C}(y) = \hat{u}_0(y)$$

Thus solution is

$$u(x,t) = \int_{\mathbb{R}} e^{2\pi i y x} \hat{u}_0(y) e^{-4\pi^2 y^2 t} dt = \int_{\mathbb{R}} \frac{u_0(y) e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy$$

$$\left( u_0 * \frac{e^{-z^2/4t}}{\sqrt{4\pi t}} \right) (x)$$

HALF LINES  $[0, \infty)$  Solve heat eq here with  
 $\uparrow$  boundary

- ① Dirichlet BC solution = 0 on boundary
- ② Neumann BC  $\partial_x$  solution = 0 on boundary

For ① Need  $X(0) = 0$ , for ②  $X'(0) = 0$

For  $T_{\frac{x}{2}}(t) = e^{-4\pi^2 \frac{x^2}{4} t}$   $X_{\frac{x}{2}}(x) = e^{\pm 2\pi i \frac{x}{2} x}$

$$\frac{X_{\frac{x}{2}} + X_{-\frac{x}{2}}}{2} = \cos(2\pi \frac{x}{2} x) \quad \frac{X_{\frac{x}{2}} - X_{-\frac{x}{2}}}{2i} = \sin(2\pi \frac{x}{2} x)$$

For odd functions on  $\mathbb{R}$ ,  $\hat{f}(\frac{x}{2}) = \int e^{-2\pi i \frac{x}{2} x} f(x) dx =$   
 $= \int (\cos(-2\pi i x \frac{x}{2}) + i \sin(-2\pi i x \frac{x}{2})) f(x) dx = i \int_{\mathbb{R}} \sin(-2\pi i x \frac{x}{2}) f(x) dx =$   
 $= -i \int_{\mathbb{R}} \sin(2\pi i x \frac{x}{2}) f(x) dx = -2i \int_0^\infty \sin(2\pi \frac{x}{2} x) f(x) dx$

For even functions on  $\mathbb{R}$ ,  $\hat{f}(\frac{x}{2}) = 2 \int_0^\infty \cos(2\pi x \frac{x}{2}) f(x) dx$

DEF  $\mathcal{F}_{\sin}(f)(\frac{x}{2}) = \int_0^\infty \sin(2\pi x \frac{x}{2}) f(x) dx$

$$\mathcal{F}_{\cos}(f)(\frac{x}{2}) = \int_0^\infty \cos(2\pi x \frac{x}{2}) f(x) dx$$

$$f(x) = 4 \mathcal{F}_c(\mathcal{F}_c(f))(x) \quad \text{because } \mathcal{F}_c(f)(\frac{x}{2}) = \mathcal{F}_c(f)(-\frac{x}{2})$$

$$f(x) = 4 \mathcal{F}_s(\mathcal{F}_s(f))(-x)$$

To solve heat eq. on  $[0, \infty)$  with DBC, ~~where~~

for IC  $u_0(x)$ ,  $u_0(0) = 0$ , extend  $u_0$  oddly to  $\mathbb{R}$

$$\begin{aligned} \text{Then } u(x, t) &= \int \hat{u}_0(\frac{x}{2}) e^{-4\pi^2 \frac{x^2}{4} t} e^{2\pi i \frac{x}{2} x} d\frac{x}{2} = \\ &= \int \frac{u_0(y)}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} dy = \int_{-\infty}^0 + \int_0^\infty = \\ &= \int_{\mathbb{R}} u_0(y) \frac{(e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t})}{\sqrt{4\pi t}} dy \end{aligned}$$

Exercise: Extend  $u_0$  evenly to  $\mathbb{R}$  and show  
that solution is + above

DFT, FFT, and SAMPLING THEOREMTime analysis  
 $f(t)$ Frequency analysis  
 $\hat{f}(\xi)$ DEF  $\ell_N^2 = \left\{ (s_n)_{n=0}^{N-1} \mid s_n \in \mathbb{C}, \langle (s_n), (\tau_n) \rangle = \sum_{n=0}^{N-1} s_n \bar{\tau}_n \right\}$ 

Let  $e_k(n) = \frac{e^{+2\pi i k n / N}}{\sqrt{N}}$  for  $k$  and  $n \in \mathbb{Z}$

$\downarrow$  vector component  
 $e_k = (e_k(n))_{n=0}^{N-1} \in \ell_N^2$

PROP  $\{e_k\}_{k=0}^{N-1}$  are an ONB for  $\ell_N^2$ 

$$\begin{aligned} \text{Proof} \quad & \text{Compute } \langle e_k, e_j \rangle = \sum_{n=0}^{N-1} \frac{e^{-2\pi i k n / N}}{\sqrt{N}} \frac{e^{-2\pi i j n / N}}{\sqrt{N}} = \\ & = \sum_{n=0}^{N-1} \frac{e^{2\pi i (j-k)n / N}}{\sqrt{N}^2} = 1 \text{ if } j=k, \text{ Thus } \|e_k\| = \sqrt{\langle e_k, e_k \rangle} = 1 \end{aligned}$$

$$\begin{aligned} \text{If } j \neq k, \text{ OBS! geometric sum } & \frac{1}{N} \sum_{n=0}^{N-1} \left( e^{2\pi i (j-k)/N} \right)^n = \\ & = \frac{1}{N} \left( \frac{1 - (e^{2\pi i (j-k)/N})^N}{1 - e^{2\pi i (j-k)/N}} \right) = 0 \quad \blacksquare \end{aligned}$$

Remark: In finite dimensional vector spaces,  $N$  dimensional space, then any set of  $N$  linearly independent vectors is a basis (span the space)

Fix  $T$  and  $N$  and look at  $f(t)$  on  $[0, NT]$  $f(n) := f(t_n) := f(nT)$ . Identify  $f$  with an element  $(f(t_n))_{n=0}^{N-1} \in \ell_N^2$ DEF Discrete Fourier transform is for  $w_k = \frac{2\pi k}{NT}$ 

$$F(w_k) := \langle f, e_k \rangle = \hat{f}(w_k) = \sum_{n=0}^{N-1} \frac{f(t_n) e^{-2\pi i k n / N}}{\sqrt{N}} = \sum_{n=0}^{N-1} \frac{f(t_n) e^{-i w_k n}}{\sqrt{N}}$$

~~REMEMBER~~

PROP (discrete inversion formula)

$$f(t_n) = \langle \hat{f}, e_n \rangle = \sum_{k=0}^{N-1} \hat{f}(w_k) e_n(k)$$

Proof (use the definitions)

$$\begin{aligned} & = \sum_{k=0}^{N-1} \left( \sum_{m=0}^{N-1} \frac{f(t_m) e^{-i w_k m}}{\sqrt{N}} \right) \frac{e^{2\pi i k n / N}}{\sqrt{N}} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} f(t_m) e^{-2\pi i k m / N} e^{2\pi i k n / N} \\ & = \sum_{m=0}^{N-1} f(t_m) \underbrace{\sum_{k=0}^{N-1} \frac{e^{-2\pi i k m / N} e^{2\pi i k n / N}}{N}}_{\langle e_n, e_m \rangle = 0 \text{ unless } m=n} \Rightarrow f(t_n) \end{aligned}$$

$\langle e_n, e_m \rangle = 0 \text{ unless } m=n$   
 If  $m=n$ ,  $\langle e_n, e_m \rangle = 1$

What happened to time? Exercise! Next time...

$$\hat{f}(w_k) = \left\langle f(t_n), \frac{e^{2\pi i k n / N}}{\sqrt{N}} \right\rangle$$

$$\hat{f} = \begin{bmatrix} \hat{f}(w_0) \\ \hat{f}(w_1) \\ \vdots \\ \hat{f}(w_{N-1}) \end{bmatrix} = \left[ \frac{e^{-2\pi i k n / N}}{\sqrt{N}} \right]_{n=0}^{N-1} \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_{N-1}) \end{bmatrix}$$

*n's → K's*  
↓  
*discrete f*

This is an  $N \times N$  matrix.

Assume  $N = 2^d$  (power of 2,  $d \in \mathbb{N}$ )

$$\hat{f}(w_k) = \frac{1}{\sqrt{N}} \left( \sum_{j=0}^{N/2-1} f(t_{2j}) e^{-2\pi i k (2j)/N} + \sum_{j=0}^{N/2-1} f(t_{2j+1}) e^{-2\pi i k (2j+1)/N} \right)$$

$$e_N^k(n) = e^{-2\pi i k n / N}, \text{ then } e_N^k(2j) = e^{-2\pi i k (2j) / N} = e_{N/2}^k(j) = e^{-2\pi i k j / (N/2)}$$

$$\Rightarrow \hat{f}(w_k) = \frac{1}{\sqrt{N}} \left( \sum_{j=0}^{N/2-1} f(t_{2j}) e_{N/2}^k(j) + e_N^k(1) \sum_{j=0}^{N/2-1} f(t_{2j+1}) e_{N/2}^k(j) \right)$$

⇒ Reduced to  $\frac{N}{2} \times \frac{N}{2}$  matrix

Repeat this ⇒  $\sim \frac{N}{2} \log(N)$  computations

If  $N = 2^{10}$  this saves 99% calculation time

$$N^2 = 2^{20} \text{ vs } (2^{10})^{10}$$

$$N^2 \text{ calc. vs } \sim \frac{N \log(N)}{\text{FFT}}$$

PROP ① If  $g \in L^1(\mathbb{R}) \Rightarrow \hat{g} \in C^\circ(\mathbb{R})$

② If  $g \in L^2(\mathbb{R})$  and  $\hat{g} \in L^1(\mathbb{R})$  then  $g$  is continuous.

Proof ①  $\lim_{\delta \rightarrow 0} \hat{g}(\xi + \delta) = \hat{g}(\xi)$  want to show this  $\forall \xi \in \mathbb{R}$

$$\lim_{\delta \rightarrow 0} \hat{g}(\xi + \delta) = \lim_{\delta \rightarrow 0} \int_{\mathbb{R}} g(x) e^{-2\pi i (\xi + \delta)x} dx =$$

$$= \left\{ \begin{array}{l} \text{Lebesgue's dominated convergence thm because} \\ |g(x) e^{-2\pi i (\xi + \delta)x}| = g(x) \text{ and } \int_{\mathbb{R}} |g(x)| dx < \infty \text{ because } g \in L^1(\mathbb{R}) \end{array} \right\} =$$

$$= \int_{\mathbb{R}} g(x) \lim_{\delta \rightarrow 0} (e^{-2\pi i (\xi + \delta)x}) dx = \int_{\mathbb{R}} g(x) e^{-2\pi i \xi x} dx$$

$$\textcircled{2} \quad g = F^{-1}(F(g))(x) = F(\hat{g})(-x)$$

By assumption  $\hat{g} \in L^1(\mathbb{R})$

By \textcircled{1}  $F(\hat{g})(x)$  is continuous  $\Rightarrow F(\hat{g})(-x)$  is also ■

PROP  $L^1([-L, L]) \supseteq L^2([-L, L])$ ,  $L > 0$

$$\begin{aligned} \int_{[-L, L]} |f(x)| dx &= \int_{[-L, L]} |f(x)| 1 dx = \langle |f|, 1 \rangle_{L^2([-L, L])} \leq \\ &\leq \| |f| \|_{L^2([-L, L])} \| 1 \|_{L^2([-L, L])} = \left( \int_{[-L, L]} |f(x)|^2 dx \right)^{1/2} \sqrt{2L} \end{aligned}$$

Thus Cauchy-Schwarz, like with vectors.

If  $f \in L^2([-L, L])$  then

$$\| f \|_{L^1([-L, L])} = \int_{[-L, L]} |f(x)| dx \leq \| f \|_{L^2([-L, L])} \sqrt{2L} < \infty$$

Hence  $L^2([-L, L]) \subset L^1([-L, L])$  ■

OBS!

like  $x^{-1/2}$  on  $(0, 1)$ , 0 on  $(-\infty, 0) \cup (1, \infty)$

$$\int_0^1 x^{-1/2} dx < \infty \quad \text{but} \quad \int_0^1 (x^{-1/2})^2 dx = \infty$$

$$f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ 0, & x \leq 1 \end{cases} \Rightarrow \int_{\mathbb{R}} |f(x)|^2 dx = \int_1^{\infty} \frac{1}{x^2} dx < \infty$$

but  $\int_1^{\infty} \frac{1}{x} dx = \infty$

SAMPLING THEOREM Let  $f \in L^2(\mathbb{R})$ , assume  $\hat{f}$  has compact support, that is  $\hat{f}(\omega) = 0 \quad \forall |\omega| \geq L$

Then  $f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2L}\right) \frac{\sin(n\pi - 2\pi t + L)}{n\pi - 2\pi t + L}$   
 only need  $\{f\left(\frac{n}{2L}\right)\}_{n \in \mathbb{Z}}$  to recover  $f$ .

Proof Idea: use Fourier transform and Fourier series all together

$$f \in L^2 \Rightarrow \hat{f} \in L^2 \quad \boxed{-L \quad i \quad L}$$

$$\| \hat{f} \|_{L^1(\mathbb{R})} = \int_{[-L, L]} | \hat{f}(\omega) | d\omega$$

By preceding prop  $\| \hat{f} \|_{L^1(\mathbb{R})} < \infty \quad \hat{f} \in L^1(\mathbb{R})$

$$\text{Fourier inversion} \quad f(t) = \int_{\mathbb{R}} e^{2\pi i \xi t} \hat{f}(\xi) d\xi = \int_{[-L, L]} e^{2\pi i \xi t} \hat{f}(\xi) d\xi =$$

= {Expand  $\hat{f}$  in a Fourier series on  $[-L, L]$ } =

$$= \int_{-L}^L e^{2\pi i \xi t} \sum_{n=-\infty}^{\infty} c_n e^{i n \pi \xi / L} d\xi$$

Fourier series expansion on  $[-L, L]$

$$c_n = \frac{1}{2L} \int_{\mathbb{R}} \hat{f}(y) e^{-i n \pi y / L} dy = \frac{1}{2L} \int_{[-L, L]} \hat{f}(y) e^{-i n \pi y / L} dy$$

↑  
because  $\hat{f}(y) = 0$  for  $|y| \geq L$

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \int_{-L}^L e^{2\pi i \xi t} \frac{e^{i n \pi \xi / L}}{2L} \int_{[-L, L]} \hat{f}(y) e^{-i n \pi y / L} dy d\xi = \\ &= \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int_{-L}^L e^{2\pi i \xi t} e^{i n \pi \xi / L} \int_{\mathbb{R}} \hat{f}(y) e^{-2\pi i ny / 2L} dy d\xi \end{aligned}$$

$$\text{Fourier inversion thm: } \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i y (-n/2L)} dy = f\left(\frac{-n}{2L}\right)$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^L e^{i \pi \xi (2t + n/L)} f\left(\frac{-n}{2L}\right) d\xi = \{\text{Reverse order}\}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^L e^{i \pi \xi (2t - n/L)} f\left(\frac{n}{2L}\right) d\xi$$

$$\int_{-L}^L e^{i \pi \xi (2t - n/L)} d\xi = \left[ \frac{e^{i \pi \xi (2t - n/L)}}{i \pi (2t - n/L)} \right]_{\xi=-L}^L =$$

$$= \frac{e^{i \pi L (2t - n/L)}}{i \pi (2t - n/L)} - \frac{e^{-i \pi L (2t - n/L)}}{i \pi (2t - n/L)} = \frac{2i \sin(\pi L (2t - n/L))}{i \pi (2t - n/L)}$$

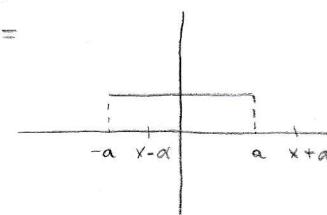
$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2L}\right) \frac{\sin(n \pi - 2\pi t L)}{n \pi - 2\pi t L}$$

□

EÖ66 Beräkna  $F = f * f(x)$  där  $f(x) = \chi_{(-a,a)}(x) = \begin{cases} 1, & |x| < a \\ 0, & \text{annars} \end{cases}$

$$f * f(x) = \int_{-\infty}^{\infty} f(t) f(x-t) dt ,$$

$$\begin{aligned} f * f(x) &= \int_{-\infty}^{\infty} \chi_{(-a,a)}(t) \chi_{(-a,a)}(x-t) dt = \int_{-a}^a \chi_{(-a,a)}(x-t) dt = \\ &= \left\{ \begin{array}{l} x-t=y, \dots dt=-dy, y \in (x+a, x-a) \end{array} \right\} = \\ &= - \int_{x-a}^{x+a} \chi(y) dy = \int_{x-a}^{x+a} \chi_{(-a,a)}(y) dy \end{aligned}$$



$$|x| > 2a \Rightarrow F(x) = 0$$

$$\left. \begin{array}{l} \text{Antag } x > 0, x < 2a \\ \text{Antag } x < 0, x > -2a \end{array} \quad \begin{array}{l} \int_{-a}^a 1 dy = 2a - x \\ \int_{-a}^{x+a} 1 dy = 2a + x \end{array} \right\} F(x) = 2a - |x|, x \in (-2a, 2a)$$

Fouriertransformera  $F(x)$ ! Minns:  $\mathcal{F}(f * f) = \mathcal{F}(f) \cdot \mathcal{F}(f)$

$$\mathcal{F}(\chi_{(-a,a)}(x)) = \frac{2 \sin(wa)}{w}$$

$$\mathcal{F}(F) = (\mathcal{F}(f))^2 = \frac{4 \sin^2(wa)}{w^2}$$

7.2.13 a) Visa att  $\int_{-\infty}^{\infty} \frac{\sin(at)}{t} \frac{\sin(bt)}{t} dt = \pi \min(a, b)$

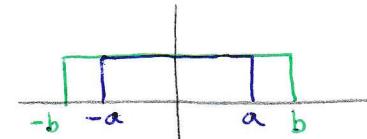
PLANCHEREL  $\langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle$ ,  $\langle f, g \rangle = \int_{-\infty}^{\infty} f \bar{g} dx$

$$\mathcal{F}\left(\frac{\sin(at)}{t}\right) = \pi \chi_{(-a,a)}(w)$$

$$\int_{-\infty}^{\infty} \underbrace{\frac{\sin(at)}{t}}_f \underbrace{\frac{\sin(bt)}{t}}_g dt = \langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle =$$

↑  
Plancherel

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi^2 \chi_{(-a,a)}(w) \chi_{(-b,b)}(w) dw$$



$$\text{Antag } a > b$$

$$\frac{1}{2\pi} \int_{-b}^b \pi^2 dw = \frac{\pi}{2} (b - (-b)) = \pi b$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(at)}{t} \frac{\sin(bt)}{t} dt = \pi \min(a, b)$$

b) Visa att  $\int_{-\infty}^{\infty} \frac{t^2}{(t^2+a^2)(t^2+b^2)} dt = \frac{\pi}{a+b}$

$$\mathcal{F}\left(\frac{1}{t^2+a^2}\right) = \frac{\pi}{a} e^{-aw}, \quad \mathcal{F}(tf(t)) = i \hat{f}'(w)$$

$$\mathcal{F}\left(\frac{t}{t^2+a^2}\right) = i \frac{d}{dw} \left( \frac{\pi}{a} e^{-aw} \right) = \begin{cases} -ia \frac{\pi}{a} e^{-aw}, & w > 0 \\ ia \frac{\pi}{a} e^{aw}, & w < 0 \end{cases}$$

KONTROLL  
derivatan  
av en jämn  
funktion  
är udda

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{t}{t^2+a^2} \frac{t}{t^2+b^2} dt &= \left\langle \underbrace{\frac{t}{t^2+a^2}}_f, \underbrace{\frac{t}{t^2+b^2}}_g \right\rangle = \frac{1}{2\pi} \langle i\hat{f}, i\hat{g} \rangle = \\
 &= \frac{1}{2\pi} i(-i) \langle \hat{f}, \hat{g} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi e^{-aw} \uparrow \quad \uparrow \quad \text{jämn} \quad \text{jämn} \\
 &\quad e^{-bw} dw = \pi \int_0^{\infty} e^{-w(a+b)} dw = \\
 &= \pi \left[ \frac{-e^{-w(a+b)}}{a+b} \right]_0^{\infty} = \pi \left( 0 - \left( -\frac{1}{a+b} \right) \right) = \frac{\pi}{a+b}
 \end{aligned}$$

EÖ12 Låt  $f(t) = \int_0^1 \sqrt{w} e^{w^2} \cos(wt) dw$

Beräkna  $\int_{-\infty}^{\infty} |f'(t)|^2 dt$

$$\begin{aligned}
 \int_{-\infty}^{\infty} |f'(t)|^2 dt &= \langle f'(t), f'(t) \rangle = \frac{1}{2\pi} \left\langle \underbrace{F(f'(t))}_{i w \hat{f}(t)}, \underbrace{F(f'(t))}_{i w \hat{f}(t)} \right\rangle = \\
 &= \frac{1}{2\pi} \langle i w \hat{f}(w), i w \hat{f}(w) \rangle = (*)
 \end{aligned}$$

$$f(t) = \int_0^1 \sqrt{w} e^{w^2} \cos(wt) dw, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwt} dw$$

$$\begin{aligned}
 f(t) &= \frac{1}{2} \int_{-1}^1 \sqrt{|w|} e^{w^2} \cos(wt) dw + 0 = \\
 &= \frac{1}{2} \int_{-1}^1 \sqrt{|w|} e^{w^2} \cos(wt) dw + \underbrace{\frac{i}{2} \int_{-1}^1 \sqrt{|w|} e^{w^2} \sin(wt) dw}_{} = 0 \\
 &= \frac{1}{2} \int_{-1}^1 \sqrt{|w|} e^{w^2} e^{itw} dw =
 \end{aligned}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \sqrt{|w|} e^{w^2} \chi_{(-1,1)}(w) e^{itw} dw$$

$$\text{Slutsats: } \hat{f}(w) = \pi \sqrt{|w|} e^{w^2} \chi_{(-1,1)}(w)$$

$$\begin{aligned}
 (*) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |w| w^2 e^{2w^2} \chi_{(-1,1)}^2(w) dw = \pi^2 = \\
 &= \frac{\pi}{2} \int_{-1}^1 |w| w^2 e^{2w^2} dw = \frac{2\pi}{2} \int_0^1 w^3 e^{2w^2} dw = \\
 &= \pi \int_0^1 w^2 (we^{2w^2}) dw = \pi \left[ w^2 \cdot \frac{e^{2w^2}}{4} \right]_0^1 - \pi \int_0^1 \frac{2w e^{2w^2}}{4} dw = \\
 &= \frac{\pi e^2}{4} - \frac{\pi}{2} \left[ \frac{e^{2w^2}}{4} \right]_0^1 = \frac{\pi e^2}{4} - \frac{\pi}{8} e^2 + \frac{\pi}{8} = \frac{\pi}{8} (e^2 + 1)
 \end{aligned}$$

7.4.2 a) Beräkna  $\mathcal{F}_s[e^{-kx}]$

$$\mathcal{F}_s[f] = \int_0^\infty f(x) \sin(wx) dx$$

$$\begin{aligned}\mathcal{F}_s[e^{-kx}] &= \int_0^\infty e^{-kx} \sin(wx) dx = \int_0^\infty e^{-kx} \left( \frac{e^{iwx} - e^{-iwx}}{2i} \right) dx = \\ &= \int_0^\infty \frac{e^{x(iw-k)} - e^{-x(iw+k)}}{2i} dx = \frac{1}{2i} \left[ \frac{e^{x(iw-k)}}{iw-k} + \frac{e^{-x(iw+k)}}{iw+k} \right]_0^\infty = \\ &= \frac{1}{2i} \left( -\frac{1}{iw-k} - \frac{1}{iw+k} \right) = \frac{1}{2i} \left( \frac{-iw-k - iw+k}{-w^2-k^2} \right) = \frac{2iw}{2i(w^2+k^2)} = \underline{\underline{\frac{w}{w^2+k^2}}}\end{aligned}$$

b) Beräkna  $\mathcal{F}_c[e^{-kx}]$ ,  $\mathcal{F}_c[f] = \int_0^\infty f(x) \cos(wx) dx$

$$\begin{aligned}\mathcal{F}_c[e^{-kx}] &= \int_0^\infty e^{-kx} \cos(wx) dx = \int_0^\infty e^{-kx} \left( \frac{e^{iwx} + e^{-iwx}}{2} \right) dx = \\ &= \int_0^\infty \frac{e^{x(iw-k)} + e^{-x(iw+k)}}{2} dx = \frac{1}{2} \left[ \frac{e^{x(iw-k)}}{iw-k} + \frac{e^{-x(iw+k)}}{-wi-k} \right]_0^\infty = \\ &= \frac{1}{2} \left( -\frac{1}{iw-k} + \frac{1}{iw+k} \right) = \frac{1}{2} \left( \frac{-wi-k + wi-k}{-w^2-k^2} \right) = \underline{\underline{\frac{k}{w^2+k^2}}}\end{aligned}$$

---

Question: When to use separation of variables?

1. Use sep. of var. to find a discrete set of " $\lambda$ 's" on a bounded set (i.e. an interval). By Sturm-Liouville problem (SLP) theory get  $L^2(I)$  ONB of solutions that you can use with the initial cond. to solve the problem.  
Solution =  $\sum \dots$
2. On  $\mathbb{R}$ ,  $[0, \infty)$ , any unbounded region, use Fourier transform or Laplace transform. ( $\mathbb{R}$ -fourier,  $[0, \infty)$ -laplace)

## THE LAPLACE TRANSFORM ch8

DEF  $\mathcal{L} f(z) = \hat{f}\left(\frac{-iz}{2\pi}\right) = \int_0^\infty f(t) e^{-zt} dt$  as long as  $f$  satisfies  $\textcircled{1}$

$\textcircled{1}$   $f(t) = 0 \quad \forall t < 0$ .  $\exists a > 0$  such that  $|f(t)| \leq e^{at}$   $\forall$  (a.e.)  $t > 0$ . Then  $\mathcal{L} f(z)$  is defined for all  $z$  with  $\Re(z) > a$   
almost everywhere

$$\begin{aligned} \text{Check: } |\mathcal{L} f(z)| &\leq \int_0^\infty |f(t)| |e^{-zt}| dt \leq \int_0^\infty e^{at} e^{-Re(z)t} dt = \\ &= \int_0^\infty e^{(a-Re(z))t} dt = \frac{1}{-(a-Re(z))} \end{aligned}$$

$\mathcal{L} f(z)$  is analytic func. for  $\Re(z)$ . let  $\theta(+)=\begin{cases} 1 & \text{for } t>0 \\ 0 & \text{for } t\leq 0 \end{cases}$

Properties Assume  $f$  satisfies  $\textcircled{1}$

$$\textcircled{1} \quad \mathcal{L}(f(x+iy)) \rightarrow 0 \text{ as } |y| \rightarrow \infty, \quad x, y \in \mathbb{R}, \quad \forall x > a$$

$$\textcircled{2} \quad \mathcal{L} f(x+iy) \rightarrow 0 \text{ as } x \rightarrow \infty, \quad \forall y \in \mathbb{R}$$

$$\textcircled{3} \quad \mathcal{L}(\theta(t-a) f(t-a))(z) = e^{-az} \mathcal{L} f(z)$$

$$\textcircled{4} \quad \mathcal{L}(e^{ct} f(t))(z) = \mathcal{L} f(z-c)$$

$$\textcircled{5} \quad \mathcal{L}(f(at))(z) = a^{-1} \mathcal{L}(f)\left(\frac{z}{a}\right)$$

$$\textcircled{6} \quad \text{If } f \text{ is continuous and piecewise } C^1 \text{ on } [0, \infty) \text{ and}$$

$$f' \text{ satisfies } \textcircled{1}, \text{ then } \boxed{\mathcal{L}(f'(t))(z) = z \mathcal{L} f(z) - f(0)}$$

$$\textcircled{7} \quad \mathcal{L}\left(\int_0^t f(s) ds\right)(z) = z^{-1} \mathcal{L}(f)(z)$$

$$\textcircled{8} \quad \mathcal{L}(tf(t))(z) = -(\mathcal{L} f)'(z)$$

$$\textcircled{9} \quad \boxed{\mathcal{L}(f * g)(z) = \mathcal{L} f(z) \mathcal{L} g(z)}$$

$$\text{Proof } \mathcal{L}f(x+iy) = \int_0^\infty f(t) e^{-(x+iy)t} dt = \int_0^\infty f(t) e^{-xt} e^{-ity} dt$$

$$\text{Let } g(t) = f(t) e^{-xt}. \text{ Then } \mathcal{L}f(x+iy) = \hat{g}\left(\frac{y}{2\pi}\right)$$

Riemann-Lebesgue Lemma  $\Rightarrow \hat{g}\left(\frac{y}{2\pi}\right) \rightarrow 0$  as  $|y| \rightarrow \infty$

$$\Rightarrow \hat{g}\left(\frac{y}{2\pi}\right) \rightarrow 0 \text{ as } |y| \rightarrow \infty$$

OBS!  $g \in L^1(\mathbb{R})$  because  $f$  satisfies (2),  $\operatorname{Re}(z) = x > a$

$$\textcircled{2} \quad |\mathcal{L}f(z)| \leq \frac{1}{\operatorname{Re}(z)-a} = \frac{1}{x-a} \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$\textcircled{3} \quad \int_0^\infty \theta(t-a) f(t-a) e^{-zt} dt = \int_a^\infty f(t-a) e^{-zt} dt = \{s=t-a\} = \\ = \int_0^\infty f(s) e^{-z(s+a)} ds = e^{-az} \mathcal{L}f(z)$$

$$\textcircled{4} \quad \mathcal{L}(e^{ct} f(t))(z) = \int_0^\infty e^{ct} f(t) e^{-zt} dt = \int_0^\infty f(t) e^{-zt+ct} dt = \\ = \int_0^\infty f(t) e^{-(z-c)t} dt = \mathcal{L}f(z-c)$$

$$\textcircled{5} \quad \mathcal{L}(f(at))(z) = \int_0^\infty f(at) e^{-zt} dt = \{s=at, ds=adt\} = \\ = \int_0^\infty f(s) e^{-zs/a} \frac{ds}{a} = a^{-1} \mathcal{L}f\left(\frac{z}{a}\right)$$

$$\textcircled{6} \quad \mathcal{L}(f')(z) = \int_0^\infty f'(t) e^{-zt} dt = \left[ f(t) e^{-zt} \right]_0^\infty - \int_0^\infty f(t) (-ze^{-zt}) dt = \\ = -f(0) + z \int_0^\infty f(t) e^{-zt} dt = -f(0) + z \mathcal{L}f(z)$$

$$\textcircled{7} \quad \text{Let } F(t) = \int_0^t f(s) ds, \text{ By } \textcircled{6} \quad \mathcal{L}(F')(z) = z \mathcal{L}(F)(z) - F(0)$$

By FTC,  $F'(t) = f(t)$ , also  $F(0) = 0$

$$\Rightarrow \mathcal{L}(f)(z) = z \mathcal{L}(F)(z) \Rightarrow \mathcal{L}(F)(z) = \frac{1}{z} \mathcal{L}(f)(z)$$

$$\textcircled{8} \quad \mathcal{L}(tf(t))(z) = \int_0^\infty f(t) t e^{-zt} dt, \quad \mathcal{L}f(z) = \int_0^\infty f(t) e^{-zt} dt$$

By DCT,  $\mathcal{L}(f)'(z) = \int_0^\infty -te^{-zt} f(t) dt$  (dominated Lebesgue conv. thm)

$$\Rightarrow \mathcal{L}(tf(t))(z) = -(\mathcal{L}f)'(z)$$

$$\textcircled{9} \quad \mathcal{L}(f*g)(z) = \mathcal{F}(f*g)\left(\frac{-iz}{2\pi}\right) = \hat{f}\left(\frac{-iz}{2\pi}\right) \hat{g}\left(\frac{-iz}{2\pi}\right) = \mathcal{L}f(z) \mathcal{L}g(z)$$

$$\text{EXTRA } \textcircled{1} \quad \{f: \mathbb{R} \rightarrow \mathbb{C}, \text{ measurable, } \int_{\mathbb{R}} |f|^p < \infty\} / \sim = L^p(\mathbb{R}) \\ \text{for } p \geq 1, \|f\|_{L^p(\mathbb{R})} = \left( \int_{\mathbb{R}} |f|^p \right)^{1/p}$$

This is a Banach (complete, normed vector) space

ONLY for  $p=2$ , is  $L^2 = \mathcal{L}^2$  a Hilbert space = Banach space with inner product  $\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f \bar{g}$

$$\textcircled{2} \quad \mathcal{L}^\infty(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{C} \text{ such that } \exists M \geq 0 \text{ with } |f(x)| \leq M \text{ a.e. } x \in \mathbb{R}\} / \sim$$

For such an  $f$ , the smallest  $M$  such that this holds is  $\|f\|_\infty = \|f\|_{L^\infty}$ ,  $\mathcal{L}^\infty$  is also a Banach space

Exercise:  $\mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{L}'(\mathbb{R}) \not\subseteq \mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$  but

### APPLICATIONS OF $\mathcal{L}$ TO ODES AND PDEs

PROP  $\mathcal{L}(f^{(k)})(z) = z^k \mathcal{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0) z^{j-1}$ ,  $f^{(0)} = f$ ,  $\forall k \geq 1$

Proof Base case:  $\mathcal{L}(f')(z) = z \mathcal{L}f(z) - f(0)$  by thm  
Set  $k=1$  in RHS.

$z \mathcal{L}f(z) - f(0) z^0 = z \mathcal{L}f(z) - f(0)$ . Proceed by assuming true for some  $k \geq 1$ . Show true for  $k+1$ .

$$\begin{aligned}\mathcal{L}(f^{(k+1)})(z) &= \mathcal{L}((f^{(k)})')(z) = z \mathcal{L}(f^{(k)})(z) - f^{(k)}(0) \\ &= z \left( z^k \mathcal{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0) z^{j-1} \right) - f^{(k)}(0) \stackrel{\text{Thm}}{=} \\ &= z^{k+1} \mathcal{L}f(z) - \sum_{j=1}^k f^{(k-j)}(0) z^j - f^{(k)}(0) = \begin{cases} k-j = (k+1)-(j+1) \\ \text{Let } l=j+1 \end{cases} = \\ &= z^{k+1} \mathcal{L}f(z) - \sum_{l=2}^{k+1} f^{(k+1-l)}(0) z^{l-1} - f^{(k)}(0), \quad \text{This is the } l=0 \text{ term in } \sum \\ &= z^{k+1} \mathcal{L}f(z) - \sum_{l=1}^{k+1} f^{(k+1-l)}(0) z^{l-1} \quad \blacksquare\end{aligned}$$

Thus for an ODE  $\sum_{k=0}^n c_k u^{(k)}(t) = f(t)$  for  $t \geq 0$ ,  $c_k$  constants  
 $\mathcal{L}$  to both sides:  $\sum_{k=0}^n c_k \mathcal{L}(u^{(k)})(z) = \mathcal{L}f(z)$   
 $\underbrace{P(z) \mathcal{L}u(z) + Q(z)}_{\substack{\uparrow \text{ polynomials}}} \Rightarrow \mathcal{L}u(z) = \frac{\mathcal{L}f(z) - Q(z)}{P(z)}$

Ex.   
Heat source  $u(0,t) = f(t)$  (heat/temp of sun at time  $t$ )  
 $u(x,0) = u_t(x,0) = 0$ ,  $\boxed{u=0}$ ,  $x \in [0, \infty)$

Use Laplace transform (not Fourier)

int variable because more  $\mathcal{L}(u_t - u_{xx})(x,z) = 0 \Rightarrow \mathcal{L}(u_t)(x,z) = \mathcal{L}(u_{xx})(x,z)$

Let  $U(x,z) = \mathcal{L}(u)(x,z)$

$$\Rightarrow zU(x,z) = U_{xx}(x,z). \text{ Basis of solutions are } a(z)e^{\sqrt{z}x} \text{ and } b(z)e^{-\sqrt{z}x}$$

$\operatorname{Re}(z) > a > 0$  is where  $\mathcal{L}$  is defined  
 $\Rightarrow$  Physically relevant solution is  $b(z)e^{-\sqrt{z}x}$

$$U(x,z) = b(z)e^{-\sqrt{z}x}, \quad U(0,z) = \mathcal{L}f(z) \Rightarrow U(x,z) = \mathcal{L}f(z)e^{-\sqrt{z}x}$$

Find  $h(x, t)$  such that  $\mathcal{L} h(x, z) = e^{-\sqrt{z}x}$

Then then  $u(x, t) = f * h(x, t)$

↑  
convolution in  $t$  variable

Because  $\mathcal{L}(f * h)(x, z) = \mathcal{L} f(x, z) \mathcal{L} h(x, z) = \mathcal{L} f(x, z) e^{-\sqrt{z}x}$

Exercise: Find  $h$ . To do this, can use  $\mathcal{L}^{-1}$

$$\mathcal{L} f(x+iy) = \int_0^\infty f(t) e^{-t(x+iy)} dt = \int_0^\infty f(t) e^{-tx} e^{-ity} dt$$

Let  $g(t) = f(t) e^{-xt}$ : Exercise:  $g \in \mathcal{L}'(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ . Use  $f$  satisfies  $\mathcal{L}$ .

Estimate  $\int_{\mathbb{R}} |g(t)| dt = \int_0^\infty |g(t)| dt$ .  $\int_{\mathbb{R}} |g(t)|^2 dt = \int_0^\infty |g(t)|^2 dt$

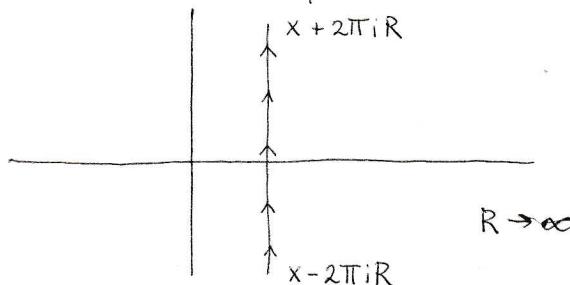
$$\mathcal{L}(f(x+iy)) = \int_{\mathbb{R}} g(t) e^{-ity} dt = \hat{g}\left(\frac{y}{2\pi}\right)$$

To simplify this, look at  $\mathcal{L} f(x+2\pi i y) = \int_{\mathbb{R}} g(t) e^{-2\pi i y t} = \hat{g}(y)$

Fourier inversion theorem, that  $g(t) = \int_{\mathbb{R}} \hat{g}(\xi) e^{2\pi i \xi t} d\xi$

Note also that  $f(t) = e^{xt} g(t)$

$$\Rightarrow f(t) = e^{xt} g(t) = e^{xt} \int_{\mathbb{R}} \hat{g}(\xi) e^{2\pi i \xi t} d\xi = \\ = e^{xt} \int_{\xi=-\infty}^{\infty} \mathcal{L} f(x+2\pi i \xi) e^{2\pi i \xi t} d\xi$$



Write this as a complex integral

$z = x + 2\pi i \xi$ ,  $\xi$  from  $-\infty$  to  $\infty$

$$f(t) = e^{xt} \int_{x-i\infty}^{x+i\infty} \mathcal{L} f(z) e^{zt} e^{-xt} \frac{dz}{2\pi i}$$

Thus 
$$f(t) = \int_{x-i\infty}^{x+i\infty} \mathcal{L} f(z) e^{zt} dz \frac{1}{2\pi i}$$

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx \text{ and } \mathcal{F}^{-1}(f)(\xi) = \mathcal{F}(f)(-\xi)$$

For  $\mathcal{L}^{-1}$ , factor of  $2\pi i$ , and our integral is vertical rather than horizontal.

$$\int_{x-i\infty}^{x+i\infty} \mathcal{L} f(z) e^{zt} dz = \lim_{R \rightarrow \infty} \int_{x-iR}^{x+iR} \mathcal{L} f(z) e^{zt} dz$$

More generally for functions  $F(z)$  defined and holomorphic for  $\operatorname{Re}(z) > a$  for some  $a > 0$ ,

$$\mathcal{L}^{-1}(F)(t) = \lim_{R \rightarrow \infty} \int_{x-iR}^{x+iR} F(z) e^{zt} dz$$

This under growth hyp. on  $F$  will be the same  $\forall x > a$

1/2  
65

DEF Let  $f$  be integ. on  $\mathbb{R}$ , then Fourier transform is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = \tilde{F}(f)$$

Fourier transform extends  $L_2$ .  $\tilde{F}: L_2 \rightarrow L_2$  is invertable

$$\text{with inverse } f(x) = \tilde{F}^{-1}(\hat{f}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi = \check{f}$$

$$\textcircled{1} \quad \hat{f}' = i\xi \hat{f} \quad \textcircled{2} \quad \widehat{xf} = i\hat{f}'$$

\textcircled{3} Plancherel thm: If  $f, g \in L^2$ , then  $\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle$

$$\Rightarrow \|\hat{f}\|^2 = 2\pi \|f\|^2$$

$$\textcircled{4} \quad \tilde{F}(f * g) = \tilde{F}(f) \tilde{F}(g)$$

$$\text{EÖ 6. a)} \quad f(t) = \frac{t}{(a^2 + t^2)^2}, \quad a > 0$$

$$\text{From table 2: } \tilde{F}\left(\frac{1}{a^2 + t^2}\right) = \frac{\pi}{a} e^{-|a|\xi}, \quad \hat{f}' = i\xi \hat{f}$$

$$\text{Note } \left(\frac{1}{a^2 + t^2}\right)' = \frac{-2t}{(a^2 + t^2)^2}$$

$$\begin{aligned} \tilde{F}\left(\frac{t}{(a^2 + t^2)^2}\right) &= \tilde{F}\left(\frac{1}{2} \left(\frac{1}{a^2 + t^2}\right)'\right) = \frac{1}{2} i\xi \tilde{F}\left(\frac{1}{a^2 + t^2}\right) = \\ &= \frac{1}{2} i\xi \frac{\pi}{a} e^{-|a|\xi} \end{aligned}$$

$$\text{b) } f(t) = \frac{1}{(a^2 + t^2)^2}$$

$$\tilde{F}\left(\frac{1}{a^2 + t^2}\right) = \int_{-\infty}^{\infty} \frac{1}{a^2 + t^2} e^{-it\xi} dt = \frac{\pi}{a} e^{-|a|\xi}$$

derivative with respect to  $a$ .

$$\int_{-\infty}^{\infty} \frac{-2a}{(a^2 + t^2)^2} e^{-it\xi} dt = -\frac{\pi}{a^2} e^{-|a|\xi} - \frac{\pi i \xi}{a} e^{-|a|\xi}$$

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{(a^2 + t^2)^2} e^{-it\xi} dt}_{\tilde{F}\left(\frac{1}{a^2 + t^2}\right)} = \left(\frac{\pi}{2a^3} + \frac{\pi i \xi}{2a^2}\right) e^{-|a|\xi}$$

$$\tilde{F}\left(\frac{1}{(a^2 + t^2)^2}\right) = \left(\frac{\pi}{2a^3} + \frac{\pi i \xi}{2a^2}\right) e^{-|a|\xi}$$

$$\text{E07} \quad \hat{f}(w) = \frac{\omega}{1+w^4} \quad \text{a)} \quad \int_{-\infty}^{\infty} t f(t) dt \quad \text{b)} \quad f'(0) = ?$$

$$\text{a)} \quad F(t f(t)) = i \hat{f}'(w) = i \left( \frac{\omega}{1+w^4} \right)' = i \left( \frac{1+w^4 - 4\omega w^3}{(1+w^4)^2} \right) = \\ = i \left( \frac{1-3w^4}{(1+w^4)^2} \right)$$

$$F(t f(t)) = \int_{-\infty}^{\infty} t f(t) e^{-i\omega t} dt = i \left( \frac{1-3w^4}{(1+w^4)^2} \right)$$

$$\text{Take } w=0 : \underbrace{\int_{-\infty}^{\infty} t f(t) dt}_{=} = i \left( \frac{1-0}{(1+0)^2} \right) = i$$

$$\text{b)} \quad F(f'(x)) = i \omega \hat{f}(w) = i \frac{\omega^2}{1+w^4} \quad \text{Inverse Fourier tr. both sides!}$$

$$f'(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2}{1+w^4} e^{i\omega x} dw$$

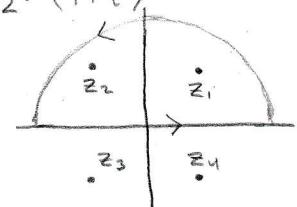
$$f'(0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2}{1+w^4} dw$$

$$g(z) = \frac{z^2}{1+z^4}, \quad \text{Singularities?} \quad 1+z^4=0$$

$g(z)$  has simple poles at points  $z_1 = e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$

$$z_2 = e^{-3i\pi/4} = \frac{1}{\sqrt{2}}(-1+i), \quad z_3 = e^{-5i\pi/4} = \frac{1}{\sqrt{2}}(-1-i)$$

$$z_4 = e^{-7i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$$



$$\int_{-\infty}^{\infty} g(z) dz = 2\pi i (\text{Res}_{z_1} g + \text{Res}_{z_2} g)$$

$$\begin{aligned} \text{Res}_{z_1} g &= \lim_{z \rightarrow z_1} g(z)(z-z_1) = \lim_{z \rightarrow z_1} \frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \cancel{(z-z_1)} = \\ &= \frac{z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{(\frac{1}{\sqrt{2}})^2 (1+i)^2}{\frac{1}{\sqrt{2}} \cdot 2 \cdot \frac{1}{\sqrt{2}} (2+2i) \cdot \frac{1}{\sqrt{2}} 2i} = \\ &= \frac{(1+i)}{4\sqrt{2}i} = \frac{1-i}{4\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{Res}_{z_2} g &= \frac{z_2^2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{(\frac{1}{\sqrt{2}})^2 (-1+i)^2}{\frac{1}{\sqrt{2}} (-2) \cdot \frac{1}{\sqrt{2}} 2i \cdot \frac{1}{\sqrt{2}} (-2+2i)} = \\ &= -\frac{-1+i}{4\sqrt{2}i} = \frac{-1-i}{4\sqrt{2}} \end{aligned}$$

$$\int_{-\infty}^{\infty} g(z) dz = 2\pi i \left( \frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}} \right) = 2\pi i \left( \frac{-2i}{4\sqrt{2}} \right) = \frac{4\pi}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

$$f'(0) = \frac{i}{2\pi} \cdot \frac{\pi}{\sqrt{2}} = \frac{i}{2\sqrt{2}}$$

$$\text{EÖ10} \quad \hat{f}(\xi) = \frac{1}{|\xi|^3 + 1}, \quad \int_{-\infty}^{\infty} |f * f'|^2 dx = ?$$

$$\int_{-\infty}^{\infty} |f * f'|^2 dx = \|f * f'\|_{L^2}^2 = \frac{1}{2\pi} \|\widehat{f * f'}\|^2 =$$

↑  
Plancherel  
↑  
(4)

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f} \hat{f}'|^2 dx = \left\{ \hat{f}' = i\xi \hat{f} \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{|\xi|^3 + 1} \cdot i\xi \frac{1}{|\xi|^3 + 1} \right|^2 = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\xi}{(|\xi|^3 + 1)^2} \right|^2 d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\xi|^2}{(|\xi|^3 + 1)^4} d\xi = \{\text{even func.}\} \\ &= \frac{2}{2\pi} \int_0^{\infty} \frac{\xi^2}{(\xi^3 + 1)^4} d\xi = \left\{ \begin{array}{l} \xi^3 + 1 = t \\ 3\xi^2 d\xi = dt \\ \xi \in (0, \infty), t \in (1, \infty) \end{array} \right\} = \frac{1}{\pi} \int_1^{\infty} \frac{1/3}{t^4} dt = \\ &= \left[ \frac{1}{9\pi} (-t^{-3}) \right]_1^{\infty} = \frac{1}{9\pi} \end{aligned}$$

$\mathcal{F}\Sigma$  for function on  $(-\pi, \pi)$ , then the  $\Sigma$  convg. to periodic extension for points in  $\mathbb{R} \setminus (-\pi, \pi)$

THE LAPLACE TRANSFORM

THM 8.5 (LIT) Let  $F(z)$  be analytic in  $\operatorname{Re}(z) > 0$

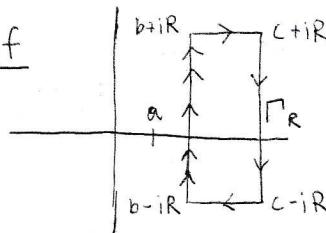
For  $b > a$ ,  $R > 0$ ,  $t \in \mathbb{R}$ , let  $f_{R,b}(t) = \frac{1}{2\pi i} \int_{b-iR}^{b+iR} F(z) e^{zt} dz$ .

Assume that for some  $\alpha > 1/2$ ,  $c > 0$ ,

$|F(z)| \leq \frac{C}{(1+|z|)^\alpha}$  for  $\operatorname{Re}(z) > a$ . Assume that for some  $b > a$ ,  $\lim_{R \rightarrow \infty} f_{R,b}(t) = f(t)$  such that  $f(t)$  satisfies  $\mathcal{D}$

Then,  $\forall c \geq b$ ;  $\lim_{R \rightarrow \infty} f_{R,c}(t) = f(t)$  and  $F(z) = \mathcal{L} f(z)$

Proof



By assumption,  $F(z)e^{zt}$  is holomorphic for  $\operatorname{Re}(z) > a$

$$\text{Thus } \int_{\Gamma_R} F(z) e^{zt} dz = 0 =$$

$$= \int_{\Gamma_R} F(z) e^{zt} dz = 2\pi i (f_{R,b}(t) - f_{R,c}(t)) + \int_{\text{top, bottom}}^{\text{c} \pm iR}$$

$$\left| \int_{b \pm iR}^{c \pm iR} F(z) e^{zt} dz \right| \leq \int_{b \pm iR}^{c \pm iR} |F(z)| |e^{zt}| dz \leq \int_{b \pm iR}^{c \pm iR} \frac{C}{(1+|z|)^\alpha} e^{\operatorname{Re}(z)t} dz \leq$$

$$\leq e^{ct} \int_{b \pm iR}^{c \pm iR} \frac{C}{(1+|z|)^\alpha} dz \leq e^{ct} (c-b) \frac{C}{(1+R)^\alpha} \xrightarrow{(1z)=\sqrt{\operatorname{Re}(z)^2+\operatorname{Im}(z)^2}\geq|\operatorname{Im}(z)|=R} 0 \text{ as } R \rightarrow \infty$$

Thus  $f_{R,c}(t) = f_{R,b}(t) - \Theta(R^{-\alpha})$  as  $R \rightarrow \infty$  "Big  $\Theta$ "

(Recall func  $f(R)$  is  $\mathcal{O}(R^\alpha)$  if  $\exists C \geq 0$  such that  $|f(R)| \leq CR^\alpha$  as  $R \rightarrow \infty$ )

Idea: we want to use the FIT somehow...

Observe that the integrand  $F(z)e^{zt}$  in defn. of  $f_{R,b}(t)$  only varies in the  $\operatorname{Im}(z) \in \mathbb{R}$

$$f_{R,b}(t) = \frac{1}{2\pi i} \int_{b-iR/2\pi}^{b+iR/2\pi} F(b+is2\pi) e^{(b+is2\pi)t} 2\pi i ds = (*)$$

$s = R/2\pi$   
 $s = -R/2\pi$

(curve is  $\gamma(s) = b + 2\pi i s \Rightarrow \gamma'(s) = 2\pi i$  ds

$$(*) = e^{bt} \int_{-R/2\pi}^{R/2\pi} F(b+is2\pi) e^{2\pi i st} ds \rightarrow f(t) \text{ as } R \rightarrow \infty$$

$\rightarrow e^{-bt} f(t) \text{ as } R \rightarrow \infty$

Let  $g_b(s) = F(b+2\pi i s)$  for  $|s| \leq \frac{R}{2\pi}$ , 0 for  $|s| > \frac{R}{2\pi}$

$g_b : \mathbb{R} \rightarrow \mathbb{C}$

$$\text{Then } \int_{-R/2\pi}^{R/2\pi} g_b(s) e^{2\pi i s t} ds = \int_{\mathbb{R}} g_b(s) e^{2\pi i s t} ds = \underbrace{\check{g}_b(t)}_{\text{Inverse FT, "g check"}}$$

Inverse FT, "g check"

$$\text{Let } R \rightarrow \infty \Rightarrow \check{g}_b(t) = e^{-bt} f(t)$$

$$\begin{aligned} \text{By the FIT, } g_b(t) &= F(b+2\pi i t) = \int_{\mathbb{R}} \check{g}_b(s) e^{-2\pi i s t} ds = \\ &= \int_{\mathbb{R}} e^{-bs} f(s) e^{-2\pi i s t} ds = \int_0^\infty f(s) e^{-s(b+2\pi i t)} ds = \\ &\quad \uparrow \\ &f \text{ satisfies } \mathcal{Q} \\ &= \mathcal{L} f(b+2\pi i t) = F(b+2\pi i t) \blacksquare \end{aligned}$$

VS Them  
 $F(f)(\xi) = F(f)(2\pi\xi)$

### APPLICATION OF $\mathcal{L}$

$x \in [0, \infty)$

$$\begin{cases} u(0, t) = f(t), & u(x, 0) = u_0(x, 0) = 0 \\ \frac{\partial u}{\partial t} = 0 \end{cases}$$

Solution is  $u(x, t) = f * h(x, t)$ , where  $\mathcal{L} h(x, z) = e^{-x\sqrt{z}}$   
 ↑  
 in t variable

We would like to find  $h$ ... We know that on  $\mathbb{R}$  solve the heat eqn.  $v(x, 0) = g(x)$   
 $\frac{\partial v}{\partial t}(x, t) = 0 \quad \forall t > 0$

$$v(x, t) = \int_{\mathbb{R}} \frac{g(y) e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy$$

$$\text{Let's compute } \mathcal{L} \left( \frac{\theta(+)}{\sqrt{4\pi t}} e^{-x^2/4t} \right) (z) = \int_0^\infty \frac{e^{-zt} e^{-x^2/4t}}{\sqrt{4\pi t}} dt =$$

$$= \int_0^\infty \frac{e^{-(\sqrt{z}t)^2 - (\frac{x}{2\sqrt{t}})^2}}{2\sqrt{\pi} \sqrt{t}} dt = *$$

{Don't complete square here  
 like before... use}

Cauchy-Schlömilch transform

$$\int_0^\infty a f((ax + b/x)^2) dx = \int_0^\infty f(y^2) dy$$

Recall, if  $f$  satisfies  $\mathcal{Q}$  then  $\forall z \in \mathbb{C}$  with  $\operatorname{Re}(z) > a$ ,  
 $f(t) e^{-zt} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$\int_0^\infty |f(t)e^{-zt}| dt < \infty \Rightarrow \text{Can move derivs} \rightarrow \int, \int \text{deriv} = \text{deriv} \int$   
 by Lebesgue Dom. Con. Thm

$$* = \int_0^\infty \frac{e^{-(\sqrt{z}t - x/2\sqrt{t})^2 - x\sqrt{z}}}{2\sqrt{\pi t}} dt =$$

$$\begin{aligned} &= \frac{e^{-x\sqrt{z}}}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-(\sqrt{z}\sqrt{t} - \frac{x}{2\sqrt{t}})^2}}{\sqrt{t}} dt = \left\{ \begin{array}{l} s = \sqrt{t} \\ ds = \frac{dt}{2\sqrt{t}} \Rightarrow 2\sqrt{t} ds = dt \end{array} \right\} = \\ &= \frac{e^{-x\sqrt{z}}}{\sqrt{\pi}} \int_0^\infty e^{-(\sqrt{z}s - \frac{x}{2s})^2} ds \end{aligned}$$

Use CS transform with "a" =  $\sqrt{z}$ , "b" =  $\frac{x}{2}$

$$\Rightarrow * = \frac{e^{-x\sqrt{z}}}{\sqrt{\pi} \sqrt{z}} \underbrace{\int_0^\infty \sqrt{z} e^{-(\sqrt{z}s - \frac{x}{2}s)^2} ds}_{\int_0^\infty e^{-y^2} dy} = \frac{\sqrt{\pi}}{2}$$

$$* = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}$$

$$\text{Thus } \mathcal{L} \left( \frac{\Theta(+)}{\sqrt{4\pi t}} e^{-x^2/4t} \right) (z) = \frac{e^{-x\sqrt{z}}}{2\sqrt{z}}$$

$$\text{By thm 8.1 (properties of } \mathcal{L} \text{): } \mathcal{L} \left( \frac{\Theta(+)}{t\sqrt{4\pi t}} e^{-x^2/4t} \right) = \int_z^\infty \frac{e^{-x\sqrt{z}}}{x} dx$$

$$\Rightarrow \mathcal{L} \left( \frac{\Theta(+)}{t\sqrt{4\pi t}} e^{-x^2/4t} \right) (z) = e^{-x\sqrt{z}}$$

Thus the solution to the heat eqn. is

$$u(x,t) = f * \frac{\Theta(+)}{t\sqrt{4\pi t}} (t) =$$

↑  
in t variable

$$= \int_{\mathbb{R}} f(t-s) \frac{e^{-x^2/4s}}{s\sqrt{4\pi s}} \Theta(+) = \int_0^t f(t-s) \frac{x e^{-x^2/4s}}{s\sqrt{4\pi s}} ds$$

$\uparrow$   
 $f=0 \text{ for } t < 0$

$\boxed{s \geq 0 \text{ because of } \Theta}$

$\boxed{t-s \geq 0 \text{ because of } f}$

Hint: for future applications, look up ~~mm~~ on a table

Example: Waving a rope,  $\begin{cases} \square u = 0 \text{ on } x \in [0, l] \\ u(x, 0) = 0 = u_t(x, 0) \\ u(l, t) = 0, u(0, t) = f(t) \end{cases}$

$[0, l]_x \times [0, \infty)_t \leftarrow \mathcal{L} \text{ here!}$

↑  
us waving rope around

$$\mathcal{L}(u)(x, z) = z^2 \mathcal{L}(u)(x, z) + \mathcal{L}(u_{xx})(x, z)$$

$$\text{Let } U = \mathcal{L}(u) \Rightarrow z^2 U + U_{xx} = 0$$

$$\Rightarrow U(x, z) = a(z)e^{zx} + b(z)e^{-zx}. \text{ For } x=0, U(0, z) = \mathcal{L}(f)(z) = F(z)$$

$$\Rightarrow a(z) + b(z) = F(z)$$

$$\text{For } x=l, U(l, z) = 0 \Rightarrow a(z)e^{lz} + b(z)e^{-lz} = 0 \Rightarrow a(z) = -b(z)e^{-2lz}$$

$$\Rightarrow b(z)(1 - e^{-2lz}) = F(z) \Rightarrow b(z) = \frac{F(z)}{1 - e^{-2lz}}$$

$$\text{Thus } U(x, z) = \frac{F(z)}{1 - e^{-2lz}} (-e^{-2lx+2xz} + e^{-zx})$$

Multiply by  $1 = \frac{e^{lz}}{e^{lz}}$

$$\Rightarrow \frac{F(z)}{e^{lz} - e^{-lz}} \left( e^{lz-x} - e^{-lz+2x} \right) = F(z) \underbrace{\frac{\sinh(z(l-x))}{\sinh(lz)}}_{\text{Find a func whose transform is this.}}$$

To proceed "by hand"

Thm 8.1 properties of  $\mathcal{L}$ :

$$\mathcal{L}(\theta(t-a)f(t-a)) = e^{-az}\mathcal{L}(fz) = e^{-az}F(z)$$

To use this:  $\frac{1}{1-e^{-2lz}} = \sum_{n=0}^{\infty} e^{-2lnz}$

What we have is  $\sum_{n \geq 0} (e^{-z(x+2ln)} - e^{-z(2l-x+2ln)})$

By thm 8.1 this is:

$$\mathcal{L} \left( \sum_{n \geq 0} \theta(t-(x+2ln)) f(t-(x+2ln)) - \theta(t-(2l(n+1)-x)) f(t-(2l(n+1)-x)) \right)$$

Thus the solution is the inside of this.

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F11

CS TRANSFORM

$$\begin{aligned} * &= \int_0^\infty af((ax - \frac{b}{x})^2) dx = \left\{ \begin{array}{l} \text{Let } t = \frac{b}{ax}, \quad dt = -\frac{b}{ax^2} dx, \quad ax = \frac{b}{t} \\ \frac{b}{x} = ta, \quad dt = -t^2 \frac{a}{b} dx \end{array} \right\} = \\ &= \int_{+\infty}^0 af\left(\left(\frac{b}{t} - ta\right)^2\right) \left(\frac{-b}{at^2}\right) dt = \int_0^\infty \frac{b}{t^2} f\left(\left(at - \frac{b}{t}\right)^2\right) dt \end{aligned}$$

$$\begin{aligned} \text{Thus } * &= \frac{1}{2} \left( \int_0^\infty af\left(\left(ax - \frac{b}{x}\right)^2\right) + \frac{b}{x^2} f\left(\left(ax - \frac{b}{x}\right)^2\right) dx \right) = \\ &= \frac{1}{2} \left( \int_0^\infty \left(a + \frac{b}{x^2}\right) f\left(\left(ax - \frac{b}{x}\right)^2\right) dx \right) \end{aligned}$$

$$\text{Let } y = ax - \frac{b}{x} \Rightarrow dy = \left(a + \frac{b}{x^2}\right) dx$$

$$\text{Thus } * = \frac{1}{2} \int_{-\infty}^\infty f(y^2) dy = \int_0^\infty f(y^2) dy \blacksquare$$

DEF A Banach space is a complete - A Cauchy  $\{x_n\} \subset X \quad \exists \lim_{n \rightarrow \infty} x_n \in X$   
 • normed  
 • vector space

Thus also a metric space  $(X, \| \cdot \|)$

Recall:  $\{x_n\}$  is Cauchy  $\Leftrightarrow \forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$  such that  
 $\forall n, m \geq N \quad \|x_n - x_m\| < \varepsilon$

$\forall x, y \in X, \quad a, b \in \mathbb{C} \quad ax + by \in X \quad \text{Vector space}$

$$\|ax\| = |a| \|x\| \quad \text{and} \quad \|x+y\| \leq \|x\| + \|y\|$$

Finite dimens:  $\mathbb{C}^n$       Infinite dimension:  $\ell^p, \quad p \geq 1$

A Hilbert space is a Banach space which also has an inner product,  $\langle \cdot, \cdot \rangle: X \times X \rightarrow \mathbb{C}$ , continuous, and  $\langle ax+by, z \rangle = \langle x, z \rangle a + b \langle y, z \rangle, \quad \langle x, y \rangle = \langle y, x \rangle$   
 and  $\|x\| = \sqrt{\langle x, x \rangle}$

Examples 1.  $\ell^2 = \left\{ (c_n)_{n \in \mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty \text{ and } \langle (c_n), (b_n) \rangle = \sum_{n \in \mathbb{Z}} c_n \bar{b}_n \right\}$

This is a Hilbert space.

2. Let  $Y \subset \mathbb{R}$ ,  $\mathcal{L}^2(Y) = \{f: Y \rightarrow \mathbb{C}, \text{ measurable } \int_Y |f|^2 < \infty\} / \sim$  a.e.  
 $\langle f, g \rangle = \int_Y f \bar{g}$ , Hilbert space!

3. Similarly,  $\mathcal{L}^p(Y)$  for  $p \geq 1$  ( $p = \infty$ ) is only a Banach  
 $\forall p \neq 2$ ,  $\|f\|_{\mathcal{L}^p(Y)} = (\int_Y |f|^p)^{1/p}$

### Properties Hilbert spaces

1.  $\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2$

2.  $|\langle x, y \rangle| \leq \|x\| \|y\|$  Cauchy-Schwarz

3. If  $\{x_k\}_{k=1}^n$  are pairwise  $\perp$  then

$$\left\| \sum_{k=1}^n x_k \right\|^2 = \sum_{k=1}^n \|x_k\|^2 \quad (\text{Pythagorean Thm.})$$

Proof : 1.  $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x+y \rangle + \langle y, x+y \rangle =$   
 $= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2$

2. Idea: use 1 somehow... Without loss of generality  
Assume  $y \neq 0$

Next without loss of generality  $\langle x, y \rangle \in \mathbb{R}$ . In any case,  
 $\langle x, y \rangle = r e^{i\theta}$ ,  $r \geq 0$ ,  $\theta \in \mathbb{R} \Rightarrow \langle x, e^{i\theta} y \rangle = r$

$$|\langle x, e^{i\theta} y \rangle| = r = |\langle x, y \rangle|$$

More over,  $\|x\| \text{ same}$ ,  $\|e^{i\theta} y\| = \|y\|$ . We could prove the inequality using  $x$  and  $z = e^{i\theta} y$   
since  $\langle x, z \rangle = r \in \mathbb{R}$

$$\text{Let } f(t) = \|x+ty\|^2 \geq 0$$

$$0 \leq \|x+ty\|^2 = \|x\|^2 + 2t\langle x, y \rangle + t^2\|y\|^2$$

This is a quadratic  $\mathbb{R}$  valued function in  $t$

$$\text{Unique minimum, } f'(t) = 2\langle x, y \rangle + 2t\|y\|^2$$

$$\text{Thus the unique minimum is when } t = -\frac{\langle x, y \rangle}{\|y\|^2}$$

$\Rightarrow 0 \leq f(t) \quad \forall t \text{ including this } \nearrow t$

$$0 \leq \|x\|^2 - \frac{2|\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^2} \Rightarrow |\langle x, y \rangle| \leq \|x\| \|y\|$$

3. Use 1 and induction.

$$\text{If } x_1 \perp x_2 \text{ then } \|x_1 + x_2\|^2 = \|x_1\|^2 + 2\operatorname{Re}\langle x_1, x_2 \rangle + \|x_2\|^2 = 0$$

Base case is true.

Assume  $\{x_k\}_{k=1}^n$  are pairwise  $\perp$  and  $x_{n+1}$  is  $\perp$  to all  $x_k$

$$\begin{aligned} \text{Induction} \quad & \left\| \sum_{k=1}^{n+1} x_k \right\|^2 = \left\| \sum_{k=1}^n x_k + x_{n+1} \right\|^2 = \left\| \sum_{k=1}^n x_k \right\|^2 + 2 \operatorname{Re} \underbrace{\left\langle \sum_{k=1}^n x_k, x_{n+1} \right\rangle}_{2 \operatorname{Re} \sum_{k=1}^n \langle x_k, x_{n+1} \rangle} + \|x_{n+1}\|^2 \\ & = \sum_{k=1}^n \|x_k\|^2 \end{aligned}$$

PROP If  $f_n \rightarrow f$  in  $L^\infty([a,b])$  then  $f_n \rightarrow f$  in  $L^p([a,b]) \forall p \geq 1$

### BESSEL'S INEQUALITY FOR HILBERT SPACES

$\{\phi_n\}_{n \in \mathbb{Z}}$  ONS (orthonormal set) in a Hilbert space,  $H$ , then

$$\forall f \in H, \sum_{n \in \mathbb{Z}} |\langle f, \phi_n \rangle|^2 \leq \|f\|^2$$

Proof Let  $g_N = \sum_{-N}^N \langle f, \phi_n \rangle \phi_n \in H$ . By pythagorus

$$\|g_N\|^2 = \sum_{-N}^N |\langle f, \phi_n \rangle|^2. \text{ Next, } 0 \leq \|f - g_N\|^2 =$$

$$= \|f\|^2 - 2 \operatorname{Re} \langle f, g_N \rangle + \|g_N\|^2 =$$

$$= \|f\|^2 - 2 \operatorname{Re} \underbrace{\langle f, \sum_{-N}^N \langle f, \phi_n \rangle \phi_n \rangle}_{\{-2 \operatorname{Re} \left( \overline{\sum_{-N}^N \langle f, \phi_n \rangle} \langle f, \phi_n \rangle \right)\}} + \sum_{-N}^N |\langle f, \phi_n \rangle|^2$$

$$\left\{ -2 \operatorname{Re} \left( \overline{\sum_{-N}^N \langle f, \phi_n \rangle} \langle f, \phi_n \rangle \right) = -2 \sum_{-N}^N |\langle f, \phi_n \rangle|^2 \right\}$$

$$\Rightarrow 0 \leq \|f\|^2 - \sum_{-N}^N |\langle f, \phi_n \rangle|^2 \Rightarrow \sum_{-N}^N |\langle f, \phi_n \rangle|^2 \leq \|f\|^2$$

Let  $N \rightarrow \infty$

DEF  $\hat{f}_n = \langle f, \phi_n \rangle$  are the Fourier coefficients of  $f \in H$  with respect to  $\{\phi_n\}$  an orthonormal set.

THM (when is an ONS actually an ONB?)  $\{\phi_n\} \subset H$  ONS

The following are equivalent (T.F.A.E)

$$\textcircled{1} \text{ If } \hat{f}_n = 0 \quad \forall n \Rightarrow f = 0$$

$$\textcircled{2} \quad \forall f \in H, \quad f = \sum \hat{f}_n \phi_n$$

$$\textcircled{3} \quad \text{Bessel's inequality is an equality}$$

Proof ( $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ )

|                   |
|-------------------|
| $1 \Rightarrow 2$ |
|-------------------|

$$\sum_{n \in \mathbb{Z}} \hat{f}_n \phi_n ? \quad \text{WLOG re-index by } \mathbb{N}, \sum_{n \in \mathbb{N}} \hat{f}_n \phi_n$$

If  $\textcircled{1}, \textcircled{2}$  or  $\textcircled{3}$  holds  
 $\Rightarrow$  they all hold  
and  $\{\phi_n\}$  is an ONB

Why is this in  $H$ ? Let  $g_N = \sum_{n=1}^N \hat{f}_n \phi_n$

For  $M = N+K$ ,  $\|g_M - g_N\|^2 = \sum_{n=N+1}^M |\hat{f}_n|^2 \rightarrow 0$  as  $N \rightarrow \infty$

because  $\sum_{n=1}^{\infty} |\hat{f}_n|^2 \leq \|f\|^2 < \infty$

Thus  $\{g_N\}_{N \geq 1}$  are a Cauchy sequence. Hence converge to  $g \in H$ . To show  $g = f \Leftrightarrow g - f = 0$ , to do that use ①

$$\langle g - f, \phi_n \rangle = \left\langle \sum_{m \geq 1} \hat{f}_m \phi_m - f, \phi_n \right\rangle = \sum_{m \geq 1} \hat{f}_m \underbrace{\langle \phi_m, \phi_n \rangle}_{\begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}} - \hat{f}_n =$$

$$= \hat{f}_n - \hat{f}_n = 0. \text{ This shows that } \overbrace{(g-f)}_n = 0 \quad \forall n$$

By ①  $\Rightarrow g - f = 0 \Leftrightarrow g = f$

2  $\Rightarrow$  3 By pythagorus:  $\left\| \sum_{n=1}^N \hat{f}_n \phi_n \right\|^2 = \sum_{n=1}^N |\hat{f}_n|^2 \leq \|f\|^2 \quad \forall N$

By Bessel's ineq. By ②  $\lim_{N \rightarrow \infty} \sum_{n=1}^N \hat{f}_n \phi_n = f$

$$\Rightarrow \lim_{N \rightarrow \infty} \left\| \sum_{n=1}^N \hat{f}_n \phi_n \right\|^2 = \|f\|^2 \Rightarrow \lim_{N \rightarrow \infty} \sum_{n=1}^N |\hat{f}_n|^2 = \|f\|^2$$

OBS: If  $\{g_N\} \in H$  and  $\lim_{N \rightarrow \infty} g_N = f \in H$  then by defn.

$$\lim_{N \rightarrow \infty} \|g_N - f\| = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \|g_N\| = \|f\|$$

To see this:  $\|f\| = \|f - g_N + g_N\| \leq \|f - g_N\| + \|g_N\| \leq \|f - g_N\| + \|f\| \quad \text{in}$

By ③

3  $\Rightarrow$  1  $\|f\|^2 = \sum_{n=1}^{\infty} |\hat{f}_n|^2$ . Hence if  $\hat{f}_n = 0 \quad \forall n \Rightarrow$   
 $\Rightarrow \|f\|^2 = 0 \Rightarrow f = 0$

IHM  $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}$  is an ONB for  $L^2(-\pi, \pi)$

Best approx thm Let  $\{\phi_n\}_{n \geq 1}$  be an ONS in  $H$

and let  $(c_n)_{n \geq 1} \subset \ell^2$ . Then  $\forall f \in H$ ,

$$\|f - \sum_{n \geq 1} c_n \phi_n\|^2 \geq \|f - \sum_{n \geq 1} \hat{f}_n \phi_n\|^2$$

equality holds iff  $c_n = \hat{f}_n \quad \forall n$

Proof idea: Compute  $\|f - \sum_{n \geq 1} \hat{f}_n \phi_n + \sum_{n \geq 1} \hat{f}_n - \sum_{n \geq 1} c_n \phi_n\|^2$

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S6EÖ19  $x(n)$  N-periodic

$$x(n) = \begin{cases} 1 & 0 \leq n \leq k-1 \\ 0 & k \leq n \leq N-1 \end{cases}$$

Using Parseval formula, find  $\sum_{m=1}^{N-1} \frac{1 - \cos\left(\frac{2\pi km}{N}\right)}{1 - \cos\left(\frac{2\pi m}{N}\right)}$

$$\text{Solution: } \hat{x}(m) = \sum_{n=0}^{N-1} x(n) w^{mn} \quad (w = e^{-i\frac{\pi^2}{N}})$$

$$\Rightarrow \hat{x}(m) = \sum_{n=0}^{k-1} w^{mn} = \begin{cases} k & m=0 \\ \frac{w^{mk}-1}{w^m-1} & m=1, 2, \dots, N-1 \end{cases}$$

$$\text{Parseval: } \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{m=0}^{N-1} |\hat{x}(m)|^2$$

$$\begin{aligned} \text{R.H.S: } \sum_{m=0}^{N-1} |\hat{x}(m)|^2 &= k^2 + \sum_{m=1}^{N-1} \frac{|e^{-i\frac{2\pi mk}{N}} - 1|^2}{|e^{-i\frac{2\pi m}{N}} - 1|^2} = \\ &= k^2 + \sum_{m=1}^{N-1} \frac{(\cos\left(\frac{2\pi mk}{N}\right) - 1)^2 + \sin^2\left(\frac{2\pi mk}{N}\right)}{(\cos\left(\frac{2\pi m}{N}\right) - 1)^2 + \sin^2\left(\frac{2\pi m}{N}\right)} = \end{aligned}$$

$$= k^2 + \sum_{m=1}^{N-1} \frac{2(1 - \cos\left(\frac{2\pi mk}{N}\right))}{2(1 - \cos\left(\frac{2\pi m}{N}\right))} = N \underbrace{\sum_{n=0}^{k-1} |x(n)|^2}_{\text{Parseval}} = k$$

$$\Rightarrow \sum_{m=1}^{N-1} \frac{1 - \cos\left(\frac{2\pi mk}{N}\right)}{1 - \cos\left(\frac{2\pi m}{N}\right)} = Nk - k^2$$

EÖ21 Find discrete Fourier transform of  $x(n) = \sin\left(\frac{\pi n}{N}\right) \quad n=0, \dots, N-1$  $x(n)$  is N-periodic.

$$\begin{aligned} \text{Solution: } w &= e^{2\pi i/N}, \quad \hat{x}(m) = \sum_{n=0}^{N-1} x(n) w^{-mn} = \\ &= \frac{1}{2i} \sum_{n=0}^{N-1} \left( e^{\frac{i\pi n}{N}} - e^{-\frac{i\pi n}{N}} \right) e^{-\frac{i2\pi mn}{N}} = \left\{ \text{let } u := e^{-i\pi/N} \right\} = \end{aligned}$$

$$= \frac{1}{2i} \sum_{n=0}^{N-1} (u^{-n} - u^n) e^{2\pi mn} = \frac{1}{2i} \sum_{n=0}^{N-1} (u^{(2m-1)n} - u^{(2m+1)n})$$

Summing geometric progressions

$$\sum_{n=0}^{N-1} (u^{(2m-1)n})^N = \frac{(u^{2m-1})^N - 1}{u^{2m-1} - 1} = -\frac{2}{u^{2m-1} - 1}$$

$$\sum_{n=0}^{N-1} u^{(2m+1)n} = -\frac{2}{u^{2m+1} - 1}$$

$$\hat{x}(m) = \frac{1}{2i} (-2) \left( \frac{1}{e^{2\pi i m} - 1} - \frac{1}{e^{-2\pi i m} - 1} \right) =$$

$$= -\frac{1}{i} \left( \frac{e^{-i\frac{\pi}{N}} - e^{i\frac{\pi}{N}}}{e^{-i2m\pi/N} + e^{i2m\pi/N} - (e^{-i\pi/N} + e^{i\pi/N})} \right) \Rightarrow$$

$$\Rightarrow \hat{x}(m) = \frac{\sin(\frac{\pi}{N})}{\cos(2m\frac{\pi}{N}) - \cos(\frac{\pi}{N})}$$

7.4.6.  $u_{xx} + u_{yy} = 0 \quad x > 0, \quad y \in (0,1)$



$$u_x(0,y) = 0$$

$$u_y(x,0) = 0$$

$$u(x,1) = e^{-x}$$

Solution:  $u(x,y) = X(x)Y(y)$

$$\text{Laplace} \Rightarrow \frac{X''(x)}{X(x)} = - \frac{Y''(y)}{Y(y)} = -\xi^2$$

$$\Rightarrow X''(x) = -\xi^2 X(x), \quad x > 0$$

$$Y''(y) = +\xi^2 Y(y), \quad y > 0$$

$$u_x(0,y) = X'(0)Y(y) = 0 \Rightarrow X'(0) = 0 \quad (*)$$

$$u_y(x,0) = X(x)Y'(0) = 0 \Rightarrow Y'(0) = 0 \quad (**)$$

Note:  $X''(x) = -\xi^2 X(x), \quad X(x) = A(\xi) \cos(\xi x) + B(\xi) \sin(\xi x)$

$$(*) \Rightarrow B(\xi) = 0$$

Similarly,  $Y(y) = C(\xi) \cosh(\xi y) + D(\xi) \sinh(\xi y)$

$$(**) \Rightarrow D(\xi) = 0$$

So,  $E(\xi) = A(\xi) C(\xi)$

$$\Rightarrow u(x,y) = \int_0^\infty E(\xi) \cosh(\xi y) \cos(\xi x) d\xi$$

Now,  $u(x,1) = e^{-x}$

$$\Rightarrow e^{-x} = \int_0^\infty E(\xi) \cosh(\xi) \cos(\xi x) d\xi = F_c(E(\cdot) \cosh(\cdot))(x) \quad (*)$$

Exercise 7.4.1.b)  $F_c(e^{-x})(\xi) = \frac{1}{\xi^2 + 1}$

$$\Rightarrow E(\xi) \cosh(\xi) = \frac{2}{\pi} \cdot \frac{1}{\xi^2 + 1} \quad (\text{applying Fourier cosine to } *)$$

$$\Rightarrow u(x,y) = \int_0^\infty \frac{2}{\pi} \frac{\cos(\xi y) \cos(\xi x)}{(1+\xi^2) \cosh(\xi)} d\xi \quad p.239 \text{ Folland}$$

$$\begin{aligned} \text{7.4.4} \quad & u_t = k u_{xx}, \quad x > 0 \\ & u(x, 0) = f(x), \quad x > 0 \\ & u(0, t) = 0, \quad t > 0 \end{aligned} \quad | \quad x(0) T(t) = 0 \Rightarrow x(0) = 0$$

Solution  $u(x, y) = X(x) T(t)$

$$\text{Heat eq: } \frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \sigma = -\frac{\pi^2}{4}$$

$$X(x) = d_1 \cos(\sqrt{\sigma} x) + d_2 \sin(\sqrt{\sigma} x)$$

$$T(t) = d_3 e^{k\sigma x}$$

$$u(x, t) = \int_0^\infty \sin(\xi x) e^{-k\xi^2 t} E(\xi) d\xi$$

$$u(x, 0) = X(x) T(0) = \int_0^\infty \sin(\xi x) E(\xi) d\xi = f(x)$$

$$\Rightarrow f(x) = F_s [E(\cdot)](x)$$

$$\Rightarrow E(\xi) = \frac{2}{\pi} F_s (f(\cdot))(\xi) \quad (\text{Inverse Fourier sine and cosine transforms})$$

$$\Rightarrow u(x, t) = \frac{2}{\pi} \int_0^\infty F_s [f](\xi) e^{-k\xi^2 t} \sin(\xi x) d\xi$$

Note  $u(x, t)$  above is expressed via  $F_s[f]$ , not  $f$  itself!

$$\text{Hint: } e^{-u^2 kt} = F_c[g_t](u) \quad \text{for } g_t(x) = \frac{1}{\sqrt{\pi kt}} e^{-x^2/4kt}$$

$$\begin{aligned} u(x, t) &= \frac{2}{\pi} \int_0^\infty F_s[f](\xi) \cdot F_c[g_t](\xi) \sin(\xi x) d\xi \\ &= \frac{2}{\pi} F_s [F_s[f](\cdot) \cdot F_c[g_t](\cdot)](x) \end{aligned}$$

Our goal is to find a function  $H$  such that

$$F_s[H] = F_s[f] F_c[g_t]$$

$$\text{then } u(x, t) = \frac{2}{\pi} F_s [F_s[H]] = H$$

by Hint (exercise 7.4.4)

$$H(x, t) = \int_0^\infty f(y) \cdot \frac{g_t(|x-y|) - g_t(x+y)}{2} dy \quad \text{where } g_t = \frac{1}{\sqrt{\pi kt}} e^{-y^2/4kt}$$

S - FUNCTION (s)

1. The  $\delta$  indicator function, often  $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

2. The  $\delta$  distribution. A distribution is a linear function whose inputs are functions  $D'(\mathbb{R}) = \{u: \mathcal{C}_0^\infty(\mathbb{R}) \rightarrow \mathbb{R} \text{ or } \mathbb{C}\}$ ,

such that

- ①  $u$  is linear,  $u(af + bg) = au(f) + bu(g)$
- ② Continuous in the sense that if  $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{C}_0^\infty(\mathbb{R})$  such that all  $\varphi_n$  are 0 off a compact set  $K$ , and  $\|\varphi_n\|_\infty \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \Rightarrow u(\varphi_n) \rightarrow 0$ .

Then  $\delta \in D'(\mathbb{R})$  which is defined by  $\delta(f) = f(0)$

Check: ①  $\delta(af + bg) = (af + bg)(0) = af(0) + bg(0) = a\delta(f) + b\delta(g)$

② If  $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{C}_0^\infty(\mathbb{R})$  and  $\|\varphi_n\|_\infty = \sup_{x \in \mathbb{R}} |\varphi_n(x)| \rightarrow 0$

$\Rightarrow \varphi_n(x) \rightarrow 0 \quad \forall x \in \mathbb{R}$ ,  $\varphi_n(0) = \delta(\varphi_n) \rightarrow 0$  as  $n \rightarrow \infty$ . So ② holds

Let  $u \in L^2(\mathbb{R})$ . Then define  $L_u \in D'(\mathbb{R})$  by  $L_u(f) = \int_{\mathbb{R}} uf$ ,

for any test function  $f \in \mathcal{C}_0^\infty(\mathbb{R})$

Exercise: Check that  $L_u$  satisfies ① and ②

For any  $L \in D'(\mathbb{R})$ ,  $L' \in D'(\mathbb{R})$  is defined by  $L'(f) = -L(f')$

Why this way?

Consider  $u \in \mathcal{C}_0^\infty(\mathbb{R})$  and  $L_u(f) = \int_{\mathbb{R}} uf$ . Thus

$$\int_{\mathbb{R}} uf' \stackrel{\substack{\uparrow \\ \text{I.P.}}}{=} - \int_{\mathbb{R}} wf = -L_u w$$

In this way, we can differentiate  $u \in L^2(\mathbb{R})$  by associating  $u$  to  $L_u \in D'(\mathbb{R})$  and differentiating  $L_u$ .

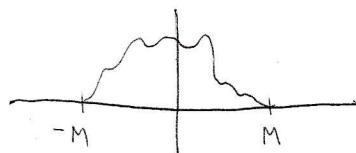
$$L^{(k)}(f) = (-1)^{(k)} L(f^{(k)}) \quad \left| \quad \text{for } L \in D'(\mathbb{R}) \right.$$

$$\hat{L}(f) = L(\hat{f}), \quad f \in \mathcal{C}_0^\infty(\mathbb{R}) \quad \left| \quad \text{(more generally } \mathcal{S}'(\mathbb{R}) \text{)}$$

tempered distributions - you can look up these

Recall:  $\mathcal{C}_0^\infty(\mathbb{R}) = \{f \in \mathcal{C}^\infty(\mathbb{R}) \text{ which have compact support}\}$

Compact support means  $\exists M > 0$  such that  $f(x) = 0$  on  $\mathbb{R} \setminus [-M, M]$



## BEST APPROX THM

Let  $\{\phi_n\}_{n \geq 1} \subset H = \text{Hilbert space}$ , be an ONS.

Let  $(c_n)_{n \geq 1} \in \ell^2$ . Then  $\forall f \in H$ ,

$$\|f - \sum_{n \geq 1} c_n \phi_n\| \geq \|f - \sum_{n \geq 1} \hat{f}_n \phi_n\| \quad \text{and equality holds iff } c_n = \hat{f}_n \forall n.$$

Proof Idea:  $\|f - \sum \hat{f}_n \phi_n + \sum \hat{f}_n \phi_n - \sum c_n \phi_n\|^2 =$

$$= \|f - \sum c_n \phi_n\|^2 = \|f - \sum \hat{f}_n \phi_n\|^2 + \|\sum \hat{f}_n \phi_n - \sum c_n \phi_n\|^2 +$$

$$+ \underbrace{2 \operatorname{Re} \langle f - \sum \hat{f}_n \phi_n, \sum \hat{f}_n \phi_n - \sum c_n \phi_n \rangle}_{\text{we will show that this is 0. Keep calm! (*)}} =$$

$$= \|f - \sum \hat{f}_n \phi_n\|^2 + \|\sum (\hat{f}_n - c_n) \phi_n\|^2 + (*) =$$

$$= \|f - \sum \hat{f}_n \phi_n\|^2 + \underbrace{\sum |\hat{f}_n - c_n|^2}_{\text{This is } \geq 0 = 0 \text{ iff } |\hat{f}_n - c_n| = 0 \Leftrightarrow \hat{f}_n = c_n \forall n} + (*)$$

Hence we only need to show that  $(*) = 0$ .

$$\begin{aligned} & \langle f, \sum (\hat{f}_n - c_n) \phi_n \rangle - \langle \sum \hat{f}_m \phi_m, \sum (\hat{f}_n - c_n) \phi_n \rangle = \\ & = \langle f, \sum (\hat{f}_n - c_n) \phi_n \rangle - \sum \hat{f}_m \langle \phi_m, \sum (\hat{f}_n - c_n) \phi_n \rangle = \\ & = \sum \overline{(\hat{f}_n - c_n)} \langle f, \phi_n \rangle - \sum \hat{f}_m \sum \overline{(\hat{f}_n - c_n)} \underbrace{\langle \phi_m, \phi_n \rangle}_{0 \text{ unless } m=n} \end{aligned}$$

$$\text{Thus } (*) = 2 \operatorname{Re} \left( \sum \overline{(\hat{f}_n - c_n)} \hat{f}_n - \sum \hat{f}_n \overline{(\hat{f}_n - c_n)} \right) = 0 \blacksquare$$

## APPLICATION / EXAMPLES

1. An ONS but not an ONB?  
Consider  $L^2(-\pi, \pi)$ ,  $\left\{ \frac{\sin(nx)}{\sqrt{\pi}} \right\}_{n \geq 1}$  is an ONS

Exercise: check that  $\|\sin(nx)\|_{L^2(-\pi, \pi)} = \sqrt{\pi}$

This is not an ONB for  $L^2(-\pi, \pi)$

Example:  $\langle 1, \frac{\sin(nx)}{\sqrt{\pi}} \rangle = \int_{-\pi}^{\pi} \frac{\sin(nx)}{\sqrt{\pi}} = 0$  because sine is odd.

Thus  $\langle 1, \frac{\sin(nx)}{\sqrt{\pi}} \rangle = 0 \quad \forall n$

The constant function 1 is not the 0 function  
By thm,  $\left\{ \frac{\sin(nx)}{\sqrt{\pi}} \right\}_{n \geq 1}$  is not an ONB

However  $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}$  is an ONB for  $L^2(-\pi, \pi)$

Application of this is STURM-LIOUVILLE PROBLEMS (SLPs)

DEF A regular Sturm Liouville problem on  $[a, b]$  is specified by:

① A "formally self-adjoint" operator  $L(f) = (rf')' + pf$ , where  $r, r'$  and  $p$  are  $\mathbb{R}$  valued and continuous on  $[a, b]$ , and  $r > 0$  on  $[a, b]$

② Self-adjoint boundary conditions

$$\begin{cases} B_1(f) = \alpha, f(a) + \alpha_1 f'(a) + \beta_1 f(b) + \beta_1' f'(b) = 0 \\ B_2(f) = 0 \end{cases}$$

$\alpha$ 's and  $\beta$ 's are constants

③ A positive, continuous  $w$  on  $[a, b]$

The SLP is to find all solutions to the boundary value problem.

$$L(f) + \lambda wf = 0, \text{ for some } \lambda \in \mathbb{C}. \quad (f \neq 0)$$

$f$  is an eigenfunction of the SLP, and the corresponding  $\lambda$  is its eigenvalue.

Ex. Let  $r=w=1$ ,  $p=0$ , then a regular SLP is to find all  $\lambda \in \mathbb{C}$  such that  $\exists f$  which satisfies  $B_1(f) = B_2(f) = 0$  and  $L(f) = f''$ ,  $L(f) + \lambda f = 0$   
 $f'' + \lambda f = 0$

Self-adjoint boundary conditions means that if  $f$  and  $g$  satisfy  $B_1(f)=0$ ,  $B_2(f)=0$ ,  $B_1(g)=0$ ,  $B_2(g)=0$ , then

$$[r(f'g - fg')]_a^b = 0$$

THM ∀ such regular SLP, all the  $\lambda \in \mathbb{R}$

② For EF  $f$  with EV  $\lambda$  and EF  $g$  with EV  $\mu \neq \lambda$

$$\Rightarrow \langle f, g \rangle_{L^2_w([a, b])} = 0$$

③  $\exists$  ONB,  $\{\phi_n\}_{n \geq 1}$  with corresponding EV  $\{\lambda_n\}_{n \geq 1}$   
 for EFs  
 with  $\lim_{n \rightarrow \infty} \lambda_n = \infty$

$L^2_w([a, b]) = \{f: [a, b] \rightarrow \mathbb{C}, \text{ measurable, such that}$

$$\int_{[a, b]} |f|^2 w \, dx < \infty\} / \sim \text{ a.e.}$$

$$\langle f, g \rangle_{L^2_w([a,b])} = \int_{[a,b]} f \bar{g} w$$

Self-adjoint comes from the fact that under these hypotheses  $\langle Lf, g \rangle_{L^2_w([a,b])} = \langle f, Lg \rangle_{L^2_w([a,b])}$

Exercise: Check this if  $f$  and  $g$  satisfies "self-adjoint" BCs.

Proof ① Assume  $L(f) + \lambda wf = 0 \Rightarrow L(f) = -\lambda wf$

$$\Rightarrow \langle Lf, f \rangle_{L^2_w} = \langle -\lambda wf, f \rangle_{L^2_w}$$

$$\text{Self-adjointness} \Rightarrow \langle Lf, f \rangle_{L^2_w} = \langle f, Lf \rangle_{L^2_w} = \\ = \langle f, -\lambda wf \rangle_{L^2_w} = -\bar{\lambda} \langle f, wf \rangle_{L^2_w}$$

$$\text{Thus } -\lambda \langle wf, f \rangle_{L^2_w} = -\bar{\lambda} \langle f, wf \rangle_{L^2_w}$$

$$\langle wf, f \rangle_{L^2_w} = \int_{[a,b]} w^2 f \bar{f} = \int_{[a,b]} w^2 |f|^2 \in \mathbb{R} \quad \text{and non-zero}$$

$$\therefore \langle f, wf \rangle_{L^2_w} = \overline{\langle wf, f \rangle_{L^2_w}}, \text{ Thus } -\lambda = -\bar{\lambda} \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

Exercise:

Consider  $\langle f, Lg \rangle_w$ , use ① to show ② in a similar way

10/2  
S7

E045  $\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0 \\ u(x, 0) = (1 - \frac{1}{2}x^2) e^{-x^2/2} \end{cases}$

$$\hat{u}_t(\xi) = k(\imath\xi)^2 \hat{u}(\xi) = -k\xi^2 \hat{u}(\xi)$$

$$\hat{u}(\xi, t) = \hat{u}(\xi, 0) e^{-k\xi^2 t}$$

To compute FT of  $u(x, 0)$ :

$$(1) x \mapsto (e^{-ax^2/2})^{\wedge}(\xi) = \sqrt{\frac{2\pi}{a}} e^{-\xi^2/2a}$$

$$a=2 \Rightarrow (x \mapsto e^{-x^2})^{\wedge}(\xi) = \sqrt{\pi} e^{-\xi^2/4}$$

$$(2) (x \mapsto -x^2 e^{-x^2})^{\wedge}(\xi) = (x \mapsto (1x)^2 e^{-x^2})^{\wedge}(\xi) =$$
$$= \frac{d^2}{d\xi^2} (x \mapsto e^{-x^2})^{\wedge}(\xi) = -\frac{d}{d\xi} (\sqrt{\pi} \frac{\xi}{2} e^{-\xi^2/4}) =$$
$$= \sqrt{\pi} \left( \frac{\xi^2}{4} - \frac{1}{2} \right) e^{-\xi^2/4}$$

(3) Combine (1)+(2)

$$\hat{u}(\xi, 0) = \sqrt{\pi} e^{-\xi^2/4} - 2\sqrt{\pi} \left( \frac{\xi^2}{4} - \frac{1}{2} \right) e^{-\xi^2/4} = \frac{\sqrt{\pi}}{2} \xi^2 e^{-\xi^2/4}$$
$$\Rightarrow \hat{u}(\xi, t) = \frac{\sqrt{\pi}}{2} \xi^2 e^{-\xi^2/4} e^{-kt^2} = \frac{\sqrt{\pi}}{2} \xi^2 e^{-(kt + \frac{1}{4})\xi^2}$$

$$\text{Set } \frac{1}{2a} = kt + \frac{1}{4} \quad \hat{u}(\xi, t) = \frac{\sqrt{\pi}}{2} \xi^2 e^{\frac{\xi^2}{2a}} =$$
$$= \frac{1}{2} \sqrt{\frac{a}{2}} \xi^2 \sqrt{\frac{2\pi}{a}} e^{-\xi^2/2a}$$

$$u(x, t) = \frac{1}{2} \sqrt{\frac{a}{2}} \left( i \frac{d}{dx} \right)^2 e^{-ax^2/2} =$$
$$= -\frac{1}{2} \sqrt{\frac{a}{2}} \frac{d}{dx} (ax e^{-ax^2/2}) =$$
$$= -\frac{1}{2} \sqrt{\frac{a}{2}} (-a + a^2 x^2) e^{-ax^2/2} =$$
$$= \left( \frac{a}{2} \right)^{3/2} (1 - ax^2) e^{-ax^2/2}$$

$$a = \frac{2}{4kt+1}, \quad u(x, t) = \frac{1}{(4kt+1)^{3/2}} \left( 1 - \frac{2x^2}{4kt+1} \right) e^{-\frac{x^2}{4kt+1}}$$

$$8.4.2 \quad f(t) = \begin{cases} 1 & , t \in (0,1) \\ 0 & , t > 1 \end{cases} \quad (f : (0, \infty) \rightarrow \mathbb{R})$$

PDE  $\begin{cases} u_t = ku_{xx} \\ u(x, 0) = 0 \\ u(0, t) = f(t) \end{cases}$

$v(x, t) = \frac{x}{\sqrt{4\pi k}} \int_0^t f(t-s) e^{-as} s^{-3/2} e^{-x^2/4ks} ds$  solves  $v_t = kv_{xx} - av$

Case 1:  $t \in (0, 1)$

$$\begin{aligned} u(x, t) &= \frac{x}{\sqrt{4\pi k}} \int_0^t s^{-3/2} e^{-x^2/4ks} ds = \left\{ \begin{array}{l} y = \sqrt{\frac{x^2}{4ks}} \\ dy = \frac{x}{\sqrt{4ks}} \left(-\frac{1}{2}\right) \frac{1}{s^{3/2}} ds \end{array} \right\} = \\ &= \frac{-2}{\sqrt{\pi}} \int_{\infty}^{\frac{x}{\sqrt{4kt}}} e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{\infty} e^{-y^2} dy = \operatorname{erfc}\left(\frac{x}{\sqrt{4\pi t}}\right) \end{aligned}$$

(p. 261 ← old book)

$$\begin{aligned} 2: t > 1: \quad u(x, t) &= \frac{x}{\sqrt{4\pi k}} \int_{t-1}^t s^{-3/2} e^{-x^2/4ks} ds = \\ &= \frac{-2}{\sqrt{\pi}} \int_{x/\sqrt{4k(t-1)}}^{\frac{x}{\sqrt{4kt}}} e^{-y^2} dy = \\ &= \frac{2}{\sqrt{\pi}} \left( \int_0^{\frac{x}{\sqrt{4kt}}} e^{-y^2} dy - \int_0^{\frac{x}{\sqrt{4k(t-1)}}} e^{-y^2} dy \right) \\ &= \operatorname{erf}\left(\frac{x}{\sqrt{4k(t-1)}}\right) - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) \end{aligned}$$

3.3.10 a) Compute  $\sum_{n=1}^{\infty} \frac{1}{n^4}$

$$f \in L^2(-\pi, \pi), \quad \{\varphi_n\} \text{ ONB}, \quad f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$$

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2$$

$$f(\theta) = \theta^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta)$$

$$\varphi_n = \left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ \frac{\cos(n\theta)}{\sqrt{\pi}}, \frac{\sin(n\theta)}{\sqrt{\pi}} \right\} \text{ ONB for } L^2(-\pi, \pi)$$

$$|\langle f, \frac{1}{\sqrt{2\pi}} \rangle|^2 = \left( \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(\theta) d\theta \right)^2 = \left( \frac{1}{\sqrt{2\pi}} \frac{\pi^2}{3} 2\pi \right)^2 = \frac{2}{9} \pi^5$$

$$\begin{aligned} \left| \langle f, \frac{\cos(n\theta)}{\sqrt{\pi}} \rangle \right|^2 &= \left( \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta \right)^2 = \\ &= \frac{1}{\pi} \left( \frac{4(-1)^n}{n^2} \pi \right)^2 = \frac{16\pi}{n^4} \end{aligned}$$

$$\|f\|^2 = \int_{-\pi}^{\pi} \theta^4 d\theta = \left[ \frac{\theta^5}{5} \right]_{-\pi}^{\pi} = \frac{2\pi^5}{5}$$

$$\|f\|^2 = \frac{2\pi^5}{9} + 16\pi \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{16\pi} \left( \frac{2\pi^5}{5} - \frac{2\pi^5}{9} \right) = \frac{\pi^4}{90}$$

$$\text{Ex 14} (*) \int_0^\infty e^{-\tau} u(t-\tau) d\tau - \int_\infty^0 e^\tau u(t-\tau) d\tau = \sqrt{3} u(t) - e^{-|t|}$$

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ -e^x & x < 0 \end{cases}$$

$$(*) \Leftrightarrow (u * f)(t) = \sqrt{3} u(t) - e^{-|t|}$$

$$\hat{u}(\xi) \hat{f}(\xi) = \sqrt{3} \hat{u}(\xi) - \frac{2}{1+\xi^2}$$

$$\hat{f}(\xi) = \int_0^\infty e^{-x} e^{-ix\xi} dx - \int_{-\infty}^0 e^x e^{ix\xi} dx =$$

$$= \frac{1}{1+i\xi} - \frac{1}{1-i\xi} = \frac{-2i\xi}{1+\xi^2}$$

$$\hat{u}(\xi) = \frac{-2}{1+\xi^2} - \frac{1+\xi^2}{-2i\xi - \sqrt{3}(1+\xi^2)} = \frac{2}{\sqrt{3}\xi^2 + 2i\xi + \sqrt{3}} =$$

$$s = i\xi \quad = \frac{2}{\sqrt{3}} - \frac{1}{s^2 - \frac{2s}{\sqrt{3}} - 1} = \frac{2}{\sqrt{3}} - \frac{1}{(s + \frac{1}{\sqrt{3}})} - \frac{1}{(s - \sqrt{3})} =$$

$$= \frac{1}{2} \frac{1}{s + \frac{1}{\sqrt{3}}} - \frac{1}{2} \frac{1}{s - \sqrt{3}} = \frac{1}{2} \frac{1}{\frac{1}{\sqrt{3}} + i\xi} + \frac{1}{2} \frac{1}{\sqrt{3} - i\xi}$$

$$\theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$[e^{-at} \theta(t)]^\wedge(\xi) = \int_{-\infty}^\infty e^{-at} \theta(t) e^{-i\xi t} dt =$$

$$= \int_0^\infty e^{-(a+i\xi)t} dt = \frac{1}{a+i\xi}$$

$$[e^{-at} (1-\theta(t))]^\wedge(\xi) = [e^{-at} \theta(-t)]^\wedge(\xi) = \frac{1}{a-i\xi}$$

$$u(t) = \frac{1}{2} e^{-\frac{1}{\sqrt{3}}t} \theta(t) + \frac{1}{2} e^{\sqrt{3}t} (1-\theta(t)) \quad \blacksquare$$

SLPs ②  $L(f) + \lambda wf = 0$  and  $L(g) + \mu wg = 0$ ,  $\lambda \neq \mu$

Show that  $\langle f, g \rangle_{L^2} = 0$

$$\text{Write } \langle Lf, g \rangle_{L^2} = \uparrow \langle f, Lg \rangle_{L^2}$$

by def. of self-adjoint BCS

$$\text{left side} = \langle -\lambda wf, g \rangle_{L^2} = -\lambda \langle wf, g \rangle_{L^2}$$

$$\text{Right side} = \langle f, -\mu wg \rangle_{L^2} = \uparrow -\mu \langle f, wg \rangle_{L^2}$$

$\mu \in \mathbb{R}$  by ①

$$\text{OBS: } \langle wf, g \rangle_{L^2} = \int_{[a,b]} wf \bar{g} = \int_{[a,b]} f \bar{wg} = \langle f, wg \rangle_{L^2}$$

$w$  is real

$$\text{Both } \langle wf, g \rangle_{L^2} = \langle f, wg \rangle_{L^2} = \langle f, g \rangle_{L^2}$$

$$\text{Thus } -\lambda \langle f, g \rangle_{L^2} = -\mu \langle f, g \rangle_{L^2} \text{ but } \lambda \neq \mu \Rightarrow \langle f, g \rangle_{L^2} = 0.$$


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Compare with linear algebra - also numerical methods

Sample a function  $\rightarrow$  vector, fcn  $\rightsquigarrow$  vector = discretized fcn.

Linear function from  $\mathbb{R}^N \rightarrow \mathbb{R}^N$  can be represented by a matrix.  $Mv = L(v)$ ,  $M$  matrix,  $l$  linear fcn.

$\lambda$  is an eigenvalue if  $\exists v \neq 0$  such that  $Mv = \lambda v$

Diagonal matrices are convenient. Symmetric matrices can be diagonalized. Analogous result here for  $L$  linear, self-adjoint as in SLP.

$\exists$  ONB  $\{\phi_n\}_{n \geq 1}$  of eigenfunctions of  $L$ , with eigenvalues  $\{\lambda_n\}_{n \geq 1}$ . Thus can write  $f \in L^2$  as  $f = \sum_{n \geq 1} \hat{f}_n \phi_n \Rightarrow L(f) = \sum_{n \geq 1} \hat{f}_n (-\lambda_n \phi_n w)$

$\therefore L$  is diagonalized as  $\begin{bmatrix} -\lambda_1 w & & & \\ & -\lambda_2 w & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$

$f$  as  $\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \end{bmatrix}$  vector wrt  $\{\phi_n\}_{n \geq 1}$

Then  $L(f) = \begin{bmatrix} -\lambda_1 w & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \end{bmatrix}$  in the basis  $\{\phi_n\}_{n \geq 1}$

*Not important for exam*

OBS: If  $L(f) + \lambda wf = 0$  and  $L(g) + \mu wg = 0$  then, if  $\mu \neq \lambda$ , most likely  $f+g$  is not an eigenfunction

$$L(f+g) = L(f) + L(g) = -\lambda wf - \mu w \underset{\substack{\uparrow \\ \text{most likely}}}{g} \neq -w(L(f+g))$$

only if  $-\lambda f - \mu g = -w(f+g)$  holds, in general false.

Ex.  $f'' + \lambda f = 0$  on  $[-\pi, \pi]$ , BC  $f(-\pi) = f(\pi)$

$\Rightarrow$  solutions  $\{e^{inx}\}$ ,  $\lambda_n = n^2$ , try for  $m \neq n$

$e^{inx} + e^{imx}$  is not a solution to the SLP (Exercise)

Ex.  $\nabla u = 0$  on  $[0, l]$ ,  $\begin{cases} u_x(0, t) = \alpha u(0, t) \\ u_x(l, t) = \beta u(l, t) \end{cases}$  BC  
 $\alpha, \beta \in \mathbb{R}$   $\begin{cases} u(x, 0) = u_0(x) \end{cases}$  IC

Check that these are self-adjoint boundary conditions by verifying that if  $f$  and  $g$  are defined on  $[0, l]$  and satisfy the BCs, then compute

$$[r(f'g - fg')]_0^l. \text{ What is } r? \text{ What is the SLP here?}$$

Separate variables in  $\nabla u = 0 \Rightarrow X''T = T'X$

$$\frac{X''}{X} = \frac{T'}{T} = \text{constant} \Rightarrow X'' = (\text{constant})X$$

$\Leftrightarrow X'' + \lambda X = 0$  for some constant  $\lambda$ . This is our SLP

From defn.  $r=1$ ,  $w=1$ . Check the BCs are self-adjoint

$$[f'g - fg']_0^l = f'(l)\bar{g}(l) - f(l)\bar{g}(l) - f'(0)\bar{g}(0) + f(0)\bar{g}(0) \quad (*)$$

$$\text{BC: } \begin{cases} f'(0) = \alpha f(0) \\ f'(l) = \beta f(l) \end{cases}, \quad \begin{cases} g'(0) = \alpha g(0) \\ g'(l) = \beta g(l) \end{cases}$$

Substitute for  $f'$ ,  $g'$ . Check that  $[f'g - fg']_0^l$  vanishes.

$$(*) = \beta f(l)\bar{g}(l) - f(l)\overline{\beta g'(l)} - \alpha f(0)\overline{g(0)} + f(0)\overline{\alpha g(0)} = 0$$

as long as  $\alpha$  and  $\beta$  are real. They are.

The theory of SLPs  $\Rightarrow$  can find  $(\exists)$   $L^2$  ONB of eigenfunctions  $X_n$ , evals  $\lambda_n$

$$X_n'' + \lambda_n X_n = 0 \Rightarrow X_n'' = -\lambda_n X_n \Rightarrow \frac{X_n''}{X_n} = -\lambda_n$$

$\Rightarrow$  up to a constant,  $T_n(t) = e^{-\lambda_n t}$

We are given  $u(x, 0) = u_0(x)$  (IC)

Each pair  $X_n(x) T_n(t)$  satisfies the homogenous heat equation  $\nabla(X_n T_n) = 0$

Thus  $\sum_{n \geq 1} a_n X_n(x) T_n(t)$  solves the homogenous heat eq.

Letting  $t=0 \Rightarrow T_n(t)=1 \forall n \Rightarrow$  let  $a_n = \langle u_0, X_n \rangle_{L^2([0, l])} = \hat{u}_0(n)$

$$\Rightarrow u(x, t) = \sum_{n \geq 1} \hat{u}_0(n) X_n(x) T_n(t)$$

Hence we just need to find the  $\{X_n\}$

$$(u_0(x) = \sum_{n \geq 1} \langle u_0, X_n \rangle_{L^2([0, l])} X_n(x))$$

① Theorem  $\Rightarrow \lambda \in \mathbb{R}$  for the SLP

$$X'' + \lambda X = 0$$

①  $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = ax + b$  for some  $a, b \in \mathbb{R}$

$$\text{BC: } X'(0) = \alpha X(0), \quad X'(l) = \beta X(l)$$

$$\begin{array}{ll} \Updownarrow & \Updownarrow \\ a = \alpha b & a = \beta (al + b) \end{array}$$

$\Leftrightarrow \alpha b = \beta (al + b)$ ,  $\alpha = \frac{\beta}{1-\beta l}$ , if this holds (depends on  $\alpha, \beta, l$ ) then there is a non-trivial  $X_0(x)$  with  $\lambda_0 = 0$ . If not, then no  $X_0(x)$  with  $\lambda_0 = 0$ .

②  $X'' + \lambda X = 0$  for  $\lambda > 0$ , write  $\lambda = \mu^2$ ,  $\mu > 0$

$$X'' = -\mu^2 X \Rightarrow \text{basis of solutions is } \{ \sin(\mu x), \cos(\mu x) \}$$

To determine which values  $\mu$  can be used the BCs

$$X(x) = a \cos(\mu x) + b \sin(\mu x), \quad X'(x) = -a\mu \sin(\mu x) + b\mu \cos(\mu x)$$

$$X'(0) = b\mu = \alpha X(0) = \alpha a \Rightarrow b = \frac{\alpha a}{\mu} \text{ since } \mu \neq 0$$

$$X'(l) = -a\mu \sin(\mu l) + b\mu \cos(\mu l) = \beta X(l) =$$

$$= \beta a \cos(\mu l) + \beta b \sin(\mu l) = \left\{ b = \frac{\alpha a}{\mu} \right\} \Rightarrow$$

$$\Rightarrow -a\mu \sin(\mu l) + \alpha a \cos(\mu l) = \beta a \cos(\mu l) + \beta \frac{\alpha a}{\mu} \sin(\mu l)$$

$$\Rightarrow (\alpha a - \beta a) \cos(\mu l) = (\alpha \mu + \beta \frac{\alpha a}{\mu}) \sin(\mu l)$$

$$\text{Since } b = \frac{\alpha a}{\mu} \Rightarrow a \neq 0$$

Divide by  $a$ , multiply by  $\mu$

$$\Rightarrow (\alpha \mu - \beta \mu) \cos(\mu l) = (\mu^2 + \beta \alpha) \sin(\mu l)$$

$$\Rightarrow \frac{\alpha \mu - \beta \mu}{\mu^2 + \beta \alpha} = \tan(\mu l) = \frac{\mu(\alpha - \beta)}{\mu^2 + \beta \alpha}$$

By the SLP theorem there is a sequence  $\{\mu_n\}_{n \geq 1}$

$$\mu_n \rightarrow \infty \text{ as } n \rightarrow \infty \text{ of solutions to the eqn.}$$

$$\tan(\mu l) = \frac{\mu(\alpha - \beta)}{\mu^2 + \beta \alpha}$$

$$\therefore \tilde{X}_n(x) = a \cos(\mu_n x) + \frac{\alpha a}{\mu_n} \sin(\mu_n x)$$

$\tilde{X}_n$  does not yet have  $\|\tilde{X}_n\|_{L^2([0, l])} = 1$

③ For  $\lambda < 0$ ,  $X'' = \nu^2 X$ ,  $\nu > 0$

Basis of solutions  $\{\sinh(\nu x), \cosh(\nu x)\}$

equivalent to

$\{e^{\nu x}, e^{-\nu x}\}$  but has the advantage of being almost like  $\{\sin, \cos\}$

"Recycle" our previous calculations!

Note  $\sinh'(\nu x) = \nu \cosh(\nu x)$ ,  $\cosh'(\nu x) = \nu \sinh(\nu x)$

$\Rightarrow$  Still get for  $X(x) = a \cosh(\nu x) + b \sinh(\nu x)$

that  $b = \frac{\alpha \alpha}{\nu}$ ,  $\tanh(\nu x) = \frac{\nu(\alpha - \beta)}{\beta \alpha - \nu^2}$

(if  $\alpha > \beta$ , then  $\frac{\nu(\alpha - \beta)}{\beta \alpha - \nu^2} \xrightarrow{\nu \rightarrow \infty} -\infty$ )

Theorem of SLPs  $\Rightarrow$  there are at most finitely many  $v_n$  which satisfy the eqn, and corresponding

$$a \cosh(v_n x) + \frac{\alpha \alpha}{v_n} \sinh(v_n x)$$

More specific information about the  $v_n$  and  $\mu_n$  is only numerically available for general  $\alpha, \beta, c \in \mathbb{R}$

---

For contrast, if BCs have  $\alpha = \beta = 0$ ,  $\begin{cases} X'(0) = 0 \\ X'(1) = 0 \end{cases}$  BCs

Exercise: Solve the SLP  $X'' + \lambda X = 0$

Find  $\{X_n\}$  and  $\{\lambda_n\}$ .

Verify that self-adjoint BCs.

$$3.3.10b \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = ?$$

Solution: PARSEVAL EQ:  $\{\psi_n\}$  ONB in  $L_2(a,b)$ ,  $f \in L_2(a,b)$

$$\text{Then } \|f\|^2 = \sum |\langle f, \psi_n \rangle|^2$$

$$\text{From table 1: } f(\theta) = \theta(\pi - |\theta|), \quad \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{(2n-1)^3}$$

$$\Rightarrow a_n = 0, \quad b_n = \begin{cases} 0, & n \text{ even} \\ \frac{8}{\pi} \frac{1}{n^3}, & n \text{ odd} \end{cases}$$

$$\{1\} \cup \{\cos(n\theta)\}_{n=1}^{\infty} \cup \{\sin(n\theta)\}_{n=1}^{\infty} \text{ normalize} \Rightarrow$$

$$\Rightarrow \left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ \frac{\cos(n\theta)}{\sqrt{\pi}} \right\}_{n=1}^{\infty} \cup \left\{ \frac{\sin(n\theta)}{\sqrt{\pi}} \right\}_{n=1}^{\infty} \text{ ONB}$$

$$|\langle f, \frac{1}{\sqrt{2\pi}} \rangle|^2 = \left| \underbrace{\int_{-\pi}^{\pi} f(\theta) d\theta}_{\pi \cdot a_0} \frac{1}{\sqrt{2\pi}} \right|^2 = 0$$

$$|\langle f, \frac{\cos(n\theta)}{\sqrt{\pi}} \rangle|^2 = \left| \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta}_{=\pi a_n} \right|^2 = 0$$

$$|\langle f, \frac{\sin(n\theta)}{\sqrt{\pi}} \rangle|^2 = \left| \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta}_{=\pi b_n} \right|^2 = |\sqrt{\pi} b_n|^2 = \begin{cases} 0, & n \text{ even} \\ \frac{64}{\pi n^6}, & n \text{ odd} \end{cases}$$

$$\text{Parseval} \Rightarrow \|f\|^2 = \frac{64}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$$

$$\|f\|^2 = \int_{-\pi}^{\pi} \theta^2 (\pi - |\theta|)^2 d\theta = 2 \int_0^{\pi} (\theta^4 - 2\pi\theta^3 + \pi^2\theta^2) d\theta =$$

$$= \frac{2\pi^5}{5} - \frac{4\pi^5}{4} + \frac{2\pi^5}{3} = \frac{\pi^5}{15}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^5}{15} \cdot \frac{\pi^6}{64} = \underbrace{\frac{\pi^6}{15 \cdot 64}}$$

3.3.7a  $f(x) = x$ ,  $[0, \pi]$ , Find the best approx in norm among all func of the form  $a_0 + a_1 \cos x + a_2 \cos 2x$

Solution: THM 3.8: If  $\{\psi_n\}$  ONS in  $L_2(0, \pi)$ ,  $f \in L_2(0, \pi)$

$$\Rightarrow \|f - \sum_{n=1}^{\infty} \langle f, \psi_n \rangle \psi_n\| \leq \|f - \sum_{n=1}^{\infty} c_n \psi_n\| \text{ for all } c_n \text{ s.t.}$$

$$\sum c_n^2 < \infty, \quad \Leftrightarrow c_n = \langle f, \psi_n \rangle$$

$$\|f - a_0 - a_1 \cos x - a_2 \cos 2x\| = \|f - a_0 \sqrt{\frac{1}{\pi}} - a_1 \sqrt{\frac{2}{\pi}} \cos x - a_2 \sqrt{\frac{2}{\pi}} \cos 2x\|$$

By thm 3.8, we achieve the minimum if:  $a_0 \sqrt{\pi} = \langle f, \frac{1}{\sqrt{\pi}} \rangle$

$$a_1 \sqrt{\frac{2}{\pi}} = \langle f, \sqrt{\frac{2}{\pi}} \cos x \rangle, \quad a_2 \sqrt{\frac{2}{\pi}} = \langle f, \sqrt{\frac{2}{\pi}} \cos 2x \rangle$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\infty} x dx = \frac{\pi}{2}$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \cos x dx = \frac{2}{\pi} \underbrace{\left[ x \sin x \right]_0^{\pi}}_{=0} - \frac{2}{\pi} \int_0^{\pi} \sin x dx = \frac{-4}{\pi}$$

$$a_2 = \frac{2}{\pi} \int_0^{\pi} x \cos 2x dx = \frac{1}{\pi} \underbrace{\left[ x \sin 2x \right]_0^{\pi}}_{=0} - \frac{1}{\pi} \int_0^{\pi} \sin 2x dx = \frac{1}{2\pi} \left[ \cos 2x \right]_0^{\pi} = 0$$

$\underbrace{\frac{\pi}{2} - \frac{4}{\pi} \cos x}_{\text{is our best approximation}}$

3.5.4 Find the eigenvalues and normalized eigenfunctions

$$f'' + \lambda f = 0, \quad f'(0) = 0, \quad f(l) = 0, \quad [0, l]$$

Solution: Thm 3.9a)  $\Rightarrow$  all  $\lambda$  are real.

$$1) \lambda = 0 \Rightarrow f'' = 0 \Rightarrow f(x) = ax + b, \quad \begin{cases} f'(0) = a = 0 \\ f(l) = 0 = b \end{cases} \Rightarrow f \equiv 0$$

$$2) \lambda < 0, \quad \lambda = (im)^2, \quad m > 0 \Rightarrow f(x) = ae^{mx} + be^{-mx}$$

$$\begin{aligned} f'(0) = 0 &\Rightarrow ma - mb = 0 \Rightarrow a = b \\ f(l) = 0 &\Rightarrow ae^{ml} + be^{-ml} = 0 \Rightarrow a = 0 \end{aligned} \quad \begin{cases} \left. \begin{aligned} a &= b \\ a &= 0 \end{aligned} \right\} \Rightarrow f \equiv 0 \end{cases} \quad \begin{matrix} \uparrow \\ \text{not interesting} \end{matrix}$$

$$3) \lambda > 0, \quad \lambda = \mu^2, \quad \mu > 0 \Rightarrow f(x) = a \cos(\mu x) + b \sin(\mu x)$$

$$f'(0) = 0 \Rightarrow b = 0$$

$$f(l) = 0 \Rightarrow a \cos(\mu l) = 0 \Rightarrow \mu = \frac{(2n-1)\pi}{2l}$$

$$\Rightarrow \lambda_k = \left( \frac{(2k-1)\pi}{2l} \right)^2, \quad f_k(x) = a_k \cos \left( \frac{(2k-1)\pi x}{2l} \right)$$

$$1 = \|f_k\|^2 = a_k^2 \int_0^l \cos^2 \left( \frac{(2k-1)\pi x}{2l} \right) dx = \frac{a_k^2}{2} \int_0^l \left( \cos \left( \frac{(2k-1)\pi x}{2l} \right) + 1 \right) dx =$$

$$= \frac{a_k^2}{2} l = 1 \Rightarrow a_k = \sqrt{\frac{2}{l}}$$

$$\lambda_k = \left( \frac{(2k-1)\pi}{2l} \right)^2, \quad f_k(x) = \sqrt{\frac{2}{l}} \cos \left( \frac{(2k-1)\pi x}{2l} \right)$$

15/2  
F14

Ex. Solve  $\begin{cases} \Delta u = F(x, t) \\ B(u) = g(t) \\ I(u) = h(x) \end{cases}$

PDE  
BC  
IC

Technique: Use superposition to deal with inhomogeneities one at a time.

- To do this: Solve   
 1.  $\Delta u = 0, B(u) = 0, I(u) = h(x)$   
 2.  $\Delta u = 0, B(u) = g(t), I(u) = 0$   
 3.  $\Delta u = F(x, t), B(u) = 0, I(u) = 0$

Call these solutions  $u_1, u_2, u_3$ . Then by superposition,  $\underline{u_1 + u_2 + u_3}$  solves the problem!

1. On a bounded interval we must be. Because of  $B(u)$  BC.  
 Example - BC are specifying  $u(0, t)$  and  $u(l, t)$ . Then homogenous BC are  $u(0, t) = u(l, t) = 0$  (Dirichlet BC)  
 self-adjoint. Use separation of variables.

Write  $X(x)T(t) \Rightarrow \Delta(XT) = T''X - X''T = 0$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = \text{const.} \Rightarrow X'' + \lambda X = 0 \quad SLP.$$

$$X(0) = X(l) = 0 \quad (BC) \Rightarrow X_n(x) = \frac{\sin\left(\frac{n\pi x}{l}\right)}{\sqrt{l/2}}, \quad \lambda_n = +\frac{n^2\pi^2}{l^2}$$

↑  
 normalized eigenfunctions.

↑  
 corresponding eigenvalues.

$$T_n(t) = e^{-n^2\pi^2 t / l^2}$$

Solution:  $u_1(x, t) = \sum_{n=1}^{\infty} \hat{h}_n X_n(x) T_n(t)$  where  $\hat{h}_n = \langle h, X_n \rangle = \int_{[0, l]} h(x) X_n(x) dx$ .

2. BC now are  $u(0, t) = \cancel{g(t)}, u(l, t) = \cancel{0}$

$\xrightarrow{0} \xleftarrow{l}$  x variable,  $\xrightarrow{t}$  t variable  
 no transforms,  $\mathcal{L}$  on  $\mathbb{R}^+$ ,  $\mathcal{F}$  on  $\mathbb{R}$

Thus use  $\mathcal{L}$  here,  $\mathcal{L}(\Delta u) = \mathcal{L}u_{xx} = \mathcal{U}_{xx}, \mathcal{U} = \mathcal{L}(u)$

$$\Rightarrow \mathcal{U}(x, z) = b(z) e^{-\sqrt{z}x}, \quad BC \Rightarrow \mathcal{U}(0, z) = b(z) = \mathcal{L}(g(z))$$

We computed  $\mathcal{L}\left(\frac{x}{\sqrt{4\pi t^3}} e^{-x^2/4t}\right) = e^{-x\sqrt{z}}$

$$\Rightarrow \mathcal{U}(x, z) = \tilde{g}(z)$$

$$\therefore u(x, t) = g * \frac{x e^{-x^2/4t}}{\sqrt{4\pi t^3}} \theta(t)$$

↑  
t variable

$$\text{Thus } u_2(x, t) = \frac{x}{\sqrt{4\pi t}} \int_0^t g(t-s) s^{-3/2} e^{-x^2/4s} \theta(s) ds$$

By assumption,  $\mathcal{L}(g)$  defined  $\Rightarrow g$  satisfies  $\mathcal{L}g = 0$  for  $s < 0$   
 $\Rightarrow g(t-s) = 0$  for  $t-s < 0 \Leftrightarrow t < s$

Presumably,  $g(t) = 0$  for  $t < 0$ .  $\theta(t) = 0$  for  $t < 0$ .

3. BC  $u(0,t) = 0 = u(l,t)$ , IC  $u(x,0) = 0$

Here can use SLP, ex.  $F(x,t) = xt$

For each  $t$ , can write  $F(x,t) = \sum_{n \geq 1} c_n(t) X_n (*)$

Here,  $c_n(t) = t \hat{X}_n$ ,  $\hat{X}_n = \langle x, X_n \rangle$ . Look for solution  $u_3$ ,

$$u_3(x,t) = \sum_{n \geq 1} w_n(t) X_n(x)$$

$$\oplus u_3(x,t) = \sum_{n \geq 1} (w_n'(t) X_n(x) - w_n(t) X_n''(x)) =$$

$$= \sum_{n \geq 1} \left( w_n'(t) + w_n(t) \frac{n^2 \pi^2}{l^2} \right) X_n(x) = F(x,t) =$$

$$= \sum_{n \geq 1} t \hat{X}_n X_n(x)$$

equate coeffs of  $X_n \Rightarrow w_n'(t) + \frac{n^2 \pi^2}{l^2} w_n(t) = t \hat{X}_n$

Solution to this ODE,  $a_n t + b_n = w_n(t)$

$$a_n + \frac{n^2 \pi^2}{l^2} (a_n t + b_n) = t \hat{X}_n \Rightarrow a_n = \frac{\hat{X}_n}{\pi^2 n^2 / l^2}$$

$$b_n = -\frac{a_n}{\pi^2 n^2 / l^2}$$

$$u_3(x,t) = \sum_{n \geq 1} w_n(t) X_n(x) \text{ does the job.}$$

Final solution:  $u_1 + u_2 + u_3$  (This is ONE solution)

Expanding  $F(x,t) = xt$  in a Fourier Series in the basis  $\{X_n\}_{n \geq 1}$ , for each  $t$ ,  $xt$  function of  $x$

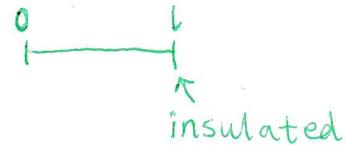
$$\Rightarrow x = \sum_{n \geq 1} \langle x, X_n \rangle X_n$$

$$\langle x, X_n \rangle = \int_0^l x \frac{\sin(n \pi x)}{\sqrt{l/2}} dx \stackrel{IP}{=} \left[ \frac{-x \cos(n \pi x)}{\frac{n \pi}{l} \sqrt{l/2}} \right]_0^l + \int_0^l \cos =$$

$$= \frac{-l^2 (-1)^n \sqrt{2}}{n \pi \sqrt{l/2}} = \langle x, X_n \rangle.$$

Ex. Solve  $\frac{\partial u}{\partial t} = x$

$$\begin{aligned}u(0, t) &= 0 \\u_x(l, t) &= 0 \\u(x, 0) &= u_0(x) = 0\end{aligned}$$



1. Solve homog. heat eqn.

Sep var.  $X'' + \lambda X = 0$ ,  $X(0) = 0$ ,  $X'(l) = 0$  self-adjoint BCs.

Cases for  $\lambda$ : ①  $\lambda = 0 \Rightarrow X(x) = ax + b$ . BC  $\Rightarrow b = 0$ ,  $a = 0 \Rightarrow N.O.$

②  $\lambda > 0$ :  $X'' = -\lambda X$ , write  $\lambda = \mu^2$ ,  $\mu > 0$

$$X(x) = a \cos(\mu x) + b \sin(\mu x) \quad BC \Rightarrow a = 0$$

$$X'(l) = b\mu \cos(\mu l) = 0 \Rightarrow \mu = \frac{(n + \frac{1}{2})\pi}{l}, n \in \mathbb{Z}, n \geq 0$$

$$\tilde{X}_n(x) = \sin\left(\frac{(n + \frac{1}{2})\pi x}{l}\right)$$

③  $\lambda < 0$ :  $X'' = \nu^2 X$ ,  $\nu > 0$ . Then

$$X(x) = a \cosh(\nu x) + b \sinh(\nu x), \quad BC \Rightarrow a = 0$$

$$X'(l) = b\nu \cosh(\nu l) = 0 \Leftrightarrow b = 0 \Rightarrow \text{no solutions.}$$

$\Rightarrow$  All solutions from case ②.

$$X_n(x) = \frac{\sin((n + \frac{1}{2})\pi x/l)}{\sqrt{l/2}}$$

$$\text{Solution: } u(x, t) = \sum_{n \geq 1} a_n X_n(x) T_n(t)$$

$$X_n'' + \lambda_n X_n = 0 \Rightarrow X_n'' = -\lambda_n X_n \Rightarrow \frac{X_n''}{X_n} = -\lambda_n = \frac{T_n''}{T_n}$$

$$\lambda_n = \frac{(n + \frac{1}{2})^2 \pi^2}{l^2}$$

$$T_n(t) = e^{-\lambda_n t} = e^{-(n + \frac{1}{2})^2 \pi^2 t / l^2}$$

$$\text{IC is } u(x, 0) = 0 \Rightarrow a_n = 0 \quad \forall n$$

For the inhomogeneous heat eqn.  $\frac{\partial u}{\partial t} = F(x, t) = x$   
independent of time

Start with this part.

Look for a solution which is independent of time.  
"Steady state solution"

$$\frac{\partial f(x)}{\partial t} = -f''(x) = x \quad . \quad f(x) = \frac{-x^3}{6} + ax + b$$

$$\text{BC} \Rightarrow b = 0$$

$$f'(l) = -\frac{l^2}{2} + a = 0 \Rightarrow a = \frac{l^2}{2} \Rightarrow f(x) = \frac{-x^3}{6} + x \frac{l^2}{2}$$

Solves the inhomogenous PDE, and the BC but not the IC.

Now solve  $\nabla^2 u = 0$ , BC, IC is  $u(x,0) = -f(x)$   
↑  
steady state

By our previous work,  $u(x,t) = \sum_{n \geq 1} -\hat{f}_n X_n(x) T_n(t)$

$$\hat{f}_n = \langle f, X_n \rangle = \int_0^L \frac{f(x) \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)}{\sqrt{L/2}} dx$$

Thus the whole solution is  $f(x) + \sum_{n \geq 1} -\hat{f}_n X_n(x) T_n(t)$

It can be good to double check that the solution really is.

---

In summary this far:

- ① On bounded intervals use superpositions to deal with inhomogeneities  $u_h + u_n$ .
- ② Understanding BCs: If only involve unknown solution to the PDE (i.e.  $u(x,t)$ ) and constants (no other functions!)  $\Rightarrow$  use SV+SLP

If BC involves some fcn of time  $\Rightarrow$  use  $\mathcal{L}$ -transform

In that case, solve  $\begin{cases} PDE = 0 \\ I(u) = 0 \\ BC(u) = g(t) \end{cases}$  use  $\mathcal{L}$  to solve



Then add this solution to  $\begin{cases} PDE = \dots \\ I(u) = \dots \\ B(u) = 0 \end{cases}$

The sum solves it all

---

Recall the defn. of  $D'(R)$ : linear functions on  $C_c^\infty(R)$

(Alt. math facts:  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ ,  $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$  etc.)  
(comes from desire for linearity.)

15/2  
S9E021 a)  $\psi_n(x) = \frac{\sin \pi x}{\pi x} e^{inx}$  are pairwise orthogonalb) Find  $c_n$  such that  $\int_{-\infty}^{\infty} \left( \frac{1}{1+x^2} - \sum_{n=-N}^N c_n \psi_n(x) \right)^2 dx$  min.

$$\text{Solution: table 2: } F\left(\frac{\sin \pi x}{\pi x}\right) = \chi_{[-1/2, 1/2]}(\xi)$$

$$\Rightarrow F(\psi_n) = F\left(\frac{\sin \pi x}{\pi x} e^{inx}\right) = \chi_{[-1/2, 1/2]}(\xi - n) = \chi_{[n-1/2, n+1/2]}(\xi)$$

$$\langle \psi_n, \psi_m \rangle = \{ \text{Plancherel} \} = \frac{1}{2\pi} \langle \hat{\psi}_n, \hat{\psi}_m \rangle =$$

$$= \int_{-\infty}^{\infty} \chi_{[n-1/2, n+1/2]}(\xi) \chi_{[m-1/2, m+1/2]}(\xi) d\xi =$$

$$= \begin{cases} 0, & m \neq n \\ \frac{1}{2\pi}, & m = n \end{cases}$$

$$\chi_{[a,b]}(x) = \begin{cases} 0, & x \in [a,b] \\ 1, & x \notin \end{cases}$$

$\Rightarrow \{\psi_n\}$  orthogonal,  $\{\hat{\psi}_n\}$  orthogonal,  $\|\hat{\psi}_n\| = 1$

$$b) \left\| \frac{1}{1+x^2} - \sum_{n=-N}^N c_n \psi_n(x) \right\|^2 = \{ \text{Plancherel} \} = \frac{1}{2\pi} \left\| \pi e^{-|\xi|} - \sum_{n=-N}^N c_n \chi_{[n-1/2, n+1/2]} \right\|^2$$

By thm 3.8. we have minimum if  $c_n = \langle \pi e^{-|\xi|}, \chi_{[n-1/2, n+1/2]} \rangle =$

$$= \pi \int_{n-1/2}^{n+1/2} e^{-|\xi|} d\xi$$

$$n=0, \quad c_0 = \pi \int_{-1/2}^{1/2} e^{-|\xi|} d\xi = 2\pi \int_0^{1/2} e^{-\xi} d\xi = 2\pi(1 - e^{-1/2})$$

$$n>0, \quad c_n = \pi \int_{n-1/2}^{n+1/2} e^{-|\xi|} d\xi = \left[ -\pi e^{-\xi} \right]_{n-1/2}^{n+1/2} = \\ = \pi \left( e^{-n+1/2} - e^{-n-1/2} \right)$$

$$n<0, \quad c_n = \pi \int_{n-1/2}^{n+1/2} e^{-|\xi|} d\xi = \pi(e^{n+1/2} - e^{n-1/2})$$

$$c_n = \begin{cases} \pi e^{-|n|} (e^{1/2} - e^{-1/2}), & n \neq 0 \\ 2\pi(1 - e^{-1/2}), & n = 0 \end{cases}$$

E022 Find sol.  $y(x)$  of  $y'' - y = 0$  s.t.  $\int_1^1 (1+x-y(x))^2 dx \min$

Solution:  $y(x) = a\cosh(x) + b\sinh(x)$

$$\cosh(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sinh(x) = \frac{e^{ix} - e^{-ix}}{2}$$

$$e_1(x) = \cosh x, \quad e_2(x) = \sinh x$$

$$\int_1^1 \cosh(x) \sinh(x) dx = \frac{1}{2} \int_1^1 \sinh(2x) dx = 0$$

$$\|1+x-y(x)\| = \left\| 1+x - a \frac{e_1(x)}{\|e_1(x)\|} - b \frac{e_2(x)}{\|e_2(x)\|} \right\| \text{orthonormal} \quad -\min$$

$$\text{By thm 2.8} \Rightarrow a = \frac{1}{\|e_1\|^2} \langle 1+x, e_1 \rangle$$

$$b = \underbrace{\frac{1}{\|e_2\|^2} \langle 1+x, e_2 \rangle}_{\sim\sim\sim\sim\sim\sim\sim\sim}$$

$$\|e_1\|^2 = \int_1^1 (\cosh(x))^2 dx = \frac{1}{2} \int_1^1 (\cosh(2x) + 1) dx = \\ = \frac{1}{2} \sinh(2) + 1$$

$$\|e_2\|^2 = \int_1^1 (\sinh(x))^2 dx = \frac{1}{2} \int_1^1 (\cosh(2x) - 1) dx = \frac{1}{2} \sinh(2) - 1$$

$$\langle 1+x, e_1 \rangle = \int_1^1 (1+x) \cosh(x) dx = \int_1^1 \cosh(x) dx + \underbrace{\int_1^1 x \cosh(x) dx}_{=0, \text{ } x\cosh(x) \text{ odd}} = \\ = [\sinh(x)]_1^1 = \sinh(1) - \sinh(-1) = 2\sinh(1)$$

$$\langle 1+x, e_2 \rangle = \int_1^1 (1+x) \sinh(x) dx = \underbrace{\int_1^1 \sinh(x) dx}_{=0, \text{ } \sinh(x) \text{ odd}} + \int_1^1 x \sinh(x) dx =$$

$$= [x \cosh(x)]_1^1 - \int_1^1 \cosh(x) = 2\cosh(1) - 2\sinh(1)$$

$$y(x) = \frac{2\sinh(1)}{\frac{1}{2}\sinh(2) + 1} \cosh(x) + \frac{2(\cosh(1) - \sinh(1))}{\frac{1}{2}\sinh(2) - 1} \sinh(x)$$

4.2.1  $\begin{cases} u_t = k u_{xx} & [0, l] \\ u(0, t) = 0 \\ u_x(l, t) = 0 \\ u(x, 0) = f(x) \end{cases}$

a) Ser. exp. for  $u(x, t)$

b)  $u(x, t) \ ?$  if  $f(x) = 50$

$$\text{Solution: } u(x, t) = X(x)T(t) \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0 \\ X'(l) = 0 \end{cases}$$

$$1) \lambda = 0 \Rightarrow X(x) = ax + b, \text{ BC} \Rightarrow a = b = 0 \Rightarrow u(x, t) = 0$$

$$2) \lambda < 0, \lambda = (i\mu)^2, \mu > 0 \Rightarrow X(x) = ae^{i\mu x} + be^{-i\mu x}$$

$$X(0) = 0 = a + b \Rightarrow b = -a, \quad X'(l) = i\mu a e^{i\mu l} + i\mu b e^{-i\mu l} = 0 \Rightarrow a = 0$$

$$3) \lambda > 0, \lambda = \mu^2, \mu > 0 \Rightarrow X(x) = a \cos(\mu x) + b \sin(\mu x)$$

$$X(0) = a = 0, X'(l) = b\mu \cos(\mu l) = 0 \Rightarrow \mu = \frac{(2n-1)\pi}{2l}$$

$$\lambda_k = \left(\frac{(2k-1)\pi}{2l}\right)^2, X_k(x) = b_k \sin\left(\frac{(2k-1)\pi x}{2l}\right), T_n(t) = c_n e^{-\lambda_n kt}$$

$$u(x,t) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{(2n-1)\pi x}{2l}\right) e^{-\left(\frac{(2n-1)\pi}{2l}\right)^2 kt} \quad (*)$$

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{(2n-1)\pi x}{2l}\right)$$

lets take scalar product by  $\sin\left(\frac{(2m-1)\pi x}{2l}\right)$

$$\int_0^l f(x) \sin\left(\frac{(2m-1)\pi x}{2l}\right) dx = d_m \int_0^l \sin^2\left(\frac{(2m-1)\pi x}{2l}\right) dx$$

$$\int_0^l \left(\sin\left(\frac{(2m-1)\pi x}{2l}\right)\right)^2 dx = \frac{l}{2} \Rightarrow d_m = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{(2m-1)\pi x}{2l}\right) dx$$

$$b) f(x) = 50, d_m = \frac{2}{l} \int_0^l 50 \sin\left(\frac{(2m-1)\pi x}{2l}\right) dx = \\ = \frac{100}{l} \frac{2l}{(2m-1)\pi} \left[-\cos\left(\frac{(2m-1)\pi x}{2l}\right)\right]_0^l = \frac{200}{(2m-1)\pi} \quad \text{Plug in in (*)}$$

3.5.5 Find the normalized eigenfunctions of the

$$\begin{cases} f'' + \lambda f = 0 \\ f'(0) = 0 \\ f'(l) = \beta f(l) \end{cases}$$

$$\text{Solution: } \lambda = 0, f(x) = ax + b, f'(0) = a = 0$$

$$f'(l) = \beta f(l) = 0$$

$$\beta \neq 0 \Rightarrow f(x) = b = 0$$

$$\beta = 0 \Rightarrow f(x) = b \quad \text{e.f.}$$

$$\|f\|^2 = \int_0^l b^2 dx = l b^2 = 1 \Rightarrow b = \frac{1}{\sqrt{l}} \Rightarrow f(x) = \frac{1}{\sqrt{l}}, \beta = 0$$

$$\lambda < 0, \lambda = (\imath\mu)^2, \mu > 0 \Rightarrow f(x) = ae^{\mu x} + be^{-\mu x}$$

$$f'(0) = \mu a - \mu b = 0 \Rightarrow a = b$$

$$f'(l) = \beta(ae^{\mu l} + ae^{-\mu l}) = a\mu e^{\mu l} - b\mu e^{-\mu l}$$

$$\Rightarrow \frac{e^{\mu l} - e^{-\mu l}}{e^{\mu l} + e^{-\mu l}} = \frac{\beta}{\mu}$$

$$\frac{f_l(x)}{\mu} = a(e^{\mu x} + e^{-\mu x}) \quad \text{when } \mu > 0 \text{ s.t. } \tanh(\mu l) = \frac{\beta}{\mu}, \beta > 0$$

$\tanh(\mu l)$

$$\|f\|^2 = a^2 \int_0^l (e^{\mu x} + e^{-\mu x})^2 dx = 4a^2 \int_0^l \sinh^2(\mu x) dx$$

$$= \frac{4a^2}{2} \int_0^l (\cosh(2\mu x) + 1) dx = 2a^2 \left(\frac{\sinh(2\mu l)}{2\mu} + l\right) = 1$$

$$\Rightarrow a = \left(\frac{\sinh(2\mu l)}{\mu} + 2l\right)^{-1/2}$$

$$\lambda > 0, \quad \lambda = \mu^2, \quad \mu > 0 \Rightarrow f(x) = a \cos(\mu x) + b \sin(\mu x)$$

$$f'(0) = 0 \Rightarrow b = 0$$

$$f'(l) = \beta a \cos(\mu l) = -\mu a \sin(\mu l) \Rightarrow \frac{\sin(\mu l)}{\cos(\mu l)} = -\frac{\beta}{\mu}$$

$$\underline{f_3(x) = a \cos(\mu x)} \quad \text{when } \mu > 0 \quad \text{s.t.} \quad \tan(\mu l) = -\frac{\beta}{\mu}$$

$$\|f\|^2 = a^2 \int_0^l \cos^2(\mu x) dx = a^2 \int_0^l \frac{\cos(2\mu x) + 1}{2} dx =$$

$$= \frac{a^2}{2} \left[ \frac{1}{2\mu} \sin(2\mu l) + \frac{1}{2} l \right] = 1 \Rightarrow$$

$$\Rightarrow a = \left( \frac{1}{4\mu} \sin(2\mu l) + \frac{l}{2} \right)^{-1/2}$$

$$\beta = 0 \quad f_1, f_3$$

$$\beta < 0 \quad f_3$$

$$\beta > 0 \quad f_2, f_3$$

Inhomogeneous wave equation on  $[0, l]$ 

$$\square u = F(x, t)$$

$$\text{BC: } u(0, t) = u(l, t) = 0$$

$$\text{IC: } u(x, 0) = \varphi(x)$$

$$u_t(x, 0) = g(x)$$

CASE 1:  $F$  is independent of  $t$ . Find steady state solution, that is  $f(x)$  such that  $\square f = F \Leftrightarrow \begin{cases} -f''(x) = F(x) \\ f(0) = f(l) = 0 \end{cases}$

Next, solve  $\square u = 0$

$$\text{BC: } u(0, t) = u(l, t) = 0$$

$$\text{IC: } u(x, 0) = \varphi(x) - f(x)$$

$$u_t(x, 0) = g(x) \quad \cancel{\text{BC}}$$

Then,  $f+u$  solves the problem.

CASE 2:  $F$  is not independent of  $t$ . OBS! Self-adjoint BCs.

Sep.var.  $\Rightarrow$  in homogenous wave eqn.  $X T'' - T X'' = 0$

$$\Leftrightarrow \frac{T''}{T} = \frac{X''}{X} = \text{constant} \rightarrow \text{SLP } X'' + \lambda X = 0, X(0) = X(l) = 0$$

$$\text{Solved this before } X_n(x) = \frac{\sin\left(\frac{n\pi x}{l}\right)}{\sqrt{l/2}}, \lambda^2 \text{ ONB } [0, l]$$

Expand, for each  $t$ ,  $F(x, t)$  as Fourier series in  $\{X_n\}$

$$\Rightarrow F(x, t) = \sum_{n \geq 1} c_n(t) X_n(x), \text{ look for solution of the form}$$

$$\sum_{n \geq 1} w_n(t) X_n(x) \stackrel{\text{wave eqn.}}{\square} \sum_{n \geq 1} w_n''(t) X_n(x) \equiv w_n(t) X_n''(x)$$

$$X_n''(x) = -\frac{n^2\pi^2}{l^2} X_n(x) \Rightarrow \sum_{n \geq 1} \left( w_n''(t) + w_n(t) \frac{n^2\pi^2}{l^2} \right) X_n(x) =$$

$$= F(x, t) = \sum_{n \geq 1} c_n(t) X_n(x) \Rightarrow \text{solve ODEs } w_n''(t) + \frac{n^2\pi^2}{l^2} w_n(t) = c_n(t)$$

$$\Rightarrow \text{Solution } w(x, t) = \sum_{n \geq 1} w_n(t) X_n(x) \text{ to } \square w = F$$

Next solve  $\square u = 0$

$$\text{BC: } u(0, t) = u(l, t) = 0$$

$$\text{IC: } u(x, 0) = \varphi(x) - w(x, 0)$$

$$u_t(x, 0) = g(x) - w_t(x, 0)$$

$w+u$  solves

the problem.

SPECIFIC EXAMPLES

$$1. F(x, t) = \sin x. \text{ Looking for } f \text{ to satisfy } \begin{cases} -f''(x) = \sin x \\ f(0) = f(l) = 0 \end{cases}$$

$$f(x) = \sin x + ax + b$$

$$f(0) = b \Rightarrow b = 0, 0 = f(l) = \sin l + al \Leftrightarrow a = \frac{-\sin(l)}{l}$$

$$\text{Thus steady state solution is } f(x) = \sin x - \frac{x \sin(l)}{l}$$

Next we solve the new IVP for homogenous  $\square u = 0$   
Using sep. var. in  $\square u = 0$ , found  $\{X_n\}$  already

$$\text{The } T_n(t) = a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right) X_n(x)$$

Find  $a_n$  &  $b_n$  using the ICs.

$$u(x,0) = \sum_{n=1}^{\infty} a_n X_n(x) = \psi(x) - f(x)$$

$$\text{Thus, } a_n = \langle \psi - f, X_n \rangle = \int_0^L (\psi(x) - f(x)) \overline{X_n(x)} dx = \int_0^L (\psi(x) - f(x)) X_n(x) dx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} X_n(x) = g(x)$$

$$\Rightarrow b_n \frac{n\pi}{L} = \langle g, X_n \rangle = \int_0^L g(x) \overline{X_n(x)} dx = \int_0^L g(x) X_n(x) dx$$

$$\Rightarrow b_n = \frac{L}{n\pi} \langle g, X_n \rangle$$

Then,  $u(x,t) + f(x)$  solves the problem.

### CASE 2. $F(x,t) = \sin(x+t)$

To deal with this,  $\sin(x+t) = \sin x \cos t + \cos x \sin t$   
Solve separately for each part.

$$\sin(x) \cos(t) = \sum_{n=1}^{\infty} \cos(t) \widehat{\sin}_n X_n(x) \text{ where}$$

$$\widehat{\sin}_n = \langle \sin(x), X_n(x) \rangle = \int_0^L \frac{\sin x \sin\left(\frac{n\pi x}{L}\right)}{\sqrt{L/2}} dx$$

$$\begin{aligned} * &= \int_0^L \underbrace{\sin(x)}_u \underbrace{\sin(bx)}_v dx = \cancel{\left[uv\right]_0^L} + \int_0^L \underbrace{\cos(x)}_u \underbrace{\cos(bx)}_v b dx = \\ &= \left[ \sin(x) \cos(bx) b \right]_0^L - \int_0^L b^2 \sin(x) \sin(bx) dx = \\ &= \sin(l) \cos(bl) b - b^2 * \end{aligned}$$

$$\Rightarrow * = \frac{b \sin(l) \cos(bl)}{1 + b^2}, \quad \text{in our case } b = \frac{n\pi}{L}$$

$$\text{So } \widehat{\sin}_n = \frac{\frac{n\pi}{L} \sin(l) (-1)^n}{\sqrt{L/2} \left( 1 + \frac{n^2\pi^2}{L^2} \right)}$$

Similarly, expand  $\cos(x) \sin(t) = \sum_{n=1}^{\infty} \sin(t) \widehat{\cos}_n X_n(x)$

$$\widehat{\cos}_n = \langle \cos(x), X_n(x) \rangle = \int_0^L \frac{\cos(x) \sin\left(\frac{n\pi x}{L}\right)}{\sqrt{L/2}} dx$$

$\heartsuit = \int_0^L \cos(x) \sin(bx) dx$ , IP twice, both using  $\cos(x)$  as  $u$ .

$$\heartsuit = \frac{\cos(l) \cos(bl) b - b}{1 - b^2} , \text{ for us } b = \frac{n\pi}{l}$$

$$\Rightarrow \heartsuit = \frac{(\cos(l) (-1)^n - 1) \frac{n\pi}{l}}{1 - \frac{n^2\pi^2}{l^2}} \Rightarrow \widehat{\cos}_n = \frac{\heartsuit}{\sqrt{2}}$$

$$\text{Solve } \square \sum_{n \geq 1} c_n(t) X_n(x) = \sum_{n \geq 1} \widehat{\sin}_n \cos(t) X_n(x) = F_1(x, t)$$

$$\sum_{n \geq 1} (c_n''(t) X_n(x) - c_n(t) X_n''(x)) = \sum_{n \geq 1} \left( c_n''(t) + \frac{n^2\pi^2}{l^2} c_n(t) \right) X_n(x)$$

equate coeffs.

$$c_n''(t) + \frac{n^2\pi^2}{l^2} c_n(t) = \widehat{\sin}_n \cos(t). \text{ Let } f(t) = \alpha \cos(t)$$

Solving eqn. ~~after the form~~

$$-\alpha \cos(t) + \alpha \alpha \cos(t) = b \cos(t) \Rightarrow -\alpha + \alpha \alpha = b \Rightarrow \alpha = \frac{b}{\alpha - 1}$$

$$\text{For us, } b = \widehat{\sin}_n, \alpha = \frac{n^2\pi^2}{l^2} \Rightarrow c_n(t) = \frac{\widehat{\sin}_n}{\frac{n^2\pi^2}{l^2} - 1} \cos(t)$$

solves the eqn. ↗

$$u_1(x, t) = \sum_{n \geq 1} c_n(t) X_n(x)$$

$$\square u_1 = F_1$$

$$\text{Similarly, solve } \square \sum_{n \geq 1} b_n(t) X_n(x) = F_2(x, t) = \sum_{n \geq 1} \widehat{\cos}_n \sin(t) X_n(x)$$

$$\Rightarrow b_n''(t) + \frac{n^2\pi^2}{l^2} b_n(t) = \widehat{\cos}_n \sin(t)$$

Let  $f(t) = \alpha \sin(t)$ . Then eqn is

$$-\alpha \sin(t) + \alpha \alpha \sin(t) = b \sin(t) \Rightarrow \alpha = \frac{b}{\alpha - 1}$$

$$\Rightarrow b_n(t) = \frac{\widehat{\cos}_n \sin(t)}{\frac{n^2\pi^2}{l^2} - 1}$$

$$u_2(x, t) = \sum_{n \geq 1} b_n(t) X_n(x) \text{ solves } \square u_2 = F_2.$$

$$\text{Let } u = u_1 + u_2, \quad \square u = F = F_1 + F_2$$

BC are satisfied by  $u$  because BC hold  $\forall X_n$ .

$$\text{IC: } u(x, 0) = u_1(x, 0) + \underbrace{u_2(x, 0)}_{=0} = u_1(x, 0) = \sum_{n \geq 1} c_n(0) X_n(x)$$

$$u_t(x, 0) = \underbrace{\frac{\partial}{\partial t} u_1(x, 0)}_{=0} + \frac{\partial}{\partial t} u_2(x, 0) = \sum_{n \geq 1} b_n'(0) X_n(x)$$

Finally solve  $\square w = 0$

$$\left. \begin{array}{l} BC: w(0,t) = w(L,t) = 0 \\ IC: \begin{cases} w(x,0) = \varphi(x) - u_1(x,0) \\ w_t(x,0) = g(x) - \partial_t u_2(x,0) \end{cases} \end{array} \right\}$$

SV, SLP for  $\{X_n\}$

$$T_n(t) = a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$w(x,t) = \sum_{n \geq 1} T_n(t) X_n(x)$$

$a_n$  and  $b_n$  come from IC.  $w(x,0) = \sum_{n \geq 1} a_n X_n(x)$ , thus  $a_n = \langle \varphi - u_1, X_n \rangle$

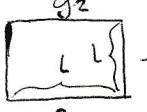
$$w_t(x,0) = \sum_{n \geq 1} b_n \frac{n\pi}{L} X_n(x) \Rightarrow$$

$$\Rightarrow \frac{b_n n\pi}{L} = \langle g - \partial_t u_2, X_n \rangle \Rightarrow b_n = \langle g - \partial_t u_2(x,0), X_n \rangle \frac{L}{n\pi}$$

Then  $u+w$  solves the problem.

{ On exam, if SLP that you've seen many times, just write the solution. If it's a problem of this type.  
Show that you know what  $\langle , \rangle$  is. }

### DIRICHLET PROBLEM ON $\square$



$$\Delta u = 0 \quad \text{inside } R = \text{Square, specified } u \text{ boundary}$$

$$\Delta u = \partial_x^2 u + \partial_y^2 u$$

We solve two problems 1.  $u(0,y) = u(l,y) = 0$

$$2. v(x,0) = v(x,l) = 0$$

$$v(0,y) = f_1(y)$$

$$v(l,y) = f_2(y)$$

$$\Delta v = 0$$

$$u(x,0) = g_1(x)$$

$$u(x,l) = g_2(x)$$

$$\Delta u = 0 \quad \text{inside } R$$

$$\text{Sep. var. } u(x,y) = X(x)Y(y)$$

$$\Delta XY = X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = \text{constant}$$

$$BC \Rightarrow X(0) = 0, X(l) = 0 \Rightarrow \text{SLP}$$

$$X'' + \lambda X = 0, X(0) = X(l) = 0$$

$$X_n(x) = \frac{\sin\left(\frac{n\pi x}{l}\right)}{\sqrt{2/2}}, \quad \frac{X_n''}{X_n} = \frac{-Y_n''}{Y_n} = -\frac{n^2 \pi^2}{l^2}$$

$$\Rightarrow Y_n'' = \frac{n^2 \pi^2}{l^2} Y_n, \quad \text{Thus } Y_n = a_n \cosh\left(\frac{n\pi x}{l}\right) + b_n \sinh\left(\frac{n\pi x}{l}\right)$$

$$u(x,y) = \sum_{n \geq 1} X_n(x)Y_n(y)$$

To find  $a_n$  and  $b_n$

$$u(x, 0) = g_1(x) = \sum_{n \geq 1} X_n(x) a_n \Rightarrow a_n = \langle g_1, X_n \rangle$$

$$u(x, l) = g_2(x) = \sum_{n \geq 1} X_n(x) (a_n \cosh(n\pi) + b_n \sinh(n\pi))$$

$$a_n \cosh(n\pi) + b_n \sinh(n\pi) = \langle g_2, X_n \rangle$$

$$\Rightarrow b_n = \frac{\langle g_2, X_n \rangle - a_n \cosh(n\pi)}{\sinh(n\pi)}$$

Next, by symmetry

$$v(x, y) = \sum_{n \geq 1} X_n(y) Y_n(x)$$

the coeffs,  $\alpha_n$  and  $\beta_n$ ,

$$\alpha_n = \langle f_1, X_n \rangle, \quad \beta_n = \frac{\langle f_2, X_n \rangle - \alpha_n \cosh(n\pi)}{\sinh(n\pi)}$$

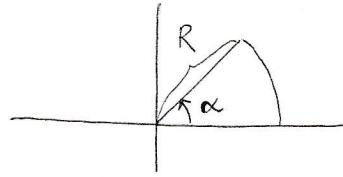
$u+v$  solves the problem.

20/2  
F16

## BESSEL FUNCTIONS ch5

Sep. vars works on product spaces

- ex. ①  $\square [0, L]_x \times \mathbb{R}_t^+$       ②  $\square [0, L]_x \times [0, w]_y$  in  $\mathbb{R}^2$   
 ③  $\square [0, \alpha]_\theta \times [0, R]_r$  polar coords, in  $\mathbb{R}^2$  looks like



It is a general fact that to solve the initial value problems

$$1. \begin{cases} \Delta u = 0 & \text{for } \vec{x} \in \Omega, \Omega \subset \mathbb{R}^n \text{ bounded domain} \\ u|_{\partial\Omega} = 0 & (\text{Dirichlet BC}) \\ u|_{t=0} = u_0(\vec{x}) \end{cases}$$

$$2. \begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = u_0(\vec{x}) \\ u_t|_{t=0} = v_0(\vec{x}) \end{cases}$$

the key is to solve SLP  $\Delta u + \lambda u = 0$ ,  $u|_{\partial\Omega} = 0$

Spectral Theorem  $\Rightarrow \exists \mathcal{L}^2$  ONB of solutions  $\{\phi_n\}$  corresponding eigenvalues  $\{\lambda_n\}$ ,  $0 < \lambda_1 \leq \lambda_2 \leq \dots$

The solution to problem 1 is  $\sum_{n \geq 1} e^{-\lambda_n t} \phi_n(\vec{x}) \hat{u}_0(n)$ ,

$$\text{where } \hat{u}_0(n) = \langle u_0, \phi_n \rangle = \int_{\Omega} u_0(x) \overline{\phi_n(x)} dx = \int_{\Omega} u_0(x) \phi_n(x) dx$$

The solution to problem 2 is  $\sum_{n \geq 1} (a_n \cos(\sqrt{\lambda_n} t) + b_n \sin(\sqrt{\lambda_n} t)) \phi_n(x)$   
 where  $a_n = \langle u_0, \phi_n \rangle$ ,  $b_n = \frac{\langle v_0, \phi_n \rangle}{\sqrt{\lambda_n}}$

Thus, solving  $\Delta u + \lambda u = 0$ ,  $u|_{\partial\Omega} = 0$  is key to solving IVPs for  $\Delta$  and  $\Delta$

$$\Delta = \partial_x^2 + \partial_y^2 = \partial_r^2 + r^{-1} \partial_r + r^{-2} \partial_\theta^2, \text{ Separate variables}$$

$$R(r)T(\theta) \Rightarrow R''T + r^{-1}R'T + r^{-2}RT'' + \lambda^2 RT = 0$$

$$r^2 R''T + rR'T + RT'' + \lambda^2 r^2 RT = 0$$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + \frac{T''}{T} + \lambda^2 r^2 = 0$$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + \lambda^2 r^2 = \underbrace{-\frac{T''}{T}}_{\text{simpler} \Rightarrow \text{ado first!}} = \text{constant } \mu^2$$

$$T'' = -\mu^2 T$$

$\theta = \alpha$

$r = R$

to have Dirichlet BC  
need  $T(0) = 0 = T(\alpha)$

Thus  $T_n(\theta) = \frac{\sin(\frac{n\pi\theta}{\alpha})}{\sqrt{\alpha/2}}$  (know from having done before)

$$\mu_n = \frac{n\pi}{\alpha}$$

Recycle into the eqn. for  $R = R_n$

$$\frac{r^2 R''}{R} + \frac{r R'}{R} - \mu_n^2 + \lambda^2 r^2 = 0$$

$$r^2 R'' + r R' - \mu_n^2 R + \lambda^2 r^2 R = 0$$

Almost Bessel's eqn. except for. To kill it,

$$f(\lambda r) = R(r), \quad x = \lambda r, \quad R'(r) = \lambda f'(x), \quad R''(r) = \lambda^2 f''(x)$$

$$\left(\frac{x}{\lambda}\right)^2 \lambda^2 f''(x) + \left(\frac{x}{\lambda}\right) \lambda f'(x) - \mu_n^2 f(x) + x^2 f(x) = 0$$

$$\underbrace{x^2 f''(x)}_{\text{purple}} + \underbrace{x f'(x)}_{\text{purple}} + \underbrace{(x^2 - \mu_n^2) f(x)}_{\text{purple}} = 0$$

This is Bessel's equation of order  $\mu_n$ .

Solve it. In general, solve Bessel eqn. of order  $v$ .

Ansatz:  $f(x) = \sum_{j \geq 0} a_j x^{j+b}$ . Sub into eqn.

$$0 = \sum_{j \geq 0} (j+b)(j+b-1) a_j x^{j+b} + \sum_{j \geq 0} a_j (j+b) x^{j+b} - \sum_{j \geq 0} v^2 a_j x^{j+b} + \sum_{j \geq 0} a_j x^{j+b+2}$$

Set the coeff's equal to 0

$$\sum_{j \geq 0} ((j+b)^2 - v^2) a_j x^{j+b} + a_j x^{j+b+2}$$

Recursive eqn. for  $a_2 \xrightarrow{\text{from}} a_0$ ,  $a_3 \xrightarrow{\text{from}} a_1$ , and so forth

$$((j+b)^2 - v^2) a_{j+2} + a_j = 0 \quad \forall j \geq 0$$

$$\Rightarrow a_{j+2} = \frac{-a_j}{(j+b)^2 - v^2} \quad \forall j \geq 0$$

Do not want both  $a_0 = 0$  and  $a_1 = 0$

Need  $x^b$  coefficient to vanish  $(b^2 - v^2) a_0 = 0$

Need  $x^{b+1}$  coeff. to vanish  $((1+b)^2 - v^2) a_1 = 0$

Convention,  $b = v$  and  $a_1 = 0$ , because it is most simple

(E.g.: What value of  $b$  solves  $(1+b)^2 - v^2 = 0$ ?)

$$\text{Claim: } a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (1+\nu)(2+\nu)\dots(k+\nu)}$$

Proof by induction

$$a_2 = \frac{-a_0}{2^2 (1!) (1+\nu)}, \quad \text{by above, } j=0, a_{2+0} = a_2$$

$$a_j = \frac{-a_{j-2}}{(j+\nu)^2 - \nu^2} = \frac{-a_{j-2}}{j(j+2\nu)}, \quad \text{set } j=2,$$

$$a_2 = \frac{-a_0}{2(2+2\nu)} = \frac{-a_0}{4(1+\nu)}$$

Assume now for  $k$ , then use  $a_j = \frac{-a_{j-2}}{j(j+2\nu)}$  to show formula for  $k+1$

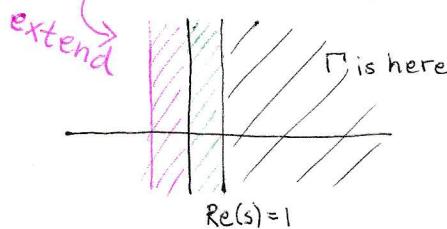
$$a_{2k+2} = \frac{-a_{2k}}{(2k+2)(2k+2+2\nu)} = \frac{-a_{2k}}{4(k+1)(k+\nu+1)}$$

$$\text{Sub formula for } a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (1+\nu)\dots(k+\nu)}$$

$$\begin{aligned} a_{2k+2} &= \frac{(-1)^{k+1} a_0}{2^2 2^{2k} (k+1)k! (k+\nu+1)(1+\nu)\dots(k+\nu)} = \\ &= \frac{(-1)^{k+1} a_0}{2^{2(k+1)} (k+1)! (1+\nu)\dots(k+1+\nu)} \quad \blacksquare \end{aligned}$$

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \quad \text{for } \operatorname{Re}(s) > 1$$

$$\Gamma(s+1) = s\Gamma(s) \quad (\text{EO: Prove this! Integrate by parts!})$$



Also  $\Gamma(1) = 1$ , Thus by

induction  $\Gamma(n+1) = n!$

Tradition (convenience): Choose  $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$

Bessel function of order  $\nu > 0$  is then defined

$$\text{as } f(x) = J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+\nu}}{n! \Gamma(n+\nu+1)} \quad \leftarrow \begin{aligned} \Gamma(k+\nu+1) &= (k+\nu) \Gamma(k+\nu) = \\ &= (k+\nu)(k-1+\nu) \cdots (1+\nu) \Gamma(1+\nu) \end{aligned}$$

Note the  $\Gamma$  has simple poles at  $-\mathbb{N} \cup \{0\}$

$\Rightarrow \frac{1}{\Gamma}$  has simple zeros.

$$\text{Thus } J_{-n}(x) = \sum_{k \geq n} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-n}}{k! \Gamma(k+n+1)}$$

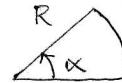
here taking order " $v = -n$ "

$$\Rightarrow J_{-n}(x) = (-1)^n J_n(x) \quad \text{for } n \in \mathbb{N}$$

Thus to have a basis of solutions to Bessel's eqn

$$2^{\text{nd}} \text{ kind of Bessel function } Y_v(x) = \frac{\cos(v\pi) J_v(x) - J_{-v}(x)}{\sin(v\pi)}$$

Then  $\{J_v, Y_v\}$  are a basis of solutions to Bessel's eqn. of order  $v$ .

Return to our problem   $\Delta u + \lambda^2 u = 0, u|_{\text{boundary}} = 0$

$$\text{Sep. vars. } u(r, \theta) = R(r)T(\theta), \text{ found } T_n(\theta) = \frac{\sin\left(\frac{n\pi\theta}{\alpha}\right)}{\sqrt{\alpha/2}}$$

Eqn. for  $R=R_n$  became Bessel eqn. of order  $\mu_n = \frac{n\pi}{\alpha}$ ,  
for  $f(x) = R(\lambda r)$ . Thus  $R_n(\lambda r) = J_{n\pi/\alpha}(\lambda r)$

What is  $\lambda$ ? Boundary conditions

This one (not  $Y_{n\pi/\alpha}$ ) because of BC at  $r=0$ ,

$$J_{n\pi/\alpha}(0) = 0, \text{ not so for } Y_{n\pi/\alpha}. J_{n\pi/\alpha}(\lambda R) = 0$$

$$\text{Choose } \lambda = \frac{\text{zero of } J_{n\pi/\alpha}}{R}$$

Fact: (comes from spectral thm) the Bessel function,  $J_v$  for  $v > 0$  has an infinite, discrete set of positive zeros.

Let  $z_{k,n}$  be the  $k^{\text{th}}$  positive zero of  $J_{n\pi/\alpha}$ .

Then  $\lambda_{n,k} = \frac{z_{k,n}}{R}$ . The eigenvalues of SLP are

$$\left\{ \frac{z_{k,n}^2}{R^2} \right\}, \text{ eigenfunctions are } \frac{J_{n\pi/\alpha}(\lambda_{n,k} r)}{\| J_{n\pi/\alpha} \|_{L^2}} \frac{\sin(\theta n\pi/\alpha)}{\sqrt{\alpha/2}}$$

$$J_{1/2}(x) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{n! \Gamma(n+\frac{1}{2}+1)} = \frac{\sqrt{2}}{\sqrt{x}} \sum_{n \geq 0} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{n! \Gamma(n+\frac{1}{2}+1)} = \sqrt{\pi} \text{ EO}$$

$$\Gamma(n+\frac{1}{2}+1) = (n+\frac{1}{2}) \Gamma(n+\frac{1}{2}) = (n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2}) \cdots \Gamma(\frac{1}{2})$$

$$2^{n+1} (n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2}) \cdots (\frac{1}{2}) = (2n+1)(2n-1) \cdots 1$$

$$2^n n! = (2n)(2n-2)(2n-4) \cdots 2$$

$$\text{Thus } J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \sum_{n \geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{\sqrt{2}}{\sqrt{\pi x}} \sin(x)$$

$$E^{\tilde{f}} : J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

20/2  
S10

3.5.10 Find eigenvalues and normalized eigenfunctions for

$$(*) (xf')' + \lambda x^{-1} f = 0, \quad f(1) = f(b) = 0, \quad b > 1$$

Expand  $g(x) = 1$  in terms of these eigenfunctions.

$$(*) xf'' + f' + \lambda x^{-1} f = 0$$

$$\{Av = \lambda v \rightsquigarrow Lf = \lambda f\}$$

$$Lf = xf'' + f', \quad Lf = -\lambda \underline{x^{-1} f} \Leftrightarrow Mf = -\lambda f \text{ in } L^2_{x^{-1}}(1, b)$$

$$\mathcal{L}w^2(1, b) = \{f : \int_1^b f^2 w(x) dx < \infty\}, \quad w(x) = x^{-1}$$

$$Mf = -\lambda f, \quad (xL f) x^{-1}, \quad M = xL f$$

$$\lambda = 0: \quad (xf')' = 0 \Rightarrow xf' = \text{const.} = c, \quad f'(x) = \frac{c}{x},$$

$$f(x) = c_1 \log(x) + c_2, \quad 0 = f(1) = c_2, \quad 0 = f(b) = c_1 \underbrace{\log(b)}_{>0} \Rightarrow c_1 = 0$$

$$\lambda \neq 0: \quad \text{Solutions are } \left\{ \begin{array}{l} f(x) = c_1 x^{r_1} + c_2 x^{r_2} \quad r_1 \neq r_2 \\ f(x) = c_1 x^{r_1} + c_2 x^{r_2} \log(x) \quad r_1 = r_2 \end{array} \right\} \text{ given in prob.}$$

$$\text{where } r_1, r_2 \text{ solve } r(r-1) + 1r + \lambda = 0 \Leftrightarrow r^2 = -\lambda$$

$$\lambda < 0: \quad \lambda = -\mu^2, \quad \mu > 0, \quad r = \pm \mu$$

$$\lambda > 0: \quad \lambda = \mu^2, \quad \mu > 0 \Rightarrow r = \pm i\mu$$

$$\lambda < 0: \quad f(x) = c_1 x^\mu + c_2 x^{-\mu}$$

$$0 = f(1) \Rightarrow c_1 = -c_2, \quad 0 = f(b) = c_1 b^\mu = c_2 b^{-\mu} \Rightarrow c_1 = c_2 = 0$$

$$\lambda > 0: \quad f(x) = c_1 x^{i\mu} + c_2 x^{-i\mu} = c_1 e^{i\mu \log x} + c_2 e^{-i\mu \log x}$$

$$0 = f(1) \Rightarrow c_1 = -c_2, \quad 0 = f(b) = c_1 (e^{i\mu \log b} - e^{-i\mu \log b}) =$$

$$= (\cos(\mu \log b) + i \sin(\mu \log b) - \cos(\mu \log b) + i \sin(\mu \log b)) c_1 =$$

$$= 2ic_1 \sin(\mu \log b)$$

$$\mu \log b = n\pi \Rightarrow \mu_n = \frac{n\pi}{\log b}$$

$$\text{Eigenfunctions: } \mu_n \rightsquigarrow f_n(x) = \overbrace{2ic_1}^{\tilde{c}_1} \sin\left(\frac{n\pi \log x}{\log b}\right)$$

$$\begin{aligned} 1 &= \|f_n\|_{L^2_{x^{-1}}(1, b)}^2 = \tilde{c}_1^2 \int_1^b \sin^2\left(\frac{n\pi \log x}{\log b}\right) x^{-1} dx = \left\{ \begin{array}{l} y = \log x \\ dy = x^{-1} dx \end{array} \right\} = \\ &= \tilde{c}_1^2 \int_0^{\log b} \sin^2\left(\frac{n\pi y}{\log b}\right) dy = \end{aligned}$$

$$= \tilde{c}_1^2 \frac{1}{2} \int_0^{\log b} \left(1 - \cos\left(\frac{n\pi y}{\log b} 2\right)\right) dy = \frac{\tilde{c}_1^2}{2} [x]_0^{\log b} =$$

$$= \frac{\tilde{c}_1^2}{2} \log b \quad \Rightarrow \tilde{c}_1 = \sqrt{\frac{2}{\log b}}$$

$$\tilde{f}_n(x) = \sqrt{\frac{2}{\log b}} \sin\left(\frac{n\pi \log x}{\log b}\right)$$

$g \in L^2(1, b; x^{-1} dx)$ ,  $\{\psi_n\}$  basis

$$g(x) = \sum_{n=1}^{\infty} \langle g, \psi_n \rangle_{L^2(x^{-1} dx)} \psi_n$$

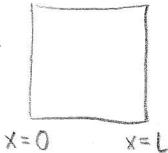
$$g(x) = 1$$

$$\langle 1, \psi_n \rangle_{L^2(x^{-1} dx)} = \int_1^b \sqrt{\frac{2}{\log b}} \sin\left(\frac{n\pi \log x}{\log b}\right) x^{-1} dx =$$

$$= \sqrt{\frac{2}{\log b}} \int_0^{\log b} \sin\left(\frac{n\pi y}{\log b}\right) dy = \sqrt{\frac{2}{\log b}} \left(\frac{-\log b}{n\pi}\right) \left[\cos\left(\frac{n\pi y}{\log b}\right)\right]_0^{\log b} =$$

$$= -\sqrt{\frac{2}{\log b}} \frac{\log b}{n\pi} (\cos(n\pi) - 1)$$

4.4.2.



$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} = 0 \\ u_x(0, y) &= 0, \quad u_x(l, y) = 0 \quad (x=0, x=l \text{ insulated}) \\ u(x, 0) &= 0, \quad u(x, l) = x \end{aligned}$$

$$u(x, y) = X(x) Y(y), \quad \Delta u = 0 \Leftrightarrow X''(x) Y(y) + X(x) Y''(y) = 0$$

$$\begin{cases} X'' + \nu^2 X = 0 \\ Y'' - \nu^2 Y = 0 \end{cases}, \quad \nu \text{ variable of separation.}$$

$$0 = u_x(0, y) = X'(0) Y(y) \Rightarrow X'(0) = 0 = X'(l)$$

$$0 = u_x(l, y) = X'(l) Y(y) \Rightarrow X'(l) = 0$$

$$0 = u(x, 0) = X(x) Y(0) \Rightarrow Y(0) = 0$$

$$X = u(x, l) = X(x) Y(l)$$

$$1) \nu^2 > 0 : \quad X(x) = A \cos(\nu x) + B \sin(\nu x)$$

$$0 = X'(0) = \nu B = 0 \Rightarrow B = 0$$

$$0 = X'(l) \Rightarrow 0 = -A \nu \sin(\nu l) \Rightarrow \nu_l = \frac{n\pi}{l}$$

$$2) \nu^2 < 0 : \quad \nu = i\mu$$

$$2) \nu^2 > 0 : \quad Y(y) = C \cosh(\nu y) + D \sinh(\nu y)$$

$$Y(y) = D \sinh(\nu y)$$

$$3) \nu = 0 : \quad X''(x) = 0, \quad Y''(y) = 0$$

$$X(x) = ax + b, \quad Y(y) = cy + d$$

$$0 = X'(0) \Rightarrow a = 0, \quad X(x) = b$$

$$0 = Y(0) \Rightarrow Y = cy$$

$$u(x, y) = \tilde{c}y + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi}{l}x\right) \sinh\left(\frac{n\pi}{l}y\right)$$

$$x = u(x, l) = \tilde{c}l + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi}{l}x\right) \sinh(n\pi)$$

$$u = \sum_{n=0}^{\infty} \frac{\langle u, v_n \rangle}{\|v_n\|^2} v_n$$

$$\begin{bmatrix} v_n \perp v_m & m \neq n \\ \left\{ \frac{v_n}{\|v_n\|} \right\} \text{ orthonormal} \end{bmatrix}$$

$$\tilde{c}l = \frac{1}{\|1\|^2} \int_0^l x = \frac{1}{l} - \frac{1}{2} l^2 = \frac{1}{2} l \Rightarrow \tilde{c} = \frac{1}{2}$$

$$\text{At } y = l, \quad v_n = \cos\left(\frac{n\pi}{l}x\right)$$

$$\underbrace{E_n \sinh(n\pi)}_{\text{constant}} = \frac{1}{\|v_n\|^2} \int_0^l x \cos\left(\frac{n\pi}{l}x\right) dx =$$

$$= \|v_n\|^{-2} \left( \frac{l}{n\pi} \left[ x \sin\left(\frac{n\pi}{l}x\right) \right] - \frac{l}{n\pi} \int_0^l \sin\left(\frac{n\pi}{l}x\right) dx \right) =$$

$$= \|v_n\|^{-2} \left( \frac{l}{n\pi} \left[ \cos\left(\frac{n\pi}{l}x\right) \right]_0^l \right) =$$

$$= \|v_n\|^{-2} \left( \frac{l}{n\pi} \right)^2 ((-1)^n - 1)$$

$$\|v_n\|^2 = \int_0^l \left( \cos^2\left(\frac{n\pi}{l}x\right) \right) dx = \frac{1}{2} \int_0^l \cos\left(\frac{2n\pi}{l}x\right) dx + 1 =$$

$$= \frac{1}{2} \int_0^l \cos\left(\frac{2n\pi}{l}x\right) dx + \frac{1}{2} \int_0^l 1 dx = \frac{l}{2}$$

$$E_n = \begin{cases} 0, & n \text{ even} \\ -\frac{4l}{\sinh(n\pi)n^2\pi^2}, & n \text{ odd} \end{cases}$$

4.2.4  $\begin{cases} u_t = k u_{xx} + R & \text{inhomogeneous heat eqn. } R-\text{const. rate} \\ \text{on } [0, l] \begin{cases} u(t, 0) = 0 \\ u(t, l) = 0 \\ u(0, x) = f(x) \end{cases} \end{cases}$

Steady state :  $k u_{xx} + R = 0 \quad u'_o(l) = 0, \quad u_o(0) = 0$

$$u(x, t) = u_o(x) + v(x, t)$$

$$u_t(x, t) = v_t(x, t), \quad u_{xx}(x, t) = u_o''(x) + v_{xx}(x, t) = -\frac{R}{k} + v_{xx}(x, t)$$

$$u_t - k u_{xx} = R - (k v_{xx}(x, t) - v_t(x, t))$$

$$u(t, 0) = u_o(0) + v(t, 0) \Rightarrow v(t, 0) = 0$$

$$u_x(t, l) = u'_o(l) + v_x(t, l) \Rightarrow v_x(t, l) = 0$$

$$f(x) = u(0, x) = u_0(x) + v(0, x), \quad v(0, x) = f(x) - u_0(x)$$

Solve: ①  $ku_0'' + R = 0$ ,  $u_0'(l) = 0$ ,  $u_0(0) = 0$

Steady state / homogenous  
heat eqn.

②  $v_t = kv_{xx}$ ,  $v(t, 0) = 0$ ,  $v_x(0, l) = 0$

$$v(0, x) = f(x) - u_0(x)$$

Sep. vars... different  $\lambda$ 's ...

22/2  
F17

- Recurrence Formulae:
1.  $(x^{-v} J_v)' = -x^{-v} J_{v+1}$
  2.  $(x^v J_v)' = x^v J_{v-1}$
  3.  $x J_v' = v J_v = -x J_{v+1}$
  4.  $x J_v' + v J_v = x J_{v-1}$
  5.  $x J_{v-1} + x J_{v+1} = 2v J_v$
  6.  $J_{v-1} - J_{v+1} = 2 J_v$

Proof 1. by defn of  $J_v$ :  $J_v(x) x^{-v} = \sum_{n \geq 0} \frac{(-1)^n x^{2n} / 2^{2n+v}}{n! \Gamma(n+v+1)}$

Differentiate termwise  $\Rightarrow (J_v x^{-v})' = \sum_{n \geq 0} \frac{(-1)^n 2^n x^{2n-1} / 2^{2n+v}}{n! \Gamma(n+v+1)} =$   
 $= \sum_{n \geq 1} \frac{(-1)^n 2^n x^{2n-1} / 2^{2n+v}}{n! \Gamma(n+v+1)} = \sum_{n \geq 1} \frac{(-1)^n x^{2n-1} / 2^{2n+v-1}}{(n-1)! \Gamma(n+v+1)}$

Change index, let  $m = n-1$ ,  $m+1 = n$

$$\begin{aligned} &\Rightarrow \sum_{m \geq 0} \frac{(-1)^{m+1} x^{2(m+1)-1} / 2^{2(m+1)+v-1}}{m! \Gamma(m+1+v+1)} = \\ &= \sum_{m \geq 0} \frac{(-1)(-1)^m x^{2m+1} / 2^{2m+v+1}}{m! \Gamma(m+v+1+1)} = \\ &= -x^{-v} \sum_{m \geq 0} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+v+1}}{m! \Gamma(m+(v+1)+1)} = -x^{-v} J_{v+1}(x) \end{aligned}$$

2. Can be done in the same way, Eō!

3. Return to 1 write differently, use product rule on left.

$$(x^{-v} J_v)' = -v x^{-v-1} J_v + x^{-v} J_v' = -x^{-v} J_{v+1}, \text{ multi by } x^{v+1}$$

$$\Rightarrow -v J_v + x J_v' = -x J_{v+1}$$

4. Can be done analogously, write out the left side of 2 using the product rule. Then use 2 to equate with right side. Eō.

5. Subtract 3 from 4.

$$\begin{aligned} &x J_v' + v J_v = x J_{v-1} \\ &-(x J_v' - v J_v) = -x J_{v+1} \\ &2v J_v = x (J_{v-1} + J_{v+1}) \end{aligned}$$

6. Add 3 and 4. Eō!

## IHM Generating function for $J_n(x)$

$\forall x$ , and all  $z \in \mathbb{C} \setminus \{0\}$

$$\left\{ \sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})} \right\}$$

Proof Expand  $e^{xz/2}$  and  $e^{-x/2z}$  in power series.

$$J_n(x) = \sum_{k \geq 0} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k! \Gamma(k+n+1)} \Rightarrow \text{Need two series on the right}$$

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!}, \quad e^{-x/2z} = \sum_{k \geq 0} \frac{(-1)^k \left(\frac{x}{2z}\right)^k}{k!}$$

OBS - both converge absolutely and uniformly on compact sets for  $x \in \mathbb{C}$ ,  $z \in \mathbb{C} \setminus \{0\}$

$$\Rightarrow \text{Multiply } e^{xz/2} e^{-x/2z} = e^{x(z - \frac{1}{z})/2} = \\ = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k \geq 0} \frac{(-1)^k \left(\frac{x}{2z}\right)^k}{k!} =$$

$$= \sum_{\substack{j \geq 0 \\ k \geq 0}} \frac{(-1)^k \left(\frac{x}{2}\right)^{j+k} z^{j-k}}{j! k!} \quad \begin{array}{l} \text{OBS - this ranges from} \\ -\infty \text{ to } \infty \end{array}$$

Let  $n = j - k$ ,  $n + 2k = j + k$   
Keep  $k$

$$\Rightarrow e^{\frac{x}{2}(z - \frac{1}{z})} = \sum_{\substack{n=-\infty \\ k \geq 0}}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k} z^n}{k! (n+k)!} = \left\{ (n+k)! = \Gamma(n+k+1) \right\} =$$

$$= \sum_{n=-\infty}^{\infty} z^n \underbrace{\sum_{k \geq 0} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k! \Gamma(k+n+1)}}_{\text{defn. of } J_n(x)}$$

$$\Rightarrow e^{\frac{x}{2}(z - \frac{1}{z})} = \sum_{n=-\infty}^{\infty} z^n J_n(x) \quad \blacksquare$$

Let  $z = e^{i\theta}$  for  $\theta \in \mathbb{R}$ .

$$\text{Then } e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta}$$

This looks rather like a Fourier series!

For each fixed  $x$ ,  $e^{i\theta}$  is  $2\pi$  periodic

Thus  $e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}$  is also  $2\pi$  periodic  $\Rightarrow$

$\Rightarrow$  Can expand in Fourier series, with  $e^{inx}$

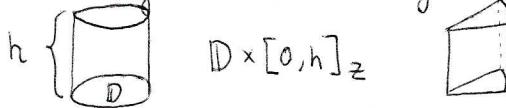
The Fourier coefficients are  $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\frac{x}{2}(2isn\theta)} e^{-inx} d\theta$

By uniqueness of these coeffs,

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\sin\theta} e^{-inx} d\theta = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)) d\theta = \\ &\quad 0, \text{ sin odd} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta) d\theta = \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta - n\theta) d\theta \end{aligned}$$

APPLICATIONS ① Solving PDEs on parts of disks 

② Solving PDEs on cylinder or wedge type regions in  $\mathbb{R}^3$



The reason is that using polar coordinates, we can separate variables  $(r, \theta)$  in  $\mathbb{R}^2$ .

Solving PDEs involving  $\Delta$  on  $\mathbb{R}^2$  or  $\mathbb{R}^3 \rightarrow$  the  $r$  part will turn into a Bessel equation.

Ex:  $\Delta u + \lambda u = 0$  on  $[0, R_0] \times [0, \alpha]$ ,  $u|_{\text{boundary}} = 0$

Solutions are  $J_n \pi / \alpha (z_{n,k} r / R_0) \sin(n\pi \theta / \alpha)$ ,  $\lambda_{n,k} = \frac{z_{n,k}^2}{R_0^2}$

The  $z_{n,k}$  is the  $k^{\text{th}}$  positive zero (root) of  $J_n \pi / \alpha$

THM 5.3 1. Let  $z_{n,k}$  be the  $k^{\text{th}}$  positive zero of

$$\begin{aligned} J_{\mu_n} \text{ for some } \mu_n \geq 0. \text{ Then } \int_0^{R_0} J_{\mu_n} \left( \frac{z_{n,k} r}{R_0} \right)^2 r dr = \\ = \frac{R_0^2}{2} J_{\mu_n+1}^2(z_{n,k}) \end{aligned}$$

Moreover  $\left\{ \frac{J_{\mu_n} (z_{n,k} r / R_0)}{\sqrt{\frac{R_0}{2}} J_{\mu_n+1} (z_{n,k})} \right\}_{k \geq 1}$  is an  $L^2$  ONB on  $[0, R_0]$  with respect to  $r dr$ .

2. Let  $w_{n,k}$  be the  $k^{\text{th}}$  positive zero (root) of

$c J_{\mu_n}(x) + x J_{\mu_n}'(x)$  then  $\{J_{\mu_n}(w_{n,k} x / R_0)\}_{k \geq 1}$  is an orthogonal basis for  $L^2([0, R_0])$  with respect to  $r dr$ .

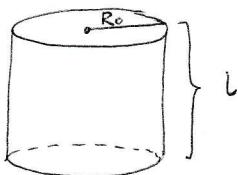
$$\text{Moreover } \int_0^{R_o} J_{\mu_n} (W_{n,k} r / R_o)^2 r dr = \frac{R_o^2}{2 W_{n,k}^2} (W_{n,k}^2 - c^2 \mu_n^2) J_{\mu_n}^2 (W_{n,k})$$

Here and above  $c$  is a constant with  $c \geq -\mu_n$   
(look up in Folland)

APPLICATIONS: Dirichlet problem in a cylinder,  $D$

$$\begin{aligned}\Delta u &= 0 \text{ in } D \\ u(r, \theta, 0) &= 0 = u(R_o, \theta, z) \\ u(r, \theta, l) &= g(r)\end{aligned}$$

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$



Use cylindrical coords  $(r, \theta, z)$

$$\Delta = \Delta_{r,\theta} + \Delta_z^2$$

Separate variables  $R(r) \phi(\theta) Z(z)$

$\Rightarrow \Delta (R\phi Z) = 0$  becomes

$$R'' \phi Z + r^{-1} R' \phi Z + r^{-2} \phi'' R Z + R \phi Z'' = 0 \quad (\div \text{ by } R \phi Z)$$

$$\underbrace{\frac{R''}{R} + r^{-1} \frac{R'}{R}}_{\Delta_{r,\theta}(R\phi)} + r^{-2} \frac{\phi''}{\phi} + \frac{Z''}{Z} = 0 \quad \Rightarrow \text{Both sides equal to constant.} = -\Lambda^2$$

$$\frac{\Delta_{r,\theta}(R\phi)}{R\phi} = -\Lambda^2$$

$$\Rightarrow \Delta_{r,\theta}(R\phi) = -\Lambda^2 R\phi \Rightarrow \Delta_{r,\theta}(R\phi) + \Lambda^2 R\phi = 0$$

In solving  $\Delta_{r,\theta}(R\phi) + \Lambda^2 R\phi = 0$ , the  $\phi$ 's also

satisfy  $\phi'' + \mu^2 \phi = 0$ .  $\phi$  must be  $2\pi$  periodic.

$\Rightarrow \phi_n(\theta) = e^{in\theta}$ . Moreover  $\mu_n^2 = n^2 \Rightarrow$  the eqn. for

$R$  is Bessel eqn. of order  $n$ , the solution

$J(\lambda_{n,k} r / R_o)$  such that the boundary conditions are satisfied.

Need  ~~$\lambda_{n,k}$~~   $J(\lambda_{n,k}) = 0 \Rightarrow \lambda_{n,k} = k^{\text{th}}$  root of  $J_n$

$$\Lambda_{n,k}^2 = \frac{\lambda_{n,k}^2}{R_o^2}, \quad \frac{-Z''}{Z} = -\Lambda_{n,k}^2$$

$$\Rightarrow Z_{n,k}(z) = a_{n,k} \cosh(\Lambda_{n,k} z) + b_{n,k} \sinh(\Lambda_{n,k} z)$$

$$Z_{n,k}(0) = 0 \Rightarrow a_{n,k} = 0$$

General solution is  $\sum_{\substack{n \in \mathbb{Z} \\ k \geq 1}} J_{n,k}(\lambda_{n,k} r / R_o) e^{in\theta} b_{n,k} \sinh\left(\frac{\lambda_{n,k}}{R_o} z\right)$

Due to the BC  $u(r, \theta, l) = g(r)$   $\Rightarrow$  indep. of  $\theta$   
 $\Rightarrow$   $\theta$  indep. solution

Let  $n=0$ . By thm (5.3)  $\{J_0(\lambda_{0,k}r/R_0)\}$  is an OB

Can expand  $g(r)$  in a Fourier-Bessel series with these

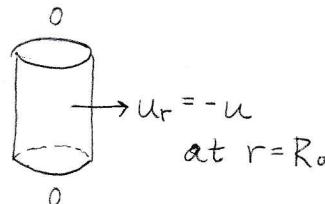
$$g(r) = \sum_{k \geq 1} J_0(\lambda_{0,k}r/R_0) \hat{g}_k, \quad \hat{g}_k = \frac{\int_0^{R_0} g(r) J_0(\lambda_{0,k}r/R_0) r dr}{\frac{R_0^2}{2} J_1(\lambda_{0,k})^2}$$

Solution  $\sum_{k \geq 1} J_0\left(\frac{\lambda_{0,k}r}{R_0}\right) \sinh\left(\frac{\lambda_{0,k}z}{R_0}\right) b_{0,k}$

$$b_{0,k} = \frac{\hat{g}_k}{\sinh\left(\frac{\lambda_{0,k}l}{R_0}\right)}$$


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Heat eqn. in a cylinder

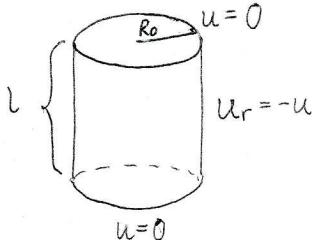


$$\nabla u = 0, \quad u|_{t=0} = g(r, \theta, z)$$

Try to solve in a similar way. To be continued...

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Ex.



$$u|_{t=0} = g(r, \theta, z) \quad \nabla u = 0$$

Sep. vars. in the heat eqn.  $u = R\phi Z T$

$$\nabla u = -\Delta_{r,\theta,z} u + \partial_t u = 0 \Rightarrow \frac{-\Delta_{r,\theta}(R\phi)}{R\phi} - \frac{Z''}{Z} + \frac{T'}{T} = 0$$

Thus both sides must be constant.  $\Delta^2$

$$-\frac{T'}{T} = \Delta^2 \text{ . Also } \frac{-\Delta_{r,\theta}(R\phi)}{R\phi} = \frac{Z''}{Z} + \Delta^2 \text{ , Also constant.}$$

BC  $\Rightarrow Z(0) = Z(L)$  , Also  $\frac{Z''}{Z} = -\mu^2$  , we have solved this before

$$Z_n(z) = \sin\left(\frac{n\pi z}{L}\right)/\sqrt{\mu_n} \text{ and } \mu_n = \frac{n\pi}{L}, \mu_n^2 = \frac{n^2\pi^2}{L^2}$$

Use this to solve for  $R, \phi$

The function  $\phi$  must be  $2\pi$  periodic because  $\theta$  and  $\theta + 2\pi$  are the same angular position on the cylinder.

$$\frac{-\Delta_{r,\theta}(R\phi)}{R\phi} = -\mu_n^2 + \Delta^2 = \frac{\lambda^2}{R_0^2} \Leftrightarrow -\Delta_{r,\theta}(R\phi) = \frac{\lambda^2}{R_0^2} R\phi$$

$$\Leftrightarrow \Delta_{r,\theta}(R\phi) + \frac{\lambda^2}{R_0^2} R\phi = 0 \quad \text{SLP on a disk}$$

$$\Rightarrow \frac{\phi''}{\phi} = \text{constant} \Rightarrow \phi_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi}} \Rightarrow \text{Bessel's eqn of order } m$$

Solution is  $J_m(\lambda r/R_0)$  , choose  $\lambda$  so that BC is satisfied.

This is for  $r=R_0$

$$J_m(\lambda) = -\frac{\lambda}{R_0} J_m'(\lambda) \Leftrightarrow R_0 J_m(\lambda) + \lambda J_m'(\lambda) = 0$$

By thm 5.3 , there are solutions  $\lambda_{m,k}$  ,  $k \in \mathbb{N}$

and  $\left\{ J_m\left(\frac{\lambda_{m,k} r}{R_0}\right) \right\}_{k \geq 1}$  are an orthogonal basis for  $L^2([0, R_0])$  with  $r dr$ .

$$\text{Let } R_{m,k}(r) = \frac{J_m\left(\frac{\lambda_{m,k} r}{R_0}\right)}{\left(\int_0^{R_0} J_m^2\left(\frac{\lambda_{m,k} r}{R_0}\right) r dr\right)^{1/2}} \quad \text{look up in thm 5.3.}$$

$$\text{Now, } -\mu_n^2 + \Delta^2 = \frac{\lambda_{m,k}^2}{R_0^2} \Rightarrow \Delta^2 = \mu_n^2 + \frac{\lambda_{m,k}^2}{R_0^2} = \frac{n^2\pi^2}{L^2} + \frac{\lambda_{m,k}^2}{R_0^2}$$

$$-\frac{T'}{T} = \Delta_{n,m,k}^2 \Rightarrow T_{n,m,k}(t) = e^{-\Delta_{n,m,k}^2 t}$$

↑  
up to constant

$$u(r, \theta, z, t) = \sum_{n,m,k} T_{n,m,k}(t) R_{m,k}(r) \Phi_m(\theta) Z_n(z) a_{n,m,k}$$

Set  $t=0 \Rightarrow T_{n,m,k}(0)=1 \quad \forall n,m,k$

$$g(r, \theta, z) = \sum_{n,m,k} a_{n,m,k} R_{m,k}(r) Z_n(z) \Phi_m(\theta) \Rightarrow$$

$$\Rightarrow a_{nmk} = \int_0^{R_0} \int_0^{2\pi} \int_0^l \overline{R_{m,k}(r)} g(r, \theta, z) \overline{Z_n(z)} \overline{\Phi_m(\theta)} r dr d\theta d\theta$$

From theory of SLPs we can expand  $g$  in such a series.

$$g(r, \theta, z) = \sum_{n,m,k} \underbrace{\langle g, R_{m,k} Z_n \Phi_m \rangle}_{a_{nmk}} R_{m,k} Z_n \Phi_m$$

## SPECIAL POLYNOMIALS ch6

Fact: Polynomials are dense in  $L^2([a,b])$

This means  $\forall f \in L^2([a,b]) \exists \{p_n\}_{n \geq 1}$  polynomials s.t.  $\lim_{n \rightarrow \infty} \|f - p_n\|_{L^2([a,b])} = 0$

Same holds for  $L_w^2([a,b])$ ,  $w=w(x) > 0$

Lemma 6.1.  $\{p_n\}_{n \geq 0}$ ,  $p_0 \neq 0$ , polynomials of degree exactly  $n$ . Then  $\forall$  polynomial  $q$  of degree  $k$ ,  $\exists \{c_j\}_{j=0}^k$  constants ( $\mathbb{C}$ ) such that  $q(x) = \sum_{j=0}^k c_j p_j(x)$

Proof By induction on the degree of  $q$ . If  $q$  is degree 0.  $\Rightarrow q = q_0 \in \mathbb{C} \Rightarrow q = q_0 = \left(\frac{q_0}{p_0}\right)p_0$  true

Assume holds for some  $k$ .

Let  $q(x)$  be degree  $k+1$ . Thus  $q(x) = \underbrace{q_{k+1} x^{k+1}}_{\in \mathbb{C} \setminus \{0\}} + \tilde{q}(x)$

$\tilde{q}(x)$  is a polynomial of degree  $k$ .

$\tilde{q}(x) = \sum_{j=0}^k c_j p_j(x)$  by induction

$p_{k+1}(x)$  is degree  $k+1 \Rightarrow p_{k+1}(x) = a_{k+1} x^{k+1} + \underbrace{\tilde{p}(x)}_{\text{polynomial of degree } k}$

$$q(x) = \frac{q_{k+1}}{a_{k+1}} p_{k+1}(x) - \frac{q_{k+1}}{a_{k+1}} \tilde{p}(x) + \tilde{q}(x)$$

$$\text{By induction } \tilde{p}(x) = \sum_{j=0}^k b_j p_j(x)$$

$$\Rightarrow q(x) = \frac{q_{k+1}}{a_{k+1}} p_{k+1}(x) + \sum_{j=0}^k \left( \frac{-q_{k+1} b_j}{a_{k+1}} + c_j \right) p_j(x)$$

□

IHM Assume  $\{p_n\}_{n \geq 0}$  are  $L^2$  ON with  $w(x) > 0$  on  $[a, b]$

Then  $\forall f \in L_w^2([a, b])$ ,  $f = \sum_{n \geq 0} \hat{f}_n p_n$ ,  $\hat{f}_n = \langle f, p_n \rangle_{L_w^2([a, b])}$   
( $p_n$ -poly degree  $n$ )

Proof Enough to show that if  $\langle f, p_n \rangle_{L^2} = 0 \quad \forall n \Rightarrow f = 0$

By the fact (thm) that polynomials are dense in  $L_w^2$

$\exists \{q_m\}_{m \geq 0}$  polys. degree  $m$  such that  $\|f - q_m\| \xrightarrow{m \rightarrow \infty} 0$

By the lemma,  $q_m = \sum_{j=0}^m c_j p_j$

$$\|f\|^2 = \langle f, f \rangle = \langle f - q_m + q_m, f \rangle = \langle f - q_m, f \rangle + \langle q_m, f \rangle =$$

$$\langle q_m, f \rangle = \left\langle \sum_{j=0}^m c_j p_j, f \right\rangle = \sum_{j=0}^m c_j \langle p_j, f \rangle = 0$$

$$|\|f\|^2| = |\langle f - q_m, f \rangle| \stackrel{\substack{\uparrow \\ \text{Cauchy Schwarz}}}{\leq} \|f - q_m\| \|f\| \xrightarrow{m \rightarrow \infty} 0$$

$$\Rightarrow \|f\|^2 = 0 \Rightarrow \|f\| = 0 \Rightarrow f = 0 \blacksquare$$

---

EÖ Prove that, given  $w > 0$  on  $[a, b]$ ,  $\exists!$  polys  $p_n$   
(unique)  
degree  $n$ , which  $L_w^2$  ON.

Hint: by induction (and linear algebra)

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### DEF LEGENDRE POLYNOMIALS

$$n^{\text{th}} \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

IHM 6.1.  $P_n$  are orthogonal in  $L^2(-1, 1)$

$\|P_n\|_{L^2(-1, 1)}^2 = \frac{2}{2n+1}$ . Thus  $\left\{ \frac{P_n}{\sqrt{\frac{2}{2n+1}}} \right\}_{n \geq 0}$  is an  $L^2$  ONB  
for  $(-1, 1)$  ( $[-1, 1]$ ).

Proof Assume  $n > m$ ,  $m \geq 0 \Rightarrow n \geq 1$

$$\begin{aligned} \langle P_n, P_m \rangle &= \int_{-1}^1 P_n(x) \overline{P_m(x)} dx = \int_{-1}^1 P_n(x) P_m(x) dx = \\ &= \frac{1}{2^n n!} \int_{-1}^1 \frac{d^n}{dx^n} (x^2 - 1)^n P_m(x) dx \end{aligned}$$

Idea: integrate by parts  $n$  times.

Do I.P. one time

$$\langle P_n, P_m \rangle = \frac{1}{2^n n!} \left( \left[ \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n P_m(x) \right]_{-1}^1 - \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n P_m(x) dx \right)$$

$(x-1)^n (x+1)^n$  vanishes to order  $n$  at  $\pm 1$

$$\Rightarrow \left[ \frac{d^{n-j}}{dx^{n-j}} ((x^2 - 1)^n) \right]_{x=-1}^1 = 0 \quad \text{for } j=1, \dots, n$$

$\Rightarrow$  each time do IP, the boundary term = 0

$$\text{Thus } \langle P_n, P_m \rangle = \frac{(-1)^n}{2^n n!} \int_{-1}^1 (x^2 - 1)^n P_m^{(n)}(x) dx$$

because degree of  $P_m = m < n$ .

$$\text{Similarly, compute } \langle P_n, P_n \rangle = \frac{(-1)^n}{2^n n!} \int_{-1}^1 (x^2 - 1)^n P_n^{(n)}(x) dx$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^{2k} \right)$$

$$\frac{d^n}{dx^n} P_n(x) = \frac{1}{2^n n!} \frac{d^{2n}}{dx^{2n}} (x^{2n}) = \frac{(2n)!}{2^n n!}$$

$$\Rightarrow \langle P_n, P_n \rangle = \frac{(2n)! (-1)^n}{(2^n n!)^2} \int_{-1}^1 (x^2 - 1)^n dx =$$

$$= \frac{2(2n)!}{(2^n n!)^2} \int_0^1 (1-x^2)^n dx \stackrel{\uparrow \text{Beta integral}}{=} 2 \frac{(2n)!}{(2^n n!)^2} \frac{\Gamma(n+1) \Gamma(\frac{1}{2})}{\Gamma(n + \frac{3}{2})} =$$

$$= 2 \frac{(2n)! n! \sqrt{\pi}}{(2^n n!)^2 (n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2}) \dots \frac{1}{2} \sqrt{\pi}} \stackrel{\uparrow \Gamma(\frac{1}{2})}{=} =$$

$$= 2 \frac{(2n)!}{2^n (n+\frac{1}{2})(n-\frac{1}{2}) \dots (\frac{1}{2}) 2^n n!} =$$

$$= \frac{(2n)! 2}{(2n+1)(2n)!} = \frac{2}{2n+1} \quad \blacksquare$$

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SII

$$4.2.5 \quad u_t = ku_{xx} + e^{-2t} \sin x \quad x \in (0, \pi) \\ u(x, 0) = u(0, t) = u(\pi, t) = 0 \quad t > 0$$

$$\text{Homogenous eqn. } \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

$$1) \lambda = 0, \quad X(x) = Ax + B, \quad 0 = X(0) = B, \quad 0 = X(\pi) = A\pi$$

$$2) -\lambda^2 > 0 \Rightarrow \lambda = i\mu \text{ for } \mu > 0$$

$$X(x) = A \cosh(\mu x) + B \sinh(\mu x) \\ 0 = X(0) = A \cosh(0) \Rightarrow A = 0, \quad 0 = X(\pi) = B \sinh(\mu\pi) \Rightarrow B = 0$$

$$3) -\lambda^2 < 0 \Rightarrow X(x) = \cos(\mu x) + B \sin(\mu x)$$

$$0 = X(0) \Rightarrow A = 0$$

$$0 = X(\pi) \Rightarrow B \sin(\mu\pi) = 0, \quad \mu_n = 1, 2, \dots$$

$$u^+(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$$

$$\sum_{n=1}^{\infty} b_n'(t) \sin(nx) = \sum_{n=1}^{\infty} -n^2 k b_n(t) \sin(nx) + e^{-2t} \sin(x)$$

$$b_n'(t) = \begin{cases} -kn^2 b_n(t), & n \neq 1 \\ -kn^2 b_1(t) + e^{-2t}, & n = 1 \end{cases}$$

$$0 = u(x, 0) = \sum_{n=1}^{\infty} b_n(0) \sin(nx) \Rightarrow b_n(0) = 0$$

$$n \neq 1, \quad b_n(t) = c_n e^{-kn^2 t}, \quad b_n(0) = 0 \Rightarrow c_n = 0$$

$$n = 1, \quad b_1'(t) e^{kt} + k e^{kt} b_1(t) = e^{-2t+kt}$$

$$\frac{d}{dt} (b_1(t) e^{kt}) = e^{t(k-2)}$$

$$e^{kt} b_1(t) = \begin{cases} \frac{e^{(k-2)t}}{k-2} + c & k \neq 2 \\ t + d & k = 2 \end{cases}$$

$$b_1(0) = 0 \Rightarrow c = \frac{-1}{k-2}, \quad d = 0$$

$$u(x, t) = \begin{cases} \frac{e^{-2t} - e^{-kt}}{k-2} \sin x & k \neq 2 \\ t e^{-kt} \sin x & k = 2 \end{cases}$$

E026 Solve eq  $\begin{cases} \sqrt{1+t} u_{xx} = u_t \\ u(0,t) = 1 \\ u(1,t) = 0 \\ u(x,0) = 1-x^2 \end{cases}$   $x \in (0,1), t > 0$

Solution:  $v(x,t) = u(x,t) + ax + b$   
 $v(0,t) = u(0,t) + 0 + b = 1 + b = 0 \quad \text{if } b = -1$   
 $v(1,t) = u(1,t) + a + b = a + b = 0 \quad \text{if } a = -b = 1$

$$\begin{cases} \sqrt{1+t} v_{xx} = v_t \\ v(0,t) = 0 \\ v(1,t) = 0 \\ v(x,0) = u(x,0) + x - 1 = x - x^2 \end{cases}$$

$v(x,t) = X(x) T(t) \Rightarrow \frac{X''}{X} = \frac{T'}{T} \frac{1}{\sqrt{1+t}} = -\lambda$

$\lambda = 0 \Rightarrow X(x) = ax + b \stackrel{BC}{\Rightarrow} a = b = 0, X(x) = 0$

$\lambda < 0, \lambda = (i\mu)^2 \Rightarrow X(x) = ae^{i\mu x} + be^{-i\mu x} \stackrel{BC}{\Rightarrow} a = b = 0, X(x) = 0$

$\lambda > 0, \lambda = \mu^2, \Rightarrow X(x) = a \cos(\mu x) + b \sin(\mu x)$

$X(0) = 0 \Rightarrow a = 0$

$X(1) = 0 \Rightarrow b \sin(\mu) = 0 \Rightarrow \mu = n\pi, n \in \mathbb{N}$

$\lambda_n = (n\pi)^2, X_n(x) = b_n \sin(n\pi x)$

$\frac{T'}{T} = -\lambda \sqrt{1+t} \Rightarrow (\ln T)' = -\lambda \sqrt{1+t} \Rightarrow \ln T = -\lambda \frac{2}{3} (1+t)^{3/2} + C$

$\Rightarrow T(t) = C e^{-\lambda \frac{2}{3} (1+t)^{3/2}}$

$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi)^2 \frac{2}{3} (1+t)^{3/2}} \sin(n\pi x)$

$v(x,0) = \sum_{n=1}^{\infty} \underbrace{c_n e^{-(n\pi)^2 \frac{2}{3}}}_{d_n} \sin(n\pi x) = x - x^2$

$d_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx =$

$= -\frac{2}{n\pi} \left[ x(1-x) \cos(n\pi x) \right]_0^1 + \frac{2}{n\pi} \int_0^1 (1-2x) \cos(n\pi x) dx =$

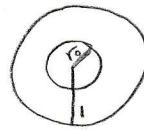
$= \frac{2}{(n\pi)^2} \underbrace{\left[ (1-2x) \sin(n\pi x) \right]_0^1}_{=0} - \frac{-4}{(n\pi)^2} \int_0^1 \sin(n\pi x) dx =$

$= \begin{cases} -\frac{8}{(n\pi)^3}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

$\Rightarrow \begin{cases} c_n = \begin{cases} -\frac{8}{(n\pi)^3} e^{-(n\pi)^2 \frac{2}{3}}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{cases}$

[ Gives the answer ]

$$\begin{cases} \Delta u = 0 \\ u_r(r_0, \theta) = 0 \\ u(1, \theta) = f(\theta) \end{cases}$$



$$\{(r, \theta), r_0 < r < 1\}$$

Find steady state solution

$$\text{Solution: } \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$u(r, \theta) = R(r) \phi(\theta) \Rightarrow R'' \phi + \frac{1}{r} R' \phi + \frac{1}{r^2} R \phi'' = 0 \Rightarrow$$

$$\Rightarrow -\frac{r^2 R'' + r R'}{R} = \frac{\phi''}{\phi} = -\lambda \quad , \quad \phi \text{ 2}\pi \text{ periodic}$$

$$1) \lambda = 0 \Rightarrow \phi(\theta) = a\theta + b, \phi \text{ 2}\pi \text{ periodic} \Rightarrow a = 0 \Rightarrow \phi = \text{constant} = h$$

$$2) \lambda < 0, \lambda = (i\mu)^2 \Rightarrow \phi(\theta) = a e^{i\mu\theta} + b e^{-i\mu\theta}$$

$$\phi \text{ 2}\pi \text{ periodic} \Rightarrow a, b = 0 \Rightarrow \phi = 0$$

$$3) \lambda > 0, \lambda = \mu^2, \phi(\theta) = a e^{i\mu\theta} + b e^{-i\mu\theta}, \phi \text{ 2}\pi \text{ periodic} \Rightarrow$$

$$\Rightarrow \mu = n \in \mathbb{N} \Rightarrow \lambda = n^2$$

$$\phi_n(\theta) = a_n e^{in\theta} + b_n e^{-in\theta} \quad n = 0, 1, \dots$$

$$-r^2 R'' - r R' + n^2 R_n = 0$$

$$x(1-x) - x + n^2 = 0 \Rightarrow \begin{cases} x_1 = n \\ x_2 = -n \end{cases} \Rightarrow R_n(r) = \begin{cases} c_n r^n + d_n r^{-n} \\ c_0 + d_0 \ln r \end{cases}$$

$$u(r, \theta) = \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n}) (a_n e^{in\theta} + b_n e^{-in\theta}) + h(c_0 + d_0 \ln r)$$

$$= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (\tilde{c}_n r^n + \tilde{d}_n r^{-n}) e^{in\theta} + \tilde{c}_0 + \tilde{d}_0 \ln r$$

$$u_r(r_0, \theta) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (n \tilde{c}_n r_0^{n-1} - \tilde{d}_n n r_0^{-n-1}) e^{in\theta} + \frac{\tilde{d}_0}{r_0} = 0$$

$$\begin{cases} n \tilde{c}_n r_0^{n-1} - \tilde{d}_n n r_0^{-n-1} = 0 \\ \tilde{d}_0 = 0 \end{cases} \Rightarrow \tilde{d}_n = r_0^{2n} \tilde{c}_n$$

$$u(r, \theta) = \sum_{n \neq 0} \tilde{c}_n (r^n + r_0^{2n} r^{-n}) e^{in\theta} + \tilde{c}_0$$

$$u(1, \theta) = \sum_{n \neq 0} \tilde{c}_n (1 + r_0^{2n}) e^{in\theta} + \tilde{c}_0 = f(\theta) \Rightarrow$$

$$\Rightarrow \left[ \tilde{c}_n = \frac{1}{1 + r_0^{2n}} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \right]$$

$$\left[ \tilde{c}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \right]$$

27/2  
F19

## Orthogonal polynomials & best approximation

forall interval  $[a, b]$  and any  $w(x) > 0$  on  $[a, b]$   $\exists!$   $L^2_w([a, b])$  ONB of polynomials.

t.ex.  $\left\{ \frac{P_n(x)}{\sqrt{\frac{2}{2n+1}}} \right\}_{n \geq 0}$  Legendre polynomials (normalized)  
on  $[-1, 1]$  with  $w(x) = 1$ .

OBS!  $\exists$  other ONB for  $L^2_w([a, b])$  just not comprised of polynomials.  
t.ex.  $\left\{ \frac{e^{inx\pi}}{\sqrt{2}} \right\}_{n \in \mathbb{Z}}$   $L^2$  ONB for  $[-1, 1]$

THM For  $f \in L^2_w([a, b])$ , let  $q(x)$  be a polynomial of degree  $n$ . Let  $\{p_j\}_{j \geq 0}$  be the  $L^2_w$  ONB of polynomials.

Let  $\hat{f}_j = \langle f, p_j \rangle$ . Then  $\|f - q\|^2 \geq \|f - \sum_{j=0}^n \hat{f}_j p_j\|^2$  with equality iff  $q = \sum_{j=0}^n \hat{f}_j p_j$ .

Proof  $q$  is degree  $n \Rightarrow$  we have proven that  $\exists \{c_j\}_{j=0}^n$   $c_j \in \mathbb{C}$ , such that  $q = \sum_{j=0}^n c_j p_j$ .

$$\begin{aligned} \|f - q\|^2 &= \|f - \sum_{j=0}^n c_j p_j\|^2 = \left\| \sum_{j=0}^{\infty} \hat{f}_j p_j - \sum_{j=0}^n c_j p_j \right\|^2 = \left\{ \begin{array}{l} \text{let } c_j = 0 \\ \forall j > n \end{array} \right\} = \\ &= \left\| \sum_{j=0}^{\infty} (\hat{f}_j - c_j) p_j \right\|^2 \stackrel{\substack{\text{pythagorus} \\ \text{and } p_j \text{ ON}}}{=} \sum_{j=0}^{\infty} |\hat{f}_j - c_j|^2 = \\ &= \sum_{j=0}^n |\hat{f}_j - c_j|^2 + \sum_{j=n+1}^{\infty} |\hat{f}_j|^2 \geq \underbrace{\left\| f - \sum_{j=0}^n \hat{f}_j p_j \right\|^2}_{\sum_{j=n+1}^{\infty} |\hat{f}_j|^2} = \end{aligned}$$

$$= \left\| \sum_{j=0}^{\infty} \hat{f}_j p_j - \sum_{j=0}^n \hat{f}_j p_j \right\|^2$$

$$\text{Equal} \Leftrightarrow \sum_{j=0}^n |\hat{f}_j - c_j|^2 = 0$$

$$\Leftrightarrow \hat{f}_j = c_j, j = 0, \dots, n \Leftrightarrow q = \sum_{j=0}^n \hat{f}_j p_j \blacksquare$$

Ex. Let  $f(x) = x^2$ , on  $[-1, 1]$ . Do Taylor exp. about 0.  
 $f(0) = f'(0) = 0 \Rightarrow$  the degree 0 Taylor polynomial is 0.

$$\hat{f}_0 = \frac{\langle f, p_0 \rangle}{\|p_0\|^2} = \int_{-1}^1 \frac{x^2}{2} dx = \frac{1}{3}.$$

$$\text{Compute } \|f - 0\|^2 = \int_{-1}^1 x^4 dx = \frac{2}{5}.$$

$$\|f - \hat{f}_0 p_0\|^2 = 2 \left( \frac{1}{5} - \frac{1}{9} \right) \text{ smaller than}$$

( $p_0$  - legendre order 0.)

$$p_0 = 1, \text{ its } L^2 \text{ norm on } [-1, 1] = \sqrt{\frac{2}{2 \cdot 0 + 1}} = \sqrt{2}$$

So better approx.

From where do the Legendre polynomials come?  
Come from solving  $\Delta u=0$  in a 3D ball.

THM 6.2 The Legendre polynomials satisfy

$$[(1-x^2) P_n'(x)]' + n(n+1) P_n(x) = 0.$$

Thus they are EFs for the SLP  $[(1-x^2) P_n']' + \lambda P_n = 0$   
with EVs  $\lambda_n = n(n+1)$ .

E0: prove using the explicit form of  $P_n$ . (exercise 6.2.1)  
and induction.

Proof different  

$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{d^n}{dx^n} x^{2k}$$

By product rule,  $\underline{((1-x^2) P_n')'} = (1-x^2)' P_n + (1-x^2) P_n'' =$   
 $= -2x P_n' + (1-x^2) P_n''$ . Compute the highest power of  $x$  here.  

$$\left( \frac{-2x (x^{2n})^{(n+1)}}{2^n n!} - \frac{x^2 (x^{2n})^{(n+2)}}{2^n n!} \right) = \frac{-2x (x^{2n-n-1}) (2n)(2n-1)\dots(2n-n)}{2^n n!} -$$
  
 $- \frac{x^{2+2n-n-2} (2n)(2n-1)\dots(2n-n)(2n-n-1)}{2^n n!} =$   
 $= \frac{-x^n (2n)! 2}{(n-1)! 2^n n!} - \frac{x^n (2n)!}{2^n n! (n-2)!} = \frac{-x^n (2n)! (2+n-1)}{2^n n! (n-1)!} =$   
 $= \frac{-x^n (2n)! (n+1)}{2^n n! (n-1)!} = \boxed{\frac{-x^n (2n)! (n+1)(n)}{2^n n! n!}}$

Compute the leading (highest) (power of  $x$ ) in  $P_n$  is  

$$\frac{1}{2^n n!} \frac{d^n}{dx^n} (x^{2n}) = \frac{(2n)(2n-1)\dots(2n-n+1)}{2^n n!} x^n = \boxed{\frac{(2n)! x^n}{2^n n! n!}}$$

Thus,  $\underline{[(1-x^2) P_n']'} + n(n+1) P_n = \underline{0 x^n} + \text{lower order}$   
 $= \text{degree} \leq n-1$

$$[(1-x^2) P_n']' + n(n+1) P_n = \{ \text{poly of order} \leq n-1 \} = \sum_{j=0}^{n-1} c_j P_j$$

The coeffs are  $\int_1^1 \left( [(1-x^2) P_n'(x)]' + (n+1)n P_n(x) \right) P_j(x) dx$   
 $= 0$  because  $P_n \perp P_j, n \neq j$

Integrate by parts twice!

$$\Rightarrow \int_1^1 [(1-x^2) P_n']' P_j = \int_1^1 P_n \underline{[(1-x^2) P_j']'} =$$

this is a poly of degree  $j < n$   
thus it is  $\sum_{k=0}^j b_k P_k$

$$= \int_1^1 P_n \sum_{k=0}^{\infty} b_k P_k = 0 \quad \text{because each } P_k \perp P_n$$

$$\text{Thus } c_j = 0 \text{ for } j = 0, \dots, n-1 \Rightarrow [(1-x^2)P_n]' + (n+1)n P_n = 0$$

THM 6.4.  $\left\{ \frac{P_{2n}}{\sqrt{\frac{2}{4n+1}}} \right\}_{n \geq 0}, \left\{ \frac{P_{2n+1}}{\sqrt{\frac{2}{4n+3}}} \right\}_{n \geq 0}$  are each

$L^2$  ONB of  $(0,1)$

Proof For  $f$  defined on  $(0,1)$  extend  $f$  to  $(-1,1)$  evenly,  $g$ , or oddly,  $h$ .

Thus because  $\left\{ \frac{P_n}{\sqrt{\frac{2}{2n+1}}} \right\}_{n \geq 0}$  are an  $L^2$  ONB

$$Q_n = \frac{P_n}{\sqrt{\frac{2}{2n+1}}} \Rightarrow g = \sum_{n \geq 0} \hat{g}_n Q_n, \quad h = \sum_{n \geq 0} \hat{h}_n Q_n$$

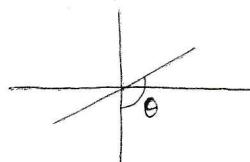
$Q_n$  is even if  $n$  is even, odd if  $n$  is odd.

Thus  $\hat{g}_n = 0 \quad \forall n \text{ odd}, \quad \hat{h}_n = 0 \quad \forall n \text{ even}. \quad f = g|_{(0,1)} = h|_{(0,1)}$

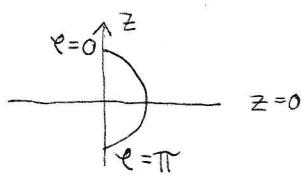
$$\Rightarrow f = \sum_{k \geq 0} \hat{g}_{2k} Q_{2k} = \sum_{j \geq 0} \hat{h}_{2j+1} Q_{2j+1}$$

■

Spherical coordinates: in  $\mathbb{R}^3$



$$r = \text{distance to the origin} = \sqrt{x^2 + y^2 + z^2}$$



$$\theta \in [0, 2\pi], \quad \varphi \in [0, \pi], \quad r \geq 0$$

$$\begin{cases} x = r \cos \theta \sin \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \varphi \end{cases}$$

Solve  $\Delta u = 0$  in a ball in  $\mathbb{R}^3$ . Separate variables.

$$\frac{\Delta(R\Theta\phi)}{R\Theta\phi} = 0 \Rightarrow \frac{R''}{R} + \frac{2R'}{Rr} + \frac{\phi'' \sin \varphi + \phi' \cos \varphi}{r^2 \sin^2 \varphi \phi} + \frac{\Theta''}{r^2 \sin^2 \varphi \Theta} = 0$$

Multiply by  $r^2 \sin^2 \varphi$

$$\Rightarrow \frac{R''r^2 \sin^2 \varphi}{R} + \frac{2R'r^2 \sin^2 \varphi}{R} + \frac{(\phi'' \sin \varphi + \phi' \cos \varphi) \sin \varphi}{\phi} = -\frac{\Theta''}{\Theta}$$

Thus both sides are constant. Call this  $m^2$

OBS  $\Theta$  is  $2\pi$ -periodic  $\Rightarrow \Theta(\theta) = e^{im\theta}, m \in \mathbb{Z}$

$$\frac{R''r^2\sin^2\theta}{R} + \frac{2R'r\sin^2\theta}{R} + \frac{\sin\theta}{\phi} (\phi''\sin\theta + \phi'\cos\theta) = m^2 \quad \text{Sep. vars again!}$$

divide by  $\sin^2\theta$

$$\Rightarrow \frac{R''r^2}{R} + \frac{2R'r}{R} + \frac{\phi''\sin\theta + \phi'\cos\theta}{\sin\theta \phi} = \frac{m^2}{\sin^2\theta}$$

$$\Rightarrow \frac{R''r^2 + 2R'r}{R} = \frac{m^2}{\sin^2\theta} = \frac{m^2}{\sin^2\theta} - \frac{\phi''\sin\theta + \phi'\cos\theta}{\sin\theta \phi}$$

$\Rightarrow$  Both sides equal a constant, call this  $\lambda$ .

Let  $s = \cos\theta \Rightarrow s \in [-1, 1]$ ,  $\theta = \arccos(s)$

$$S(s) = S(\cos\theta) = \phi(\theta)$$

$$\phi'(\theta) = S'(s) (-\sin\theta), \quad \phi''(\theta) = -sS'(s) + S''(s)\sin^2\theta$$

$$\frac{m^2}{\sin^2\theta} - \frac{\phi''}{\phi} - \frac{\phi'\cos\theta}{\phi\sin\theta} = \lambda$$

$$\Rightarrow \frac{\phi''}{\phi} = \frac{-sS' + (1-s^2)S''}{s} \quad \text{and} \quad \frac{\phi'\cos\theta}{\phi\sin\theta} = \frac{-\sin\theta\cos\theta S'}{s\sin\theta} = \frac{-sS'}{s}$$

$$\sin^2\theta = 1-s^2 \Rightarrow$$

$$\Rightarrow \frac{m^2}{1-s^2} - \left( \frac{-sS' + (1-s^2)S''}{s} - \frac{sS'}{s} \right) = \lambda, * \text{ by } s$$

$$\frac{sm^2}{1-s^2} - ((1-s^2)S')' - \lambda s = 0$$

If  $m=0$ , this eqn. is satisfied by  $P_n = S$ ,  $\lambda = n(n+1)$

For  $m \in \mathbb{Z} \setminus \{0\}$ , same eqn. for  $\pm m$ , Solution

$$P_n^{(m)}(s) := (1-s^2)^{|m|/2} \frac{d^{|m|}}{ds^{|m|}} P_n(s) \text{ solves the eqn.}$$

$$\lambda_n = n(n+1)$$

$$\Rightarrow R''r^2 + 2rR' = \lambda_n R = n(n+1)R \quad \text{Euler eqn.}$$

$$R(r) = r^x \Rightarrow x^2 + x - \lambda_n = 0 \Rightarrow x = -\frac{1}{2} \pm \sqrt{(n+\frac{1}{2})^2}$$

Do not want  $R(0) = \infty \Rightarrow$  take  $x=n \Rightarrow R(r) = r^n$

Up to normalization, solutions,

$$u_{m,n}(r, \theta, \phi) = r^n e^{im\theta} P_n^{(m)}(\cos\theta)$$

E028  $u_t - u_{xx} = t \sin x$   $x \in (0,1)$   
 $u(0,t) = u(1,t) = 0$   $t > 0$ .  
 $u(x,0) = \sin(2\pi x)$

Ansatz: (4.16)  $\begin{cases} u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x) \\ F(x,t) = t \sin(x) = \sum_{n=1}^{\infty} t \sin(n\pi x) \end{cases}$

$$a_n = \frac{1}{\frac{1}{2}} \int_0^1 \sin(x) \sin(n\pi x) dx = 2 \int_0^1 (\cos((1-n\pi)x) - \cos((1+n\pi)x)) dx =$$

$$= \left[ \frac{\sin(1-n\pi x)}{1-n\pi} - \frac{\sin(1+n\pi x)}{1+n\pi} \right]_0^1 = \frac{(-1)^n \sin(1)}{1-n\pi} - \frac{(-1)^n \sin(1)}{1+n\pi} =$$

$$= (-1)^n \sin(1) \frac{2n\pi}{1-n^2\pi^2}$$

$$u_t - u_{xx} = t \sin(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} (b_n'(t) + (n\pi)^2 b_n(t)) \sin(n\pi x) = \sum_{n=1}^{\infty} t \sin(n\pi x)$$

$$\Rightarrow b_n' = b_n(t) + (n\pi)^2 b_n(t), \quad t > 0.$$

$$e^{(n\pi)^2 t} (b_n'(t) + (n\pi)^2 b_n(t)) = e^{(n\pi)^2 t} a_n t$$

$$\frac{d}{dt} (e^{(n\pi)^2 t} b_n(t))$$

$$e^{(n\pi)^2 t} b_n(t) = a_n \int e^{(n\pi)^2 t} t = a_n \left( \frac{t e^{(n\pi)^2 t}}{(n\pi)^2} - \int \frac{e^{(n\pi)^2 t}}{(n\pi)^2} dt \right) + c_n =$$

$$= \frac{t a_n e^{(n\pi)^2 t}}{(n\pi)^2} - \frac{a_n e^{(n\pi)^2 t}}{(n\pi)^4} + c_n$$

$$b_n(t) = c_n \underbrace{e^{-(n\pi)^2 t}}_{\substack{\text{Solutions decay} \\ \text{good!} \Downarrow \\ \text{varme avtar...}}} + \frac{a_n}{(n\pi)^2} \left( t - \frac{1}{(n\pi)^2} \right)$$

Solutions decay  
good!  
varme avtar...

$$u(x,0) = \sin(2\pi x)$$

$$\sum_{n=1}^{\infty} \left( c_n - \frac{a_n}{(n\pi)^4} \right) \sin(n\pi x) = \sin(2\pi x)$$

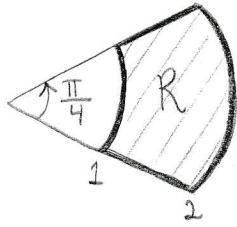
$$\begin{cases} c_n = \frac{a_n}{(n\pi)^4} & \text{when } n \neq 2 \\ c_n = \frac{a_n}{(n\pi)^4} + 1 & \text{when } n=2 \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x) = b_1(t) \sin(\pi x) + b_2(t) \sin(2\pi x) +$$

$$+ \sum_{n=3}^{\infty} b_n(t) \sin(n\pi x)$$

$$b_n(t) = \begin{cases} \frac{(-1)^n \sin(1) 2\pi n}{1+n^2\pi^2} e^{-(n\pi)^2 t} + \frac{(-1)^n \sin(1)}{(n\pi)^2 (1+n^2\pi^2)} \left( t - \frac{1}{(n\pi)^2} \right), & n \neq 2 \\ \left( \frac{(-1)^n \sin(1) 4\pi}{1+4\pi^2} + 1 \right) e^{-(n\pi)^2 t} + \frac{(-1)^n \sin(1)}{(2\pi)^2 (1+4\pi^2)} \left( t - \frac{1}{4\pi^2} \right), & n = 2 \end{cases}$$

$$\text{EÖ30} \quad \Delta u = 0, \quad R = \{(r, \theta) \mid 0 < \theta < \pi/4, 1 < r < 2\}$$



$$\begin{aligned} u &= 0 \text{ when } r = 1 \\ u &= 0 \text{ when } \theta = 0 \\ u &= r-1 \text{ when } \theta = \pi/4 \\ u_r &= 0 \text{ when } r = 2. \end{aligned}$$

$$\Delta u(r, \theta) = \frac{1}{r} \partial_r (r \partial_r u) + \frac{1}{r^2} \partial_{\theta\theta}^2 u$$

$$\begin{aligned} u(1, \theta) &= 0, \quad \partial_r u(2, \theta) = 0 \\ u(r, 0) &= 0, \quad u(r, \pi/4) = r-1 \end{aligned}$$

$$\text{Sep. vars. } u(r, \theta) = R(r) \Phi(\theta)$$

$$\Leftrightarrow \frac{1}{r} (r R') \Phi + \frac{1}{r^2} R \Phi'' = 0$$

$$\Leftrightarrow -\frac{r(rR')'}{R} = \frac{\Phi''}{\Phi} = \lambda^2$$

$$\text{Two problems, (I) } -r \frac{d}{dr} \left( r \frac{dR}{dr} \right) = \lambda^2 R, \quad R(1) = 0, \quad R'(2) = 0$$

$$(II) \quad \Phi'' = \lambda^2 \Phi, \quad \Phi(0) = 0$$

$$(I) \quad r^2 R'' + r R' + \lambda^2 R = 0 \quad \left( r(t(r)) = e^{t(r)} \right)$$

$$t: R \rightarrow \mathbb{R}, \quad t(r) = \log(r), \quad \left[ \partial_t = \partial_r \left( \frac{\partial r}{\partial t} \right) = r \frac{\partial}{\partial r} \right]$$

$$T(t) = R(e^t), \quad T'(t) = r R', \quad T''(t) = r^2 R'' + r R'$$

$$T''(t) = -\lambda^2 T(t), \quad T(0) = 0, \quad T'(\log 2) = 0$$

$$T_n(t) = A_n \sin(\lambda_n t) + B_n \cos(\lambda_n t)$$

$$T(0) = 0 \Rightarrow B_n = 0, \quad T'(\log 2) = 0 \Rightarrow \lambda_n^2 = (n + \frac{1}{2}) \frac{\pi^2}{\log 2}$$

$$R(r) = T(\log(t(r))) \Rightarrow R_n(r) = A_n \sin(\lambda_n \log(r))$$

$$(II) \quad \Phi'' = \lambda^2 \Phi \Rightarrow \phi_n(\theta) = a_n \cosh(\lambda_n \theta) + b_n \sinh(\lambda_n \theta)$$

$$\phi(0) = 0 \Rightarrow a_n = 0$$

$$u(r, \theta) = \sum_{n=0}^{\infty} E_n R_n(r) \phi_n(\theta) = \sum_{n=0}^{\infty} E_n \sinh(\lambda_n \theta) \sin(\lambda_n \log r)$$

$$t(r) = \log r \Rightarrow u(r, \pi/4) = u(e^t, \pi/4) = \sum_{n=0}^{\infty} E_n \underbrace{\sinh(\lambda_n \frac{\pi}{4})}_{\text{constant}} \sin(\lambda_n t) = e^t - 1$$

$\{e_n = \sin(\lambda_n t)\}$  orthogonal system in  $L^2(0, \log 2)$

$$\begin{aligned} \sinh(\lambda_n \pi/4) E_n \|e_n\|^2 &= \int_0^{\log 2} u(e^t, \pi/4) e_n(t) dt, \quad \|e_n\|^2 = \log 2 \\ &= \int_0^{\log 2} (e^t - 1) \sin(\lambda_n t) dt = - \left[ (e^t - 1) \frac{\cos(\lambda_n t)}{\lambda_n} \right]_0^{\log 2} + \int_0^{\log 2} \frac{e^t \cos(\lambda_n t)}{\lambda_n} dt \end{aligned}$$

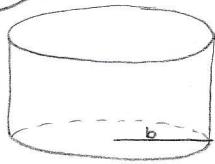
$$\lambda_n = \left(n + \frac{1}{2}\right) \frac{\pi}{\log 2}$$

$$\left[ (e^t - 1) \frac{\cos(\lambda_n t)}{\lambda_n} \right]_0^{\log 2} = \left( (e^{\log 2} - 1) \cos((n + \frac{1}{2})\pi) - (1 - 1) \right) = 0$$

$$\int_0^{\log 2} \frac{e^t \cos(\lambda_n t)}{\lambda_n} dt = \frac{1}{\lambda_n} \left[ \frac{e^t (\cos(\lambda_n t) + \lambda_n \sin(\lambda_n t))}{\lambda_n^2 + 1} \right]_0^{\log 2} = \frac{1}{\lambda_n (\lambda_n^2 + 1)} (2(\lambda_n (-1)^n - 1))$$

$$E_n = \frac{2(\lambda_n (-1)^n - 1)}{\sinh(\lambda_n \pi/4) \lambda_n (\lambda_n^2 + 1)} \frac{2}{\log 2}$$

5.5.1



$$u_t - \frac{1}{r} \partial_r(r \partial_r u) + \frac{1}{r} \partial_{\theta\theta} u = 0 \quad (k=1)$$

$$u(r, \theta, 0) = A$$

$$\text{At } r=b, [u_r + cu = 0] \quad (\text{Newton's law of cooling})$$

$$\text{Solution. } u(r, \theta, t) = R(r) \phi(\theta) T(t)$$

$$\text{Sep. vars. } \lambda, \mu, \quad \phi(\theta) + \lambda^2 \phi(\theta) = 0 \quad (2\pi\text{-periodic } \phi)$$

$$r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \lambda^2) R(r) = 0 \quad (\text{Bessel eqn.})$$

$$T'(t) = -\mu^2 T(t)$$

$$\Rightarrow R_\lambda(r) = c_\lambda J_\lambda(\mu r) \quad (\text{for each } \lambda \text{ we get a func. of } \mu)$$

$$R'(b) + c R(b) = 0 \quad (\text{from Newton's law of cooling})$$

$$\sum_{n,k} \underbrace{(c_{nk} \cos(n\theta) + d_{nk} \sin(n\theta))}_{\theta\text{-variable}} \underbrace{J_n(\mu_k r)}_{r\text{-variable}} e^{-\mu_k t} \underbrace{e^{-\mu_k t}}_{t\text{-variable}} \quad \left( \frac{d\theta dr}{\text{measure}} \right)$$

THM 5.4 :  $\lambda_{kn}$  are positive zeros of  $J_n(x)$

$$\text{Then } \left\{ J_n \left( \frac{\lambda_{kn} r}{b} \right) \cos(n\theta) \right\} \cup \left\{ J_n \left( \frac{\lambda_{kn} r}{b} \right) \sin(n\theta) \right\}$$

is a (orthogonal) basis in  $L^2(D)$

$$D = \bigcirc \rightarrow$$

The  $n^{\text{th}}$  Legendre polynomial:  $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} ((x^2 - 1)^n)$

$$P_0(x) = 1, \quad P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2 - 1) = \frac{1}{2} \cdot 2x = x$$

$$P_2(x) = \frac{1}{4 \cdot 2} \frac{d^2}{dx^2} ((x^2 - 1)^2) = \frac{1}{2} (3x^2 - 1)$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x), \quad P_4(x) = \frac{1}{8} (35x^4 - 30x + 3), \dots$$


---

6.2.6. Expand func.  $x^2, x^3, x^4$  in series of Legendre poly.

$$\text{Solution: 1)} \quad x^2 = \sum_{n=0}^{\infty} c_n P_n(x) = \left\{ \begin{array}{l} c_n = 0 \\ n > 2 \end{array} \right\} = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) =$$

$$= c_0 + c_1 x + c_2 \frac{1}{2} (3x^2 - 1)$$

$$x^2: \quad 1 = \frac{3}{2} c_2 \Rightarrow c_2 = \frac{2}{3}, \quad x: 0 = c_1, \quad 1: 0 = c_0 - \frac{1}{2} c_2$$

$$\Rightarrow c_0 = \frac{1}{2} c_2 = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

$$\underbrace{x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)}$$

$$x^3 = c_0 + c_1 x + c_2 \frac{1}{2} (3x^2 - 1) + c_3 \frac{1}{2} (5x^3 - 3x)$$

$$x^3: \quad 1 = \frac{5}{2} c_3 \Rightarrow c_3 = \frac{2}{5}$$

$$x^2: \quad 0 = \frac{3}{2} c_2 \Rightarrow c_2 = 0$$

$$x: 0 = c_1 - \frac{3}{2} c_3 \Rightarrow c_1 = \frac{3}{2} c_3 = \frac{3}{2} \cdot \frac{2}{5} = \frac{3}{5}$$

$$1: 0 = c_0 - \frac{1}{2} c_2 \Rightarrow c_0 = \frac{1}{2} c_2 = 0$$

$$\underbrace{x^3 = \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x)}$$

$$\text{Do as above for } x^4 \Rightarrow x^4 = -\frac{3}{35} P_0(x) + \frac{30}{35} P_1(x) + \frac{8}{35} P_4(x)$$


---

The  $n^{\text{th}}$  Hermite polynomial:  $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$

$$H_0(x) = 1, \quad H_1(x) = -e^{-x^2} (-2xe^{-x^2}) = 2x$$

$$H_2(x) = 4x^2 - 2, \dots$$

$\{H_n\}_{n=0}^{\infty}$  orthonormal basis  $L^2(\mathbb{R})$ ,  $w = e^{-x^2}$

$$\|H_n\|_{L^2_w}^2 = 2^n n! \sqrt{\pi}$$

6.4.5 Expand  $e^{\alpha x}$  series of Hermite poly.

$$e^{\alpha x} = \sum_{n=0}^{\infty} c_n H_n(x) = \|H_m\|^2$$

$$\langle e^{\alpha x}, H_m \rangle_{L^2_w} = \sum_{n=0}^{\infty} c_n \langle H_n, H_m \rangle_{L^2_w} = c_m \underbrace{\langle H_m, H_m \rangle_{L^2_w}}$$

$$\Rightarrow c_m = \frac{1}{\|H_m\|^2} \langle e^{\alpha x}, H_m \rangle_{L^2_w}$$

$$\langle e^{\alpha x}, H_m \rangle_{L^2_w} = \int_{-\infty}^{\infty} e^{\alpha x} (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) e^{-x^2} dx =$$

$$= (-1)^n \int_{-\infty}^{\infty} e^{\alpha x} \frac{d^n}{dx^n} (e^{-x^2}) dx = \left\{ \begin{array}{l} n \text{ times} \\ \text{integration by parts} \end{array} \right\} =$$

$$= (-1)^n \int_{-\infty}^{\infty} (-1)^n a^n e^{-x^2} e^{\alpha x} dx = (*)$$

$$\left( (-1)^n \int_{-\infty}^{\infty} e^{\alpha x} \frac{d^n}{dx^n} (e^{-x^2}) dx = (-1)^n \left[ e^{\alpha x} \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2}) \right]_{-\infty}^{\infty} - (-1)^n \int_{-\infty}^{\infty} a e^{\alpha x} \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2}) dx \right)$$

$$(*) = a^n \int_{-\infty}^{\infty} e^{-x^2 + \alpha x - \frac{a^2}{4} + \frac{a^2}{4}} dx = a^n e^{\frac{a^2}{4}} \int_{-\infty}^{\infty} e^{-(x - \frac{a}{2})^2} dx =$$

$$= a^n e^{\frac{a^2}{4}} \int_{-\infty}^{\infty} e^{-x^2} dx = a^n e^{\frac{a^2}{4}} \sqrt{\pi}$$

$$c_n = \frac{1}{2^n n! \sqrt{\pi}} a^n e^{\frac{a^2}{4}} \sqrt{\pi} = \frac{a^n e^{\frac{a^2}{4}}}{2^n n!}$$

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{a^n e^{\frac{a^2}{4}}}{2^n n!} H_n(x)$$

Let  $\alpha > -1$ . Laguerre poly.  $L_n^\alpha(x) = \frac{x^{-\alpha}}{n!} e^x \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x})$

$\{L_n^\alpha(x)\}_{n=0}^{\infty}$  orthonormal basis in  $L_w^2(0, +\infty)$ ,  $w = e^{-x}$

$$\|L_n^\alpha\|_{L_w^2}^2 = \frac{\Gamma(n+\alpha+1)}{n!}$$

$$\alpha = 0, L_0(x) = 1$$

$$L_1(x) = 1 - x$$

$$\|L_n(x)\|_{L_w^2} = \frac{\Gamma(n+1)}{n!} = 1$$

$$L_2(x) = 1 - 2x + \frac{x^2}{2}$$

E026 Find poly.  $P(x)$  with degree 2 such that

$$\int_0^{\infty} |Tx - P(x)|^2 e^{-x} dx \text{ minimum.}$$

$$\text{Solution: } P(x) = \sum_{n=0}^2 c_n L_n(x)$$

Since  $L_n$  ON in  $L_w^2(0, \infty)$   $\Rightarrow L_n(x) e^{-x/2}$  ON in  $L_2$

$$\text{Indeed, } \langle L_n(x) e^{-x/2}, L_m(x) e^{-x/2} \rangle_{L^2} = \int_0^{\infty} L_n(x) L_m(x) e^{-x} dx =$$

$$= \langle L_n, L_m \rangle_{L_w^2} = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

$$\int_0^\infty |\sqrt{x} - \sum_{n=0}^2 c_n L_n(x)|^2 e^{-x} dx = \int_0^\infty |\sqrt{x} e^{-x/2} - \sum_{n=0}^2 c_n L_n(x) e^{-x/2}|^2 dx =$$

$$= \|\sqrt{x} e^{-x/2} - \sum c_n L_n e^{-x/2}\|_{L_2}^2$$

THM 3.8  $\Rightarrow c_n = \langle \sqrt{x} e^{-x/2}, L_n(x) e^{-x/2} \rangle_{L_2}$

(Let  $\{e_n\}$  orthogonal set in  $L_2(D)$ ,  $c_n, \sum c_n^2 < \infty$   
 $f \in L_2(D)$ , then  $\|f - \sum \langle f, e_n \rangle e_n\| \leq \|f - \sum c_n e_n(x)\|$ )

$$c_n = \int_0^\infty x^{\frac{1}{2}} L_n(x) e^{-x} dx$$

$$c_0 = \int_0^\infty x^{\frac{1}{2}} \cdot 1 \cdot e^{-x} dx = \Gamma\left(\frac{3}{2}\right) = \left\{ \Gamma(t+1) = t \Gamma(t) \right\} = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

defn.

$$c_1 = \int_0^\infty x^{\frac{1}{2}} (1-x) e^{-x} dx = \Gamma\left(\frac{3}{2}\right) - \Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}\right) - \frac{3}{2} \Gamma\left(\frac{3}{2}\right) =$$

$$= -\frac{1}{2} \Gamma\left(\frac{3}{2}\right) = -\frac{\sqrt{\pi}}{4}$$

$$c_2 = \int_0^\infty x^{\frac{1}{2}} (1-2x+\frac{x^2}{2}) e^{-x} dx = \Gamma\left(\frac{3}{2}\right) - 2 \Gamma\left(\frac{5}{2}\right) + \frac{1}{2} \Gamma\left(\frac{7}{2}\right) =$$

$$= \Gamma\left(\frac{3}{2}\right) - 2 \Gamma\left(\frac{5}{2}\right) + \frac{1}{2} \frac{5}{2} \Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}\right) - \frac{3}{4} \Gamma\left(\frac{5}{2}\right) =$$

$$= \Gamma\left(\frac{3}{2}\right) - \frac{3}{4} \cdot \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = -\frac{1}{8} \Gamma\left(\frac{3}{2}\right) = -\frac{\sqrt{\pi}}{16}$$

$$P(x) = \frac{\sqrt{\pi}}{2} L_0(x) - \frac{\sqrt{\pi}}{4} L_1(x) - \frac{\sqrt{\pi}}{16} L_2(x) =$$

$$\left\{ = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{4} (1-x) - \frac{\sqrt{\pi}}{16} \left(1-2x+\frac{x^2}{2}\right) \right.$$

THM 6.8 The solution to the Dirichlet problem in the unit 3-sphere,  $\Delta u = 0$ ,  $u|_{r=1} = f(\theta, \varphi)$ , is

$$u(r, \theta, \varphi) = \sum_{\substack{m \in \mathbb{Z} \\ n \geq 0}} c_{mn} r^n e^{im\theta} P_n^{(ml)}(\cos \varphi) \quad \text{with coeffs}$$

$$\begin{aligned} c_{mn} &= \int_0^\pi \int_0^{2\pi} \frac{f(\theta, \varphi) e^{-im\theta} P_n^{(ml)}(\cos \varphi)}{2\pi \|P_n^{(ml)}(\cos \varphi)\|^2} d\theta \sin \varphi d\varphi = \left\{ s = \cos \varphi \right\} = \\ &= \int_{-1}^1 \int_0^{2\pi} \frac{f(\theta, \arccos(s)) e^{-im\theta} P_n^{(ml)}(s)}{2\pi \|P_n^{(ml)}(s)\|^2} d\theta ds \end{aligned}$$

OBS!  $P_m^{(ml)} = 0$  for  $|m| > n$  so those terms vanish

Proof 1.  $\left\{ \frac{e^{im\theta}}{\sqrt{2\pi}} \right\}_{m \in \mathbb{Z}}$  is an  $L^2$  ONB for  $[0, 2\pi]$ ,  $d\theta$

2.  $\frac{P_n^{(0l)}(s)}{\|P_n^{(0l)}(s)\|}$  is an  $L^2$  ONB for  $[-1, 1]$ .

Thus  $\frac{P_n^{(ml)}(s)}{\|P_n^{(ml)}(s)\|}$  is a basis for  $L^2([-1, 1])$ , an ONB!

Thus  $\forall f \in L^2([0, 2\pi]_\theta \times [-1, 1]_s)$  can expand  $f$  using

$$Y_{mn}(\theta, s) = \frac{e^{im\theta}}{\sqrt{2\pi}} \frac{P_n^{(ml)}(s)}{\|P_n^{(ml)}(s)\|} \Rightarrow f(\theta, \arccos(s)) = \sum_{\substack{m \in \mathbb{Z} \\ n \geq |ml|}} c_{mn} Y_{mn}(\theta, s)$$

Moreover, by construction,  $\Delta(r^n Y_{mn}(\theta, s)) = 0 \quad \forall m, n$

By superposition  $u$  satisfies  $\Delta u = 0$

Finally,  $u(1, \theta, \varphi) = \sum_{\substack{m \in \mathbb{Z} \\ n \geq |ml|}} c_{mn} Y_{mn}(\theta, \cos \varphi) = f(\theta, \varphi)$

$$(P_n^{(ml)}(s) = (1-s^2)^{|ml|/2} \frac{d^{|ml|}}{ds^{|ml|}} P_n(s))$$

One can compute that  $\|P_n^{(ml)}\|^2 = \frac{(n+m)!}{(n-m)!} \frac{2}{2n+1}$  for  $n \geq |ml|$

### Polynomials on $L_w^2(\mathbb{R})$

DEF Hermite polynomial of degree  $n$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

PROP  $\forall n \geq 0$ ,  $\frac{d^n}{dx^n} e^{-x^2} = p_n(x) e^{-x^2}$ , where  $p_n(x)$  is a polynomial of degree  $n$ .

Proof By induction on  $n$ . True for  $n=0$  and  $p_0=1$

Assume for  $n \geq 0$

$$\begin{aligned} \text{Thus } \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) &= \frac{d}{dx} \left( \frac{d^n}{dx^n} e^{-x^2} \right) = \frac{d}{dx} \left( p_n(x) e^{-x^2} \right) = \\ &= p_n'(x) e^{-x^2} + p_n(x) (-2x) e^{-x^2} = (p_n'(x) - 2x p_n(x)) e^{-x^2} \end{aligned}$$

degree  
 $n-1$       degree  
degree  $n+1$

$\Rightarrow p_{n+1}(x)$  is indeed degree  $n+1$  and is equal  
to  $p_n'(x) - 2x p_n(x)$   $\blacksquare$

---

$$\text{COR } H_{n+1}(x) = -H_n'(x) + 2x H_n(x)$$

Proof  $e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = p_n'(x) - 2x p_n(x)$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{x^2} p_n(x) e^{-x^2} = (-1)^n p_n(x)$$

$$\Rightarrow (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = (-1)^n p_n'(x) - 2x \underbrace{(-1)^n p_n(x)}_{H_n(x)}$$

$$H_n(x) = (-1)^n p_n(x) \Rightarrow H_n'(x) = (-1)^n p_n'(x)$$

$$\Rightarrow (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = -H_n'(x) + 2x H_n(x) = H_{n+1}(x) \blacksquare$$

EÖ: Use this recursive formula to obtain an explicit (non-recursive) for  $H_n(x)$

---

THM 6.11  $\{H_n\}_{n \geq 0}$  are  $L^2_w(\mathbb{R})$  orthogonal with

$$w(x) = e^{-x^2}, \text{ Moreover } \|H_n\|_{L^2_w(\mathbb{R})}^2 = 2^n n! \sqrt{\pi}$$

Proof Idea 1: Assume  $n > m \geq 0$

$$\begin{aligned} \text{Consider } \langle H_n, H_m \rangle_w &= \int_{\mathbb{R}} H_n(x) \overline{H_m(x)} e^{-x^2} dx = \\ &= \int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = \int_{\mathbb{R}} (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} H_m(x) e^{-x^2} dx \\ &\quad \text{Integrate by parts to move these derivs onto } H_m. \end{aligned}$$

Do this because  $H_m$  is degree  $m < n \Rightarrow \frac{d^n}{dx^n} H_m = 0$

Thus we need to show that the boundary terms vanish when we do integration by parts.

I. by P. one time:

$$\langle H_n, H_m \rangle_w = \left[ (-1)^n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} H_m(x) \right]_{-\infty}^{\infty} + (-1)^{n+1} \int_{\mathbb{R}} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} H_m'(x) dx$$

$$\lim_{x \rightarrow \pm\infty} \left( \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H_m(x) = \lim_{x \rightarrow \pm\infty} p_{n-1}(x) H_m(x) e^{-x^2} = 0$$

Similarly  $\forall j=0, 1, \dots, n$ ,  $\lim_{x \rightarrow \pm\infty} \left( \frac{d^{n-j}}{dx^{n-j}} e^{-x^2} \right) H_m^{(j-1)}(x) = 0$  (\*)

I.P. once  $\Rightarrow \langle H_n, H_m \rangle_w = (-1)^{n+1} \int_{\mathbb{R}} \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2}) H_m'(x) dx$

Again  $\Rightarrow \underbrace{\left[ (-1)^{n+1} \frac{d^{n-2}}{dx^{n-2}} (e^{-x^2}) H_m'(x) \right]_{-\infty}^{\infty} + (-1)^{n+2} \int_{\mathbb{R}} \frac{d^{n-2}}{dx^{n-2}} (e^{-x^2}) H_m''(x) dx}_{=0}$

By (\*) each time we do I.P. boundary terms vanish

$$\Rightarrow \langle H_n, H_m \rangle_w = (-1)^{2n} \int_{\mathbb{R}} e^{-x^2} \underbrace{H_m^{(n)}(x)}_{=0} dx \quad (\text{do I.P. } n \text{ times})$$

Finally, by the same process,

$$\langle H_n, H_n \rangle_w = (-1)^{2n} \int_{\mathbb{R}} e^{-x^2} \underbrace{H_n^{(n)}(x)}_{\text{this is } \frac{d^n}{dx^n} (x^n \text{ term in } H_n)} dx$$

The  $x^n$  term in  $H_n(x)$  is  $2^n x^n$

Eo Use corollary to prove this  $\uparrow$  by induction

$$\Rightarrow \frac{d^n}{dx^n} (H_n(x)) = \frac{d^n}{dx^n} (2^n x^n) = 2^n n!$$

$$\text{Thus } \langle H_n, H_n \rangle_w = (-1)^{2n} \int_{\mathbb{R}} e^{-x^2} 2^n n! dx = \sqrt{\pi} 2^n n! \blacksquare$$

THM 6.12  $\left\{ \frac{H_n}{\sqrt{2^n n! \sqrt{\pi}}} \right\}_{n \geq 0}$  are an  $L^2_w(\mathbb{R})$  ONB  
with  $w(x) = e^{-x^2}$

Proof Let  $f \in L^2_w(\mathbb{R})$ . Assume that  $\langle f, h_n \rangle_w = 0 \quad \forall n \geq 0$

$$h_n(x) = \frac{H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}. \quad \text{We claim that } f = 0.$$

$$\int_{\mathbb{R}} e^{-2\pi i tx} f(x) e^{-x^2} dx = \star$$

$$= \int_{\mathbb{R}} \sum_{n \geq 0} \frac{(-2\pi i tx)^n}{n!} f(x) e^{-x^2} dx = \sum_{n \geq 0} \int_{\mathbb{R}} \frac{(-2\pi i tx)^n}{n!} f(x) e^{-x^2} dx$$

Any polynomial  $p(x)$  of degree  $n$  can be written as

$$p(x) = \sum_{j=0}^n c_j h_j(x), \quad c_j \in \mathbb{C}, \quad j=0, \dots, n$$

$$\Rightarrow \langle f, p \rangle_{L^2_w(\mathbb{R})} = \int_{\mathbb{R}} f(x) \sum_{j=0}^n \overline{c_j} h_j(x) e^{-x^2} dx =$$

$$= \sum_{j=0}^n c_j \int_{\mathbb{R}} f(x) h_j(x) e^{-x^2} dx$$

$$\Rightarrow \forall n \quad \int_{\mathbb{R}} \frac{(-2\pi i tx)^n}{n!} f(x) e^{-x^2} dx = 0 \quad \forall n \geq 0 \Rightarrow \star = 0$$

OBS!  $\star = (\langle f(x) e^{-x^2} \rangle^{\wedge}(t))$ . By the FIT  $f(x) e^{-x^2} = \tilde{f}^{-1}(\star) = 0$

### THM 6.13 Generating function for $H_n$ .

$$e^{2xz - z^2} = \sum_{n \geq 0} \frac{H_n(x) z^n}{n!}$$

Proof Idea: multiply both sides by  $e^{-x^2}$ . Thus want to

show  $\underbrace{e^{-x^2 + 2xz - z^2}}_{e^{-(x-z)^2}} = \sum_{n \geq 0} e^{-x^2} \frac{H_n(x) z^n}{(n!)}$  Looks like a Taylor series expansion about  $z=0$ .

expand in a taylor series about  $z=0$

The coeffs are  $\frac{d^n}{dz^n} (e^{-(x-z)^2}) \Big|_{z=0}$ . Let  $u=x-z$ ,  $\frac{du}{dz} = -1$

$$\frac{d^n}{dz^n} (e^{-(x-z)^2}) \Big|_{z=0} = \frac{d^n}{du^n} e^{-u^2} \Big|_{u=x} (-1)^n = \frac{d^n}{dx^n} e^{-x^2} (-1)^n$$

$$\Rightarrow e^{-(x-z)^2} = \sum_{n \geq 0} (-1)^n \underbrace{\left( \frac{d^n}{dx^n} e^{-x^2} \right)}_{\text{This is almost like } H_n(x) \text{ but } e^{x^2} \text{ is missing...}} \frac{z^n}{n!} = \sum_{n \geq 0} e^{-x^2} H_n(x) \frac{z^n}{n!}$$

This is almost like  $H_n(x)$  but  $e^{x^2}$  is missing...

$$\text{Laguerre} \quad L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x})$$

### Best approximation by Polynomials

Ex. Find the polynomial of at most degree 1 which minimizes  $\int_0^2 |e^x - p(x)|^2 dx$  degree 1!

We know  $\exists! L^2$  ONB of polys on  $[0, 2]$ ,  $\{P_n\}_{n \geq 0}$

Best approx is, for  $f(x) = e^x$ ,  $\hat{f}_0 P_0 + \hat{f}_1 P_1$

$$\hat{f}_0 = \langle f, P_0 \rangle = \int_0^2 e^x \overline{P_0(x)} dx$$

$$\hat{f}_1 = \langle f, P_1 \rangle = \int_0^2 e^x \overline{P_1(x)} dx$$

$\therefore$  Find  $P_0$  and  $P_1$  and then compute  $\hat{f}_0$  and  $\hat{f}_1$ .

$$\int_0^2 P_0^2 dx = 2 P_0^2 = 1 \Rightarrow P_0^2 = \frac{1}{2} \Rightarrow P_0 = \frac{1}{\sqrt{2}}$$

$$\langle P_0, P_1 \rangle = 0 = \int_0^2 \frac{1}{\sqrt{2}} (ax+b) dx = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[ \frac{ax^2}{2} + bx \right]_0^2 = 2a+2b=0 \Rightarrow b=-a$$

$$P_1(x) = a(x-1)$$

$$\int_0^2 P_1(x)^2 dx = a^2 \int_0^2 \underbrace{(x-1)^2}_{x^2-2x+1} dx = a^2 \left[ \frac{x^3}{3} - x^2 + x \right]_0^2$$

$$a^2 \left( \frac{8}{3} - 4 + 2 \right) = a^2 \frac{2}{3} = 1 \Rightarrow a^2 = \frac{3}{2} \Rightarrow a = \sqrt{\frac{3}{2}}$$

$$\text{Thus } P_1(x) = \sqrt{\frac{3}{2}}(x-1)$$

$$\text{Compute coeffs, } \hat{f}_0 = \int_0^2 e^x \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}} (e^2 - 1)$$

$$\hat{f}_1 = \int_0^2 e^x \sqrt{\frac{3}{2}}(x-1) dx = \sqrt{\frac{3}{2}} 2e^2$$

On exam:  
don't need to  
compute these  
integrals

1. Polynomial of degree  $\leq 1$  to minimize  $\int_0^\infty (e^{-2x} - P(x))^2 e^{-x} dx$

This is  $L^2$  best approx problem on  $[0, \infty)$  with weight  $w(x) = e^{-x}$

Look for our  $L^2_w(0, \infty)$  ONB of polynomials. Only need  $P_0$  and  $P_1$ .

$$\int_0^\infty P_0^2 e^{-x} dx = 1 \Leftrightarrow P_0^2 \underbrace{\int_0^\infty e^{-x} dx}_{=1} = 1 \Rightarrow \text{need } P_0 = 1$$

$$\text{Find } P_1 \Rightarrow \int_0^\infty (ax+b) \cdot 1 \cdot e^{-x} dx = 0 \quad \text{so that } P_1 \perp P_0$$

$$\int_0^\infty b e^{-x} dx = b, \quad a \int_0^\infty x e^{-x} dx = a \left[ \frac{x e^{-x}}{-1} \right]_0^\infty - a \int_0^\infty \frac{e^{-x}}{-1} dx = a$$

$$\Rightarrow a+b=0 \Rightarrow b=-a, \quad P_1 = a(x-1),$$

$$\text{Find } a \text{ by requiring } \|P_1\|_{L^2}^2 = 1$$

$$\int_0^\infty P_1^2 e^{-x} dx = a^2 \int_0^\infty (x-1)^2 e^{-x} dx = a^2 \int_0^\infty (x^2 - 2x + 1) e^{-x} dx = a^2 = a = 1$$

Compute for  $f(x) = e^{-2x}$

$$\hat{f}_0 = \int_0^\infty f(x) P_0 e^{-x} dx = \int_0^\infty e^{-3x} dx = \frac{1}{3}$$

$$\hat{f}_1 = \int_0^\infty f(x) P_1 e^{-x} dx = \int_0^\infty e^{-3x} (x-1) dx = \frac{1}{9}$$

{ If short of time  
don't care about  
calculating these  
integrals }

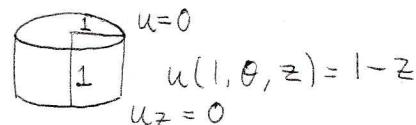
Best approx is  $\hat{f}_0 P_0 + \hat{f}_1 P_1$

2. Bounded solution to  $\nabla^2 u = 0, u(x, 0) = \frac{\cos x}{1+x^2}$

OBS - homogenous solution to the initial value problem  
for the heat eqn. is

$$\int_{\mathbb{R}} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} \frac{\cos y}{1+y^2} dy \quad (\text{convolution of initial data and the function})$$

3.  $\Delta u + 2u = 0$  in



A little harder

Separate variables, write  $u = R\Theta Z$

$$\frac{R''}{R} + r^{-1} \frac{R'}{R} + \frac{r^{-2} \Theta''}{\Theta} + \frac{Z''}{Z} + Z = 0$$

Both sides must be constant.

$$-\frac{Z''}{Z} - Z = \text{constant} \quad Z'(0) = 0 = Z(1)$$

Sines and cosines and a positive constant. If need to, do the cases  $-\frac{Z''}{Z} = \lambda$ , check the cases  $\lambda=0, \lambda<0, \lambda>0$ .

$Z'(0)=0 \Rightarrow$  no sine part. Up to a constant factor

$$Z(z) = \cos(\mu z) \quad \text{Need } Z(1) = 0 \Rightarrow \cos(\mu) = 0$$

$$\Rightarrow \mu = (n + 1/2)\pi, n \geq 0$$

$$Z_n(z) = \cos((n + 1/2)\pi z) \quad \text{and} \quad -\frac{Z_n''}{Z_n} - 2 = \mu_n^2 - 2 > 0 \quad \forall n \geq 0$$

$$\Rightarrow \frac{R''}{R} + \frac{r^{-1}R'}{R} + \frac{r^{-2}\Theta''}{\Theta} = -\frac{Z_n''}{Z_n} - 2 = \mu_n^2 - 2$$

$$\Rightarrow \frac{\Theta''}{\Theta} = r^2(\mu_n^2 - 2) - \frac{r^2 R''}{R} - \frac{r R'}{R} = \text{constant}$$

$\frac{\Theta''}{\Theta}$  is a constant,  $\Theta$  is  $2\pi$  periodic

Up to constant factor,  $\Theta_m(\theta) = e^{im\theta}, m \in \mathbb{Z}$

$\frac{\Theta''}{\Theta} = -m^2$  (If you don't know this immediately, do the cases  $\frac{\Theta''}{\Theta} = \lambda \Leftrightarrow \Theta'' = \lambda \Theta$ . Check for solutions with  $\lambda=0, \lambda<0, \lambda>0$ , and  $2\pi$  periodic)

"I know this is the solution because I have solved it before"

Let  $\lambda_n = \sqrt{\mu_n^2 - 2}$  Equation for  $R$

$$-m^2 = r^2 \lambda_n^2 - \frac{r^2 R''}{R} - \frac{r R'}{R} \Rightarrow r^2 R'' + r R' - (m^2 + r^2 \lambda_n^2) R = 0$$

Almost Bessel eqn. Let  $f(x) = R(\lambda_n r)$ ,  $x = \lambda_n r \Rightarrow$

$\Rightarrow$  eqn. for  $f$  is  $x^2 f''(x) + x f'(x) - (m^2 + x^2) f(x) = 0$  Modified Bessel eqn.

Solution is  $I_{|m|}(x)$ , same eqn. for  $m > 0, m \leq 0$

Thus  $R_{m,n}(r) = I_{|m|}(\lambda_n r)$

Do we really need all the  $\Theta_m(\theta) = e^{im\theta}$ ?

No, because  $\Theta$  independent data. Only need  $m=0$

$\Rightarrow$  Only need  $I_0(\lambda_n r)$

Put them together  $\Rightarrow u(r, \theta, z) = \sum_{n \geq 0} I_0(\lambda_n r) \cos((n+1/2)\pi z) a_n$

Boundary condition  $\Rightarrow$  want  $\sum_{n \geq 0} I_0(\lambda_n) \cos((n+1/2)\pi z) a_n = 1-z$

Thus expand  $1-z$  in the orthogonal basis

$$\left\{ \cos((n+1/2)\pi z) \right\}_{n \geq 0} \text{ on } (0,1)$$

$$\Rightarrow a_n = \frac{1}{I_0(\lambda_n)} \int_0^1 \frac{\cos((n+1/2)\pi z)(1-z) dz}{\| \cos((n+1/2)\pi z) \|_{L^2(0,1)}^2} \leftarrow \text{this} = \frac{1}{2}$$

(Don't need to calculate)  
if don't remember

$\downarrow \text{hard!}$

$$(I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \theta} d\theta)$$

~~Do not want  $I_{m1}$  because  $\uparrow$  at  $r=0$~~   
Only use  $I_{m1}$ , nice and smooth at  $r=0$ .

---

If we keep all the  $\theta$  part,

$$a_{m,n} = \frac{1}{I_{m1}(\lambda_n)} \int_{\theta=0}^{2\pi} \int_{z=0}^1 \frac{\cos((n+1/2)\pi z)(1-z)e^{-im\theta}}{2\pi(1/2)} dz d\theta$$

OBS!  $a_{m,n} = 0 \quad \forall m \neq 0$ . So  $a_{m,n}$  as above is totally fine as answer!

It's also ok to skip  $\theta$  from the beginning (sep. vars  $u = ZR$ ) because  $\theta$  independent.

4. Solve  $\square u = 0$  in  $D = \{x^2 + y^2 < 1\}$

$$u(1, \theta, t) = \sin \theta, \quad u(r, \theta, 0) = u_t(r, \theta, 0) = 0$$

Separate  $\theta$  from  $r$  and  $t \Rightarrow v(r, t) \Theta(\theta)$ .

Want then  $v(1, t) = 1, \quad \Theta(\theta) = \sin \theta$

Look for a steady state solution ~~to the eqn with  $R(1) = 1$~~   
~~to the eqn with  $R(1) = 1$~~

After separate vars in  $\square u = 0 \Rightarrow$

$$\Rightarrow \square(v(r, t) \Theta(\theta)) = \square(v(r, t) \sin \theta) = 0$$

$$\Leftrightarrow \frac{\Theta \square_{r,t} v(r, t) - r^2 \Theta''(\theta) v}{v \Theta} = 0 \Rightarrow \frac{\square_{r,t} v(r, t)}{v(r, t)} - \frac{r^2 \Theta''}{\Theta} = 0$$

$$\frac{(\partial_t^2 - \partial_r^2 - r^{-1} \partial_r) v(r,t)}{v(r,t)} - \frac{r^{-2} \theta''}{\theta} = 0$$

$$\frac{\theta''}{\theta} = \frac{\sin''(\theta)}{\sin \theta} = -1$$

$$\frac{(\partial_t^2 - \partial_r^2 - r^{-1} \partial_r) v(r,t)}{v(r,t)} + r^{-2} = 0 \quad \text{Multiply by } r^2 \Rightarrow$$

$$\Rightarrow \frac{r^2 (\partial_t^2 - \partial_r^2 - r^{-1} \partial_r) v(r,t)}{v(r,t)} + 1 = 0$$

Look for  $R(r)$  to satisfy this eqn. and which has  $R(1)=1$ .

$$\Rightarrow \frac{r^2 (-\partial_r^2 - r^{-1} \partial_r) R}{R} + 1 = 0$$

$$\Rightarrow -r^2 R'' - r R' + R = 0$$

Euler eqn. Look for  $R(r) = r^x$

$$-(x)(x-1) - x + 1 = 0 \Rightarrow x = \pm 1, \text{ don't want } R(0) = \frac{1}{0}$$

take  $x=1 \Rightarrow R(r) = r$  does the job.

Now, look for solution  $w(r,t)$  to solve

$$r^2 \square_{r,t} w = -w, \quad w(1,t) = 0$$

$$w(r,0) = -r \quad \Rightarrow \quad v(r,t) = R(r) + w(r,t) = r + w(r,t)$$

Separate variables  $w = R(r) T(t)$

$$\frac{r^2 T''}{T} - \frac{r^2 R''}{R} - \frac{r R'}{R} = -1 \quad \Rightarrow$$

$$\Rightarrow r^{-2} - \frac{R''}{R} - \frac{R'}{rR} = \frac{-T''}{T} = \text{constant} = \lambda^2$$

$$\Rightarrow -R + r^2 R'' + r R' + \lambda^2 r^2 R = 0$$

Let  $J(x) = R(\lambda r) \Rightarrow$  eqn. is

$$x^2 J''(x) + x J'(x) + (x^2 - 1) J = 0 \quad \text{Bessel's eqn. of order } 1$$

Solutn. is  $J_1 \Rightarrow R(\lambda r) = J_1(\lambda r)$ . Need  $R(1) = 0 \Rightarrow$

$\Rightarrow J_1(\lambda) = 0$ . Let  $\lambda_n = n^{\text{th}}$  positive zero of  $J_1$

$$-\frac{T''}{T} = \lambda_n^2 \Rightarrow T_n(t) = a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)$$

$w(r,t) = \sum_{n \geq 1} T_n(t) J_1(\lambda_n r)$  and IC determine  $a_n$  and  $b_n$ . Want  $w(r,0) = -r$

$$\Rightarrow \sum_{n \geq 1} a_n J_1(\lambda_n r) = -r \Rightarrow a_n = \frac{\int_0^1 -r J_1(\lambda_n r) r dr}{\| J_1(\lambda_n r) \|_{L^2}^2}$$

$$b_n = \frac{1}{\lambda_n} \int_0^1 -J_1(\lambda_n r) r dr$$

Solution is  $(w(r,t) + r) \sin \theta = u(r, \theta, t)$

5.  $\hat{f}(\xi) = \begin{cases} \frac{1}{\sqrt{1-\xi^2}} & \text{for } |\xi| < 1, \xi \neq 0 \\ 0 & \text{else} \end{cases}$  (On exam: FT like Beta)

Compute  $f'(0)$  and  $\int_{\mathbb{R}} f(x) \frac{\sin(2x)}{x} e^{inx} dx$ ,  $n \in \mathbb{N}$

$$\text{FIT: } f'(0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{f'(\xi)} d\xi \Big|_{x=0}$$

$$\widehat{f'(\xi)} = i\xi \hat{f}(\xi)$$

$$f'(0) = \frac{1}{2\pi} \int_{\mathbb{R}} i\xi \hat{f}(\xi) d\xi = 0 \text{ by oddness!}$$

$$\mathcal{F} \left( f \frac{\sin(2x)}{x} \right)(-n) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(n-y) \left( \frac{\sin(2x)}{x} \right)^*(y) dy =$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(n-y) \chi_{(-2,2)}(y) dy = \frac{1}{2\pi} \int_{-2}^2 \hat{f}(n-y) dy =$$

$$\left\{ \hat{f}(n-y) = 0 \text{ if } |n-y| > 1 \Leftrightarrow y < n-1 \text{ or } y > n+1 \right\}$$

$$= \frac{1}{2\pi} \int_{\max(-2, n-1)}^{\min(2, n+1)} f(n-y) dy = \frac{1}{2\pi} \int_{\max(-2, n-1)}^{\min(2, n+1)} \frac{1}{\sqrt{|n-y|}} dy = \begin{cases} 0, & n \geq 3 \\ \frac{1}{4}, & n=2 \\ \frac{\sqrt{2}}{4}, & n=1 \end{cases}$$

ok to stop here.

## THEORY LIST

THM  $f$   $2\pi$  periodic,  $f$  pw cont. on  $\mathbb{R}$ , and  $\forall x \in \mathbb{R}$  right and left limits of  $f$  and  $f'$  are finite  
 $\exists c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$\text{Then } \lim_{N \rightarrow \infty} S_N(x) = \frac{1}{2} (f(x_-) + f(x_+)) \quad \forall x \in \mathbb{R}$$

Hint: Folland ch 2.3.

Proof fix a point  $x \in \mathbb{R}$

$$S_N(x) = \sum_{n=-N}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx} = \sum_{n=-N}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny+inx} dy = \\ = \{y = t+x\} = \sum_{n=-N}^N \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt = \int_{-\pi}^{\pi} f(t+x) \frac{1}{2\pi} \sum_{n=-N}^N e^{int} dt = *$$

$$D_N(t) = \frac{1}{2\pi} \sum_{n=-N}^N e^{int} = \frac{1}{2\pi} + \sum_{n=1}^N \frac{1}{\pi} \cos(nt)$$

$$D_N(t) \text{ is even}, \quad \int_{-\pi}^{\pi} D_N(t) dt = 1, \quad \int_{-\pi}^0 D_N(t) dt = \int_0^{\pi} D_N(t) dt = \frac{1}{2}$$

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \sum_{n=0}^{2N} e^{int} = \frac{1}{2\pi} e^{-iNt} \cdot \frac{1 - e^{i(2N+1)t}}{1 - e^{it}}$$

$$* = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt$$

$$\text{Initial statement} \Leftrightarrow \lim_{N \rightarrow \infty} |S_N(x) - \frac{1}{2}(f(x_-) + f(x_+))| = 0$$

$$\lim_{N \rightarrow \infty} |S_N(x) - \int_{-\pi}^0 D_N(t) f(x_-) dt - \int_0^{\pi} D_N(t) f(x_+) dt| = 0$$

$$\left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \int_{-\pi}^0 D_N(t) f(x_-) dt - \int_0^{\pi} D_N(t) f(x_+) dt \right| \xrightarrow[N \rightarrow \infty]{} 0$$

$$\left| \int_{-\pi}^0 D_N(t) (f(t+x) - f(x_-)) dt + \int_0^{\pi} D_N(t) (f(t+x) - f(x_+)) dt \right| =$$

$$= \left| \int_{-\pi}^0 \frac{e^{-int} - e^{-i(N+1)t}}{2\pi(1 - e^{it})} (f(t+x) - f(x_-)) dt + \int_0^{\pi} \frac{e^{-int} - e^{i(N+1)t}}{2\pi(1 - e^{it})} (f(t+x) - f(x_+)) dt \right|$$

$$g(t) = \begin{cases} \frac{f(t+x) - f(x_-)}{1 - e^{it}}, & t < 0 \\ \frac{f(t+x) - f(x_+)}{1 - e^{it}}, & t > 0 \end{cases}$$

$$\lim_{t \rightarrow 0^-} \frac{f(t+x) - f(x_-)}{1 - e^{it}} = \frac{f'(x_-)}{-i e^{i0}} = i f'(x_-), \quad \lim_{t \rightarrow 0^+} \frac{f(t+x) - f(x_+)}{1 - e^{it}} = i f'(x_+)$$

if  $f'(x_-)$  and  $f'(x_+)$  exist, then  $g(t)$  is bounded.

$$\lim \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(-n-1)t} g(t) dt \right| = \lim_{N \rightarrow \infty} |c_n(g) - c_{-N-1}(g)| = *$$

$g$  is bounded, then its Fourier coeffs are decreasing

by Bessels inequality  $\Rightarrow * = 0$ , that finishes the Proof.

THM Låt  $f$   $2\pi$  periodisk funktion med  $f \in C^2(\mathbb{R})$  ( $2^{\text{nd}}$  derivative continuous)

Fourierkoeff  $c_n$  till  $f$  och  $c_n'$  till  $f'$  uppfyller  $c_n' = in c_n$

$$\text{Proof } f(x) = \sum c_n e^{inx}, \quad f'(x) = \sum c_n' e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad c_n' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx$$

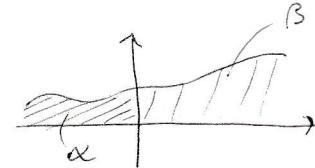
$$\Rightarrow \text{integrate by parts, } c_n' = \frac{1}{2\pi} \int_{-\pi}^{\pi} -f(x)(-in) e^{-inx} dx = in c_n$$

■

THM Big bad convolution approximation thm.

$$\text{Let } g \in L^1(\mathbb{R}) \text{ s.t. } \int_{\mathbb{R}} g(x) dx = 1$$

$$\text{Define } \alpha = \int_{-\infty}^0 g(x) dx, \quad \beta = \int_0^{\infty} g(x) dx$$



Assume  $f$  pw continuous on  $\mathbb{R}$ ,  $\exists$  left & right limits of it at points of  $\mathbb{R}$

Assume ①  $f$  is bounded or ②  $g$  vanishes outside a bounded interval.

$$\text{Let for } \varepsilon > 0, \quad g_\varepsilon(x) = \frac{g(x/\varepsilon)}{\varepsilon}$$

$$\text{Then } \lim_{\varepsilon \rightarrow 0} f * g_\varepsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}$$

$$\text{PROOF } \lim_{\varepsilon \rightarrow 0} \left( \int_{\mathbb{R}} (f(x-y) g_\varepsilon(y) dy) \right) = (\alpha f(x+) + \beta f(x-)) = 0$$

$$\Leftrightarrow \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 f(x+) g(y) dy - \int_0^{\infty} f(x-) g(y) dy = 0$$

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^0 f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 f(x+) g(y) dy = 0 \quad (\text{A})$$

$$\text{and } \lim_{\varepsilon \rightarrow 0} \int_0^{\infty} f(x-y) g_\varepsilon(y) dy - \int_0^{\infty} f(x-) g(y) dy \quad (\text{B})$$

Proof of (A), Want to show  $\forall \delta > 0 \exists \varepsilon_0$  s.t.  $\forall \varepsilon < \varepsilon_0$

$$\left| \int_{-\infty}^0 f(x-y) g_\varepsilon(y) dy - \int_{-\infty}^0 f(x+) g_\varepsilon(y) dy \right| < \delta$$

$\int_{-\infty}^0 g(y) dy = \int_{-\infty}^0 g\left(\frac{y}{\varepsilon}\right) \frac{dy}{\varepsilon} = \int_{-\infty}^0 g_\varepsilon(y) dy$  allows to replace  $g$  to  $g_\varepsilon$  in the formula above.

$$\int_{-\infty}^0 g_\varepsilon(y) (f(x-y) - f(x+)) dy$$

$$\text{By def, } \lim_{y \uparrow 0} f(x-y) - f(x+) = 0$$

Remark:  $y < 0$

$$\Rightarrow \exists y_0 \text{ s.t. } \forall y \in (y_0, 0) , |f(x-y) - f(x+)| < \tilde{\delta}$$

$$\begin{aligned} \Rightarrow \left| \int_{y_0}^0 (f(x-y) - f(x+)) g_\varepsilon(y) dy \right| &\leq \int_{y_0}^0 |f(x-y) - f(x+)| |g_\varepsilon(y)| dy \leq \\ &\leq \tilde{\delta} \int_{y_0}^0 |g_\varepsilon(y)| dy \leq \tilde{\delta} \|g\| \end{aligned}$$

(and make  $\tilde{\delta} \|g\|$  as small as possible by taking  $\tilde{\delta} \leq \frac{\delta}{\|g\|}$ )

Note:  $\|g\| > 1$ ,  $\|g\| = \int_{\mathbb{R}} |g(y)| dy \geq \left| \int_{\mathbb{R}} g(y) dy \right| = 1$  (defn.)

Need to estimate

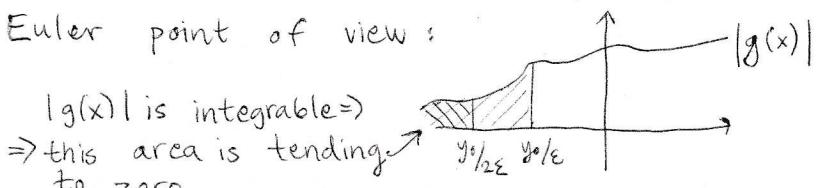
$$\int_{-\infty}^{y_0} (f(x-y) - f(x+)) g_\varepsilon(y) dy$$

Assum ①:  $f$  is bounded  $\Rightarrow \exists M > 0$  s.t.  $|f(x)| < M \quad \forall x$

$$|f(x-y) - f(x+)| \leq |f(x-y)| + |f(x+)| \leq 2M$$

$$\begin{aligned} \text{Remains to estimate } &\int_{-\infty}^{y_0} |(f(x-y) - f(x+))| |g_\varepsilon(y)| dy \leq \\ &\leq 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)| dy = 2M \int_{-\infty}^{y_0/\varepsilon} |g(x)| dx \quad \text{can be made arbitrary small.} \end{aligned}$$

Euler point of view:



Weierstraß point of view:

$$\int_{-\infty}^0 |g(y)| dy < \infty \Leftrightarrow \lim_{R \rightarrow -\infty} \int_{-R}^0 |g(y)| dy - \int_{-R}^0 |g(y)| dy = 0$$

$$\Rightarrow \lim_{R \rightarrow -\infty} \int_{-\infty}^R |g(y)| dy = 0 ,$$

$$\lim_{\varepsilon \rightarrow 0} \frac{y_0}{\varepsilon} = -\infty \Rightarrow \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{y_0/\varepsilon} |g(y)| dy = 0$$

②  $g$  vanishes outside a bounded interval.

$$(*) \int_{y_0}^0 |f(x-y) - f(x+)| |g_\varepsilon(y)| dy < \frac{\delta}{2} \text{ still holds true}$$

Need to estimate  $\int_{-\infty}^{y_0}$

$$\text{Again } \lim_{\varepsilon \downarrow 0} \frac{y_0}{\varepsilon} = -\infty$$

By assumption on  $g$ ,  $\exists R > 0$  s.t.  $g(x) = 0$  for all  $|x| > R$

$$\text{May choose } \varepsilon \text{ s.t. } \frac{y_0}{\varepsilon} < -R \Rightarrow \text{p.8 } \varepsilon_0 = -\frac{y_0}{R}$$

instead of  $-\frac{1}{Ry_0}$  FEL

$$\Rightarrow \forall \varepsilon \in (0, \varepsilon_0), \frac{y_0}{\varepsilon} < -R \quad (\tilde{x} = \frac{y}{\varepsilon})$$

$$(**) \int_{-\infty}^{y_0} |f(x-y) - f(x+)| |g_\varepsilon(y)| dy \stackrel{\downarrow}{=} \int_{-\infty}^{y_0/\varepsilon} |f(x - \varepsilon \tilde{x}) - f(x+)| |g(\tilde{x})| d\tilde{x}$$

$$= 0 \text{ because } g(\tilde{x}) = 0 \quad \forall \tilde{x} \in (-\infty, y_0/\varepsilon)$$

By (\*) and (\*\*),  $\int_{-\infty}^0$  can be made arbitrary small as  $\varepsilon \rightarrow 0$ .

### Plancherel thm and Parseval formula

$$\text{THM } f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}), \hat{f}(\tilde{x}) = \int_{\mathbb{R}} f(y) e^{-2\pi i y \tilde{x}} dy$$

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle, \quad \| \hat{f} \|^2 = \| f \|^2$$

$$\text{Proof } \langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{2\pi i \tilde{x} x} \hat{f}(\tilde{x}) d\tilde{x} \right) \overline{g(x)} dx =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{e^{2\pi i \tilde{x} x}} \hat{f}(\tilde{x}) \overline{g(x)} d\tilde{x} dx =$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} g(x) e^{-2\pi i \tilde{x} x} dx \right) \hat{f}(\tilde{x}) d\tilde{x} =$$

$$= \int_{\mathbb{R}} \overline{\hat{g}(\tilde{x})} \hat{f}(\tilde{x}) d\tilde{x} = \langle \hat{f}, \hat{g} \rangle$$

### THM Sampling thm

$$f \in L^2(\mathbb{R}), \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

Assume  $L > 0$  and  $\hat{f}(\xi) = 0 \quad \forall \xi \in \mathbb{R}$  for  $|\xi| > L$

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(\pi n - tL)}{\pi n - tL}$$

Proof  $\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{inx/L}, c_n = \frac{1}{2L} \int_{-L}^L e^{-inx} \hat{f}(x) dx$

$$\text{FIT} \Rightarrow f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixt} \hat{f}(x) dx = \frac{1}{2\pi} \int_{-L}^L e^{ixt} \hat{f}(x) dx =$$

$$= \frac{1}{2\pi} \int_{-L}^L e^{ixt} \sum_{-\infty}^{\infty} c_n e^{inx/L} dx$$

$$c_n = \frac{1}{2L} \int_{-L}^L e^{-inx/L} \hat{f}(x) dx = \frac{1}{2L} \int_{\mathbb{R}} e^{ix(-\pi n/L)} \hat{f}(x) dx = \{\text{FIT}\} =$$

$$= \frac{2\pi}{2L} f\left(-\frac{n\pi}{L}\right)$$

$$\text{Together} \Rightarrow f(t) = \frac{1}{2\pi} \int_{-L}^L e^{ixt} \sum_{-\infty}^{\infty} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{inx/L} dx =$$

$$= \{\text{interchange } \sum \text{ & } \int\}$$

$$= \frac{1}{2L} \sum_{-\infty}^{\infty} f\left(\frac{\pi n}{L}\right) \int_{-L}^L e^{x(it - in\pi/L)} dx =$$

$$= \sum_{-\infty}^{\infty} f\left(\frac{\pi n}{L}\right) \frac{\sin(Lt - n\pi)}{Lt - n\pi}$$

### THM 3 equivalent conditions to be an ONB in Hilbert space.

$\{\phi_n\}_{n \in \mathbb{N}}$  orthogonal in Hilbert space, then the following 3 conditions are equivalent.

① Every  $f \in H$  s.t.  $\langle f, \phi_n \rangle = 0 \quad \forall n \in \mathbb{N} \Rightarrow f = 0$

②  $\forall f \in H \Rightarrow f = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n$

③  $\|f\|^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2$

Proof ①  $\Rightarrow$  ② Bessel's inequality  $\sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq \|f\|^2 < \infty$

$$\sum_{n \geq N} |\langle f, \phi_n \rangle|^2 \rightarrow 0 \text{ as } N \rightarrow \infty$$

$g_N := \sum_{n=1}^N \langle f, \phi_n \rangle \phi_n$ , Want to prove  $g_N$  is Cauchy.  $M > N$

$$\|g_N - g_M\|^2 = \left\| \sum_{n=N+1}^M \langle f, \phi_n \rangle \phi_n \right\|^2 = \sum_{n=N+1}^M |\langle f, \phi_n \rangle|^2 \leq$$

$$\leq \sum_{n=N+1}^{\infty} |\langle f, \phi_n \rangle|^2 \rightarrow 0 \text{ as } N \rightarrow \infty \Rightarrow g_N \text{ is a Cauchy seq.}$$

$$\lim_{N \rightarrow \infty} g_N = \lim_{N \rightarrow \infty} \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n = g$$

By ①, we need to show that  $\langle f-g, \phi_n \rangle = 0 \forall n \in \mathbb{N}$

$$\text{or } \langle f, \phi_n \rangle = \langle g, \phi_n \rangle$$

$$\text{Indeed, } \langle g, \phi_n \rangle = \left\langle \sum_{m \geq 1} \langle g, \phi_m \rangle \phi_m, \phi_n \right\rangle = \sum_{m \geq 1} \langle g, \phi_m \rangle \langle \phi_m, \phi_n \rangle =$$

$$= \langle g, \phi_n \rangle$$

$$\Rightarrow \langle f-g, \phi_n \rangle = 0 \quad \forall n \in \mathbb{N} \stackrel{(1)}{\Rightarrow} f-g=0 \Rightarrow f=g = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \phi_n$$

$$\textcircled{2} \Rightarrow \textcircled{3} \quad f = \lim_{N \rightarrow \infty} g_N \Rightarrow \|f - g_N\|^2 \rightarrow 0 \text{ as } N \rightarrow \infty$$

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 &\leq \{ \text{Bessel} \} \leq \|f\|^2 = \|f - g_N + g_N\|^2 \leq \xrightarrow{\text{Triangle}} \\ &\leq \|f - g_N\|^2 + \|g_N\|^2 = \|f - g_N\|^2 + \sum_{n=1}^N |\langle f, \phi_n \rangle|^2 \leq \\ &\quad \text{Pyth (use only when finite nbr of terms)} \\ &\leq \|f - g_N\|^2 + \underbrace{\sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2}_{\rightarrow 0 \text{ as } N \rightarrow \infty} \end{aligned}$$

$$\sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 \leq \|f\|^2 = \|f - g_N\|^2 + \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 \quad \forall N$$

Apply sandwich thm:  $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$

$$\left[ \begin{array}{l} a_n \leq b_n \leq c_n \\ \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = b \end{array} \right] \Rightarrow \lim_{n \rightarrow \infty} b_n = b$$

$$\Rightarrow \|f\|^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2$$

③  $\Rightarrow$  ① Assume that for some  $f$ ,  $\langle f, \phi_n \rangle = 0 \quad \forall n \in \mathbb{N}$

$$\textcircled{3} \Rightarrow \|f\|^2 = \sum_{n \in \mathbb{N}} |\langle f, \phi_n \rangle|^2 = 0 \Rightarrow f=0$$

IHM Best approximation thm

$\{\phi_n\}_{n=1}^{\infty}$  orthogonal in  $H$ , let  $f \in H$

$$\|f - \sum_{n \in N} \langle f, \phi_n \rangle \phi_n\| \leq \|f - \sum_{n \in N} c_n \phi_n\| \quad \forall c_n, \{c_n\}_{n \in N} \in \ell^2$$

$$"\iff" \text{ s.t. } c_n = \langle f, \phi_n \rangle \quad \forall n \in N$$

Remark: E021, 22, 36

Finite-dim case:  $\mathbb{R}^3$   $\phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \phi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\text{then } \forall f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \mathbb{R}^3, \|f - \langle f, \phi_1 \rangle \phi_1 - \langle f, \phi_2 \rangle \phi_2\| \leq \|f - \sum_{i=1}^3 c_i \phi_i\|$$

$$\left\| \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} - \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ f_2 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ f_3 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} f_1 - c_1 \\ f_2 - c_2 \\ f_3 \end{pmatrix} \right\|$$

Proof  $g := \sum_n \hat{f}_n \phi_n, \hat{f}_n = \langle f, \phi_n \rangle, e := \sum c_n \phi_n$

$$\|f - e\|^2 = \|f - g + g - e\|^2 = \|f - g\|^2 + \|g - e\|^2 + \underbrace{2 \operatorname{Re} \langle f - g, g - e \rangle}_{\text{Claim} = 0}$$

$$\text{Indeed, } \langle f - g, g - e \rangle = \langle f, g \rangle - \langle f, e \rangle - \langle g, g \rangle + \langle g, e \rangle =$$

$$= \sum \bar{\hat{f}}_n \langle f, \phi_n \rangle - \sum \bar{c}_n \langle f, \phi_n \rangle - \sum \hat{f}_n \langle \phi_n, \sum \hat{f}_m \phi_m \rangle +$$

$$+ \sum \hat{f}_n \langle \phi_n, \sum c_m \phi_m \rangle =$$

$$= \sum |\hat{f}_n|^2 - \sum \bar{c}_n \hat{f}_n - \sum |\hat{f}_n|^2 + \sum \hat{f}_n \bar{c}_n = 0$$

$$\|f - e\|^2 = \|f - g\|^2 + \|g - e\|^2 \geq \|f - g\|^2 \text{ with } "\iff" \text{ iff } \|g - e\|^2 = 0$$

$$g - e = \sum (\hat{f}_n - c_n) \phi_n \Rightarrow \|g - e\|^2 = \sum |\hat{f}_n - c_n|^2$$

$$\Rightarrow |\hat{f}_n - c_n| = 0 \quad \forall n \Leftrightarrow c_n = \hat{f}_n \quad \forall n \in N$$

THM Låt  $f, g$  egenfunk till regulärt SLP i  $[a, b]$  med  $w \equiv 1$   
 $\lambda, \mu$  egenv till  $f$  resp  $g$

Då ①  $\lambda, \mu \in \mathbb{R}$

$$\textcircled{2} \quad \lambda \neq \mu \Rightarrow \int_a^b f(x) \overline{g(x)} dx = 0$$

Proof Defn:  $Lf + \lambda f = 0 \Leftrightarrow Lf = -\lambda f$

L self-adjoint  $\Rightarrow \langle Lf, f \rangle = \langle f, Lf \rangle$

$$\text{Def: } \langle Lf, f \rangle = \int_a^b Lf \overline{f} dx$$

$$\text{har } -\lambda \int_a^b |f(x)|^2 dx = -\bar{\lambda} \int_a^b |f(x)|^2 dx \Leftrightarrow \lambda = \bar{\lambda}$$

$$\textcircled{2} \quad \langle Lf, g \rangle = \langle f, Lg \rangle$$

$$\langle Lf, g \rangle = -\lambda \langle f, g \rangle = \langle f, Lg \rangle = \langle f, -\mu g \rangle = -\bar{\mu} \langle f, g \rangle = -\mu \langle f, g \rangle$$

$$\lambda \neq \mu \Rightarrow \langle f, g \rangle = 0 \quad \text{Def: } \langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx = 0 \quad \blacksquare$$

THM Generating functions for  $J_n(x)$

$$\forall x, \forall z \neq 0 : \sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}$$

Remark: (Folland, p 134)

Proof Taylor expansion:  $e^{xz/2} = \sum_{j \geq 0} \frac{(\frac{xz}{2})^j}{j!}, \quad e^{-x/2z} = \sum_{k \geq 0} \frac{(-\frac{x}{2z})^k}{k!}$

both converge absolutely & uniformly on compact subsets of  $\mathbb{C} \setminus \{0\}$

$$\Rightarrow e^{\frac{x}{2}(z - \frac{1}{z})} = \sum_{j+k \geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j! k!} = \begin{bmatrix} j=n+k, n=j-k \\ n=-\infty, \dots, \infty \\ j+k=n+2k \end{bmatrix} =$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{(n+k)! k!} \cdot \chi_{[n+k \geq 0]}$$

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \cdot \frac{1}{k! \Gamma(k+n+1)}$$

Properties of  $\Gamma$ : • for  $n+k \in \mathbb{N} \cup \{0\}$ ,  $\Gamma(n+k+1) = (n+k)!$

$$\frac{1}{\Gamma(k+n+1)} = \begin{cases} 1/(k+n)!, & k+n \geq 0 \\ 0, & k+n < 0 \end{cases}$$

•  $\Gamma$  is meromorphic with poles at  $\mathbb{N} \cup \{0\}$

•  $1/\Gamma$  is entire with zeros  $= \mathbb{N} \cup \{0\}$

$$\left( \frac{1}{\Gamma(m)} = 0, m \in \mathbb{Z}, m \leq 0 \right)$$

$$\text{Now } \sum_{n=-\infty}^{\infty} J_n(x) z^n = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{k! \Gamma(k+n+1)} =$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{k! (k+n)!} \chi_{[n+k \geq 0]}$$

# EXAM EXERCISES

April 2016. 5.  $\begin{cases} u_t = u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(0, x) = x^2 e^{-x^2} \end{cases}$

$$\hat{u}_t(t, \xi) = -\xi^2 \hat{u}(t, \xi) \quad \text{solve ODE } \Rightarrow \hat{u}(t, \xi) = a(\xi) e^{-\xi^2 t}$$

Boundary  $\Rightarrow a(\xi) = \widehat{x^2 e^{-x^2}}(\xi)$

$$\Rightarrow \hat{u}(t, \xi) = \widehat{x^2 e^{-x^2}}(\xi) e^{-\xi^2 t}$$

Note  $\widehat{e^{-\xi^2 t}}(x) = (4\pi t)^{-1/2} e^{-x^2/4t}$  Invers Fourier & convolution  $\Rightarrow$

$$\Rightarrow u(t, x) = \int_{\mathbb{R}} (4\pi t)^{-1/2} e^{-(x-y)^2/4t} \cdot y^2 e^{-y^2} dy \quad \blacksquare$$

March 2012 3.

Find a function  $f: (0, \pi) \rightarrow \mathbb{R}$  such that

(a)  $\int_0^\pi |f(x) - \sum_{n=1}^5 a_n \sin(nx)|^2 dx$  is minimized for  $a_n = \frac{1}{n}$

(b)  $f(x) = \sum_{n=1}^5 a_n \sin(nx)$

Solution: suppose know  $f(x)$  already

Note  $\{\sin(nx)\}_{n=1}^5$  is orthogonal

If ~~find~~ the integral is minimized on  $a_i$ ,  $\begin{cases} a_1 = \langle f, \sin x \rangle / \langle \sin x, \sin x \rangle \\ a_2 = \langle f, \sin 2x \rangle / \langle \sin 2x, \sin 2x \rangle \\ \vdots \\ a_5 = \langle f, \sin 5x \rangle / \langle \sin 5x, \sin 5x \rangle \end{cases}$

Examples of solution:

$$*f(x) = \left( \sum_{n=1}^5 a_n \sin(nx) \right) + \sqrt{\frac{42}{42}} \sin(42x)$$

$$*f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) = \frac{\pi - x}{2}$$

March 2016 8.

$$H = \{(x, y) \in \mathbb{R}^2 : y \geq 0, x^2 + y^2 \leq 1\}$$



Find all  $\lambda > 0$  and  $f \neq 0$  such that

$$\begin{cases} f_{rr} + r^{-1} f_r + r^{-2} f_{\theta\theta} = -\lambda f & \text{on } H \\ f = 0 & \text{on } \partial H \end{cases}$$

Solution: Folland ch5.5

$$f(r, \theta) = R(r) \Theta(\theta)$$

$$\Rightarrow r^2 \frac{R''}{R} + r \frac{R'}{R} + r^2 \lambda = -\frac{\Theta''}{\Theta} = \text{const.}$$

$$\text{Boundary} \rightarrow \Theta(0) = \Theta(\pi) = 0 \Rightarrow \Theta_n(\theta) = \sin(n\theta) \Rightarrow \frac{\Theta_n''}{\Theta_n} = -n^2$$

$$\Rightarrow r^2 \frac{R''}{R} + r \frac{R'}{R} + r^2 \lambda = n^2 \Leftrightarrow r^2 R'' + r R' + (r^2 \lambda - n^2) R = 0$$

Looks like Bessel but not exactly,  $\lambda > 0 \Rightarrow \lambda = \mu^2$ ,  $\mu > 0$

$$R(r) = f(\mu r) \Rightarrow R'(r) = \mu f'(\mu r), R''(r) = \mu^2 f''(\mu r)$$

$$x^2 f''(x) + x f'(x) + (x^2 - n^2) f(x) = 0$$

$$\Rightarrow f(x) = J_n(x) \Rightarrow R_n(r) = J_n(\mu r)$$

Boundary conditions:  $R_n(1) = 0$

$J_n(\mu) = 0 \Rightarrow$  Let  $\mu = \mu_{n,k}$  be the  $k^{\text{th}}$  positive zero of  $J_n$

$$\Rightarrow u_{n,k}(r, \theta) = J_n(\mu_{n,k} r) \sin(n\theta), \quad \lambda_{n,k} = \mu_{n,k}^2, \quad n, k \in \mathbb{N}$$

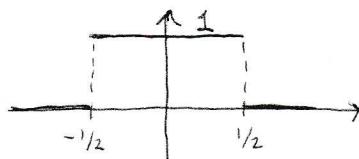
August 2014 2. Almost

Find Fourier tr. of  $S(\omega) = \frac{1 - e^{i\omega}}{\omega}$

$$\text{Solution: } S(\omega) = e^{i\omega/2} \cdot \frac{e^{-i\omega/2} - e^{i\omega/2}}{\omega} =$$

$$= -2ie^{i\omega/2} \frac{\sin(\omega/2)}{\omega}$$

$$\boxed{\left( \frac{\sin(\frac{\omega}{2})}{\omega} \right)^{\wedge} = \frac{1}{2} \chi_{[-1/2, 1/2]}}$$



Folland p. 223 #12

#3

$$\left( e^{iw/2} \frac{\sin(w/2)}{w} \right)^n = \frac{1}{2} \chi_{[-1,0]}$$

$$\Rightarrow (\mathcal{S}(w))^n(\bar{z}) = -i\chi_{[-1,0]}$$

January 2010, 5.

Show that  $\theta^4 - \theta^2 \cdot 2\pi^2 = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(n\theta)}{n^4} - \frac{7\pi^4}{15}$

for  $\theta \in (-\pi, \pi)$

Solution:  $\theta^2 = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} (-1)^k \frac{1}{n^2} \cos(n\theta), \quad \theta \in (-\pi, \pi)$

THM from Folland on integration

$$F(\theta) = \frac{\theta^3}{3} - \frac{\pi^2 \theta}{3} = c_0 + 4 \sum_n \frac{(-1)^{n+1}}{n^3} \sin(n\theta)$$

where  $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) d\theta$

Integrating once more

$$\frac{\theta^4}{12} - \frac{\pi^2 \theta^2}{6} = c_0 + 4 \sum_n \frac{(-1)^{n+1}}{n^4} \cos(n\theta)$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{\theta^4}{12} - \frac{\pi^2 \theta^2}{6} \right) d\theta = -\frac{7}{180} \pi^4, \quad \theta \in (-\pi, \pi)$$

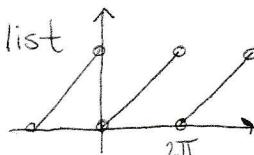
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Hint: finding sum like  $\sum_n \frac{1}{n^2}, \sum_n \frac{1}{n^4}, \sum_n \frac{(-1)^n}{n^4}$

is done like this :

- 1) Expand some function (usually given) in Fourier series on some interval
- 2) Substitute certain values of  $\theta$  in Fourier series.
- 3) Engineer instinct: check where it breaks!!!

Apply thm 2.1 / first from theory list



August 2016 6

$$\begin{cases} (1+t) u_t = u_{xx} & 0 < x < 2, t > 0 \\ u(0, t) = 0 & \text{(I)} \\ u(2, t) = 0 & \text{(II)} \\ u(x, 0) = 2x & \text{(III)} \end{cases}$$

Solution:  $u(x, t) = f(t) g(x)$

$$(1+t) \frac{f'(t)}{f(t)} = \frac{g''(x)}{g(x)} = \text{const.}$$

$$(\text{II}, \text{III}) \quad g''(x) = \lambda g(x), \quad g(0) = g(2) = 0$$

$$g_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad \lambda_n = -\frac{n^2\pi^2}{4} \quad (g_n(x) \text{ are normalized})$$

$$(1+t) \frac{f'_n(t)}{f_n(t)} = -\frac{n^2\pi^2}{4} \quad \text{notice } (\log f_n(t))' = \frac{f'_n(t)}{f_n(t)}$$

$$\Rightarrow f_n(t) = a_n (1+t)^{-n^2\pi^2/4}, \quad a_n \in \mathbb{R}$$

$$\Rightarrow f_n(t) g_n(x) = \sin\left(\frac{n\pi x}{2}\right) a_n (1+t)^{-n^2\pi^2/4}$$

$$(\text{III}) \quad f_n(0) = a_n \quad / \quad \sum_n \sin(n\pi x/2) a_n = 2x \quad \text{on } 0 < x < 2$$

$$a_n = \int_0^2 2x \sin\left(\frac{n\pi x}{2}\right) dx = \frac{8(-1)^{n+1}}{n\pi}$$

$$\underline{\underline{a_0 = \frac{1}{2} \int_0^2 2x dx = 2}}$$

$$\Rightarrow u(x, t) = \cancel{a_0} + \sum_{n \geq 1} \frac{8(-1)^{n+1}}{n\pi} (1+t)^{-n^2\pi^2/4}$$

Hint: how solve with inhomogenous boundary / initial condition?