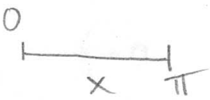
Ex



$0 < t < \infty$

söker $u(x,t)$

värmeekv.:

Ekvationens ordning:
Högsta ordningen på ingående derivator

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

begynnelsevillkor:

$$u(x,0) = \varphi(x)$$

randvillkor:

$$u(0,t) = u(\pi,t) = 0$$

randvärdesproblem

I (variabelseparation)

söker lösn av DE vilka har en enkel form

$$u(x,t) = X(x)\,T(t)$$

sätter in i ekv:

$$T'(t)\,X(x) = X''(x)\,T(t)$$

dela med XT:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

konstant

beror endast på t

beror endast på x

=> allt ober av både x och t

lös X-ekv:

$$X'' + \lambda X = 0 \,, \quad 0 < x < \pi$$

$$X(0) = 0\,; \; X(\pi) = 0$$

kar. ekv

$$p^2 + \lambda = 0$$

$$p = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$$

allmän lösn till X-ekv

$$X(x) = A e^{i\sqrt{\lambda}x} + B e^{-i\sqrt{\lambda}x}$$

för att hitta A, B och λ använder vi randvillk.

$$x=0: \quad Ae^{i\sqrt{\lambda}\cdot 0} + Be^{-i\sqrt{\lambda}\cdot 0} = 0 \qquad (1)$$

$$x=\pi: \quad Ae^{i\sqrt{\lambda}\pi} + Be^{-i\sqrt{\lambda}\cdot\pi} = 0 \qquad (2)$$

(1): $A+B=0; \quad B=-A$

$$X(x) = A\left(e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}\right)$$

(2): $A\left(e^{i\sqrt{\lambda}\pi} - e^{-i\sqrt{\lambda}\pi}\right) = 0$

mult med $e^{i\sqrt{\lambda}\pi}$:

$$e^{2\pi i\sqrt{\lambda}} = 1; \quad \cancel{2\pi i}\sqrt{\lambda} = \cancel{2\pi i}\underset{\text{heltal}}{n} \Rightarrow \lambda = n^2, \quad \lambda = 0,1,4,9,\ldots$$

$\lambda = 0$ dåligt: $X(x) = 0$

$$\Rightarrow \lambda_n = n^2, \quad n = 1,2,3,\ldots$$

$$X_n(x) = A\left(e^{inx} - e^{-inx}\right) = 2iA\sin(nx)$$

lös T-ekv.

$$\frac{T'(t)}{T(t)} = -\lambda = -n^2 \Rightarrow T'(t) + n^2T(t) = 0$$

$$T_n(t) = T_n(0)\,e^{-n^2t}$$

II (kombinera enkla lösn för att lösa randvärdesprobl)

söker $u(x,t)$ som en summa av enkla lösn med okända koeff.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\cdot e^{-n^2t}$$

A_n hittas ur begynnelsevillkoret:

$$t=0: \quad u(x,0) = \varphi(x)$$

$$\sum_{n=1}^{\infty} A_n \sin(nx) \underbrace{e^{-n^2 \cdot 0}}_{=1} = \varphi(x)$$

multiplicera med $\sin kx$ och integrera $\int_0^{\pi}$:

$$\sum_{n=1}^{\infty} A_n \int_0^{\pi} \sin nx \sin kx \, dx = \int_0^{\pi} \sin kx \, \varphi(x) \, dx$$

$$\int_0^{\pi} \sin nx \sin kx \, dx = \int_0^{\pi} \frac{1}{2}\left(\cos((n-k)x) - \cos((n+k)x)\right) dx$$

$$\int_0^{\pi} \cos((n+k)x) dx = \left[\frac{\sin((n+k)x)}{(n+k)}\right]_0^{\pi} = 0$$

$$\int_0^{\pi} \cos((n-k)x) \, dx = \left[\frac{\sin((n-k)x)}{(n-k)}\right]_0^{\pi} = 0 \quad k \neq 0$$

$$= \pi \qquad k = n$$

i summan bara termen med $n=k$ överlever

$$\frac{\pi}{2} A_k = \int_0^{\pi} \sin kx \, \varphi(x) \, dx$$

$$\Rightarrow \quad A_k = \frac{2}{\pi} \int_0^{\pi} \sin kx \, \varphi(x) \, dx$$

Trigonometriska F-serier

F-serie av 2π-periodisk fkt

$$\varphi(\theta) \qquad -\infty < \theta < \infty \, ; \quad \varphi(\theta + 2\pi) = \varphi(\theta)$$

$$]-\pi, \pi[\quad \text{grundperioden}$$

vi vill uttrycka $\varphi(\theta)$ som kombination av de "enklaste" per. fkt.

$e^{in\theta}$

$n = 0, \pm 1, \pm 2, \ldots$

$$\varphi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

hitta koeff c_n ?

multiplicera med $e^{-ik\theta}$ och integr. fr. $-\pi$ till π

$$\int_{-\pi}^{\pi} \varphi(\theta) e^{-ik\theta} d\theta = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{in\theta} e^{-ik\theta} d\theta = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta =$$

$$= \left[\frac{e^{i(n-k)\theta}}{n-k} \right]_{0}^{\pi} = 0 \quad , \quad n \neq k$$

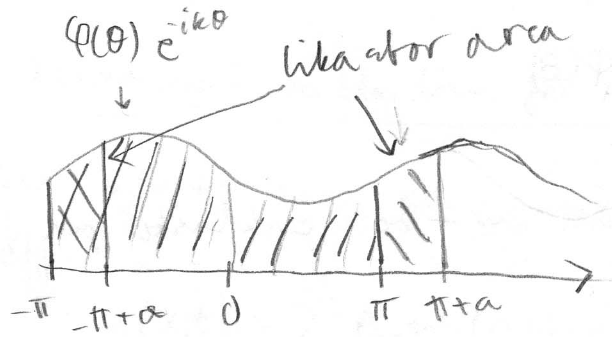
om $n = k$

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta = \int_{-\pi}^{\pi} d\theta = 2\pi$$

$$\int_{-\pi}^{\pi} e^{-ik\theta} \varphi(\theta) d\theta = c_k 2\pi \Rightarrow \boxed{c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\theta) e^{-ik\theta} d\theta}$$ exp F-serie (komplex)

c_k: F-koeff av φ m.a.p trig. syst. $e^{in\theta}$

__Anm:__ ist för integr. fr. $-\pi$ till π kan man integrera på vilket intervall som helst med längden 2π

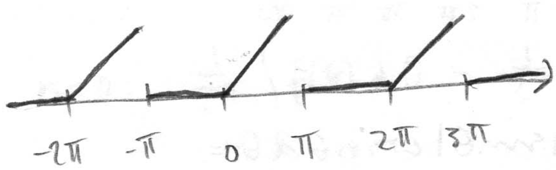


$\boxed{2.1.4}$ $$f(\theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ \theta, & 0 < \theta < \pi \end{cases}$$

[ett annat trig. syst: ta $\sin(n\theta)$, $\cos(n\theta)$ som enkla per. fkt int för $e^{in\theta}$

$$\varphi(\theta) = \sum \left(a_n \sin(n\theta) + b_n \cos(n\theta)\right)$$

$$\Rightarrow \begin{cases} a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)\cos(n\theta)\,d\theta \\ b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)\sin(n\theta)\,d\theta \end{cases}$$ Trig. F-serie]



ber. $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{2\pi}\int_0^{\pi} \theta\cos(n\theta)\,d\theta =$

$n \neq 0$

$$= \frac{1}{2\pi}\int_0^{\pi} \theta\left(\frac{\sin(n\theta)}{n}\right)' d\theta = \frac{1}{2\pi}\left(-\int_0^{\pi} \underbrace{\theta'}_{=1} \frac{\sin n\theta}{n}\,d\theta + \underbrace{\frac{\pi \sin n\pi}{n} - \frac{0 \sin n \cdot 0}{n}}_{=0}\right)$$

$$= -\frac{1}{2\pi n}\int_0^{\pi} \sin n\theta\,d\theta = \frac{1}{2\pi n^2}\left[\cos n\theta\right]_0^{\pi} = \frac{1}{2\pi n^2}\left(\cos n\pi - \cos 0\right)$$

$$\cos n\pi = \begin{cases} 1, & n \text{ jämnt} \\ -1, & n \text{ udda} \end{cases} = (-1)^n$$

$$a_n = \frac{1}{2\pi n^2}\left((-1)^n - 1\right) = \begin{cases} -\frac{1}{\pi n^2}, & n \text{ udda} \\ 0, & n \text{ jämnt} \end{cases}$$

$$n = 0: \quad a_0 = \frac{1}{2\pi}\int_0^{\pi} \theta\,d\theta = \left[\frac{1}{2\pi}\cdot\frac{\theta^2}{2}\right]_0^{\pi} = \frac{\pi^2}{4\pi} = \frac{\pi}{4}$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta = \frac{1}{2\pi}\int_{0}^{\pi} \theta \sin n\theta \, d\theta =$$

$n \neq 0$

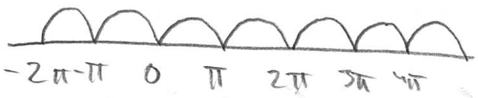
$$= \frac{1}{2\pi}\int_{0}^{\pi} \theta \left(-\frac{\cos n\theta}{n}\right)' d\theta = \frac{1}{2\pi n}\left[\int_{0}^{\pi} \cos n\theta \, d\theta - \pi\cos n\pi + 0\cos n\cdot 0\right]$$

$$= -\frac{1}{2n}(-1)^n + \cancel{\frac{1}{2\pi n}\Big[\sin n\theta\Big]_0^{\pi}} = \frac{1}{2n}(-1)^{n+1} = b_n$$

$$\sum \frac{(-1)^{n+1}}{2n}\sin n\theta + \frac{\pi}{4} + \sum \frac{-1}{n^2\pi}\cos n\theta$$

n udda

$\boxed{2.1.8}$ $f(\theta) = |\sin\theta|$



$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin\theta|\cos n\theta \, d\theta =$$

jämn

$$= \frac{1}{2\pi}\int_{0}^{\pi} 2\sin\theta\cos n\theta \, d\theta = \frac{1}{\pi}\int_{0}^{\pi} \sin\theta\cos n\theta \, d\theta =$$

$$\left[\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \left(\sin((n+1)\theta) + \sin((1-n)\theta)\right)$$

$$\int_{0}^{\pi} \sin k\theta \, d\theta = \left[-\frac{\cos k\theta}{k}\right]_0^{\pi}, \quad k \neq 0 \qquad = 0,\ k = 0$$

$$= -\frac{\cos k\pi + \cos 0}{k} = \frac{(-1)^{k+1} + 1}{k}$$

$$a_n = \frac{1}{2\pi}\left[\frac{-(-1)^{1-n}+1}{1-n} - \frac{-(-1)^{n+1}+1}{n+1}\right]$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin\theta| \sin n\theta \, d\theta$$

$\boxed{2.1.14}$

$$f(\theta) = \begin{cases} 1, & -a < \theta < a \\ -1, & 2a < \theta < 4a \\ 0 & \text{annars} \end{cases}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta = \frac{1}{2\pi}\int_{-a}^{a}\cos n\theta\, d\theta - \frac{1}{2\pi}\int_{2a}^{4a}\cos n\theta \, d\theta =$$

$$\overset{n\neq 0}{=} \frac{1}{2\pi}\left[\frac{\sin n\theta}{n}\right]_{-a}^{a} - \frac{1}{2\pi}\left[\frac{\sin n\theta}{n}\right]_{2a}^{4a} =$$

$$= \frac{1}{2\pi n}\left(\sin(na) - \sin(-na)\right) - \frac{1}{2\pi n}\left(\sin(4a) - \sin(2a)\right)$$

$$n=0: \quad \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta = \frac{1}{2\pi}\int_{-a}^{a} d\theta - \frac{1}{2\pi}\int_{2a}^{4a} d\theta = 0$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = \frac{1}{2\pi}\int_{-a}^{a}\sin n\theta\, d\theta - \frac{1}{2\pi}\int_{2a}^{4a}\sin n\theta\, d\theta$$

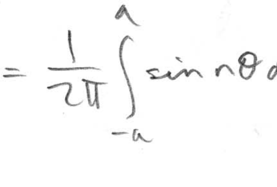
$$\overset{n\neq 0}{=} \frac{1}{2\pi}\left[\frac{\cos n\theta}{n}\right]_{-a}^{a} - \frac{1}{2\pi}\left[\frac{\cos n\theta}{n}\right]_{2a}^{4a} = -\frac{1}{2\pi n}\left(\cos 4na - \cos 2na\right)$$

Konvergens av F-serier

Vi sökte koeff c_k: $\sum_{k=-\infty}^{\infty} c_k e^{ik\theta} = \varphi(\theta)$

Vad ska vi kräva från $\varphi(\theta)$ så att serien konv. med summan $\varphi(\theta)$

Vilka är "bra" fkt?



Styckvis kontin fkt

Intervallet där φ är definierad delas i delintervall på vilka φ är kontin. I brytpkter θ_k måste ensidiga gr.v. existera

$$\varphi(\theta_k - 0) = \lim_{\substack{\delta \to 0 \\ \delta < 0}} \varphi(\theta_k + \delta)$$

$$\varphi(\theta_k + 0) = \lim_{\substack{\delta \to 0 \\ \delta > 0}} \varphi(\theta_k + \delta)$$

$\varphi(\theta)$ är styckvis glatt om φ är styckvis kontin och på varje delintervall har en kontin deriv

<u>SATS</u> (konvergenssats)

Om $\varphi(\theta)$ är styckvis glatt så konv dennas F-serie för alla θ

$\sum c_n e^{in\theta} = \varphi(\theta)$ om φ är kontin i θ

$\sum c_n e^{in\theta_k} = \left(\varphi(\theta_k + 0) + \varphi(\theta_k - 0)\right)/2$ θ_k - brytpkt

$f(\theta)$ kontin. och styckvis glatt

$\Rightarrow f'(\theta)$ har F-serie med koeff $c_n' = inc_n$

Bessel: $\sum |nc_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(\theta)|^2 d\theta < \infty$, b.l.a. $nc_n \to 0$

Låt f kontin.

f' kont.

f' styckvis glatt

c_n'' - F-koeff av f''

$c_n'' = (in)^2 c_n$

Bessel:

$$\sum |c_n''|^2 = \sum |n^2 c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f''(\theta)|^2 d\theta < \infty$$

$n^2 c_n \to 0$

f kontin

f' kontin

$f^{(k)}$ kontin och styckvis glatt

$c_n^{(k)}$ - F-koeff av $f^{(k)}$

$c_n^{(k)} = (in)^k c_n$

Bessel: $\sum_n |c_n^{(k)}|^2 = \sum_n |n^k c_n|^2 < \infty$, $n^k c_n \to 0$

Ju glattare fkt är desto snabbare går F-koeff mot 0.

Fourier 080201 Fr Lv2

Parceval, Plancherel, F-transf. i L^2

Parceval: $f, g \in L^1, L^2$; $\hat{f}, \hat{g} \in L^1, L^2$

$$\langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle \qquad \langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

$$f = g, \quad \underset{\|f\|^2}{\underset{\|}{\langle f, f \rangle}} = \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \langle \hat{f}, \hat{f} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi =$$

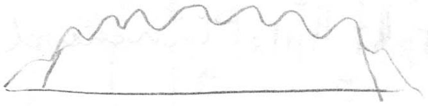
$$= \frac{1}{2\pi} \|\hat{f}\|^2$$

Vi utvidgar $\mathcal{F}$ till fkt i L^2

Låt $f \in L^2$, $\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$

approximera f med "goda" fkt f_n

$$f_n \in L^1, L^2; \; \hat{f}_n \in L^1, L^2 : \|f_n - f\| \to 0$$



$$\Rightarrow \|f_n - f_m\| \underset{m,n \to \infty}{\longrightarrow} 0 \qquad \Big(\|f_n - f_m\| \le \underbrace{\|f_n - f\|}_{\to 0} + \underbrace{\|f_m - f\|}_{\to 0}\Big)$$

om Parceval till $f_n - f_m$

$$\|f_n - f_m\|^2 = \frac{1}{2\pi} \|\hat{f}_n - \hat{f}_m\|^2 \to 0$$

SATS

Om en följd av fkt i L^2 är en Cauchyföljd så konvergerar den $\Rightarrow \exists\, \varphi \in L^2 : f_n \to \varphi$

φ kallas $\mathcal{F}$-transf av f och bet. $\hat{f}$

Vi började med

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx, \text{ men för } f \in L^2 \text{ divergerar } \int$$

och man går genom Parceval

Alternativt:

① $\hat{f}(\xi) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(x) e^{-\varepsilon x^2} e^{-ix\xi} dx$, konvergerar, $f \in L^2$

② $\hat{f}(\xi) = \lim_{R \to \infty} \int_{-R}^{R} f(x) e^{-ix\xi} dx$, konv.

$\langle f_n, g_n \rangle = \frac{1}{2\pi} \langle \hat{f}_n, \hat{g}_n \rangle$ $\quad f_n, g_n$ "goda"

$f_n \to f$, $g_n \to g$ i normen, $f, g \in L^2$

$\Rightarrow \langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle$; $\|f\|^2 = \frac{1}{2\pi} \|\hat{f}\|^2$ Plancherel-formler

$\boxed{\text{EÖ 6}}$ a) $\frac{t}{(t^2+a^2)^2} \overset{\mathcal{F}}{\supset} ?$

10: $\frac{1}{(t^2+a^2)} \overset{\mathcal{F}}{\supset} \frac{\pi}{a} e^{-a|\xi|}$

$$\left(\frac{1}{t^2+a^2}\right)' = -\frac{2t}{(t^2+a^2)^2} \overset{\mathcal{F}}{\supset} i\xi \frac{\pi}{a} e^{-a|\xi|}$$

$$\Rightarrow \frac{t}{(t^2+a^2)^2} \overset{\mathcal{F}}{\supset} \frac{-i\xi\pi}{2a} e^{-a|\xi|}$$

b) $\frac{1}{(t^2+a^2)^2} = f(t)$

Lösning 1:

10: $\frac{1}{t^2+a^2} \overset{\mathcal{F}}{\supset} \frac{\pi}{a} e^{-a|\xi|}$

$$f(t) = \frac{1}{t^2+a^2} \cdot \frac{1}{t^2+a^2}$$

8: $f \cdot g \overset{\mathcal{F}}{\supset} \frac{1}{2\pi} \hat{f} * \hat{g}$

$$g = f, \quad \frac{1}{(t^2+a^2)^2} = \frac{1}{t^2+a^2} \cdot \frac{1}{t^2+a^2} \overset{\mathcal{F}}{\supset} \frac{1}{2\pi} \frac{\pi}{a} e^{-a|\xi|} * \frac{\pi}{a} e^{-a|\xi|} =$$

$$= \frac{\pi}{2a^2} e^{-a|\xi|} * e^{-a|\xi|} = \frac{\pi}{2a^2} \int_{-\infty}^{\infty} e^{-a|\eta|} e^{-a|\xi-\eta|} \, d\eta =$$

$$\left[\varphi * \psi (\xi) = \int_{-\infty} \varphi(\eta) \psi(\xi-\eta) \, d\eta \right]$$

$$= \frac{\pi}{2a^2} \left(\int_{-\infty}^{\xi} e^{-a|\eta|} e^{-a(\xi-\eta)} \, d\eta + \int_{\xi}^{\infty} e^{-a|\eta|} e^{a(\xi-\eta)} \, d\eta \right)$$

$$\int_{-\infty}^{\xi} e^{-a|\eta|} e^{-a\xi+a\eta} \, d\eta = \int_{-\infty}^{0} e^{a\eta} e^{-a\xi+a\eta} \, d\eta + \int_{0}^{\xi} e^{-a|\eta|} e^{-a\xi+a\eta} \, d\eta =$$

$$= \underbrace{\left[\frac{1}{2a} e^{a(2\eta-\xi)} \right]_{-\infty}^{0}}_{= \frac{1}{2a} e^{-a\xi}} + \begin{cases} \int_0^{\xi} e^{-a\eta-a\xi+a\eta} \, d\eta = \xi e^{-a\xi}, & \xi > 0 \\ \int_0^{\xi} e^{a\eta-a\xi+a\eta} \, d\eta = e^{-a\xi} \cdot \frac{1}{2a}\left(e^{-a\xi}-1\right) & \xi < 0 \end{cases}$$

Lösning 2:

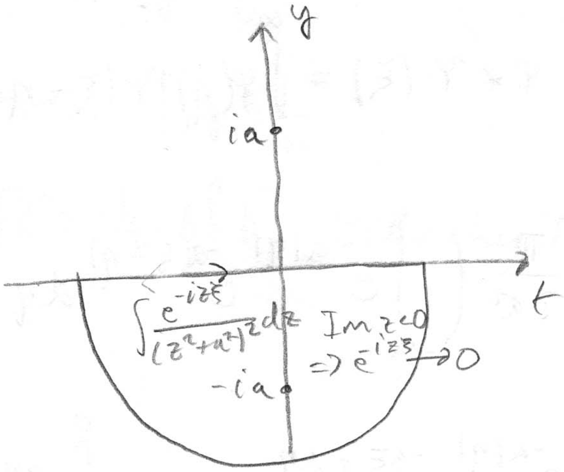
$$\frac{\partial}{\partial a}\left(\frac{1}{t^2+a^2}\right) = \frac{-2a}{(t^2+a^2)^2}$$

$$\frac{\partial}{\partial a}\cdot\frac{\pi}{a}e^{-a|\xi|} = \int_{-\infty}^{\infty} e^{-it\xi}\cdot\frac{1}{t^2+a^2}\,dt$$

$$-\frac{\pi}{a^2}e^{-a|\xi|} - \frac{\pi|\xi|}{a}e^{-a|\xi|} = \int_{-\infty}^{\infty} e^{-it\xi}\frac{-2a}{(t^2+a^2)^2}\,dt = -2a\,\widehat{\left(\frac{1}{(t^2+a^2)^2}\right)}$$

$$\Rightarrow \widehat{\left(\frac{1}{(t^2+a^2)^2}\right)} = \frac{\pi e^{-a|\xi|}}{a^2}\left(\frac{1}{a} - 1\right)$$

Lösning 3:



$$f(t) = \frac{1}{(t^2+a^2)^2}$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-it\xi}}{(t^2+a^2)^2}\,dt$$

$\xi < 0$: $z = t + iy$

c)
$$\frac{t}{(t^2+1)(t^2+2t+5)} = \frac{A+Bt}{t^2+1} + \frac{C+Dt}{t^2+2t+5} = \frac{(A+Bt)(t^2+2t+5)+(C+Dt)(t^2+1)}{(t^2+1)(t^2+2t+5)}$$

$$\left[\Rightarrow \quad C = \frac{1}{2},\ A = \frac{1}{10},\ B = \frac{1}{5},\ D = -\frac{1}{5}\right]$$

$$= \frac{\frac{1}{10} + \frac{1}{5}t}{t^2+1} - \frac{\frac{1}{2} + \frac{1}{5}t}{t^2+2t+5}$$

$$\frac{1}{10}\cdot\frac{1}{t^2+1} \overset{10}{\supset} \frac{1}{10}\frac{\pi}{1}e^{-|\xi|} = \frac{\pi}{10}e^{-|\xi|}$$

$$\frac{1}{5}\cdot\frac{t}{t^2+1} \overset{10,6}{\supset} \frac{i}{5}\left(\pi e^{-|\xi|}\right)' = \frac{\pi i}{5}\cdot\begin{cases} -e^{-\xi} & \xi>0 \\ e^{\xi} & \xi<0\end{cases} = -\frac{\pi i}{5}e^{-|\xi|}\operatorname{sgn}\xi$$

$$\frac{\frac{1}{2}+\frac{t}{5}}{t^2+2t+5} = \frac{\frac{1}{2}+\frac{t}{5}}{(t+1)^2+2^2} = \frac{\frac{1}{2}+\frac{t+1}{5}-\frac{1}{5}}{(t+1)^2+2^2} =$$

$$= \frac{3}{10}\frac{1}{(t+1)^2+2^2} + \frac{1}{5}\frac{t+1}{(t+1)^2+2^2}$$

$$\frac{1}{t^2+2^2} \supset \frac{\pi}{2}e^{-2|\xi|} \qquad t\to t+1$$

$$\frac{1}{(t+1)^2+2^2} \overset{2}{\supset} \frac{\pi}{2}e^{-2|\xi|}e^{i\xi}$$

$$\frac{t+1}{(t+1)^2+2^2}$$

$$\frac{t}{t^2+2^2} \overset{6}{\supset} i\left(\frac{\pi}{2}e^{-2|\xi|}\right)' = -i\pi e^{-2|\xi|}\operatorname{sgn}\xi$$

$$\Rightarrow \frac{t+1}{(t+1)^2+2^2} \supset i\pi e^{-2|\xi|}\operatorname{sgn}\xi\, e^{i\xi}$$

d) $e^{-a|t|}\sin bt \overset{\mathcal{F}}{\supset} ?$

11: $e^{-a|t|} \supset 2a\left(\xi^2+a^2\right)^{-1}$

$$e^{-a|t|}\sin bt = \frac{1}{2i}\Big[\underset{c=b}{e^{-a|t|}e^{ibt}} - \underset{c=-b}{e^{-a|t|}e^{-ibt}}\Big]$$

$$e^{-a|t|}\sin bt \supset \frac{1}{\cancel{2}i}\cdot\cancel{2}a\left[\left((\xi-b)^2+a^2\right)^{-1} - \left((\xi+b)^2+a^2\right)^{-1}\right]$$

$\boxed{E\bar{O}7}$ $f(t)$ har $\hat{f}(\omega) = \frac{\omega}{1+\omega^4}$

a) $\int_{-\infty}^{\infty} t f(t)\, dt =$

$$f \supset \frac{\omega}{1+\omega^4}$$

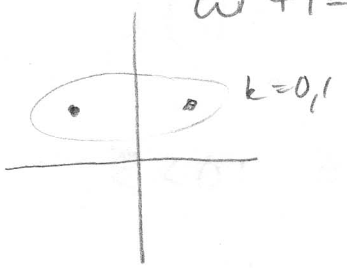
$$t f(t) \overset{\mathcal{F}}{\supset} i\left(\frac{\omega}{1+\omega^4}\right)' = i\left(\frac{1+\omega^4 - 4\omega^4}{(1+\omega^4)^2}\right) = i\,\frac{1-3\omega^4}{(1+\omega^4)^2}$$

$$\int_{-\infty}^{\infty} t f(t)\, dt = \int_{-\infty}^{\infty} t f(t) e^{-it\cdot 0} = \widehat{tf(t)}\,(0) = i \cdot 1 = i$$

$[E\bar{O}8?]$ b) $\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{(1+\omega^4)^2} d\omega =$

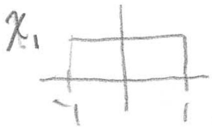
poler i öhp:

$\omega^4 + 1 = 0;\ \omega = e^{i(2k+1)\pi/4},\quad k = 0, 1, 2, 3$



$$= \frac{1}{2\pi} \cdot 2\pi i \left(\operatorname*{Res}_{e^{i\frac{\pi}{4}}} g(\omega) + \operatorname*{Res}_{e^{\frac{3\pi i}{4}}} g(\omega) \right)$$

$\boxed{E\bar{O}9}$ $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx = \langle \frac{\sin x}{x}, \frac{1}{x^2+1} \rangle \overset{\text{Plancherel}}{=} \frac{1}{2\pi}\langle \widehat{\frac{\sin x}{x}}, \widehat{\frac{1}{x^2+1}} \rangle =$



$$= \frac{1}{2\pi}\langle \pi\chi_1(\xi), \pi e^{-|\xi|} \rangle = \frac{\pi}{2}\int_{-1}^{1} e^{-|\xi|} d\xi = \pi\left(1 - \frac{1}{e}\right)$$

$\boxed{E\bar{O}11}$ $f(t) = \int_0^2 \frac{\sqrt{\omega}}{1+\omega} e^{i\omega t} dt \overset{\text{invers}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{i\omega t} dt$

$$\hat{f}(\omega) = \begin{cases} 0, & \omega \notin]0,2[\\ 2\pi \frac{\sqrt{\omega}}{1+\omega}, & \omega \in]0,2[\end{cases}$$

a) $\int_{-\infty}^{\infty} f(t)\cos t \, dt = ? = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left(e^{it} + e^{-it}\right) dt =$

$$= \frac{1}{2}\Big(\underset{\overset{\parallel}{0}}{\hat{f}(-1)} + \underset{\overset{\parallel}{\pi}}{\hat{f}(1)}\Big) = \pi \cdot \frac{1}{2} = \frac{\pi}{2}$$

b) $\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \frac{1}{2\pi}\int_0^2 (2\pi)^2 \frac{\omega}{(1+\omega)^2} d\omega$

Varianter av F-transf.

$f(x) \quad x > 0$

fortsätter fkt till neg. x

① Jämna forts $\quad f(-x) = f(x)$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = \int_{-\infty}^{\infty} \underbrace{f(x)\cos \xi x}_{j \cdot j = j} dx - \underbrace{i\int_{-\infty}^{\infty} f(x)\sin \xi x \, dx}_{=0} =$$

jämn · udda = udda

$$= 2\int_0^{\infty} f(x)\cos \xi x \, dx = 2\,\mathcal{F}_c(f)$$

Inversion:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi = \frac{1}{2\pi}\cdot 2\int_{\infty}^{\infty} \mathcal{F}_c(f)(\xi) e^{ix\xi} d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \underset{j}{\mathcal{F}_c(f)(\xi)}\,(\underset{j}{\cos \xi x} + i \underset{\downarrow 0}{\underset{u}{\sin \xi x}})\, d\xi = \frac{2}{\pi}\int_0^{\infty} \mathcal{F}_c[f](\xi) \cos \xi x \, d\xi$$

② Udda funkt. $\quad f(-x) = -f(x)$

$$\hat{f}(\xi) = \underbrace{\int_{-\infty}^{\infty} \underset{u}{f(x)}\, \underset{j}{\cos \xi x}\, dx}_{=0} \overset{}{\underset{=u}{}} - i \int_{-\infty}^{\infty} \underset{u}{f(x)}\, \underset{u}{\sin \xi x}\, dx \underset{=j}{} =$$

$$= -2i \int_{-\infty}^{\infty} f(x) \sin \xi x \, dx = -2i\, \mathcal{F}_s(f)$$ udda fkt av ξ

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \mathcal{F}_s(f)(\xi) \sin \xi x \, d\xi$$

Derivatan och trigon. F-transf.

$f(x)$, $x \in [0, \infty[\qquad \mathcal{F}_c(f)$; $\mathcal{F}_s(f)$

$\mathcal{F}_c(f') = ?$, $\quad \mathcal{F}_s(f') = ?$

$$\mathcal{F}_c(f') = \int_0^{\infty} f'(x) \cos x\xi \, dx = \Big[f(x) \cos x\xi \Big]_0^{\infty} +$$

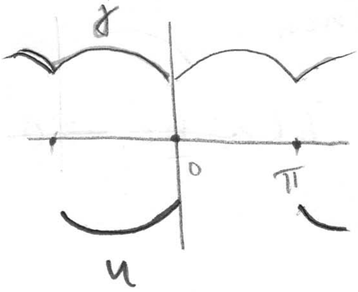
$$+ \xi \int_0^{\infty} f(x) \sin x\xi \, dx = \xi \int_0^{\infty} f(x) \sin x\xi \, dx - f(0) \qquad ??$$

$$= \xi \mathcal{F}_s(f) - f(0)$$

$$\mathcal{F}_s(f') = \int_0^{\infty} f'(x) \sin(x\xi)\, dx = \underbrace{\Big[f(x) \sin(x\xi) \Big]_0^{\infty}}_{0} - \xi \int_0^{\infty} f(x) \cos x\xi \, dx =$$

$$= -\xi \mathcal{F}_c(f)$$

F-serie på ett godtyckligt intervall.



$f(\theta)$:

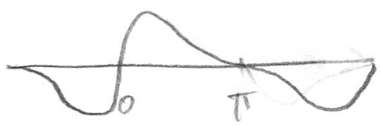
$\sum b_n \sin n\theta$

$\frac{a_0}{2} + \sum a_n \cos n\theta$

cos-serie: konvergerar mot $f(\theta)$ i alla pkter

sin-serie: konv. mot $f(\theta)$, $\theta \neq 0, \pm\pi, \ldots$; 0, $\theta = 0 \pm \pi$

om $f(0) = f(\pi) = 0$ konv F-serie mot f i alla pkter θ.



om $f(0) = f(\pi) = 0$ konv sin-serie som önskat.

$\Rightarrow$ använd sin-serie

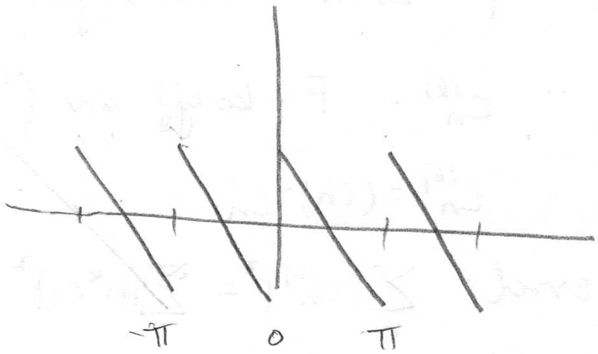
tillbaka till cos-serie.

Vi antar f glatt på $]0,\pi[$ och deriv. cos-serien

$f'(\theta) \rightarrow \sum a_n(-n) \sin n\theta$

vart konvergerar?

F-serie för f' konv. mot $f'(\theta)$ i $\theta \neq 0, \pi, \ldots$ och mot 0 i $\theta = 0, \pi, \ldots$



Endast om man vet att $f'(0) = f'(\pi) = 0$

$\Rightarrow$ F-serien för deriv konv mot f' överallt

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \quad t>0, \quad 0<x<\pi,$$

randvillk: $u(0,t) = u(\pi,t) = 0$

beg : $u(x,0) = f(x)$

söker lösn på formen:

$$u(x,t) = \sum_1^\infty T_n(t) \sin nx$$

$$T_n(t) = b_n e^{-tn^2}$$

b_n - sin - F-koeff av $f(x)$

$$u(x,t) = \sum_{n=1}^\infty b_n e^{-tn^2} \sin nx$$

antar $f(x)$ kontin; $f(0) = f(\pi) = 0$

① F-serie för $u(x,t)$ konv.

Weierstrass test: $\sum |b_n e^{-tn^2}| < \infty$!

$$\sum |b_n e^{-tn^2}| \le \Big(\underbrace{\sum |b_n|^2}_{<\infty}\Big)^{1/2} \Big(\sum e^{-2tn^2}\Big)^{1/2}$$

Cauchy-Schwarz Bessel

② Det är tillåtet att deriv. F-serie termvis

a) t-deriv.

$$\sum b_n e^{-tn^2} n^2 \sin(nx)$$

Konv. likformigt:

$$\sum |b_n e^{-tn^2} n^2| \le \Big(\sum |b_n|^2\Big)^{1/2} \Big(\sum e^{-2tn^2} n^4\Big)^{1/2} \quad \text{(Weierstrass)}$$

b) x-deriv analogt

Sätter in i ekv och kollar att den är uppfylld.

③ Randvillkoren

④ Begynnelsevillk är uppfyllda

$u(x,t) \to f(x), \quad t \to 0$

F-serie på godt. intervall

$f(x)$ perioden $2l$

variabelbyte.

$x = \frac{l}{\pi}\theta; \quad \varphi(\theta) = f(x) = f(\frac{l}{\pi}\theta)$

$\varphi(\theta)$ är 2π-periodisk?

$\varphi(\theta) - \varphi(\theta + 2\pi) = f(\frac{l}{\pi}\theta) - f(\frac{l\theta}{\pi} + \frac{2\pi l}{\pi}) = \underbrace{f(\frac{l}{\pi}\theta) - f(\frac{l}{\pi}\theta + 2l)}_{=0}$

Skriver F-serie för $\varphi(\theta)$:

$\varphi(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}; \quad c_n = \frac{1}{2\pi}\int^{\pi} \varphi(\theta) e^{-in\theta} d\theta$

$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\frac{x\pi}{l}}, \quad \theta = \frac{x\pi}{l}, \; d\theta = \frac{\pi}{l}dx \quad c_n = \frac{1}{2\pi}\cdot\frac{\pi}{l}\int_{-l}^{l} f(x) e^{-in\frac{\pi x}{l}} dx =$

$= \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\frac{\pi x}{l}} dx$

Trigonom. F-serie

$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi}{l} + \sum b_n \sin n\pi$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi}{l}x)$

Fourier 080128 Må L

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)$ på intervallet $(0, l)$

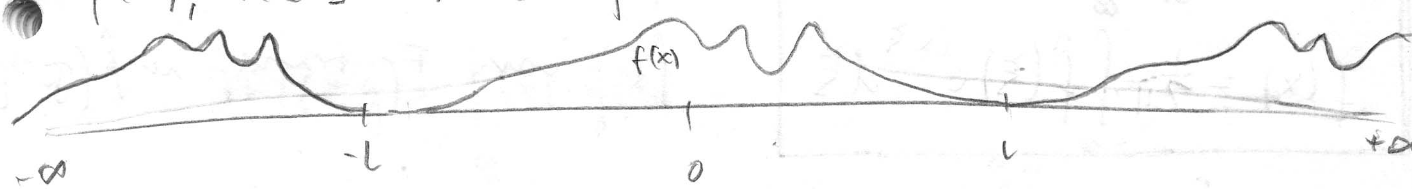
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

F-integral eller F-transformation

$f(x)$, $x \in]-\infty, \infty[$ $f(x) \to 0$, snabbt, $x \to \pm\infty$



tills $f(x)$ är litet när $|x| > l$, vi ändrar f utanför $]-l, l[$ så att f blir periodisk. Skriver F-serie för omdef. f. I $]-l, l[$ konv serie mot f.

$l \to \infty \Rightarrow$ nya formler på hela axeln

på $]-l, l[$

$$f(x) = \frac{1}{2l}\sum_{n=-\infty}^{\infty} c_{n,l}\, e^{in\frac{\pi x}{l}}$$

$$c_{n,l} = \int_{-l}^{l} f(x)\, e^{-in\frac{\pi x}{l}}\,dx$$

$$\Delta\xi = \frac{\pi}{l}$$

$$\xi_n = n\Delta\xi = \frac{n\pi}{l}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$f(x) = \sum_{-\infty}^{\infty} c_{n,l} e^{ix\xi_n}$$

$$c_{n,l} = \int_{-l}^{l} f(x) e^{-ix\xi_n} dx \underset{l\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} f(x) e^{-ix\xi_n} dx = \hat{f}(\xi_n)$$

$$f(x) = \frac{1}{2l} \sum c_{n,l} e^{ix\xi_n} \overset{l\to\infty}{\longrightarrow} \sum \hat{f}(\xi_n) \frac{1}{2l} e^{ix\xi_n} =$$

$$= \frac{1}{2\pi} \sum_{-\infty}^{\infty} \hat{f}(\xi_n) e^{ix\xi_n} \underbrace{\Delta \xi}_{\xi_{n+1}-\xi_n} \to \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi$$

DEF

$$\boxed{\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi \end{aligned}}$$

$\hat{f}$ - F-transf. av f $(\mathcal{F}[f])$

$f(x)$ inv. F-transf. av $\hat{f}$ $(\mathcal{F}^{-1}[\hat{f}])$

$$f \overset{\mathcal{F}}{\supset} \hat{f}$$

Klasser (Rum) av fkt för Fourier

$L^1 = L^1(\mathbb{R})$ består av fkt för vilka $\int_{-\infty}^{\infty} |f(x)| dx < \infty$

$L^2 = L^2(\mathbb{R})$ ——— " ——— $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$

Ex $f(x) = \frac{1}{x}$, $x > 1$ $\notin L^1$ $\in L^2$

Ex $f(x) = \frac{1}{\sqrt{x}}$ $0 < x < 1$ $\notin L^2$ $\in L^1$

$f \in L^2$ $\|f(x)\| = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{1/2}$ normen f (i L^2)

$f, g \in L^2$ $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$ skalärprod. f, g

$$\langle f, g \rangle = \|f\|^2$$

följden $f_n \in L^2$ konvergerar mot f i normen om

$\|f - f_n\| \underset{n\to\infty}{\to} 0$

Egenskaper hos $\mathcal{F}$

1) Om $f \in L^1$, så finns F-transf. (integral konv)

mer: $\hat{f}(\xi)$ kontin, $\to 0$, $|\xi| \to \infty$

2) (faltning) convolution: $f * g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy =$

$= \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy = g * f$

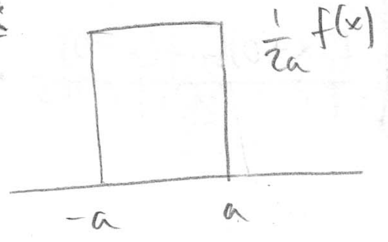
$\mathcal{F}(f*g) = \hat{f}(\xi) \cdot \hat{g}(\xi)$

$\mathcal{F}(f*g) = \int_{-\infty}^{\infty} (f*g)(x)\, e^{-ix\xi}\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\, g(y)\, e^{-i(x-y)\xi} e^{-iy\xi}\, dy dx$

$= \int_{-\infty}^{\infty} f(z)\, e^{-iz\xi}\, dz \cdot \int_{-\infty}^{\infty} g(y)\, e^{-iy\xi}\, dy = \hat{f}(\xi)\hat{g}(\xi)$

3) $\mathcal{F}$ - linjär $\quad \alpha f + \beta g \overset{\mathcal{F}}{\to} \alpha \hat{f} + \beta \hat{g}$

Ex



$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx =$

$= \frac{1}{2a}\int_{-a}^{a} 1 \cdot e^{-ix\xi}\, dx = \frac{1}{2a}\left[\frac{1}{-i\xi} e^{-ix\xi}\right]_{-a}^{a} =$

$= \frac{1}{2a}\frac{1}{-i\xi}\left(e^{-ia\xi} - e^{ia\xi}\right) = \frac{\sin a\xi}{a\xi}$

Ex $f(x) = e^{-x^2/2}$ $\qquad -xe^{-x^2/2} = \frac{d}{dx}(e^{-\frac{x^2}{2}})$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ix\xi} dx$$

$$\hat{f}'(\xi) = \int_{-\infty}^{\infty} e^{-x^2/2}(-ix) e^{-ix\xi} dx = i\int_{-\infty}^{\infty} (e^{-x^2/2})' e^{-ix\xi} dx =$$

$$= -\xi \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ix\xi} dx = -\xi \hat{f}(\xi)$$

$$\Rightarrow \hat{f}'(\xi) = -\xi \hat{f}(\xi) \qquad \frac{d\hat{f}}{\hat{f}} = -\xi d\xi$$

löser: $\hat{f}(\xi) = Ce^{-\xi^2/2}$

$$C = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}}$$

$$\widehat{\left(e^{-\frac{x^2}{2}}\right)} = \sqrt{\frac{2}{\pi}} e^{-\xi^2/2}$$

SATS (konvergenssatsen 1)

$f \in L^1$, $\hat{f} \in L^1$ f styckvis kontin

$$\Rightarrow \mathcal{F}^{-1}(\hat{f}) \equiv \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi = \begin{cases} f(x), & f \text{ kontin. i } x \\ \frac{f(x+0)+f(x-0)}{2} & \text{annars} \end{cases}$$

SATS (konvergenssatsen 2)

$f \in L^1$, styckv. kontin

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} e^{-\varepsilon\xi^2} d\xi = \begin{cases} f(x), & f \text{ kontin} \quad f \text{ kontin i } x \\ \frac{f(x+0)+f(x-0)}{2} & f \text{ inte kontin i } x \end{cases}$$

4) $f(x-a) \supset \int_{-\infty}^{\infty} f(x-a)\, e^{-ix\xi} dx = [x-a=y] =$

$= \int_{-\infty}^{\infty} f(y)\, e^{-i(y+a)} dy = e^{-ia\xi} \hat{f}(\xi)$

$f(x)e^{iax} \supset \hat{f}(\xi - a)$

5) $f'(x) \supset \int_{-\infty}^{\infty} f'(x) e^{-ix\xi} dx = -\int_{-\infty} f(x) (e^{-ix\xi})' dx = i\xi \hat{f}(\xi)$

$x f(x) \overset{\mathcal{F}}{\supset} \int_{-\infty}^{\infty} f(x) \underbrace{x e^{-ix\xi}}_{i\frac{\partial}{\partial\xi}(e^{-ix\xi}} dx = i\frac{\partial}{\partial \xi}\left(\int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx \right) = i \hat{f}'(\xi)$

$f, g;\ f, g, \hat{f}, \hat{g} \in L^1 \cap L^2$

$2\pi \langle f, g\rangle = 2\pi \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx = 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi\, \bar{g}(x) dx$

$= [e^{ix\xi} = \overline{e^{-ix\xi}}] = \int_{-\infty} \hat{f}(\xi)\, 2\pi \int_{-\infty}^{\infty} \overline{g(x)\, e^{-ix\xi}}\, dx\, d\xi =$

$= \int_{-\infty}^{\infty} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi = \langle \hat{f}, \hat{g} \rangle$

$2\pi \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$

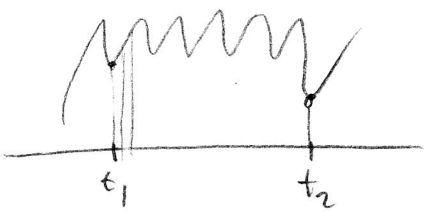
Fourier- och signalanalys

$$f(t) \qquad -\infty < t < \infty$$

Vi vill ha $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t)\,dt \approx \int_{0}^{T} e^{-it\omega} f(t)\,dt \approx$

(vi antar att $f(t)$ är VÄLDIGT liten för $t \notin]0,T[$)

$$\approx \sum e^{-it_n\omega} f(t_n)\,\Delta t$$ **sampling**



SATS (samplingsteoremet)

Antag att signalen $f(t)$ är frekv-begr.:

$$\hat{f}(\omega) = 0, \quad |\omega| > \Omega$$

Då kan man hitta signalen exakt genom

i tidpkt: $t_n = \frac{n\pi}{\Omega}$; $f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \underbrace{\frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}}_{=1 \text{ när } \Omega t - n\pi = 0}$

$$\left[\frac{\sin x}{x} \xrightarrow[x\to 0]{} 1\right]$$

BEVIS

Utveckla $\hat{f}(\omega)$, $\omega \in]-\Omega, \Omega[$ i F-serie

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_{-n} e^{\frac{-in\pi\omega}{\Omega}}$$ (− ändrar inget, ty fr. −∞ till ∞)

$$c_{-n} = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{\frac{in\pi\omega}{\Omega}}\,d\omega = \frac{1}{2\Omega}\frac{\pi}{\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{\frac{in\pi\omega}{\Omega}}\,d\omega =$$

$$= \frac{\pi}{\Omega}\cdot\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{\frac{in\pi\omega}{\Omega}}\,d\omega}_{\mathcal{F}^{-1}(\hat{f}(\omega))} = \frac{\pi}{\Omega} f\left(\frac{n\pi}{\Omega}\right) = \frac{\pi}{\Omega} f(t_n)$$

Invers Fourier

$$f(t) = \frac{\pi}{\lambda\Omega}\sum f\left(\frac{n\pi}{\lambda\Omega}\right)\mathcal{F}^{-1}\left(e^{-\frac{in\pi\omega}{\lambda\Omega}}\hat{g}_\lambda(\omega)\right) \overset{②}{=} \frac{\pi}{\lambda\Omega}\sum f\left(\frac{n\pi}{\lambda\Omega}\right) g_\lambda\left(t-\frac{n\pi}{\lambda\Omega}\right)$$

?

$$= \frac{\pi}{\lambda\Omega}\sum f\left(\frac{n\pi}{\lambda\Omega}\right)\frac{\cos\left(\frac{n\pi}{\lambda\Omega}-t\right)-\cos\left(\frac{n\pi}{\Omega}-\lambda t\right)}{\pi(\lambda-1)\Omega\left(\frac{n\pi}{\lambda\Omega}-t\right)^2}$$

Värmeledn.-ekv.

$u'_t = k u''_{xx}$ $k>0$, $-\infty<x<\infty$, $t>0$

$u(x,0) = f(x)$

hitta $u(x,t)$

$U(\xi,t) = \mathcal{F}_x(u(x,t))$

$\Rightarrow U'_t = -k\xi^2 U$

$\Rightarrow U(\xi,t) = e^{-tk\xi^2}U(\xi,0) = e^{-tk\xi^2}\hat{f}(\xi)$.

$$u(x,t) = \mathcal{F}^{-1}\left(e^{-tk\xi^2}\hat{f}(\xi)\right) \overset{⑦}{=} \mathcal{F}^{-1}\left(e^{-tk\xi^2}\right) * f(x) =$$

⑨: $e^{-a\frac{x^2}{2}} \overset{\mathcal{F}}{\supset} \sqrt{\frac{2\pi}{a}}\, e^{-\frac{\xi^2}{2a}}$

$a = ?$ $tk = \frac{1}{2a}$ $a = \frac{1}{2tk}$

$\sqrt{\frac{a}{2\pi}}\, e^{-a\frac{x^2}{2}} \supset e^{-\frac{\xi^2}{2a}}$

$\sqrt{\frac{1}{4tk\pi}}\, e^{-\frac{x^2}{4tk}} \supset e^{-tk\xi^2}$

$$= \frac{1}{\sqrt{4kt\pi}}\, e^{-\frac{x^2}{4tk}} * f(x) = \frac{1}{\sqrt{4tk\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy$$

2) $u'_t = k u''_{xx}$, $x>0$, $t>0$, $u(x,0) = f(x)$, $\underline{u(0,t) = e^{-t}}$

randvillkoret

Om den givna fkt i randvillkoret är $\neq 0$ är villkoret **inhomogent** annars **homogent**.

!!!!!!

Man får **endast** använda Fouriermetoden (serie, transform) om randvillkoren i variabeln i vilken man gör Fourier är homogena.

Om randvillk. inte är homogena gör man ett förber.-steg. Transformerar randvillk till homogena.

Hitta någon (enkel) fkn $v(x,t)$ som satisf. randvillk.

Sök lösn på formen:

$$u(x,t) = v(x,t) + w(x,t)$$

där $w(x,t)$ är den nya sökta fkt. w satisf. hom. randvillk.

$$v(x,t) = e^{-t}e^{-x}$$

$$u(x,t) = e^{-t}e^{-x} + w(x,t)$$

$$u_t' = -e^{-t}e^{-x} + w_t'$$

$$u_{xx}'' = e^{-t}e^{-x} + w_{xx}''$$

$$u_t' - ku_{xx}'' = w_t' - kw_{xx}'' - (k+1)e^{-t}e^{-x}$$

$$w_t' - kw_{xx}'' = (k+1)e^{-t}e^{-x}$$

$$u(x,0) = e^{-x} + w(x,0); \quad w(x,0) = f(x) - e^{-x}$$

$$u(0,t) = e^{-t} + w(0,t) = e^{-t}; \quad \boxed{w(0,t) = 0} \text{ hom. randvillk}$$

Löser probl. för w:

fortsätter $w(x,t)$ på hela axeln: (randvillk m. fkt ⇒ udda) (m. der. ⇒ jämn)

$$w_t' - kw_{xx}'' = (k+1)e^{-t}e^{-|x|}\operatorname{sgn}x$$

$$w(x,0) = \operatorname{sgn} f(|x|) - e^{-|x|}\operatorname{sgn}x$$

$$W(\xi,t) = \mathcal{F}_x(w(x,t)): \quad W_t' + k\xi^2 W = (k+1)e^{-t}\mathcal{F}(e^{-|x|}\operatorname{sgn}x)$$

(5): $f'(x) \supset i\xi\hat{f}(\xi)$

$$e^{-|x|}\operatorname{sgn}x = -\left(e^{-|x|}\right)'$$

$$\widehat{e^{-|x|}\operatorname{sgn}x} = -\widehat{\left(e^{-|x|}\right)'} = -i\xi\,\widehat{e^{-|x|}} \overset{(11)}{=} -i\xi \cdot \frac{2}{\xi^2+1} =$$

$$= -\frac{2i\xi}{\xi^2+1}$$

$$W'_t + k\xi^2 W = (k+1)e^{-t}\mathcal{F}\left(e^{-|x|}\operatorname{sgn}x\right) = -(k+1)e^{-t}\frac{2i\xi}{\xi^2+1}$$

- beg.-villkor Fourier:

$$W(\xi,0) = \mathcal{F}\left(\operatorname{sgn}x f(|x|) - e^{-|x|}\operatorname{sgn}x\right)$$

- Löser ODE (hittar $W(\xi,t)$) $\Rightarrow$ Invers F-transf.

Utan fortsättn:

Med $\mathcal{F}_s, \mathcal{F}_c$

Om randvillk har formen $w(0,t)=0 \Rightarrow \mathcal{F}_s$

—————— " —————— $w'_x(0,t)=0 \Rightarrow \mathcal{F}_c$

- $W(\xi,t) = \mathcal{F}_s(w(x,t))$

$\mathcal{F}_s(w'_t) = W'_t$

$$\left[\begin{array}{l} \mathcal{F}_c(f') = \xi\mathcal{F}_s(f) - f(0) \\ \mathcal{F}_s(f') = -\xi\mathcal{F}_c(f) \end{array}\right]$$

- $\mathcal{F}_s(w''_{xx}) = \mathcal{F}_s\left((w'_x)'_x\right) \overset{f=w'_x}{=} -\xi\mathcal{F}_c(w'_x) \overset{f=w}{=}$

$$= -\xi\left(\xi\mathcal{F}_s(w) - w(0,t)\right) = -\xi^2\mathcal{F}_s(w)$$

$\mathcal{F}_s$-transf. problem:

$$W'_t + k\xi^2 W = (k+1)e^{-t}\,\mathcal{F}_s(e^{-x})$$

$$W(\xi,0) = \mathcal{F}_s\left(f(x) - e^{-x}\right)$$

$$\hat{f}(\omega) = \frac{\pi}{\Omega} \sum f(t_n)\, e^{-\frac{in\pi\omega}{\Omega}}$$

invers F

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{i\omega t}\, d\omega =$$

$$= \frac{1}{2\pi}\frac{\pi}{\Omega} \sum_n f(t_n) \underbrace{\int_{-\Omega}^{\Omega} e^{-\frac{i\pi\omega n}{\Omega}} e^{i\omega t}\, d\omega}_{\text{beräknas}}$$

V.S.V.

DFT, FFT

$f(t_0) \ldots f(t_n) \ldots$ – diskreta värden (antar att signalen liten för $t > T$ och $t < 0$)

F-transformera diskreta data

$$\hat{f}\left(\frac{2\pi m}{T}\right) = \int_0^T e^{-2\pi i m \frac{t}{T}} f(t)\, dt \underset{\text{ersätt med integralsumma}}{\approx} \sum e^{-\frac{2\pi i m n}{N}} f\left(\frac{nT}{N}\right) \cdot \frac{T}{N} =$$

delar $[0,T]$ i N delintervall

$t_n = \frac{nT}{N}$, $n = 0, 1, \ldots, N-1$

$$= \frac{T}{N} \sum_{n=0}^{N-1} e^{-\frac{2\pi i n m}{N}} f(t_n)$$ DFT av följden $a_n = f(t_n)$, $n = 0, \ldots, N-1$

$$\hat{a}_m = \sum_{n=0}^{N-1} e^{-\frac{2\pi i m n}{N}} a_n$$ DFT av ändl. följd a_n

$\frac{T}{N}$?

Hur många mult för att ber. $\hat{a}_m$?

N för en $\hat{a}_m$

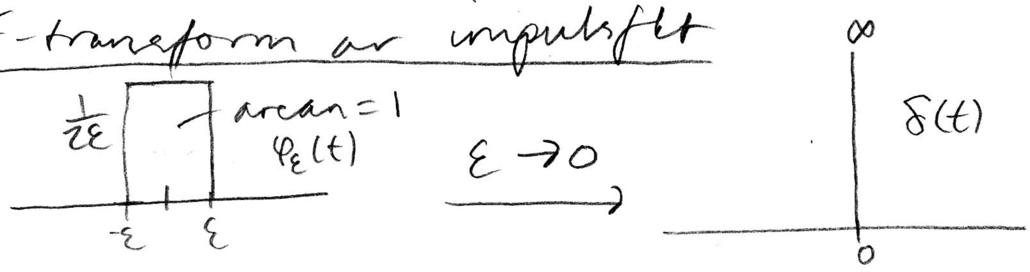
N^2 för alla N $\hat{a}_m$

FFT mest eff. för $N = 2^k$

1 MHz $\quad e^{i\omega t}\varphi(t)$ $\quad$ Amplitudmodulation

FM 100 MHz $\quad e^{i\omega\varphi(t)t}$ $\quad$ Frekvens- — " —

<u>F-transform av impulsfkt</u>



som vanliga fkt konvergerar den inte

om man har en "god" fkt $f(t)$:

$$\int_{-\infty}^{\infty} f(t)\varphi_\varepsilon(t)\,dt = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t)\,dt =$$

$$= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}(f(t)-f(0))\,dt + \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(0)\,dt \xrightarrow{\varepsilon\to 0} 0 + f(0) = f(0)$$

Man antar $\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$

$\delta(t-t_0)$ - impuls i tidpkt t_0

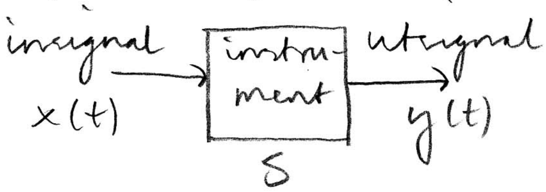
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

bl.a. $f(t) = e^{-i\omega t}$

$$\hat{\delta}(\omega) = \int e^{-i\omega t}\delta(t)\,dt = e^{-i\omega 0} = 1$$

$$\widehat{\delta(t-t_0)}(\omega) = \int e^{-i\omega t}\delta(t-t_0)\,dt = \underline{\underline{e^{-i\omega t_0}}}$$

Linjära dynamiska system



$S: x(t) \longmapsto y(t)$

S linjärt: $\alpha x_1 + \beta x_2 \longrightarrow \alpha y_1 + \beta y_2$

syst är tidsinvariant om syst inte ändrar sina egenskaper med tiden

om $S: x(t) \longrightarrow y(t)$

då $S: x(t-t_0) \longrightarrow y(t-t_0)$

om $x(t) = \delta(t)$ så kallas $h(t)$ impulssvar

SATS

$S: x(t) \longmapsto x * h(t)$

$$x(t) = \int x(\tau)\,\delta(t-\tau)\,d\tau = x * \delta$$

linj

$$Sx(t) = \int x(\tau)\, S\delta(t-\tau)\,d\tau = \int x(\tau)\,h(t-\tau)\,d\tau =$$

$$= x * h$$

Vi tar $x(t) = e^{-i\omega t}$

$$\Rightarrow y(t) = e^{-i\omega t} * h = \int_{-\infty}^{\infty} e^{-i\tau\omega}\,h(t-\tau)\,d\tau = e^{-it\omega}\underbrace{\hat{h}(\omega)}_{}$$

frekv-karakteristik av S

allmänt

$S: x(t) \longmapsto x * h$

$\mathcal{F}$ $\hat{x}(\omega) \longmapsto \hat{x}(\omega) \cdot \hat{h}(\omega)$

$\boxed{\text{EÖ 16}}$ $x_0(t) = \frac{1}{1+t^2} \longmapsto y_0(t) = \frac{t}{(4+t^2)^2}$

bestäm impulssvar, systemfkt och svaret på $\cos\omega t$

$$\frac{1}{1+t^2} \longmapsto \frac{t}{(4+t^2)^2}$$

$$\widehat{\frac{1}{1+t^2}} = \frac{\pi}{1} e^{-|\omega|} = \pi e^{-|\omega|} = \hat{x}_0$$

$$\widehat{\frac{t}{(4+t^2)^2}} = ? \; ; \quad \frac{t}{(4+t^2)^2} = -\frac{1}{2}\frac{d}{dt}\left(\frac{1}{4+t^2}\right)$$

$$\widehat{y_0(t)} = -\frac{1}{2} i\omega \frac{\pi}{2} e^{-2|\omega|} . \quad \hat{x} \cdot \hat{h} = \hat{y}$$

$$\Rightarrow \quad \hat{h} = \frac{\hat{y}_0}{\hat{x}_0} = \frac{-\frac{\pi i\omega}{4} e^{-2|\omega|}}{\pi e^{-|\omega|}} = -\frac{i\omega e^{-|\omega|}}{4}$$

$h(t) = ?$

$$h(t) = \left(-\frac{1}{4\pi}\frac{1}{1+t^2}\right)' = + \frac{1}{\cancel{4}_2\pi}\frac{1}{(1+t^2)^2} \cdot \cancel{2}t =$$

$$= \frac{1}{2\pi}\frac{t}{(1+t^2)^2}$$

$x(t) = \cos\omega t; \quad y(t) = ? = \cos\omega t * h$

$$x(t) = \cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$y(t) = \frac{1}{2}\left(\hat{h}(\omega)e^{i\omega t} + \hat{h}(-\omega)e^{-i\omega t}\right) =$$

$$= \frac{1}{2}\frac{-i\omega}{4} e^{-|\omega|}\left(e^{i\omega t} - e^{-i\omega t}\right) = \frac{\omega}{4} e^{-|\omega|} \sin\omega t$$

Systemet är kausalt om $y(t)$ endast beror på $x(\tau)$, $\tau \le t$. Syst är kausalt om $h(t) = 0$, $t < 0$

SATS

Systemet är stabilt

$\iff$

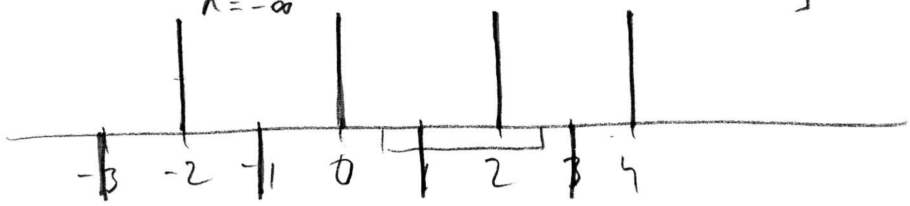
$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

[Systemet är stabilt om

$|x(t)| \le A \Rightarrow |y(t)| \le K\Delta$ för ngt K]

$\boxed{\text{EÖ18}}$ $\dfrac{1}{4+t^2} \longmapsto e^{-2t^2}$

$$x(t) = \sum_{n=-\infty}^{\infty} \left[2\delta(t-2n) - \delta(t-2n-1) \right]$$



$$y(t) = \sum_{-\infty}^{\infty} c_k e^{ik\pi t} \; ; \quad c_k = ?$$

$$x(t) = \sum_{-\infty}^{\infty} d_k e^{ik\pi t}$$

Söker d_k - F-koeff.

$$d_k = \frac{1}{2} \int_{1/2}^{5/2} e^{-ik\pi t} x(t)\, dt = \frac{1}{2}\left(e^{-ik\pi \cdot 1} + 2e^{-ik\pi \cdot 2} \right) = \frac{1}{2}\left((-1)^{k+1} + 2\right)$$

$$x(t) = \sum \frac{1}{2}\left((-1)^{k+1} + 2\right) e^{ik\pi t}$$

$$y(t) = S(x(t)) \overset{\text{lin}}{=} \sum \frac{1}{2}\left((-1)^{k+1} + 2\right) S\left(e^{-ik\pi t}\right) =$$

$$= \sum \frac{1}{2}\left((-1)^{k+1} + 2\right) e^{-itk\pi} \cdot \hat{h}(-ik\pi)$$

$\hat{h} = ?$ $\quad \widehat{\frac{1}{4+t^2}} \cdot \hat{h} = \widehat{e^{-2t^2}}$ $\quad$ 9, $a=4$

$$\frac{\pi}{2} e^{-2|\omega|} \hat{h}(\omega) = \sqrt{\frac{2\pi}{4}}\, e^{-\frac{\omega^2}{8}}$$

$$\hat{h}(\omega) = e^{-\frac{\omega^2}{8}} \sqrt{\frac{2}{\pi}}\, e^{2|\omega|}$$

$$y(t) = \sum \frac{1}{2}\left((-1)^{k+1} + 2\right) \sqrt{\frac{2}{\pi}}\, e^{-\frac{k^2\pi^2}{8}}\, e^{2|k|\pi}\, e^{itk\pi}$$

Sampling

1) $f(t)$ signal, bandbegränsad

$\hat{f}(\omega) = 0 \,;\; |\omega| \leq \Omega$

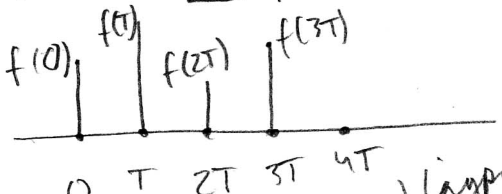
Vi samplar signalen med frekv $\alpha = \frac{2\pi}{T} \geq 2\Omega$

$\iff$ med perioden T

$f(nT)$

Vi betraktar samplade signalen som impulståg.

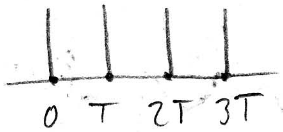
$$F(t) = \sum f(t)\,\delta(t - Tn)$$



Bevisa att om vi lågpass-filtrerar $F(t)$ med avhuggn.-frekv. Ω och multiplicerar med T, så får vi tillbaka $f(t)$:

$$F(t) = f(t) \cdot S(t)$$

$$S(t) = \sum_{-\infty}^{\infty} \delta(t - Tn)$$



$S(t)$ är T-periodisk

Utveckla S i F-serie

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-i\frac{t}{T}2\pi n} S(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-i\frac{t}{T}2\pi n}\,\delta(t)\,dt = \frac{1}{T}$$

$$\Rightarrow S(t) = \sum_{-\infty}^{\infty} \frac{1}{T} e^{in\frac{2\pi t}{T}}$$

$$F(t) = f(t)\,S(t) = \frac{1}{T} f(t) \sum e^{i\frac{2\pi}{T}nt}$$

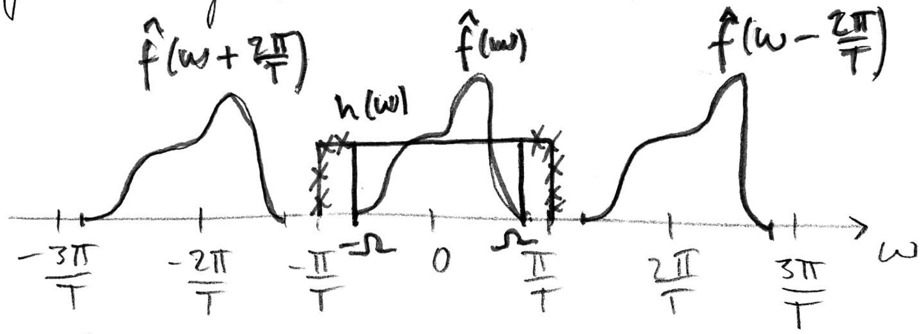
$$\hat{F}(\omega) = \frac{1}{T}\int_{-\infty}^{\infty} f(t) \sum e^{i\frac{2\pi}{T}nt} e^{-i\omega t}\,dt = \frac{1}{T}\sum \int_{-\infty}^{\infty} f(t)\, e^{-it(\omega - \frac{2\pi}{T}n)}\,dt =$$

$$= \frac{1}{T}\sum_{n} \hat{f}\left(\omega - \frac{2\pi}{T}n\right)$$

Filtrering

$$h(\omega) = \begin{cases} 0, & |\omega| > \Omega \\ 1, & |\omega| \leq \Omega \end{cases}$$

filtrering består av mult av $\hat{F}$ med h



$$\hat{F}(\omega) = \frac{1}{T}\hat{f}(\omega)$$

<u>s. 235, 8 översampling:</u>

Samplingsteorem: om $\hat{f}(\omega) = 0$, $|\omega| > \Omega$

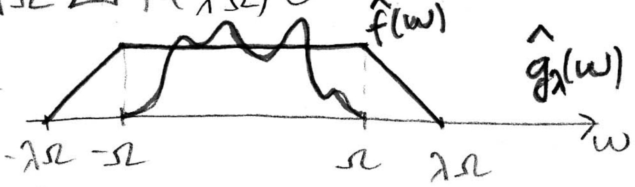
$$\Rightarrow f(t) = \sum f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}.$$

termerna går mot 0 långsamt när $n \to \pm\infty$

$\lambda > 1$, sampla med frekv. $\lambda\Omega$

$$\hat{f}(\omega) = \frac{\pi}{\lambda\Omega} \sum f\left(\frac{n\pi}{\lambda\Omega}\right) e^{-i\frac{n\pi\omega}{\lambda\Omega}}$$

$\hat{g}_\lambda(\omega) =$



$$\hat{f}(\omega) = \hat{f}(\omega)\,\hat{g}_\lambda(\omega)$$

$$\hat{f}(\omega) = \frac{\pi}{\lambda\Omega} \sum f\left(\frac{n\pi}{\lambda\Omega}\right) e^{-\frac{in\pi\omega}{\lambda\Omega}}\, \hat{g}_\lambda(\omega)$$

Geometri i $\mathbb{C}^d$

$\mathbb{C}^d$ består av vektorer:

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \qquad x_j \in \mathbb{C}$$

Operationer:

$$\mathbf{y} = (y_1, y_2, \ldots, y_d)$$

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_d + y_d)$$

$$\alpha \mathbf{x} = (\alpha x_1, \ldots, \alpha x_d)$$

Skalärprodukten (inre produkten):

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 \overline{y_1} + \ldots + x_d \overline{y_d}$$

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$$

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$$

$$\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$$

$$\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$$

Unitärt rum!

DEF

normen av $\mathbf{x}$:

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \sum |x_j|^2$$

$$\|\mathbf{x}\| = \left(\sum |x_j|^2 \right)^{\frac{1}{2}}$$

DEF

$\mathbf{x}, \mathbf{y}$ **ortogonala** om:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

$\langle f, g \rangle$; $\|f\| = \sqrt{\langle f, f\rangle}$

$$\boxed{|\langle f, g\rangle| \leq \|f\| \cdot \|g\|}$$

$$\boxed{\|f+g\| \leq \|f\| + \|g\|}$$

φ_n ortonormalt system

$c_n = c_n(f) = \langle f, \varphi_n \rangle$

$$\boxed{\sum |c_n|^2 \leq \|f\|^2}$$

SATS

Konvergens

φ_n bas om:

1) $\sum \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2} \varphi_n = f$

2) $\sum \left| \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|} \right|^2 = \|f\|^2$ Parseval

3) Alla $\langle f, \varphi_n \rangle = 0 \Rightarrow f = 0$

Ex på baser

1) $L^2]-\pi, \pi[$, $\varphi_n(x) = e^{inx}$, $n = 0, \pm 1, \ldots$

Kolla: är $\{\varphi_n\}$ ortonormalt?

$\langle e^{inx}, e^{imx} \rangle$, $m \neq n$

$$\int_{-\pi}^{\pi} e^{inx} \overline{e^{imx}}\, dx = \int_{-\pi}^{\pi} e^{ix(n-m)}\, dx = \left[\frac{i}{m-n} e^{ix(n-m)} \right]_{-\pi}^{\pi} = 0$$

Ortogonalt!

Normerat?

$$\|e^{inx}\|^2 = \int_{-\pi}^{\pi} |e^{inx}|^2\,dx = \int_{-\pi}^{\pi} 1\cdot dx = 2\pi \Rightarrow \|\varphi_n\| = \sqrt{2\pi}$$

För att få ett normerat syst.:

$$\psi_n = \frac{\varphi_n}{\|\varphi_n\|} = \frac{e^{inx}}{\sqrt{2\pi}} \quad ; \quad \{\psi_n\} \text{ ortonormalt syst.}$$

2) $L^2]-\pi,\pi[$; $\varphi_n = \cos nx$, $n = 0, 1, \ldots$

$\varphi_n = -\sin nx$, $n = -1, -2, \ldots$

för att kolla att syst ortogonalt:

$$\langle \cos nx, \sin mx \rangle \overset{?}{=} 0$$

$$\left.\begin{aligned} \langle \cos nx, \cos mx \rangle &\overset{?}{=} 0 \\ \langle \sin nx, \sin mx \rangle &\overset{?}{=} 0 \end{aligned}\right\} n \neq m$$

för att kolla φ_n normerade:

$$\|\cos nx\|^2 = \int_{-\pi}^{\pi} |\cos nx|^2\,dx = \int_{-\pi}^{\pi} \cos^2 nx\,dx =$$

$$= \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2}\,dx = \begin{cases} 2\pi, & n = 0 \\ \pi, & n \neq 0 \end{cases}$$

för att normera:

$$\psi_n = \frac{\varphi_n}{\|\varphi_n\|}$$

3) $L^2]0,\pi[$

a) $\varphi_n = \sin n\varphi$, $n = 1, 2, 3, \ldots$

b) $\varphi_n = \cos n\varphi$, $n = 0, 1, 2, \ldots$

Kolla att syst ortogonala, icke ortonormala

Hur normera syst? 3.3: 4, 6, 7

$\boxed{3.3.1}$ $f_n \to f$ i norm

g

Visa: $\langle f_n, g\rangle \to \langle f, g\rangle$

$$|\langle f_n, g\rangle - \langle f, g\rangle| \xrightarrow{??} 0$$

$$|\langle f_n - f, g\rangle| \overset{C.S.}{\leq} \underbrace{\|f_n - f\|}_{\downarrow\, 0} \cdot \underbrace{\|g\|}_{\text{konst}} \to 0$$

Om man endast vet att:

$\langle f_n, g\rangle \to \langle f, g\rangle \quad \forall \; g$ följer ej $f_n \to f$

$\boxed{3.3.9}$ Låt φ_n vara en <u>ortonormal bas</u>

f, g

Visa att:

$$\sum \langle f, \varphi_n\rangle \langle \varphi_n, g\rangle = \langle f, g\rangle$$

<u>Bevis:</u> $\sum \langle f, \varphi_n\rangle \varphi_n = f$, ty φ_n bas

skal. prod. med g:

$$\sum \langle f, \varphi_n\rangle \langle \varphi_n, g\rangle = \langle f, g\rangle$$

$\boxed{3.3.10}$ a) $\sum_1^\infty \frac{1}{n^4}$

$f(\theta) = \theta^2, \quad -\pi < \theta < \pi$

$$\frac{\pi^2}{3} + 4\sum \frac{(-1)^n}{n^2} \cos n\theta$$

$$\langle f, \sin n\theta\rangle = 0$$

$$\langle f, 1\rangle = \langle f, \cos 0\cdot\theta\rangle = \frac{\pi^2}{3}\cdot 2\pi$$

$$\frac{\langle f, \cos n\theta\rangle}{\|\cos n\theta\|^2} = \frac{4\cdot(-1)^n}{n^2} \; ; \quad \|\cos n\theta\| = \sqrt{\pi}, \; n\neq 0$$
$$\sqrt{2\pi}, \; n = 0$$

$$\frac{\pi^2}{3} + 4\sum \frac{(-1)^n}{n^2}\cos n\theta = \sum \frac{\langle f, \cos n\theta\rangle}{\|\cos n\theta\|^2}\cos n\theta$$

$$\sum \left|\frac{\langle f, \cos n\theta\rangle}{\|\cos n\theta\|}\right|^2 = \|f\|^2 = \int_{-\pi}^{\pi}\theta^4\,d\theta = \frac{2\pi^5}{5}$$

$$\frac{\left(\frac{2\pi^3}{3}\right)^2}{2\pi} + \sum\left(\frac{4}{n^2}\sqrt{\pi}\right)^2 = \frac{2\pi^5}{5}$$

$$\frac{2}{9}\pi^5 + 16\pi\sum_1^{\infty}\frac{1}{n^4} = \frac{2\pi^5}{5}$$

$$\Rightarrow 16\pi\sum\frac{1}{n^4} = \frac{18\pi^5}{45} - \frac{10\pi^5}{45} = \frac{8\pi^5}{45}$$

$$\Rightarrow \sum_1^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

b) $\sum_1^{\infty}\frac{1}{(2n-1)^6} = ?$

17: $f(\theta) = \theta(\pi - |\theta|) \quad]-\pi, \pi[$

$$f(\theta) = \frac{8}{\pi}\sum\frac{\sin((2n-1)\theta)}{(2n-1)^3} = \sum\frac{\langle f, \sin((2n-1)\theta)\rangle}{\|\sin((2n-1)\theta)\|^2}\cdot\sin((2n-1)\theta)$$

$$\frac{\langle f, \sin((2n-1)\theta)\rangle}{2\pi} = \frac{8}{\pi}\frac{1}{(2n-1)^2}$$

Parceval:

$$\int_{-\pi}^{\pi} f^2(\theta)\,d\theta = \sum\left|\frac{\langle f, \sin((2n-1)\theta)\rangle}{\|\sin((2n-1)\theta)\|}\right|^2$$

$$= \left(\frac{8}{\pi}\sqrt{2\pi}\right)^2 \sum \frac{1}{(2n-1)^6} = \frac{128}{\pi} \sum \frac{1}{(2n-1)^6} = \int_{-\pi}^{\pi} \theta^2(\pi - |\theta|)^2 \, d\theta =$$

$$= 2\int_0^{\pi} \theta^2(\pi-\theta)^2 \, d\theta = 2\int_0^{\pi} (\theta^2\pi^2 - 2\pi\theta^3 + \theta^4) \, d\theta =$$

$$= 2\left[\frac{\pi^2\theta^3}{3} - \frac{2\pi\theta^4}{4} + \frac{\theta^5}{5}\right]_0^{\pi} = 2\pi^5\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = 2\pi^5\left(\frac{10-15+6}{30}\right) =$$

$$= \frac{\pi^5}{15}$$

$\boxed{3.3.11}$ f 2π-period. reell

Visa att $\langle f, f' \rangle = 0$

Utveckla f i F-serie map. syst $\varphi_n = \begin{cases} \cos n\theta / \|\cos n\theta\| \\ \sin n\theta / \|\sin n\theta\| \end{cases}$

$$f(\theta) = \sum \langle f, \varphi_n \rangle \varphi_n$$

$$\langle f', \cos n\theta \rangle = \int_{-\pi}^{\pi} f'(\theta) \cos n\theta \, d\theta \overset{p.i.}{=} n \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta =$$

$$= n \langle f, \sin n\theta \rangle$$

$$\langle f', \sin n\theta \rangle \overset{p.s.s.}{=} -n \langle f, \cos n\theta \rangle$$

$$\langle f(\theta), f'(\theta) \rangle = \sum \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|} \cdot \frac{\langle f', \varphi_n \rangle}{\|\varphi_n\|} =$$

$$= \sum \frac{\langle f, \sin n\theta \rangle}{\|\sin n\theta\|} \cdot \frac{\langle f', \sin n\theta \rangle}{\|\sin n\theta\|} +$$

$$+ \sum \frac{\langle f, \cos n\theta \rangle}{\|\cos n\theta\|} \cdot \frac{\langle f', \cos n\theta \rangle}{\|\cos n\theta\|} =$$

$$= \sum \frac{\langle f, \sin n\theta \rangle \cdot (-n) \langle f, \cos n\theta \rangle}{\underbrace{\| \sin n\theta \|^2}_{=\pi}} + \sum \frac{\langle f, \cos n\theta \rangle \cdot n \langle f, \sin n\theta \rangle}{\underbrace{\| \cos n\theta \|^2}_{=\pi}}$$

$$= 0$$

$$\int_{-\pi}^{\pi} f(\theta) f'(\theta)\, d\theta = \int_{-\pi}^{\pi} \frac{1}{2}\left[(f^2)'\right] d\theta = \frac{1}{2}\left[f^2\right]_{-\pi}^{\pi} = 0$$

$\boxed{3.4.2}$ $\varphi_0(x) = 1$, $\varphi_1(x) = x$, $\varphi_2(x) = x^2$ $\quad]0, \infty[$

$w(x) = e^{-x}$; $L^2_w\,]0, \infty[$

Ortogonalisera!

1) Kolla fkt ej ortogonala:

$$\langle \varphi_0, \varphi_1 \rangle_w = \int_0^\infty x e^{-x}\, dx = 1$$

$$\langle \varphi_0, \varphi_2 \rangle_w = \int_0^\infty 1 \cdot x^2 \cdot e^{-x}\, dx = 2$$

$$\langle \varphi_1, \varphi_2 \rangle_w = \int_0^\infty x \cdot x^2 e^{-x}\, dx = 6$$

$f_0 = \varphi_0$, $f_1(x) = ax + b$, $f_2(x) = Ax^2 + Bx + C$

f_0, f_1, f_2 - ortonormerat

$$0 = \langle f_1, f_2 \rangle_w = \langle ax + b, 1 \rangle_w = a \langle x, 1 \rangle_w + b \langle 1, 1 \rangle_w =$$

$$= a \int_0^\infty x e^{-x}\, dx + b \int_0^\infty e^{-x}\, dx = a + b = 0 \Rightarrow b = -a$$

Välj $b = 1 \Rightarrow a = -1$

$$\tilde{f}_1 = -x+1 \Rightarrow f_1 = \frac{\tilde{f}_1}{\|\tilde{f}_1\|}$$

$$\|\tilde{f}_1\|^2 = \int_0^\infty (-x+1)^2 e^{-x}\,dx = \int_0^\infty (x^2-2x+1)e^{-x}\,dx = 2! - 2\cdot 1! + 1! = 1$$

$$\Rightarrow f_1 = \tilde{f}_1 = -x+1$$

$$\tilde{f}_2 = Ax^2 + Bx + 1$$

$$\langle \tilde{f}_2, f_0 \rangle = 0$$

$$\langle \tilde{f}_2, f_1 \rangle = 0$$

$$\langle \tilde{f}_2, f_0 \rangle = \langle Ax^2+Bx+1, 1 \rangle = \int_0^\infty (Ax^2+Bx+1)e^{-x}\,dx =$$

$$= A\cdot 2! + B + 1 = 0 \Rightarrow B = -2A$$

$$\langle \tilde{f}_2, f_1 \rangle = \langle Ax^2 - 2Ax + 1, -x+1 \rangle =$$

$$= \int_0^\infty (Ax^2-2Ax+1)(-x+1)e^{-x}\,dx$$

Hitta ett polynom av högst grad 2, $P(x)$,
s.a. $\|P(x) - x^3\|$ minimeras

Appr.-satsen: om vi har ort. syst $f_0, f_1, f_2, \ldots, f_N$
så ges bästa appr. $\|F - \sum c_k f_k\|$ av c_k - F-koeff av F map f_k

Vi söker $P(x) = c_0 f_0 + c_1 f_1 + c_2 f_2$

$$c_0 = \langle x^3, f_0 \rangle, \quad c_1 = \langle x^3, f_1 \rangle, \quad c_2 = \langle x^3, f_2 \rangle$$

3.3.7

c) $a\cos x + b\sin x$

$$\langle \cos x, \sin x \rangle \overset{??}{=} 0$$

$$\int_0^\pi \cos x \sin x \, dx = \left[\frac{\sin^2 x}{2} \right]_0^\pi = 0 \checkmark$$

normera; BAS

Sturm-Liouville-problemet

Ekvation på intervallet $]a,b[$

$$(f'r(x))' + b(x)f + \lambda w(x)f$$

r, b, w - reella fkt

$w>0$ - viktfkt.

Om interv. är oändligt $\Rightarrow$ probl är singulärt

Vi betraktar reguljära probl (ändl interv.)

Vi kräver: r, b, w - begr. på $]a,b[$

$$r, w > c_0 > 0$$

Randvillk:

① separerade: $\begin{cases} \alpha_1 f(a) + \beta_1 f'(a) = 0 \\ \alpha_2 f(b) + \beta_2 f'(b) = 0 \end{cases}$ $\alpha_j, \beta_j \in \mathbb{R}$

② periodiska $\begin{cases} f(a) = f(b) \\ f'(a) = f'(b) \end{cases}$

Det reg St-L-probl. är S-L-ekv på ändligt intervall med "goda" f, b, w och med sep. eller per. randvillk

Om några av ovanstående villk ej är uppfyllda (oändl interv. / dåliga koeff. ⇒ sing. probl)

λ - Spektralparametern

DEF

λ kallas **egenvärde** till S-L-probl.
om ekv + randvillk har en lösn $f = f_\lambda(x)$
som inte är nollfkt. Sådan f_λ kallas
egenfunktion som motsvarar egenvärdet λ

SATS

För ett reg. S-L-probl. finns oändligt många egenvärden vilka formar en följd som $\to +\infty$
$n \to +\infty$ motsv. egenfkt
$f_{\lambda_n} \equiv f_n$ är ortogonala mot varandra i $L^2_w\,]a,b[$
och formar en bas i $L^2_w\,]a,b[$

Kunna bevisa:

1) egenvärdena är reella

2) egenfkt är ortogonala

Ex

$$f'' + \lambda f = 0, \qquad 0 \le x \le \pi$$

$f(0) = 0$ $\qquad r = 1$

$f'(\pi) = 0$ $\qquad b(x) = 0$

$\qquad w = 1$

$f'' + \lambda f = 0$; kar. ekv.: $k^2 + \lambda = 0 \Rightarrow k = \pm\sqrt{-\lambda}$

3 fall:

1. $\lambda < 0$, $\lambda = -\beta^2 \Rightarrow k = \pm\beta$ allm. lösn:

$$f = A\cosh\beta x + B\sinh\beta x$$

$$f(0) = A = 0 \Rightarrow f(x) = B\sinh\beta x$$

$f(\pi) = B\beta\cosh(\beta\pi) = 0$ omöjligt $\Rightarrow$ Inga neg. λ

2. $\lambda = 0 \Rightarrow k = 0$; allm. lösn:

$$f = A + Bx$$

$$f(0) = A = 0 \Rightarrow f(x) = Bx$$

$f'(\pi) = B = 0$ omöjligt $\Rightarrow \lambda \neq 0$

3. $\lambda > 0$, $\lambda = \beta^2 \Rightarrow k = \pm i\beta$; allm. lösn:

$$f = A\sin\beta x + B\cos\beta x$$

$$f(0) = B = 0$$

$f'(\pi) = A\beta\cos\beta\pi = 0 \Rightarrow \cos\beta\pi = 0$; $\beta\pi = (n + \frac{1}{2})\pi$

$\Rightarrow \beta = n + \frac{1}{2}$

$f_n(x) = A\sin((n + \frac{1}{2})x)$ egenfkt.

Systemet $x^k \in \mathbb{C}^d$ kallas ett <u>ortogonalt system</u> om $\langle x^k, x^j \rangle = 0 \quad \forall k \neq j$

Systemet är <u>ortonormalt</u> om ortogonalt och dessutom, $\|x^k\| = 1 \quad \forall k$

Låt oss ha x^k, $k = 1, \dots, m$ med ett ortogonalt system

$$\alpha_k = \frac{\langle y, x^k \rangle}{\|x^k\|^2} \quad \text{F-koeff av } y \text{ map syst } x^k$$

och om $m = d$

$$y = \sum \alpha_k x^k \quad \text{F-serie av } y \text{ map syst. } x^k$$

BEVIS

Om x^k ortogonala $\Rightarrow$ linj ober.

om $\sum c_k x^k = 0$; skal. mult. med x^j

$$\sum c_k \langle x^k, x^j \rangle = 0$$

bara termen $k = j$ överlever

$$\Rightarrow c_j \langle x^j, x^j \rangle = 0 \Rightarrow c_j = 0$$

$\Rightarrow x^k$ bildar bas

$\Rightarrow y = \sum \alpha_k x^k$; vi hittar α_k.

mult. med x^j:

$$\langle y, x^j \rangle = \sum \alpha_k \langle x^k, x^j \rangle$$

bara termen $k = j$ överlever:

$$\langle y, x^j \rangle = \alpha_j \langle x^j, x^j \rangle = \alpha_j \|x^j\|^2$$

[Om x^k ortonormalt: $\alpha_k = \langle y, x^k \rangle$]

Pythagoras:

Om x^k ortonormalt:

$$\|y\|^2 = \sum |\alpha_k|^2$$

BEVIS

$y = \sum \alpha_k x^k$ mult med y

$$\langle y, y\rangle = \langle \sum \alpha_k x^k, \sum \alpha_j x^j\rangle = \sum_{j,k} \alpha_k \bar{\alpha}_j \langle x^k, x^j\rangle =$$

bara $k=j$ överlever $= \sum_k \alpha_k \bar{\alpha}_k \langle x^k, x^k\rangle = \sum_k |\alpha_k|$

3 fundamentala olikheter

I. Cauchy-Schwartz:

$x, y \in \mathbb{C}^d: \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

1) Om $\langle x, y\rangle = 0$ ok!

antar $\langle x, y\rangle \neq 0$

2) Om vi ersatter y med θy, $|\theta| = 1$

$\Rightarrow |\langle x, \theta y\rangle| = |\langle x, y\rangle|, \quad \|\theta y\| = \|y\|$

Välj θ: $\langle x, \theta y\rangle > 0$

$\arg\theta = \arg\langle x, y\rangle$

3) Antag då $\langle x, y\rangle > 0$

$0 \leq \|x + ty\|^2 = \langle x + ty, x + ty\rangle = \langle x, x\rangle + t\langle y, x\rangle$

$+ t\langle x, y\rangle + t^2\langle y, y\rangle = \|x\|^2 + 2t\langle x, y\rangle + \|y\|^2 \geq 0 \quad \forall t \in \mathbb{R}$

$\Rightarrow$ (teori för kvadr polynom) $\langle x, y\rangle^2 \leq \|x\|^2 \|y\|^2$

II Triangelolikheten

alla x, y: $\|x+y\| \leq \|x\| + \|y\|$

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle$$

$$+ \langle y,y \rangle \leq \|x\|^2 + |\langle x,y \rangle| + |\langle y,x \rangle| + \|y\|^2$$

$$\overset{C-S}{\leq} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

III Bessel-olikheten

Om $x^1, \ldots, x^m$ är ett ortonormalt syst

$$\alpha_k = \langle y, x^k \rangle \Rightarrow \sum |\alpha_k|^2 \leq \|y\|^2$$

Betrakta: $z = y - \sum_1^m \alpha_k x^k$

$$0 \leq \|z\|^2 = \langle y - \sum \alpha_k x^k, y - \sum \alpha_k x^k \rangle =$$

$$= \langle y, y \rangle - \sum \alpha_k \langle x^k, y \rangle - \sum \overline{\alpha_k} \langle y, x^k \rangle +$$

$$+ \langle \sum \alpha_k x^k, \sum \alpha_k x^k \rangle = \|y\|^2 - \sum \alpha_k \overline{\alpha_k} - \sum \overline{\alpha_k} \alpha_k$$

$$+ \sum |\alpha_k|^2 = \|y\|^2 - \sum |\alpha_k|^2 \geq 0$$

Hilbertrum

$L^2(I)$, $I =]a, b[$

$L^2(I)$ består av fkt $f(x)$ med kompl. värden

$$\int_a^b |f(x)|^2 dx < \infty$$

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

har alla egenskaper som skalär-prod ovan

$\|f\|^2 = \int_a^b |f(x)|^2 dx = \langle f, f \rangle \Rightarrow$ De 3 olikheterna gäller

$w(x) > 0$; viktfkt.

$L^2_w(I)$ består av $f(x)$: $\int_a^b |f(x)|^2 w(x)\,dx < \infty$

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}\,w(x)\,dx$$

$$\|f\|_w^2 = \langle f, f\rangle_w = \int_a^b |f(x)|^2 w(x)\,dx$$

$\langle f, g\rangle_w$ har alla egenskaper för sk.-prod

$\Rightarrow$ De 3 olikheterna gäller

$\Omega \subset \mathbb{R}^2$, $L^2(\Omega)$ består av $f(x,y)$, $(x,y) \in \Omega$:

$$\int_\Omega |f(x,y)|^2\,dx\,dy < \infty$$

$$\langle f, g\rangle = \int_\Omega f(x,y)\overline{g(x,y)}\,dx\,dy$$

$$\|f\|^2 = \langle f, f\rangle = \int_\Omega |f(x,y)|^2\,dx\,dy$$

Olikheterna gäller!

l^2 består av de oändl. följderna $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$ av kompl. tal s.a:

$$\sum |x_n|^2 < \infty$$

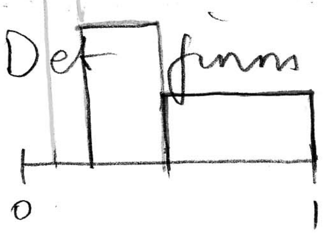
$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum x_n \overline{y_n}$$

$$\|\mathbf{x}\|^2 = \sum |x_n|^2$$

Olikheterna gäller!

<u>Alla dessa rum har dim ∞!</u>

Det finns oändl. syst. f_n som är linj. ober.



$$f_1(x) = \begin{cases} 0 & x < \frac{1}{2} \\ \sqrt{2} & x > \frac{1}{2} \end{cases}$$

$$f_2(x) = \begin{cases} 0 & x < \frac{1}{4},\ x > \frac{1}{2} \\ \sqrt{4} & \frac{1}{4} < x < \frac{1}{2} \end{cases}$$

$$f_3(x) = \begin{cases} 0 & x < \frac{1}{8},\ x > \frac{1}{4} \\ \sqrt{8} & \frac{1}{8} < x < \frac{1}{4} \end{cases}$$

$\vdots$

Konvergens i Hilbertrum

$f_n \in L^2$; f_n konvergerar mot $f \in L^2$ i normen om $\|f - f_n\|^2 \xrightarrow[n\to\infty]{} 0$

DEF

f_n är en **Cauchyföljd** om $\|f_n - f_m\| \xrightarrow[m,n\to\infty]{} 0$

SATS

i alla 4 ex. är rummen "fullständiga"
$\Rightarrow$ Varje Cauchy-följd konvergerar

Fourierserier i Hilbertrum

L^2; φ_n ortogonalt syst. $[\langle \varphi_j, \varphi_k \rangle = 0 \quad j \neq k]$

ortonormalt $[\|\varphi_j\|^2 = 1]$

$f \in L^2 \Rightarrow$ F-koeff map syst φ_n:

$$\alpha_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} \quad \text{om ortonormalt } \langle f, \varphi_n \rangle$$

serien: $\sum \alpha_n \varphi_n$ - F-serien av f map syst φ_n

SATS

Fourier-serien konvergerar alltid i normen

BEVIS [Viktigt!]

Tar ngt N:

$\sum_{n=1}^{N} \alpha_n \varphi_n$, Bessel:

$\sum_{n=1}^{N} |\alpha_n|^2 \leq \|f\|^2$

N godt. serien $\sum_{n=1}^{\infty} |\alpha_n|^2$ konvergerar

$\Rightarrow \sum_{N}^{\infty} |\alpha_n|^2 \xrightarrow[N\to\infty]{} 0$

$\Rightarrow \sum_{N}^{\infty} |\alpha_n|^2 - \sum_{M}^{\infty} |\alpha_n|^2 \xrightarrow[M,N\to\infty]{} 0$

$\Rightarrow \sum_{N}^{M} |\alpha_n|^2 \xrightarrow[M,N\to\infty]{} 0$

$s_N = \sum_{n=1}^{N} \alpha_n \varphi_n$; $\|s_N - s_M\|^2 = \|\sum_{N+1}^{M} \alpha_n \varphi_n\|^2 \overset{\text{Pyth.}}{=} \sum_{N+1}^{M} |\alpha_n|^2 \to 0$

$\Rightarrow s_N$ är en C-följd $\Rightarrow s_N$ konv.
serien konv.

Fråga: $\sum_{1}^{\infty} \alpha_n \varphi_n \overset{??}{=} f$

SATS

Följande 3 egenskaper för ortonormala syst. φ_n är ekv.:

1) $\forall f \in L^2$ gäller $\sum \alpha_n \varphi_n = f$

2) $\sum |\alpha_n|^2 = \|f\|^2$ (Parseval)

3) alla $\alpha_n = 0 \Rightarrow f = 0$

Om någon (och därmed alla) egenskap gäller så kallas syst av fkt φ_n en <u>ortonormal bas</u> i L^2

Sats 3.4, s. 77

Ex på icke ortogonala syst.

$L^2(0,1)$, $\varphi_n(x) = x^{n-1}$, $n = 1, 2, \ldots$

<u>Gram-Schmidt ortogonalisering</u>

Vi har ett syst $\varphi_1, \varphi_2, \ldots$ icke-ortogonalt
Vill transformera till ett ortonormalt syst.

<u>Steg 1:</u> $\psi_1 = \frac{\varphi_1}{\|\varphi_1\|}$; ψ_1 normerad; $\|\psi_1\| = 1$

<u>Steg 2:</u> $\psi_2 = ?$; söker $\tilde{\psi}_2 = \varphi_2 - \alpha\psi_1$

anpassar α: $\tilde{\psi}_2 \perp \psi_1$

$\langle \tilde{\psi}_2, \psi_1 \rangle = 0$; $\langle \varphi_2 - \alpha\psi_1, \psi_1 \rangle = 0$

$\Rightarrow \langle \varphi_2, \psi_1 \rangle = \alpha \langle \psi_1, \psi_1 \rangle = \alpha$

$\alpha = \langle \varphi_2, \psi_1 \rangle$ ger $\tilde{\psi}_2$

$\psi_2 = \frac{\tilde{\psi}_2}{\|\tilde{\psi}_2\|}$

<u>Steg 3:</u> $\psi_3 = ?$ söker $\tilde{\psi}_3 = \varphi_3 - \alpha\psi_1 - \beta\psi_2$

anpassar α, β: $\langle \tilde{\psi}_3, \psi_1 \rangle = 0$; $\langle \tilde{\psi}_3, \psi_2 \rangle = 0$

$\langle \hat{\psi}_3, \psi_1 \rangle = \langle \varphi_3 - \alpha\psi_1 - \beta\psi_2, \psi_1 \rangle =$

$$= \langle \varphi_3, \psi_1 \rangle - \alpha \langle \psi_1, \psi_1 \rangle - \beta \langle \psi_2, \psi_1 \rangle \Rightarrow \alpha = \langle \varphi_3, \psi_1 \rangle$$

pss: $\beta = \langle \varphi_3, \psi_2 \rangle$

O.S.V.

<u>Bästa approx.</u>

ett syst φ_n, $n = 1, \ldots, N$

$\varphi_n \in L^2$; $f \in L^2$ ortonormalt

Vi vill hitta den linj-komb. av φ_n, $F = \sum_1^N \beta_n \varphi_n$,
vilken minimerar approximationsfelet: $\| F - f \|$

SATS

Den bästa appr. ges av $\beta_n = \alpha_n = \langle f, \varphi_n \rangle$

BEVIS

$$G = f - \sum_{n=1}^{N} \alpha_n \varphi_n$$

$G \perp \varphi_n$, $n = 1, \ldots, N$: $\langle G, \varphi_k \rangle = \underset{\alpha_k}{\underset{\|}{\langle f, \varphi_k \rangle}} - \sum \alpha_n \langle \varphi_n, \varphi_k \rangle =$

$= \alpha_k - \alpha_k = 0$ bara $n=k$ överlever

$$\| f - F \|^2 = \| f - \sum_{n=1}^{N} \alpha_n \varphi_n + \sum_{n=1}^{n} \alpha_n \varphi_n - \sum_{n=1}^{N} \beta_n \varphi_n \|^2 =$$

$$= \| G + \sum_1^N (\alpha_n - \beta_n) \varphi_n \|^2 \overset{\text{Pyth}}{=} \| G \|^2 + \| \sum_1^N (\alpha_n - \beta_n) \varphi_n \|^2 =$$

$= \| G \|^2 + \sum | \alpha_n - \beta_n |^2$ minimeras för $\alpha_n = \beta_n$

Ex på S-L-problem

$$f'' + 2f + \lambda f = 0 \qquad x \in]0, \pi[$$

$$f'(0) = 0; \quad f(\pi) + \alpha f'(\pi) = 0$$

$$w(x) = 1; \quad p(x) = 2$$

1) Söker allmänna lösn:

$$k^2 + 2 + \lambda = 0 \Rightarrow k = \pm\sqrt{-2-\lambda}$$

a) $-2-\lambda < 0$

b) $-2-\lambda = 0$

c) $-2-\lambda > 0$

c) $-2-\lambda > 0 \Rightarrow -2-\lambda = \beta^2 \Rightarrow k = \pm\beta$

$$f(x) = A\cosh\beta x + B\sinh\beta x$$

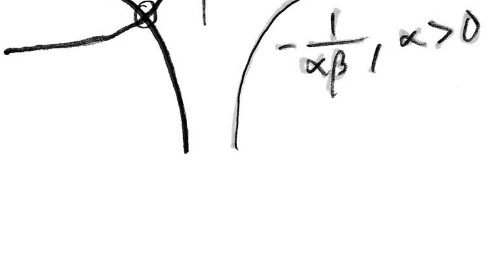
Randvillk:

$$f'(0) = 0: \quad A\beta\sinh(0) + B\beta\cosh(0) = 0 \Rightarrow B = 0$$

$$f(\pi) + \alpha f'(\pi) = A(\cosh\beta\pi + \alpha\beta\sinh\beta\pi) = 0$$

$$\overset{A\neq 0}{\Rightarrow} \cosh\beta\pi + \alpha\beta\sinh\beta\pi = 0$$

$$\Rightarrow -\frac{1}{\alpha\beta} = \frac{\sinh\beta\pi}{\cosh\beta\pi} = \tanh\beta\pi$$



för $\alpha > 0$, $\nexists$ lösn

$\alpha < 0$, $\exists$ en rot: egenf. $A\cosh\beta x$

β rot till

b) $-2-\lambda=0$

$k=0 \Rightarrow f=Ax+B$

$f'(0)=A=0 \Rightarrow f=B$

$f(\pi)+\alpha f'(\pi)=A=0 \Rightarrow f=0$

$\lambda=-2$ ej egenv.

a) $-2-\lambda<0 \Rightarrow -2-\lambda=-\beta^2 \Rightarrow k=\pm i\beta$

$\Rightarrow f(x)=A\cos\beta x+B\sin\beta x$

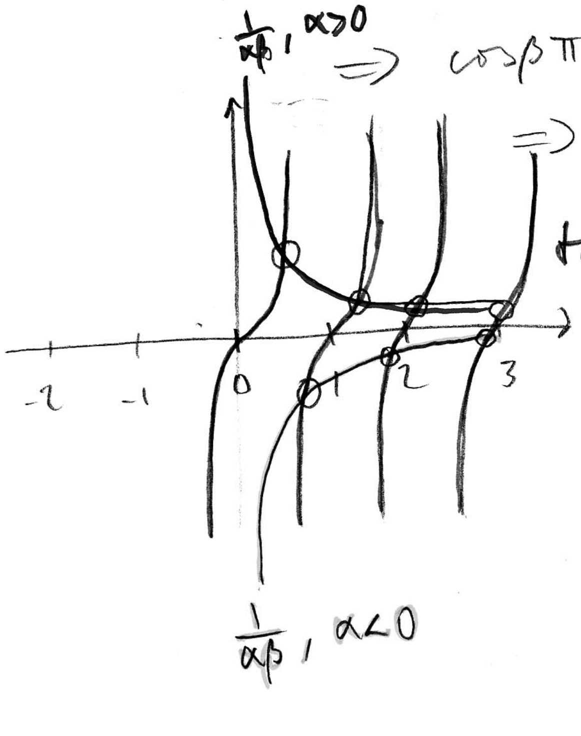
$f'(0)=-A\beta\sin 0+B\beta\cos 0=B\beta=0 \Rightarrow B=0$

$\Rightarrow f(x)=A\cos\beta x$

$f(\pi)+\alpha f'(\pi)=A(\cos\beta\pi-\alpha\beta\sin\beta\pi)=0$

$\Rightarrow \cos\beta\pi=\alpha\beta\sin\beta\pi$

$\Rightarrow \frac{1}{\alpha\beta}=\tan\beta\pi$



oändligt många $\beta=\beta_k$

$\lambda_k=\beta_k^2-2,\quad f_k(x)=\cos\beta_k x$

Ev. normera:

$\|A\cos\beta_k x\|=1 \ldots$

$f'' + 2f' + \lambda f = 0 \qquad 0<x<1, \quad f(0)=0;\ f(1)=0$

$$\boxed{(r(x)f')' + pf + \lambda w f = 0}$$

transformera till S-L

$$f'' + 2f' = e^{-2x}(e^{2x}f')'$$

$$e^{-2x}(e^{2x}f')' + \lambda f = 0$$

$$\Rightarrow \underbrace{(e^{2x}}_{r(x)}f')' + \lambda \underbrace{e^{2x}}_{w(x)}f = 0$$

$$f'' + 2f' + \lambda f = 0 \Rightarrow k^2 + 2k + \lambda = 0$$

$$(k+1)^2 + \lambda - 1 = 0$$

$$\Rightarrow k = -1 \pm \sqrt{1-\lambda}$$

a) $1-\lambda > 0$

b) $1-\lambda = 0$

c) $1-\lambda < 0$

a) $1-\lambda = \beta^2 \Rightarrow k = -1 \pm \beta$

$$f(x) = e^{-x}(A\cosh\beta x + B\sinh\beta x)$$

$$f(0) = A = 0 \Rightarrow f(x) = Be^{-x}\sinh\beta x$$

$f(1) = e^{-1}B\sinh\beta = 0 \Rightarrow$ Inga egenvärden

b) $1-\lambda=0 \Rightarrow k=-1$ dubbelrot

$\Rightarrow f(x)=(Ax+B)e^{-x}$

$\left.\begin{array}{l} f(0)=B=0 \\ f(1)=A=0 \end{array}\right\} \Rightarrow$ inga icke-triv. lösn.

c) $1-\lambda=-\beta^2 \Rightarrow k=-1\pm i\beta$

$\Rightarrow f(x)=e^{-x}(A\cos\beta x+B\sin\beta x)$

$f(0)=A=0$

$f(1)=e^{-1}B\sin\beta=0 \Rightarrow \beta_n=n\pi,\ n=1,2,3,\ldots$

$\lambda_n=1+\beta_n^2=1+n^2\pi^2$

egenfkt:

$f_n=A_n e^{-x}\sin(n\pi x)$

normerade:

$$A_n=\frac{1}{\sqrt{\|e^{-x}\sin n\pi x\|_w^2}}=\left(\int_0^1 e^{-2x}\sin^2(n\pi x)\underbrace{e^{2x}}_{w(x)}dx\right)^{-\frac{1}{2}}=$$

$$=\left(\int_0^1 \sin^2(n\pi x)\,dx\right)^{-\frac{1}{2}}=\left(\int_0^1\frac{1-\cos(2n\pi x)}{2}dx\right)^{-\frac{1}{2}}=$$

$$=\left(\left[\frac{x-\frac{\sin(2n\pi x)}{2n\pi}}{2}\right]_0^1\right)^{-\frac{1}{2}}=\sqrt{2}$$

$\Rightarrow f_n=\sqrt{2}\,e^{-x}\sin(n\pi x)$

$$\underbrace{(x f')'}_{r(x)} + \lambda \underbrace{\frac{1}{x}}_{\boxed{w(x)}} f = 0 , \quad f(1) = f(b) = 0 , \; b > 1$$

Euler-ekv: $x^2 f'' + x f' + \lambda f = 0$

Allm. lösn: $f(x) = A x^{r_1} + B x^{r_2}$

r fås ur: $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$
$(x = e^t \ldots)$

a) $-\lambda > 0$, $\lambda = -\beta^2 \Rightarrow r = \pm\beta$

Allm. lösn: $f(x) = A x^\beta + B x^{-\beta}$

$f(1) = 0 = A + B \Rightarrow B = -A \Rightarrow f = A(x^\beta - x^{-\beta})$

$f(b) = 0 = A(b^\beta - b^{-\beta}) = 0$ omöjligt $\Rightarrow$ inga egenv. $-\lambda > 0$

b) $\lambda = 0 \Rightarrow r_{1,2} = 0$ dubbelrot

Allm. lösn: $f(x) = A \ln x + B$

$f(1) = B = 0$
$f(b) = A \ln b = 0$ omöjligt (ty $b > 1$) $\Rightarrow \lambda = 0$ ej egenv.

c) $-\lambda < 0$; $\lambda = \beta^2 \Rightarrow r_{1,2} = \pm i\beta$

$f(x) = A x^{i\beta} + B x^{-i\beta}$; $\left[x^{i\beta} = e^{\ln x \, i\beta} \right]$

$f(x) = A e^{\ln x \, i\beta} + B e^{-\ln x \, i\beta}$

$f(1) = A + B = 0 \Rightarrow B = -A \Rightarrow f(x) = A(e^{\ln x \, i\beta} - e^{-\ln x \, i\beta}) =$
$= 2iA \sin(\ln x \, \beta)$

$$f(b) = 2iA\sin(\beta\ln b) = 0$$

$$\Rightarrow \sin(\beta\ln b) = 0 \Rightarrow \beta\ln b = n\pi, \quad n = 1,2,3,\ldots$$

$$\Rightarrow \beta = \frac{n\pi}{\ln b}$$

$$\Rightarrow f(x) = \underbrace{2iA}_{A_n}\sin\left(\frac{n\pi}{\ln b}\ln x\right), \quad \lambda_n = \beta_n^2 = \left(\frac{n\pi}{\ln b}\right)^2$$

Normera:

$$A_n = \frac{1}{\sqrt{\left\|\sin\left(\frac{n\pi}{\ln b}\ln x\right)\right\|^2}} = \left(\int_1^b \sin^2\left(\frac{n\pi}{\ln b}\ln x\right)\frac{1}{x}\,dx\right)^{-\frac{1}{2}} =$$

$$= \left(\int_1^b \frac{1-\cos\left(\frac{2n\pi}{\ln b}\ln x\right)}{2x}\,dx\right)^{-\frac{1}{2}} = \left(\frac{1}{2}\left[\ln x - \frac{\sin\left(\frac{2n\pi}{\ln b}\ln x\right)}{\frac{2n\pi}{\ln b}}\right]_1^b\right)^{-\frac{1}{2}}$$

$$= \left(\frac{1}{2}(\ln b)\right)^{-\frac{1}{2}} = \sqrt{\frac{2}{\ln b}}$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\ln b}}\sin\left(\frac{n\pi}{\ln b}\ln x\right)$$

F-metoden för PDE

De 3 viktigaste ex.:

1. Värmeledn.-ekv.
2. Vågekv.
3. Laplace (Poisson)

1. $u'_t - Lu = f(x,t), \quad t > 0, \quad x \in]a,b[$ (ändligt)

Lu - 2:a ordn diff-oper. i x

$$Lu = (r(x)u'_x)'_x + p(x)u$$

Vanligast:

$$Lu = ku''_{xx} + p(x)u$$

Randvillkoren:

$$B_a(u(x,t)) = \varphi_a(t)$$

$$B_b(u(x,t)) = \varphi_b(t)$$

Vanligast:

$$\begin{cases} u(a,t) = \varphi_a(t) \\ u(b,t) = \varphi_b(t) \end{cases}$$

eller

$$u'_x(a,t) = \varphi_a(t)$$

eller

$$u'_x(a,t) + 2u(a,t) = \varphi_a(t)$$

Beg. villk:

$$u(x,0) = g(x)$$

Om $\varphi_a, \varphi_b = 0$: Randvillk homogena

Om $f(x,t) \equiv 0$: ekv homogen

F-metod **ENDAST** med randvillk som är hom i den var.!

Algoritmen

0) (Förberedelsesteget)

Kolla om randvillk är hom.

Om ja ⟶ nästa steg.

Om nej: Transformera probl. till hom. probl.

Hitta enkel fkt $v(x,t)$ vilken satisf randvillk.

Ex $u(a,t) = A$
$u(b,t) = B$, söker $v(x,t)$ som en linj fkt: $v(x,t) = A + \frac{B-A}{b-a}(x-a)$

Ex $u(a,t) = A$
$u'(b,t) = B$

$$v(x,t) = A - B\sin\left(\frac{x-a}{b-a}\pi\right)\cdot\frac{b-a}{\pi}$$

$$v(x,t) = A + B(x-a)$$

Efter vi valt hjälpfkt $v(x,t)$, söker vi lösn på formen:

$$u(x,t) = v(x,t) + w(x,t)$$

sätter in i ekv och i beg-villk. Vi får ekv och beg-villk för w. Randvillk för w är hom.

Ex ekv $u'_t - u''_{xx} = x$, $0 < x < 1$

$\left.\begin{matrix} u(0,t) = 1 \\ u'_x(1,t) = 1 \end{matrix}\right\}$ Randvillk $u(x,0) = \sin\frac{2x}{\pi}$ beg-villk

hjälpfkt: $v(x,t) = 1 + x$

$u = 1 + x + w(x,t)$, $v'_t = 0$; $v''_{xx} = 0$

$$\Rightarrow w'_t(x,t) - w''_{xx}(x,t) = x$$

$$u(x,0) = v(x,0) + w(x,0) = \sin\frac{2x}{\pi}$$

$$\Rightarrow w(x,0) = \sin\frac{2x}{\pi} - 1 - x$$

$\left\{\begin{matrix} w(0,t) = 0 \\ w'_x(1,t) = 0 \end{matrix}\right.$ hom. randvillk

1) (Variabelseparation)

Man separerar **ALLTID** variabler i homogena ekv.

$\underbrace{W(x)}_{\text{viktfkt.}} u'_t = L u(x,t)$

Sök lösn till ekv. på formen:

$$u(x,t) = X(x)\,T(t)$$

sätt in i ekv.

Ex $e^x u'_t = u''_{xx} + 2u$

$$e^x X(x)T'(t) = X''(x)T(t) + 2X(x)T(t)$$

dela med $W(x)\,X(x)\,T(t)$

$$\frac{T'(t)}{T(t)} = \frac{X''(x) + 2X(x)}{e^x X(x)} \overset{\text{trolleri}}{=} -\lambda$$

$$T' + \lambda T = 0$$

$$X'' + 2X + \lambda e^x X(x) = 0$$

$\boxed{\text{Efter separation, löser man den ekv för vilken man har hom. randvillk}}$

Börja med X-ekv. Randvillk. tas ur randvillk för urspr. probl.

t.ex. om vi (kanske efter förberedelse-steget) har randvillk

$$\left.\begin{array}{l} u(0,t) = 0 \\ u'_x(1,t) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} X(0) = 0 \\ X'(1) = 0 \end{array} \qquad [u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0]$$

$\Rightarrow$ S-L

$$X'' + 2X + \lambda \underbrace{e^x}_{w(x)} X, \quad \begin{cases} X(0) = 0 \\ X'(1) = 0 \end{cases}$$

2) Löser S-L-probl. Hittar egenv. λ_n och egenf. (bättre: normerade)

3) Söker lösn

2 fall:

1. Det enkla fallet: ekv (efter steg 0) är **hom.**

$$w u_t' - L u = 0 \quad , \quad u(x,0) = g(x)$$

randvillk hom.

Lös T-ekv.:

$$T_n' + \lambda_n T_n = 0 \qquad [\text{en T-ekv för varje } \lambda_n]$$

$$\Rightarrow T_n(t) = T_n(0)\, e^{-\lambda_n t}$$

$T_n(0)$ söks ur beg.-villk: $u(x,0) = g(x)$

Söker:

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)\, T_n(t)$$

sätter in i beg.-villk:

$$u(x,0) = \sum_{n=1}^{\infty} X_n(x)\, T_n(0) = g(x)$$

utvecklar $g(x)$ i F-serie map basen $X_n(x)$ med vikten $w(x)$.

$$g(x) = \sum_{1}^{\infty} \langle g, X_n \rangle_w\, X_n(x) \qquad \underline{(\text{normerade } X_n)}$$

$$\Rightarrow T_n(0) = \langle g, X_n \rangle_w$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} X_n(x)\, e^{-t\lambda_n} \langle g, X_n \rangle_w + v(x,t)$$

2. Det komplicerade fallet: $f(x,t) \neq 0$

Söker lösn på formen:

$$u(x,t) = \sum X_n(x) T_n(t) \quad \text{med okända } T_n(t)$$

Sätter in i ekv:

$$w(x) u'_t - Lu = f(x,t)$$

$$w(x) \sum X_n(x) T'_n(t) - \sum L X_n(x) \cdot T_n(t) = f(x,t)$$, L och w

Använder: $L X_n + \lambda_n w X_n = 0$ (X_n egenfkt till S-L)

$$\sum w(x) \left(T'_n(t) + \lambda_n T_n(t) \right) X_n(x) = f(x,t)$$

Mult med $X_k(x)$ och integr. i x:

$$\sum \int_a^b w(x) X_n(x) X_k(x)\, dx \cdot (T'_n + \lambda_n T_n) = \int_a^b f(x,t) X_k(x)\, dx$$

egenfkt av S-L ortog. ⇒ endast termen n=k överlever

$$T'_k + \lambda_k T_k = f_k(t) =$$

$$= \int_a^b f(x,t) X_k(x)\, dx$$

Ett vanl. t-ekv. med beg.-villk.: $T_k(0) = \langle g, X_k \rangle_w$

$$T_k(t) = T_k(0) e^{-\lambda_k t} + \int_0^t e^{-\lambda_k (t-\tau)} f_k(\tau)\, d\tau$$

$\boxed{EÖ26}$ $\sqrt{1+t}\, u''_{xx} = u'_t$; $0<x<1$, $t>0$

$u(0,t) = 1$, $u(1,t) = 0$

$u(x,0) = 1 - x^2$

$v(x,t)$: $v(0,t) = 1$, $v(1,t) = 0$

Välj ex.:

$v(x,t) = 1 - x$ hjälpfkt

Sök lösn på formen:

$u(x,t) = v(x,t) + w(x,t)$

$u''_{xx} = v''_{xx} + w''_{xx} = w''_{xx}$

$u'_t = v'_t + w'_t = w'_t$

Ny ekv:

$\sqrt{1+t}\, w''_{xx} = w'_t$

$u(x,0) = v(x,0) + w(x,0) = 1 - x^2$

$$\Rightarrow \begin{cases} w(x,0) = x - x^2 \\ w(0,t) = 0 \\ w(1,t) = 0 \end{cases}$$

Steg 1

$w(x,t) = X(x)\,T(t)$

Ekv:

$\sqrt{1+t}\, X''(x)\,T(t) = T'(t)\,X(x)$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{1}{\sqrt{1+t}}\,\frac{T'(t)}{T(t)} = -\lambda$$

$X''(x) + \lambda X(x) = 0$ $\quad\quad\quad\quad$ $X(0) = 0,\ X(1) = 0$

$T'(t) + \lambda\sqrt{1+t}\,T(t) = 0$

<u>Steg 2</u>

$r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$

I) $-\lambda > 0$ Inga egenvärden

II) $\lambda = 0$ —— " ——

III) $-\lambda < 0$: $\lambda_n = (n\pi)^2, \quad n = 1, 2, \ldots$

$X_n = \sin(n\pi x)\cdot\sqrt{2}$

$w(x,t) = \sum X_n(x)\, T_n(t)$

Enkla fallet: Lös $T_n' + \lambda_n\sqrt{1+t}\, T_n = 0$

$$\frac{dT_n}{T_n} = -(n\pi)^2\sqrt{1+t}\,dt$$

$$\int \frac{1}{T_n}\,dT_n = \int -(n\pi)^2\sqrt{1+t}\,dt$$

$$\ln T_n = -(n\pi)^2(1+t)^{\frac{3}{2}}\cdot\frac{2}{3} + C_n'$$

$$T_n(t) = C_n e^{-\frac{2(n\pi)^2(1+t)^{3/2}}{3}}$$

Bestäm C_n ur beg.-villk

$$w(x,0) = \sum X_n(x)\, e^{-\frac{2(n\pi)^2}{3}} C_n = x - x^2$$

Mult. med $X_k(x)$ och integrera:

$$\sum \langle X_n, X_k\rangle e^{-\frac{2(n\pi)^2}{3}} C_n = \langle x-x^2, X_k\rangle$$

$n=k$ överlever:

$$\langle X_k, X_k\rangle e^{-\frac{2(n\pi)^2}{3}} C_n = \langle x-x^2, X_k\rangle$$

$$\Rightarrow \quad C_n = e^{\frac{2(n\pi)^2}{3}} \frac{\langle x-x^2, X_k\rangle}{\langle X_k, X_k\rangle_{=1}} = e^{\frac{2(n\pi)^2}{3}} \langle x-x^2, X_k\rangle$$

$$u(x,t) = 1-x+\sum \sin(n\pi x)\sqrt{2}\, e^{-(n\pi)^2 \frac{2}{3}(1+t)^{3/2}} \cdot e^{\frac{2(n\pi)^2}{3}} \langle x-x^2, \sqrt{2}\sin(n\pi x)\rangle$$

$\boxed{\text{EÖ 52}}$

$$u''_{xx}+1 = \frac{1}{4}u''_{tt}, \quad 0<x<2, \quad t>0$$

$$u(0,t)=0, \quad u(2,t)=2$$

$$u(x,0)=x-x^2, \quad u'_t(x,0)=0$$

$$v(x,t); \; v(0,t)=0, \quad v(2,t)=2$$

$$v(x,t)=x$$

$$u(x,t)=v(x,t)+w(x,t)$$

$$w''_{xx}+1=\frac{1}{4}w''_{tt}, \quad w(x,0)=-x^2, \quad w'_t(x,0)=0$$

$$w(x,t)=X(x)T(t)$$

variabelsep. i hom. ekv.

$$X''(x)T(t)+\cancel{X} = \frac{X(x)T''(t)}{4}$$

$$\frac{X''}{X} = \frac{1}{4}\frac{T''}{T} = -\lambda$$

$$X''(x)+\lambda X(x)=0, \quad X(0)=0; \; X(2)=0, \quad \text{S-L-probl.}$$

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2$$

$$X_n(x) = \sin\frac{n\pi}{2}x, \quad n=1,2,\dots$$

$w(x,t) = \sum X_n(x) T_n(t)$

Sätter in i ekv.

$$\sum X''_n T_n + 1 = \frac{1}{4} \sum X_n T''_n$$

$$\sum (X'' T_n - \frac{1}{4} X_n T_n'') = -1$$

använder S-L-ekv.:

$$\sum (-\lambda_n T_n - \frac{1}{4} T_n'') X_n = -1$$

mult med X_k int.:

$$T''_k + 4\lambda_k T_k = 4 \langle 1, X_k \rangle$$

$$T_n(0) = ? \, ; \; w(x,0) = \sum X_n(x) T_n(0) = -x^2$$

mult/int

$$T_k(0) = \langle -x^2, X_k \rangle$$

$$T_n(0) = ? \, ; \; w'_t(x,0) = \sum X_n(x) T'_n(0) = 0$$

mult/int

$$T'_k(0) = 0$$

$$\frac{1}{4} T''_k + \left(\frac{k\pi}{2}\right)^2 T_k = \langle 1, X_k \rangle \, , \quad T_k(0) = \langle -x^2, X_k \rangle$$

$$T'_k(0) = 0$$

$$T_k = T_{kh} + T_{kp}$$

hom.-/part.-lösn

$p^2 + 4\left(\frac{k\pi}{2}\right)^2 = 0$ - kar ekv

$p = \pm ik\pi$

$T_{k_h} = A\cos(k\pi t) + B\sin(k\pi t)$

T_{kp} sök som konst: $T_{kp} = C_k$

$$4\left(\frac{k\pi}{2}\right)^2 C_k = 4\langle 1, X_k\rangle$$

$$\Rightarrow C_k = \frac{4\langle 1, X_k\rangle}{(k\pi)^2}$$

$$T_k = A_k\cos(k\pi t) + B_k\sin(k\pi t) + \frac{4\langle 1, X_k\rangle}{(k\pi)^2}$$

$t = 0$:

$$T_k(0) = A_k + \frac{4\langle 1, X_k\rangle}{(k\pi)^2} = \langle -x^2, X_k\rangle$$

$$\Rightarrow A_k = \langle -x^2, X_k\rangle - \frac{4\langle 1, X_k\rangle}{(k\pi)^2}$$

$$T_k'(0) = B_k k\pi = 0 \Rightarrow B_k = 0$$

$$\Rightarrow T_k = \cos(k\pi t)\left(\langle -x^2, X_k\rangle - \frac{4\langle 1, X_k\rangle}{(k\pi)^2}\right) + \frac{4\langle 1, X_k\rangle}{(k\pi)^2}$$

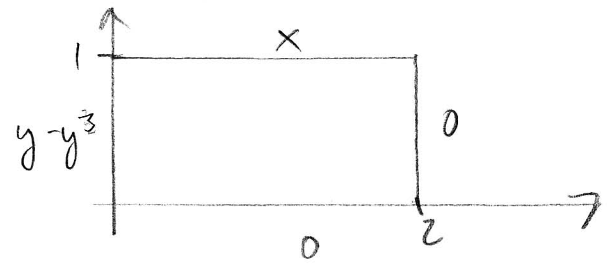
$$u(x,t) = x + \sum X_n(x)\left(\left(\langle -x^2, X_k\rangle - \frac{4\langle 1, X_k\rangle}{(k\pi)^2}\right)\cos k\pi t + \frac{4\langle 1, X_k\rangle}{(k\pi)^2}\right)$$

$\boxed{\text{E025}}$

$u''_{xx} + u''_{yy} = y, \quad 0 < x < 2, \quad 0 < y < 1$

$u(x,0) = 0, \quad u(x,1) = x$

$u(0,y) = y - y^3, \quad u(2,y) = 0$



Förberedelsesteget för Laplace-ekv i dim 2, 3, ...

<u>2 varianter:</u>

1) som tidigare, sök $v(x,y)$ som satisf. randvillk

<u>**jobbigt**</u>

2) Dela problem i 2 (motsv. 3) probl. med hom. randv. i alla var. utom en

Sök $u(x,y)$ på formen

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

där $u_1(x,y)$ satisf.

$$u''_{1xx} + u''_{1yy} = y, \quad u(x,0) = 0, \quad u(x,1) = 0$$ hom randv i y

$$u(0,y) = y - y^3, \quad u(2,y) = 0$$

$u_2(x,y)$:

$$u''_{2xx} + u''_{2yy} = 0$$

$$u_2(x,0) = 0, \quad u_2(x,1) = x$$

$$u_2(0,y) = 0, \quad u_2(2,y) = 0$$ hom randv. i x

$$u''_{1xx} + u''_{1yy} = 0$$

$$u_1(x,y) = X(x)\,Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$$X'' + \lambda X = 0$$

$$Y'' - \lambda Y = 0$$

Y-ekv först, ty randvillk i y hom.

$$Y(0) = 0, \quad Y(1) = 0$$

$$p^2 - \lambda = 0 \Rightarrow p = \pm\sqrt{\lambda}$$

3 fall, $\underbrace{\lambda > 0, \lambda = 0}_{\text{inga egenv.}}$, $\lambda < 0$, $\lambda = -\beta^2$

$$Y(y) = A\sin\beta y + B\cos\beta y$$

ur randvillk:

$$\beta_n = n\pi; \quad \lambda = -n^2\pi^2$$

$$Y_n(y) = \sqrt{\frac{2}{\pi}}\sin(n\pi y)$$

Söker lösn:

$$u_1(x,y) = \sum X_n Y_n$$

sätt in i ekv och randvillk

$$\sum(X_n'' Y_n + X_n Y_n'') = y$$

använd S-L-probl för att bli av med Y_n''

$$\sum(X_n'' Y_n + X_n \lambda_n Y_n) = y$$

$$\sum (X_n'' - n^2\pi^2 X_n) Y_n = y$$

skal-mult med Y_k

$$\Rightarrow \quad X_k'' - k^2\pi^2 X_k = \frac{\langle y, Y_k \rangle}{\underbrace{\langle Y_k, Y_k \rangle}_{=1}} = \langle y, Y_k \rangle$$

randvillk:

$$u_1(0,y) = \sum X_n(0) Y_n(y) = y - y^3$$

$$u_1(2,y) = \sum X_n(2) Y_n(y) = 0$$

$$X_k(0) = \langle y - y^3, Y_k \rangle$$

$$X_k(2) = 0$$

$$X_k = X_{kh} + X_{kp}$$

X_{kh}: $\quad X_k'' - k^2\pi^2 X_k = 0$

$$p = \pm k\pi$$

$$X_{kh} = A_k \cosh(k\pi x) + B_k \sinh(k\pi x)$$

X_{kp}: ansatt $\quad X_{kp} = C_k$

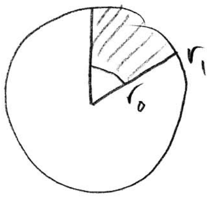
$$-k^2\pi^2 C_k = \langle y, Y_k \rangle$$

$$\Rightarrow \quad C_k = -\frac{\langle y, Y_k \rangle}{k^2\pi^2}$$

$$X_k = A_k \cosh(k\pi x) + B_k \sinh(k\pi x) + C_k$$

$$\left.\begin{aligned} X_k(0) &= A_k + C_k = \langle y - y^3, Y_k \rangle \\ X_k(2) &= A_k \cosh(2k\pi) + B_k \sinh(2k\pi) + C_k = 0 \end{aligned}\right\} \text{hitta } A_k, B_k$$

Cirkelformiga områden.



$$r_0 < r < r_1$$
$$\theta_0 < \theta < \theta_1$$

$$u''_{xx} + u''_{yy} = 0$$

$$u(x,y) = 0, \quad \theta = \theta_0$$
$$u(x,y) = 1, \quad \theta = \theta_1$$
$$u(x,y) = 0, \quad r = r_0$$
$$u(x,y) = 1, \quad r = r_1$$

$$u''_{xx} + u''_{yy} = u''_{rr} + \frac{1}{r} u'_r + \frac{1}{r^2} u''_{\theta\theta} = 0$$

Söker:

$u(r,\theta)$ som $u_1(r,\theta) + u_2(r,\theta)$

där

$$u''_{1rr} + \frac{1}{r} u'_{1r} + \frac{1}{r^2} u''_{\theta\theta} = 0$$
$$u_1(r,\theta_0) = u_1(r,\theta_1) = 0$$
$$u_1(r_0,\theta) = 0, \quad u_1(r_1,\theta) = 1$$

$$u''_{2rr} + \frac{1}{r} u'_{2r} + \frac{1}{r^2} u''_{2\theta\theta} = 0$$
$$u_2(r,\theta_0) = 0, \quad u_2(r,\theta_1) = 1$$
$$u_2(r_0,\theta) = u(r_1,\theta) = 0$$

$$u_1(r,\theta) = R(r)\,\Theta(\theta)$$

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0$$

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$r^2 R'' + r R' - \lambda R = 0$$

$$\Theta'' + \lambda \Theta = 0$$

$$\begin{cases} \Theta(\theta_0) = 0 \\ \Theta(\theta_1) = 0 \end{cases}$$

$$p^2 + \lambda = 0 \Rightarrow p = \pm\sqrt{-\lambda}$$

$$\left.\begin{array}{l} -\lambda < 0 \\ \quad = 0 \\ \quad > 0 \end{array}\right.\left.\begin{array}{l} \\ \\ \end{array}\right\} \text{inga egenv.}$$

$$-\lambda = -\beta^2 \Rightarrow p = \pm i\beta$$

$$\Theta(\theta) = A \sin\beta(\theta - \theta_0) + B\cos\beta(\theta - \theta_0)$$

$$\beta_n = \frac{\pi n}{\theta_1 - \theta_0} \; ; \; \lambda_n = \left(\frac{\pi n}{\theta_1 - \theta_0}\right)^2$$

$$\Theta_n(\theta) = \sqrt{\frac{2}{\theta_1 - \theta_0}} \sin\left(\frac{\pi n}{\theta_1 - \theta_0}(\theta - \theta_0)\right)$$

$$u_1(r,\theta) = \sum_1^\infty \Theta_n(\theta) R_n(r)$$

$$r^2 R_n'' + r R_n' - \left(\frac{\pi n}{\theta_1 - \theta_0}\right)^2 R = 0$$

$$u_1(r_0, \theta) = \sum_1 \Theta_n(\theta) R_n(r_0) = 0 \Rightarrow R_n(r_0) = 0$$

$$u_1(r_1, \theta) = 1 = \sum_1 \Theta_n(\theta) R_n(r_1)$$ skal-mult Θ_k

$$\Rightarrow R_n(r_1) = \langle 1, \Theta_n \rangle$$

$R_n = r^\alpha$

$\alpha = \pm\sqrt{\lambda_n}$

$R_n(r) = A r^{\sqrt{\lambda_n}} + B r^{-\sqrt{\lambda_n}}$

$$\begin{cases} R_n(r_0) = A r_0^{\sqrt{\lambda_n}} + B r_0^{-\sqrt{\lambda_n}} = 0 \\ R_n(r_1) = A r_1^{\sqrt{\lambda_n}} + B r_1^{-\sqrt{\lambda_n}} = \langle 1, \Theta_n \rangle \end{cases}$$

u_2: $\quad r^2R'' + rR' - \lambda R = 0\,; \quad R(r_0) = R(r_1) = 0$

$\Theta'' + \lambda\Theta = 0$

$R(r) = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}} \quad , \quad \lambda > 0\,,\ \lambda < 0\,,\ \lambda = 0$

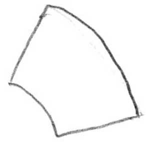
egenv. finns endast för $\lambda < 0\,,\ \lambda = -\beta^2$

$R(r) = Ar^{i\beta} + Br^{-i\beta}$

$r^{i\beta} = e^{i\beta \ln r}$

$r^2R'' + r' + \lambda R = 0$

$R(r_0) = 0;\ R(r_1) = 0$



Eulerekv.

$p^2 + \lambda = 0 \Rightarrow p = \pm\sqrt{-\lambda}$

$-\lambda > 0 \Rightarrow -\lambda = \beta^2;$

Allm. lösn: $R(r) = Ar^\beta + Br^{-\beta}$

$R(r_0) = Ar_0^\beta + Br_0^{-\beta} = 0$

$R(r_1) = Ar_1^\beta + Br_1^{-\beta} = 0$

Algebra: för att syst ska ha lösn $A, B \neq 0,0$ måste det $=0$

$$\begin{vmatrix} r_0^\beta & r_0^{-\beta} \\ r_1^\beta & r_1^{-\beta} \end{vmatrix} = r_0^\beta r_1^{-\beta} - r_0^{-\beta} r_1^\beta = r_0^\beta r_1^{-\beta} - (r_0^\beta r_1^{-\beta})^{-1} = 0$$

endast om $r_0^\beta r_1^{-\beta} = 1$

Inga egenv. $\lambda < 0$

$-\lambda = 0$

Allm. lösn:

$R(r) = A\ln r + B$

$\begin{cases} A\ln r_0 + B = 0 \\ A\ln r_1 + B = 0 \end{cases} \Rightarrow \lambda = 0$ ej egenv.

$(rR')' + \frac{\lambda}{r}R$

$-\lambda < 0 \Rightarrow -\lambda = -\beta^2$

$R(r) = Ar^{i\beta} + Br^{-i\beta}$

Randvillk:

$r_0^{i\beta} r_1^{-i\beta} - r_0^{-i\beta} r_1^{i\beta} = 0 \Rightarrow (r_0 r_1^{-1})^{2i\beta} = 1 \Rightarrow e^{2i\beta\ln(r_0 r_1^{-1})} = 1$

$$\Rightarrow 2i\beta \ln(r_0 r_i^{-1}) = 2n\pi i$$

$$\Rightarrow \beta = \frac{n\pi}{\ln(r_0 r_i^{-1})}$$

Transf. till S-L-probl.:

dela med r:

$$(rR')' + \lambda \frac{1}{r} R = 0$$

$$\Rightarrow \text{Viktfkt } w(r) = \frac{1}{r}$$

Besselfunktioner

r, θ – polära koord.

t – tiden

$$u''_{tt} = c^2 \Delta u = c^2(u''_{xx} + u''_{yy}) = c^2\left(u''_{rr} + \frac{1}{r}u'_r + \frac{1}{r^2}u''_{\theta\theta}\right)$$

$$u(b, \theta, t) = 0$$

$$u(r, \theta, 0) = \varphi(r, \theta)$$

$$u'_t(r, \theta, 0) = \psi(r, \theta)$$

Variabelsep.:

$$u(r, \theta, t) = R(r)\,\Theta(\theta)\,T(t),$$ vi räknar för $c = 1$:

$$T''R\Theta = R''\Theta T + \frac{1}{r}R'T\Theta + \frac{1}{r^2}RT\Theta''$$

dela med $TR\Theta$:

$$\frac{T''}{T} = \frac{R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''}{R\Theta} = -\mu^2$$

T-ekv

$r\theta$-ekv

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\mu^2$$

mult. med r^2:

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \mu^2 r^2 = -\frac{\Theta''}{\Theta} = \nu^2$$

$T'' + \mu^2 T = 0; \quad t > 0$

$\Theta'' + \nu^2\Theta = 0; \quad 0 < \theta < 2\pi; \quad \Theta(2\pi) = \Theta(0); \quad \Theta'(2\pi) = \Theta'(0)$

$R'' + \frac{1}{r}R' + R(\mu^2 r^2 - \nu^2) = 0; \quad R(b) = 0$

Börja med Θ-ekv.:

$\Theta(\theta) = Ae^{i\nu\theta} + Be^{-i\nu\theta}; \quad \Theta(2\pi) = Ae^{2\pi i\nu} + Be^{-2\pi i\nu}; \quad \Theta(0) = A + B$

$\Theta'(0) = i\nu(A+B); \quad \Theta'(2\pi) = i\nu(Ae^{2\pi i\nu} + Be^{-2\pi i\nu})$

det = 0:

$e^{2\pi i\nu} = 1 \Rightarrow \nu = \text{heltal}$

egenv. ν^2, $\nu = 0, \pm 1, \pm 2$

?

$\Theta_\nu(\theta) = e^{i\nu\theta}$

R-ekv

dela med r

$(rR')' - \frac{\nu^2}{r}R + \mu^2 rR = 0 \qquad w(r) = r$

$0 < r < b; \quad R(b) = 0; \quad R(0)$ ändlig

variabelbyte:

$r = \frac{x}{\mu}$

Besselekvation

$\Rightarrow \boxed{x^2 f''(x) + xf'(x) + (x^2 - \nu^2)f(x) = 0}; \quad f(0)$ ändl

$f(\mu b) = 0$

$J_\nu(x)$ - den lösn som är begr. i 0. J_ν är Besselfkt av ordn ν

$Y_\nu(x)$ - Weberfkt.

Trumvibr. forts

$R(r) = J_\nu(\mu r)$ lösn till R-ekv.

μ söks ur randvillk

$J_\nu(\mu b) = 0 \Rightarrow \mu b$ måste vara nollst till J_ν

Varje J_ν har ∞ många nollst: $\lambda_1, \lambda_2, \ldots$

$\mu_{\nu,n} = \frac{\lambda_n^\nu}{b}$ egenv. för S-L-probl. för R

$\nu = 0, \pm 1, \pm 2, \ldots \; ; \; n = 1, 2, \ldots$

T-ekv:

$$T_{\nu,n}'' + \mu_{\nu,n}^2 T_{\nu,n} = 0$$

$$T_{\nu,n}(t) = A_{\nu,n} \sin(\mu_{\nu,n} t) + B_{\nu,n} \cos(\mu_{\nu,n} t)$$

$$u(r,\theta,t) = \sum_\nu \sum_n T_{\nu,n}(t) e^{i\nu\theta} R_{\nu,n}(r)$$

$A_{\nu,n}$, $B_{\nu,n}$ söks ur beg. villk

$$u(r,\theta,0) = \varphi(r,\theta) = \sum_\nu \sum_n T_{\nu,n}(0) e^{i\nu\theta} R_{\nu,n}(r) =$$

$$= \sum_\nu \sum_n B_{\nu,n} e^{i\nu\theta} R_{\nu,n}(r)$$

Mult med $e^{-is\theta}$ och int. i θ. Bara termen $\nu = s$ överlever

$$\Rightarrow \sum_n B_{s,n} R_{s,n}(r) \| e^{is\theta} \|^2 = \int_0^{2\pi} \varphi(r,\theta) e^{-is\theta} d\theta$$

ty ej normerade

Mult med $R_{s,k}(r)$ med vikten r:

$$R_{\nu,n}(r) = J_\nu\left(\frac{\lambda_n^\nu}{b} r\right)$$

Besselfkt $R_{\nu,n}(r), (n=1,2,\ldots)$ bildar bas i $L^2_r\,]0,b[$

bara termen $n=k$ överlever.

$$B_{s,k}\,\|R_{s,k}\|^2\,\|e^{is\theta}\|^2 = \int_0^{2\pi}\int_0^{b} \varphi(r,\theta)\, e^{-is\theta}\, J_s\!\left(\frac{\lambda_k^s r}{b}\right) r\,dr\,d\theta$$

$$\|e^{is\theta}\|^2 = \int_0^{2\pi} |e^{is\theta}|^2\, d\theta = 2\pi$$

$$\left\|J_\nu\!\left(\frac{\lambda_n^\nu}{b} r\right)\right\|^2 = \frac{b^2}{2}\left(J_{\nu+1}(\lambda_n^\nu)\right)^2$$

$A_{\nu,n}$ fås ur andra beg.-villk:

$$u_t'(r,\theta,0) = \Psi(r,\theta) = \sum_\nu \sum_n \mu_{\nu,n} A_{\nu,n}\, e^{i\nu\theta} R_{\nu,n}(r)$$

$\boxed{5.2.4}$ Visa att: $\int_0^x s J_0(s)\,ds = x J_1(x)$ och $\int_0^x J_1(s)\,ds = 1 - J_0(x)$

(5.14), $\nu=1$: $(x J_1(x))' = x J_0(x)$

$$\Rightarrow \int_0^x s J_0(s)\,ds = \int_0^x (s J_1(s))'\,ds = \big[s J_1(s)\big]_0^x = J_1(x)\cdot x$$

$$\int_0^x J_1(s)\,ds = -\int_0^x (J_0)'\,ds = -\big[J_0\big]_0^x = J_0(0) - J_0(x) = 1 - J_0(x)$$

$\big[$(5.13), $\nu=0$: $J_0'(x) = -J_1(x)\big]$

$\boxed{5.2.5}$

$$\int_0^x s^2 J_1(s)\,ds = [(5.14),\ \nu=2] = \int_0^x (s^2 J_2(s))'\,ds = \big[s^2 J_2(s)\big]_0^x =$$

$$= x^2 J_2(x) = [(5.17),\ \nu=1] = x(2J_1(x) - x J_0(x))$$

$\boxed{5.2.6}$ $\int_0^x J_3(s)\,ds = \int_0^x s^2\left(\frac{1}{s^2}J_3(s)\right)ds = \left[(5.13),\ \nu=2\right] =$

$= -\int_0^x s^2\left(\frac{1}{s^2}J_2(s)\right)'ds = -\left[\frac{1}{\cancel{s^2}}J_2(s)\cdot\cancel{s^2}\right]_0^x + \int_0^x \frac{2\cancel{s}}{s^{\cancel{2}}}J_2(s)\,ds =$

$= -J_2(x) + 2\int_0^x \frac{1}{s}J_1(s)\,ds = -J_2(x) + 2\left[\frac{1}{s}J_1\right]_0^x =$

$= -J_0(x) + 2\frac{1}{x}J_1(x) - \underbrace{\lim_{x\to 0}\frac{1}{x}J_1(x)}$

$\frac{J_1(x)}{x} = \frac{J_1(x) - J_1(0)}{x-0} \xrightarrow[x\to 0]{} J_1'(0) = \frac{1}{2}$

$\boxed{5.2.8}$ Visa $\int_0^x s^3 J_0(s)\,ds = (x^3 - 4x)J_1(x) + 2x^2 J_0(x)$

$\int_0^x s^3 J_0(s)\,ds = \int_0^x s^2\cdot s J_0(s)\,ds = \left[(5.14),\ \nu=1\right] = \int_0^x s^2\cdot(sJ_1(s))'ds =$

$= \left[s^2\cdot s\,J_1(s)\right]_0^x - \int_0^x 2s\cdot sJ_1(s)\,ds = \left[(5.13),\ \nu=1\right] =$

$= x^3J_1(x) - 2\left(\left[s^2J_0(s)\right]_0^x - \int_0^x 2sJ_0(s)\,ds\right) = \left[(5.14),\ \nu=1\right] =$

$= x^3J_1(x) - 2\left(x^2J_0(x) - x^2 - 2xJ_1(x)\right) =$

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - in\theta}\, d\theta \qquad (5.22)$$

$J_{-n}(x) = (-1)^n J_n(x)$, n heltal

$\sum_{-\infty}^{\infty} J_n(x) z^n$, $z \neq 0$ $= e^{\frac{x}{2}(z - \frac{1}{z})}$

Genererande fkt för Bessel

$\boxed{5.2.9}$ $J_0(x) + 2\sum_1^{\infty} J_{2n}(x) = 1$

Ta gener. fkt med $z=1$

$$\sum_{-\infty}^{\infty} J_n(x) = 1$$

$n=0$ räknas 1 gång

n jämna räknas 2 ggr

n udda tar ut varandra

$$\Rightarrow \quad J_0 + 2\sum_1^{\infty} J_{2n} = 1$$

$\boxed{5.2.11}$ $J_0(x)^2 + 2\sum_1^{\infty} J_n(x)^2 = 1$

Väljer $z = e^{i\theta}$, $\theta \in]0, 2\pi[$

$$\underbrace{\sum_{-\infty}^{\infty} J_n(x) e^{in\theta}}_{\text{F-serie}} = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})} = \underbrace{e^{ix\sin\theta}}_{\text{summan}}$$

Parseval: $\sum (J_n(x))^2 = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|e^{ix\sin\theta}|^2}_{=1} d\theta = 1$

SATS (5.3)

a) för $u(r\ldots)=0$, $r=b$

b) för $\beta u(r\ldots)+\beta' u_r'(r\ldots)=0$, $r=b$, $\beta'\neq 0$

a) $\nu \geq 0$, $b>0$, $w(x)=x$

Låt λ_k^ν vara pos. noller $k=1,2,\ldots$

$J_\nu(x)$; $\varphi_k(x)=J_\nu\left(\lambda_k^\nu \frac{x}{b}\right)$

$\varphi_k(x)$ är en bas i $L_w^2\,]0,b[$, $\|\varphi_k\|^2=\frac{b^2}{2}J_{\nu+1}(\lambda_k^\nu)$

b) $c=\frac{b\beta}{\beta'}$; anta $c\geq -\nu$

$\tilde{\lambda}_k^\nu$ pos nollst till:

$cJ_\nu(x)+xJ_\nu'(x)$

b') $c>-\nu$: $\psi_k(x)=J_\nu\left(\frac{\tilde{\lambda}_k^\nu x}{b}\right)$ bas i $L_w^2\,]0,b[$

och ska satisf. randvillk

b'') $c=-\nu$: för att få en bas ska man lägga till en fkt $\psi_0=x^\nu$

$$\|\psi_k\|_w^2=\frac{b^2\left(\tilde{\lambda}_k^{\nu 2}-\nu^2+c^2\right)}{2\tilde{\lambda}_k^{\nu 2}}J_\nu(\tilde{\lambda}_k^\nu)^2$$

$$\|\psi_0\|_w^2=\frac{b^{2\nu+2}}{2\nu+2}$$

Fourier 080225 MaLv6

$f(x)$ på $]0,b[$, fall a, $\nu=0$; $f(x)=x^2$

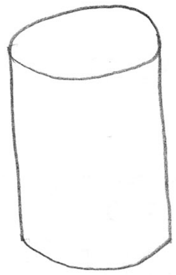
$$f(x)=\sum c_k J_0\left(\lambda_k \frac{x}{b}\right)$$

$$c_k=\frac{\langle f, J_0(\frac{\lambda_k}{b}x)\rangle_w}{\| J_0(\frac{\lambda_k}{b}x)\|_w^2} \leftarrow \text{ur } 5.5.3$$

$$\langle f, J_0(\tfrac{\lambda_k}{b}x)\rangle_w=\int_0^b x^2 J_0\left(\frac{\lambda_k x}{b}\right)x\,dx \overset{s=\frac{\lambda_k}{b}}{=} \left(\frac{b}{\lambda_k}\right)^4\int_0^b s^3 J_0(s)\,ds$$

Fourier 080229 Fr Lub.

$\boxed{5.5.3}$



$x^2+y^2 \leq 1$

$u(x,y,z,0) = ax+b$

b) $u'_z = 0$ på "lock" och "botten"

$u(x,y,z,t) = 0$, då $x^2+y^2=1$

Pol. koord.:

$u'_t - \Delta u = 0, \quad r \leq 1$

$u(r,\theta,z,0) = ar\cos\theta + b$

$u'_z = 0$ på ändytor

$u(1,\theta,z,t) = 0$

$$u'_t - u''_{rr} - \frac{1}{r}u'_r - \frac{1}{r^2}u''_{\theta\theta} - u''_{zz} = 0$$

u - ober. av z $\Rightarrow$ $u = u(r,\theta,t)$, $u(r,\theta,0) = ar\cos\theta + b$

$u(1,\theta,t) = 0$

Ekv.:

$$u'_t - u''_{rr} - \frac{1}{r}u'_r - \frac{1}{r^2}u''_{\theta\theta}$$

Söker $u = v + w$

$w(r,\theta,0) = ar\cos\theta$

$v(r,\theta,0) = b$

Sök w:

$w(r,\theta,0) = ar\cos\theta$

$w(1,\theta,t) = 0$

$$w'_t - w''_{rr} - \frac{1}{r}w'_r - \frac{1}{r^2}w''_{\theta\theta} = 0$$

Vi tittar på beg.-villkor och försöker söka lösn på samma form (ber. på θ som $\cos\theta$)

$w(r,\theta,t) = h(r,t)\cos\theta$

$$\left.\begin{array}{l} h(r,0) = ar \\ h(1,t) = 0 \end{array}\right\} \qquad h'_t - h''_{rr} - \frac{1}{r}h'_r + \frac{1}{r^2}h = 0$$

Var.-sep.:

$h = R(r)\,\Theta(\theta)$

$$RT' - R''T - \frac{1}{r}R'T + \frac{1}{r^2}RT = 0$$

$$\Rightarrow \quad \frac{T'}{T} = \frac{R'' + \frac{1}{r}R'}{R} - \frac{1}{r^2} = -\mu^2$$

$$\Rightarrow \quad \begin{cases} T' + \mu^2 T = 0 \\ R'' + \frac{1}{r}R' + \left(\mu^2 - \frac{1}{r^2}\right)R = 0, \quad R(1) = 0 \end{cases}$$

Allm. Bessel: $R'' + \frac{1}{r}R' - \frac{\nu^2}{r^2}R + \mu^2 R$ jmf för att hitta ν

Vårt fall: $\nu = 1 \Rightarrow$ Lösn är J_1

Sats 5.3 a), $\nu = 1$, $b = 1$

$R_k(r) = J_1(\lambda_k r)$ - egenf. där λ_k är pos. nollst. till $J_1(\lambda)$

$\mu_k = \lambda_k$; $\|R_k\|_w^2 = \frac{1}{2}J_2(\lambda_k)^2$

$\Rightarrow h(r,t) = \sum R_k(r)\,T_k(t)$

Enkla fallet: T-ekv fungerar:

$T_k' + \mu_k^2 T_k = 0$

$T_k = e^{-\mu_k^2 t}\,T_k(0)$

$T_k(0)$ söks ur beg.-villk:

$h(r,0) = \sum J_1(\mu_k r)\,T_k(0) = ar$

Skal.-mult med $J_1(\mu_n r)$ $(w = r)$:

$$\Rightarrow \sum \langle J_1(\mu_k r), J_1(\mu_n r)\rangle_w T_k(0) = \langle ar, J_1(\mu_n r)\rangle_w$$

Bara k = n överlever:

$$\|J_1(\mu_n r)\|^2 T_n(0) = \int_0^1 ar^2 J_1(\mu_n r)\,dr = \begin{bmatrix} s = \mu_n r \\ ds = \mu_n dr \end{bmatrix} =$$

$$= \frac{a}{\mu_n^3}\int_0^{\mu_n} s^2 J_1(s)\,ds = \left[(5.14),\ \nu = 2\right] = \frac{a}{\mu_n^3}\int_0^{\mu_n} \left(s^2 J_2(s)\right)' ds =$$

$$= \frac{a}{\mu_n^3}\Big[s^2 J_2(s)\Big]_0^{\mu_n} = \frac{a}{\cancel{\mu_n^3}}\left(\cancel{\mu_n^2} J_2(\mu_n)\right) = \frac{a J_2(\mu_n)}{\mu_n}$$

$$\Rightarrow T_n(0) = \frac{2a}{\mu_n J_2(\mu_n)}$$

$$w = \sum \frac{2a}{\mu_n J_2(\mu_n)} \cdot e^{-\mu_n^2 t} J_1(\mu_n r)\cos\theta$$

Problemet för v:

$$v_t' - v_{rr}'' - \frac{1}{r}v_r' - \frac{1}{r^2}v_{\theta\theta}''$$

$$\left.\begin{aligned} v(r,\theta,0) &= b \\ v(1,\theta,t) &= 0 \end{aligned}\right\}$$ ober. av θ, sök lösn ober. av θ

$$\Rightarrow \text{Ekv.: } v_t' - v_{rr}'' - \frac{1}{r}v_r' = 0$$

Var.-sep: $v(r,t) = R(r)T(t)$

$$\frac{T'}{T} = \frac{R'' + \frac{1}{r}R'}{R} = -\mu^2$$

$$\begin{cases} T' + \mu^2 T = 0 \\ R'' + \frac{1}{r}R' + \mu^2 R = 0 \end{cases}$$

Vår R-ekv är Bessel med $\nu=0$

5.3, $\nu=0$, $b=1$, a)

$R_k(r) = J_0(\lambda_k r)$, λ_k pos. nollor till J_0

$\mu_k = \lambda_k$

$v(r,t) = \sum J_0(\lambda_k r)\, T_k(t)$

$T_k' + \mu_k^2 T_k = 0 \Rightarrow T_k = e^{-\mu_k^2 t}\, T_k(0)$

Beg.-villk:

$v(r,0) = \sum J_0(\lambda_k r)\, T_k(0) = b$

Skal.-mult med $J_0(\lambda_n r)$ (vikt r):

$$\sum \langle J_0(\lambda_k r), J_0(\lambda_n r)\rangle_r\, T_k(0) = \langle b, J_0(\lambda_n r)\rangle_r$$

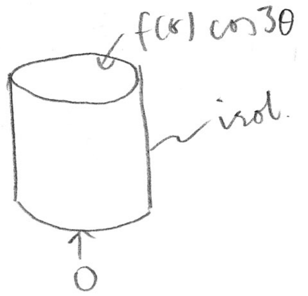
$$\Rightarrow \underbrace{\|J_0(\lambda_n r)\|_r^2\, T_n(0)}_{=\frac{1}{2}J_1(\lambda_n)^2 T_n(0)} = b\int_0^1 J_0(\lambda_n r)\, r\, dr = \begin{bmatrix} s = \lambda_n r \\ ds = \lambda_n dr \end{bmatrix} =$$

$$= \frac{b}{\lambda_n^2}\left[s J_1(s) \right]_0^{\lambda_n} = \frac{b}{\lambda_n^2}\lambda_n J_1(\lambda_n) = \frac{b J_1(\lambda_n)}{\lambda_n}$$

$$\Rightarrow T_n(0) = \frac{2b}{\lambda_n J_1(\lambda_n)}$$

MOD

$\boxed{5.5.6}$



$0 \le r \le 1$

$0 \le z \le 1$

$u''_{rr} + \frac{1}{r}u'_r + \frac{1}{r^2}u''_{\theta\theta} + u''_{zz} = 0$

$u'_r(1,\theta,z) = 0$

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$u(r,\theta,0)=0$

$u(r,\theta,1)=f(r)\cos 3\theta$

Sök lösn som $u(r,\theta,z)=v(r,z)\cos(3\theta)$

$$\Rightarrow v''_{rr}+\frac{1}{r}v'_r-\frac{9}{r^2}v+v''_{zz}=0$$

$v(r,0)=0$

$v_r(1,0)=0$

$v(r,1)=f(r)$

Var.-sep: $v(r,z)=R(r)Z(z)$

$$\Rightarrow \frac{R''+\frac{1}{r}R'-\frac{9}{r^2}R}{R}=-\frac{Z''}{Z}=-\mu^2$$

$$R''+\frac{1}{r}R'-\frac{9}{r^2}R+\mu^2R=0$$

$$Z''-\mu^2Z=0$$

R-ekv: Bessel med $\nu=3$

$\Rightarrow R(r)=J_3(\mu_k r)$,

$R(1)=0 \Rightarrow \mu_k=\lambda_k$ (pos. nollst till J_3)

$\boxed{5.3}$: b) $\nu=3$, $b=1$, $c=0$ $(>-3=-\nu)$

$$\|J_3(\lambda_k r)\|_r^2=\frac{\lambda_k^2-9}{2\lambda_k^2}J_3(\lambda_k)^2$$

$Z''-\mu^2Z=0 \Rightarrow p=\pm\lambda_k$

$Z=A\cosh\lambda_k z+B\sinh\lambda_k z$

$v(r,z)=\sum(A_k\cosh\lambda_k z+B_k\sinh\lambda_k z)J_3(\lambda_k r)$

$v(r,0)=\sum A_k J_3(\lambda_k r)=0 \Rightarrow A_k=0$

$$\Rightarrow v = \sum B_k \sinh(\lambda_k z) J_3(\lambda_k r)$$

$$v(r,1) = \sum B_k \sinh\lambda_k J_3(\lambda_k r) = f(r)$$

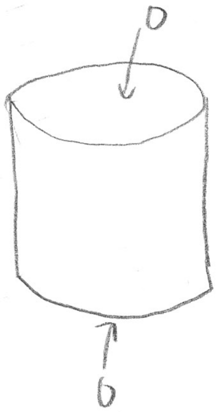
Skal.-mult $J_3(\lambda_n r)$ (vikt $= r$):

$$\|J_3(\lambda_n r)\|_r^2 B_n \sinh\lambda_n = \langle f(r), J_3(\lambda_n r)\rangle_r =$$

$$= \int_0^1 f(r) J_3(\lambda_n r)\, r\, dr$$

$$\Rightarrow \quad B_n = \frac{2\lambda_n^2 \int_0^1 r f(r) J_3(\lambda_n r)\, dr}{(\lambda_n^2 - 9) J_3(\lambda_n)^2 \sinh\lambda_n}$$

$$u(r,\theta,z) = \cos 3\theta \sum_1^\infty \frac{2\lambda_n^2 \int_0^1 r f(r) J_3(\lambda_n r)\, dr}{(\lambda_n^2 - 9) J_3(\lambda_n)^2 \sinh(\lambda_n)} \sinh(\lambda_n z) J_3(\lambda_n r)$$



$0 \le r \le 1$

$0 \le z \le 1$

$$u''_{rr} + \frac{1}{r} u'_r + \frac{1}{r^2} u''_{\theta\theta} + u''_{zz} = 0$$

$$u(r,\theta,0) = u(r,\theta,1) = 0$$

$$u(1,\theta,z) = \sin(\pi z)\cos\theta$$

Sök $u(r,\theta,z) = v(r,z)\cos\theta$

$$v(r,0) = v(r,1) = 0$$

$$v(1,z) = \sin(\pi z)$$

$$v''_{rr} + \frac{1}{r} v'_r - \frac{1}{r^2} v + v''_{zz} = 0$$

Separera: $\displaystyle \frac{R'' + \frac{1}{r}R' - \frac{1}{r^2}R}{R} = -\frac{Z''}{Z} = -\mu^2$

Hom randvillk i Z-ekv (!!!)

$$Z'' - \mu^2 Z = 0 \Rightarrow p = \pm\mu$$

$$Z(0) = 0, \quad Z(1) = 0$$

...... $Z_k(z) = \sin(k\pi z)$, egenv. $\mu_k^2 = -k^2\pi^2$, $k = 1, 2, \ldots$

R-ekv:

$$R_k'' + \frac{1}{r}R_k' - \frac{1}{r^2}R_k - k^2\pi^2 R_k = 0 \qquad (\nu = 1)$$

Jmf allm. Bessel: olika tecken

$$R'' + \frac{1}{r}R' - \frac{\nu^2}{r^2} + \mu^2 R = 0$$

Modifierade Bessel-ekv.:

$$f''(x) + \frac{1}{x}f'(x) - \frac{\nu^2}{x^2}f(x) - f(x) = 0$$

Lösn som är begr i 0 (modif. Besselfkt av ordn ν)

$I_\nu(x)$

$$\Rightarrow R_k(r) = I_\nu(k\pi r), \quad \nu = 1$$

$$R_k(r) = A_k I_1(k\pi r)$$

$$v(r, z) = \sum \sin(k\pi z)\, A_k I_1(k\pi r)$$

$$v(1, z) = \sum \sin(k\pi z)\, A_k I_1(k\pi) = \sin \pi z$$

$$\Rightarrow A_k = \begin{cases} \frac{1}{I_1(\pi)} & k = 1 \\ 0 & k \neq 1 \end{cases}$$

$$\Rightarrow v(r, z) = \sin(\pi z) \frac{I_1(\pi r)}{I_1(\pi)}$$

$$\Rightarrow u(r, \theta, z) = \frac{I_1(\pi r)}{I_1(\pi)} \sin(\pi z) \cos\theta$$

Ortogonala polynom (kap. 6)

$I =]a,b[$, $w(x) \geq 0$ viktfkt

antar att $\int_I w(x)|x|^n \, dx < \infty$, alla $n \geq 0$

Introd. sk-prod.

$$\langle f, g \rangle = \int_I w(x) f(x) \overline{g(x)} \, dx$$

$\varphi_0 = 1$

$\varphi_1 = x$ $\rightarrow$ Ortogonalisera med Gramm-Schmidt

$\vdots$ Kommer till $P_0, P_1, \dots, P_n, \dots$

$\varphi_n = x^n$ som ortogonala med varandra

$\vdots$ $\langle P_j, P_k \rangle_w = 0, \; j \neq k$

$P_0 = \varphi_0 = 1$ (polynom grad 0)

$$P_1 = \varphi_1 - \frac{\langle \varphi_1, P_0 \rangle}{\|P_0\|^2} P_0 = x - \frac{\langle x, 1 \rangle_w}{\|1\|_w^2} \quad \text{(polyn. grad 1)}$$

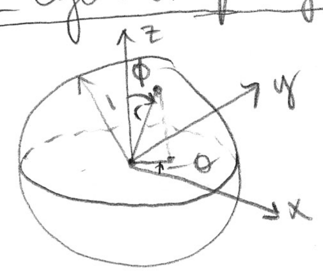
$$P_2 = \varphi_2 - \frac{\langle \varphi_2, P_0 \rangle_w}{\|P_0\|} P_0 - \frac{\langle \varphi_2, P_1 \rangle_w}{\|P_1\|_w^2} P_1 \quad \text{(polyn. grad 2)}$$

$\vdots$

P_n - polynom grad n

$\{P_n\}$ - system av ortogonala polynom på I med vikt w.

Legendrepolynom



$x = r \sin\phi \cos\theta$

$y = r \sin\phi \sin\theta$

$z = r \cos\phi$

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$$\Delta u = u''_{xx} + u''_{yy} + u''_{zz} = \frac{1}{r^2}(r^2 u'_r)'_r + \frac{1}{r^2 \sin\phi}(u'_\phi \sin\phi)'_\phi + \frac{1}{r^2\sin^2\phi} u''_{\theta\theta}$$

$\Delta u(r,\theta,\phi) = 0$ (Laplace ekv), $0 < r < 1$

$u(1,\theta,\phi) = f(\theta,\phi)$

u ändl då $r = 0$

$0 \le \theta \le 2\pi$ u 2π-per i θ

$0 \le \phi \le \pi$; u ändl. för $\phi = 0$, $\phi = \pi$

$$u(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

$$\frac{1}{r^2}(r^2 R'_r)'\,\Theta\Phi + \frac{1}{r^2}\frac{1}{\sin\phi}(\Phi'\sin\phi)'_\phi R\Theta + \frac{1}{r^2\sin^2\phi}\Theta'' R\Phi$$

Dela med $R\Theta\Phi \frac{1}{r^2\sin^2\phi}$:

$$r^2\sin^2\phi\left(\frac{R''}{R} + \frac{2R'}{rR}\right) + \sin\phi\,\frac{(\Phi'\sin\phi)'}{\Phi} = -\frac{\Theta''}{\Theta} = m^2$$

$\Theta'' + m^2\Theta = 0$, $\quad \Theta(0) = \Theta(2\pi)$

$\Theta'(0) = \Theta'(2\pi)$

$$\frac{r^2R'' + 2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{(\Phi'\sin\phi)'}{\Phi\sin\phi} = \lambda \qquad R(0) \text{ ändl.}$$

Börja med Θ-ekv:

$p^2 = \pm\sqrt{-m^2} \Rightarrow \Theta_m(\theta) = e^{im\theta}$, $m = 0, \pm 1, \ldots$

Φ-ekv:

$s = \cos\phi$! $\quad -1 \le s \le 1$

$\Phi(\phi) = S(s) = S(\cos\phi)$

$$((1-s^2)S')' - \frac{m^2 S}{1-s^2} + \lambda S = 0$$

$-1 < s < 1$, $S(-1), S(1)$ ändl.

Legendre-ekv. av ordn m, $m = 0, 1, 2, \ldots$

SATS

1) Egenfkt är polynom: $P_n^m(s)$, $n = m, m+1, \ldots$
   P_n^m är generalis. Legendre-polynom av ordn m.
   P_n^0 Legendre-polynom.
2) P_n^m för fixt m är en bas i $L^2]-1,1[$ (vikt $w=1$)
3) Egenv. $n(n+1) = \lambda_n$

Man får Leg.-polynom genom ortog av $\varphi_n = x^n$ på $[-1,1]$

Legendre, forts.:

$m \neq 0$: Assoc. Legendre-fkt

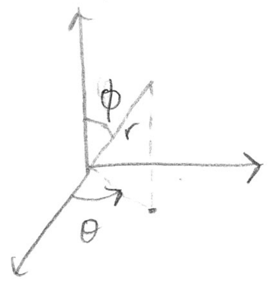
$$S_n(s) = P_n^{|m|}(s); \quad n = |m|, |m|+1, \ldots$$

$$\lambda_n = n(n+1)$$

Bas i $L^2\,]-1,1[$ med vikten 1

$$\Rightarrow \Phi_n^m(\phi) = P_n^{|m|}(\cos\phi)$$

Ortogonala m. vikten $W(\phi) = \sin\phi$



R-ekv.:

$$r^2R'' + 2rR' - n(n+1)R = 0$$

$$u(r,\theta,\phi) = \sum_{-\infty<m<\infty} \sum_{n \geq |m|} e^{im\theta} P_n^{|m|}(\cos\phi) R_{mn}(r)$$

Lös R-ekv.:

$$p^2 + p - n(n+1) = 0$$

$$\left(p + \frac{1}{2}\right)^2 = \frac{1}{4} + n(n+1) = \left(n + \frac{1}{2}\right)^2$$

$$\Rightarrow p = \begin{cases} n \\ -n-1 \end{cases}$$ leder till obegr. lösn.

$$\Rightarrow R_n(r) = r^n$$

$$\Rightarrow u(r,\theta,\phi) = \sum_{m,n} e^{im\theta} P_n^{|m|}(\cos\phi)\, r^n A_{nm}$$

$$u(1,\theta,\phi) = f(\theta,\phi)$$

$$u(1,\theta,\phi) = \sum_{m,n} e^{im\theta} P_n^{|m|}(\cos\phi) A_{nm}$$

Skal.-mult med $e^{ik\theta}$, endast $m=k$ överlever:

$$\sum_n 2\pi P_n^{|k|}(\cos\phi) A_{kn} = \int_0^{2\pi} f(\theta,\phi) e^{-ik\theta}\, d\theta$$

Skal.-mult med $P_\ell^{|k|}(\cos\phi)$ (vikt $\sin\phi$); bara $n=\ell$ överlever:

$$\underbrace{2\pi\int_0^\pi |P_\ell^{|k|}(\cos\phi)|^2 \sin\phi\, d\phi}_{\frac{(\ell+|k|)!\,2}{(\ell-|k|)!(2\ell+1)}} \cdot A_{k\ell} = \int_0^{2\pi}\int_0^\pi f(\theta,\phi)\, e^{-ik\theta} P_\ell^{|k|}(\cos\phi)\sin\phi\, d\phi\, d\theta$$

$$\Rightarrow \quad A_{k\ell} = \ldots$$

Funktionerna: $Y_\ell^k(\theta,\phi) = e^{-ik\theta}P_\ell^{|k|}(\cos\phi)$ sfäriska fkter.

Bas i L^2(sfären)

Legendre-polynom:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right) \quad \text{pol. av grad } n$$

Udda då n udda, jämn då n jämn.

$$((1-x^2)P_n')' + n(n+1)P_n = 0, \quad -1<x<1 \qquad \text{(sing S-L)}$$

$$\sum_0^\infty P_n(x)\cdot z^n = (1-2xz+z^2)^{-\frac12} \qquad \text{(Sats 6.5)}$$

Följdsats 6.1: $P_n(1)=1$, $P_n(-1)=(-1)^n$ $\quad x=1$ i

$$\sum P_n(1)z^n = (1-2z+z^2)^{-\frac12} = ((1-z)^2)^{-\frac12} = \frac{1}{1-z} = \sum_0^\infty z^n$$

Fourier-Legendre-serie:

$$f(x),\ x\in\,]-1,1[\,:\ f(x)=\sum c_n P_n(x); \quad c_n = \frac{\langle f, P_n\rangle}{\|P_n\|^2}$$

Rekurrent formel för Legendre:

$$P_{n+1}' - P_{n-1}' = (2n+1)P_n$$

$$P_{2k-1}(0) = 0; \quad P_{2k}(0) = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}$$

$\boxed{6.2.7}$

$$f(x) = \begin{cases} 1, & 0<x<1 \\ -1, & -1<x<0 \end{cases}$$

$$c_n = \frac{\langle f, P_n \rangle}{\|P_n\|^2}, \qquad \|P_n\|^2 = \frac{2}{2n+1}$$

$$\langle f, P_n \rangle = \int_{-1}^{1} \underbrace{f(x) P_n(x)}_{\text{udda}} dx = \begin{cases} 0 & n \text{ jämnt} \\ 2\int_0^1 f(x) P_n(x) dx & n \text{ udda} \end{cases}$$

$$2\int_0^1 \underbrace{f(x) P_n(x)}_{\text{udda}} = 2\int_0^1 P_n(x)\,dx \overset{\text{Rekurrent. } \cancel{(n\neq 0)}}{=} 2\int_0^1 \frac{P_{n+1}' - P_{n-1}'}{(2n+1)} dx =$$

$$= \frac{2}{2n+1}\Big[P_{n+1} - P_{n-1}\Big]_0^1 = \frac{2}{2n+1}\Big(\underset{1}{\underset{\|}{P_{n+1}(1)}} - \underset{1}{\underset{\|}{P_{n-1}(1)}} - P_{n+1}(0) + P_{n-1}(0)\Big)$$

$n=2k+1$
$n+1=2k+2$
$n-1=2k$

$$= \frac{2}{4k+3}\left(\frac{(-1)^{k+2}(2k+2)!}{2^{2k+2}((k+1)!)^2} + \frac{(-)^k (2k)!}{2^{2k}(k!)^2}\right)$$

$\boxed{6.2.8}$

$$f(x) = \begin{cases} x, & x>0 \\ 0, & x<0 \end{cases} \quad \text{utveckla i F-L-serie}$$

$$c_n = \frac{\langle f, P_n \rangle}{\|P_n\|^2}; \quad \|P_n\|^2 = \frac{2}{2n+1}$$

$$\langle f, P_n \rangle = \int_0^1 x P_n(x)\,dx = \frac{1}{2n+1}\int_0^1 x(P_{n+1}' - P_{n-1}')\,dx =$$

$$= \underbrace{\frac{1}{2n+1}\Big[x(P_{n+1} - P_{n-1})\Big]_0^1}_{=0} - \frac{1}{2n+1}\int_0^1 (P_{n+1} - P_{n-1})\,dx =$$

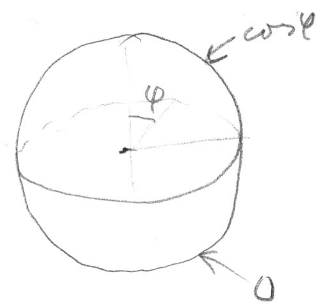
$$\overset{n>1}{=} -\frac{1}{2n+1}\int_0^1 \left(\frac{P_{n+2}' - P_n'}{2n+3} - \frac{P_n' - P_{n-2}'}{2n-1}\right) dx =$$

$$= \frac{1}{2n+1}\left(\frac{P_{n+2}(1) - P_{n+2}(0) - P_n(1) + P_n(0)}{2n+3} - \frac{P_n(1) - P_n(0) - P_{n-2}(1) + P_{n-2}(0)}{2n-1} \right)$$

Beräkna $n=0$, $n=1$ separat

$\boxed{6.3.1}$

$$\Delta u = 0, \quad r<1$$
$$u(1,\theta,\varphi) = \cos\varphi, \quad 0<\varphi<\frac{\pi}{2}$$
$$u(1,\theta,\varphi) = 0 \qquad \varphi > \frac{\pi}{2}$$



$$\Delta u = u''_{rr} + \frac{2}{r}u'_r + \frac{1}{r^2\sin\varphi}(u'_\varphi \sin\varphi)'_\varphi + \frac{1}{r^2\sin^2\varphi}u''_{\theta\theta}$$

Antag u ober av θ:

$$u''_{rr} + \frac{2}{r}u'_r + \frac{1}{r^2\sin\varphi}(u'_\varphi\sin\varphi)'_\varphi = 0$$

$$u(1,\varphi) = f(\varphi) = \begin{cases} \cos\varphi & 0<\varphi<\frac{\pi}{2} \\ 0 & \varphi>\frac{\pi}{2} \end{cases}$$

Lösn ändl. i $\varphi = 0, \pi$

Var.-sep:

$$u = R(r)\,\Phi(\varphi)$$

$$R''\Phi + \frac{2}{r}R'\Phi + \frac{1}{r^2\sin\varphi}(\Phi'\sin\varphi)'R = 0$$

$$\frac{r^2R''+2rR'}{R} = -\frac{1}{\sin\varphi}\frac{(\Phi'\sin\varphi)'}{\Phi} = \lambda$$

Homogent randvillk i $\varphi \Rightarrow$ Lös Φ-ekv.

$$\frac{1}{\sin\varphi}(\Phi'\sin\varphi)' + \lambda\Phi = 0$$
$$\Rightarrow (\Phi'\sin\varphi)' + \lambda\underbrace{\sin\varphi}_{w}\Phi = 0$$

$s = \cos\varphi$; $\quad S(s) = \Phi(\varphi)$

$\Rightarrow \quad ((1-s^2)S')' + \lambda S = 0 \;, \quad -1 < s < 1$

S ändlig i ± 1

Egenfkt $\quad S_n(s) = P_n(s)$, $\quad n = 0, 1, \ldots$

Egenv. $\quad \lambda_n = n(n+1)$

Enkla fallet, lös R-ekv.:

$r^2 R_n'' + 2r R_n' - n(n+1) R_n = 0$

$u(r,\varphi) = \sum R_n(r) P_n(\cos\varphi)$

$R_n = A_n r^n \quad$ (visat tidigare)

$\Rightarrow u(r,\varphi) = \sum A_n r^n P_n(\cos\varphi)$

Skal-mult med $P_k(\cos\varphi)$ ($r=1$, vikt $\sin\varphi$)

$$\sum A_n \langle P_n(\cos\varphi), P_k(\cos\varphi)\rangle_{\sin\varphi} = \langle \cos\varphi, P_k(\cos\varphi)\rangle_{\sin\varphi}$$

Endast $n=k$ överlever:

$$A_k \underbrace{\|P_k\|^2}_{\frac{2}{2k+1}} = \int_0^{\pi/2} \cos\varphi \, P_k(\cos\varphi) \sin\varphi \, d\varphi = [\cos\varphi = s] =$$

$$= \int_0^1 s P_k(s) \, ds$$

$\boxed{6.3.2}$ $\quad \Delta u = 0, \quad r < 1$

$u(1,\theta,\varphi) = \cos^2\theta \sin^2\varphi$; $\quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$

Sök $u(r,\theta,\varphi) = u_1 + u_2$

$\Delta u_1 = 0, \quad u_1(1,\theta,\varphi) = \frac{1}{2}\sin^2\varphi$

$\Delta u_2 = 0, \quad u_2(1,\theta,\varphi) = \frac{1}{2}\cos 2\theta \sin^2\varphi$

u_1: randvillk ober av θ, hoppas lösn är ober. av θ, löser som 1

u_2?

Vi söker lösn på formen

$$u_2(r,\theta,\varphi) = v(r,\varphi)\cos 2\theta$$

Sätt in i ekv.

$$v''_{rr}\cos 2\theta + \frac{2}{r}v'_r\cos 2\theta + \frac{1}{r^2\sin\varphi}(v'_\varphi \sin\varphi)'_\varphi \cos 2\theta + \frac{1}{r^2\sin^2\theta}v(-4\cos 2\theta) = 0$$

$\cos 2\theta$ kan förkortas $\Rightarrow$ antagande riktigt

$$\Rightarrow \quad v''_{rr} + \frac{2}{r}v'_r + \frac{1}{r^2\sin\varphi}(v'_\varphi\sin\varphi)'_\varphi - \frac{4}{r^2\sin^2\varphi}v = 0$$

Var.-sep:

$$v = R(r)\,\Phi(\varphi)$$

$$R''\Phi + \frac{2}{r}R'\Phi + \frac{1}{r^2\sin\varphi}(\Phi'\sin\varphi)'R - \frac{4}{r^2\sin^2\varphi}R\Phi = 0$$

$$\Rightarrow \quad \frac{r^2R'' + 2rR'}{R} = \frac{\frac{4\Phi}{\sin^2\varphi} - \frac{1}{\sin\varphi}(\Phi'\sin\varphi)'}{\Phi} = \lambda$$

Φ först:

$$(\Phi'\sin\varphi)' - \frac{4}{\sin\varphi}\Phi + \lambda\sin\varphi\,\Phi = 0$$

$$\boxed{s = \cos\varphi}$$

$$((1-s^2)S')' - \frac{2^2 S}{1-s^2} + \lambda S = 0$$

Assoc. Legendre, $m=2$

$$\Rightarrow S_n(s) = P_n^2(s), \quad \lambda_n = n(n+1)$$

$$v(r,\varphi) = \sum R_n(r)P_n^2(\cos\varphi)$$

$$R_n(r) = A_n r^n$$

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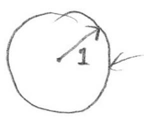
$\boxed{6.3.3}$ $\quad \Delta u = 0, \quad r > 1$



$u(1,\theta,\varphi) = f(\theta,\varphi)$

$u \to 0, \; r \to \infty$

Skillnaden i R-ekv.

$$r^2R'' + 2rR' - n(n+1)R = 0, \quad R \to 0 \text{ i } \infty$$

$$R(r) = \begin{cases} r^n \\ r^{-n-1} \leftarrow \text{Välj denna!!!} \end{cases}$$

Hermite- och Laguerrepolynom

Hermite:

$I =]-\infty,\infty[; \; w = e^{-x^2}$

$H_n(x)$

$$\underbrace{\left(e^{-x^2}H_n'\right)' + \lambda_n e^{-x^2}H_n = 0}_{\text{Hermiteekv.}}$$

$\lambda_n = 2n$

Laguerre:

$I =]0,\infty[; \; w_\alpha = x^\alpha e^{-x}, \; \alpha > -1$

$L_n^\alpha(x)$

$$\underbrace{\left(x^{\alpha+1}e^{-x}L_n^{\alpha\,\prime}\right)' + \lambda_n x^\alpha e^{-x}L_n^\alpha = 0}_{\text{Laguerre-ekv.}}$$

$\lambda_n = n$

$\boxed{\text{Formar bas i } L^2_w}$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\underbrace{h_n(x) = e^{-\frac{x^2}{2}}H_n(x)}$$

Hermite-fkt bildar bas i $L^2]-\infty,\infty[$, med $w = 1$

$\|H_n\|_w^2 = 2^n n!\sqrt{\pi}$

$H_n' = 2nH_{n-1}$

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(x^{\alpha+n}e^{-x}\right)$$

$$\|L_n^\alpha\|_w^2 = \frac{\Gamma(n+\alpha+1)}{n!}; \quad \Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$$

(H_n)

$$\underbrace{\sum_0^\infty H_n(x)\frac{z^n}{n!} = e^{2xz-z^2}}_{\text{Genererande fkt}}$$

(L_n^α)

$$\underbrace{\sum_0^\infty L_n^\alpha(x)\, z^n = \frac{e^{-\frac{xz}{1-z}}}{(1-z)^{\alpha+1}}}_{\text{Genererande fkt}}$$

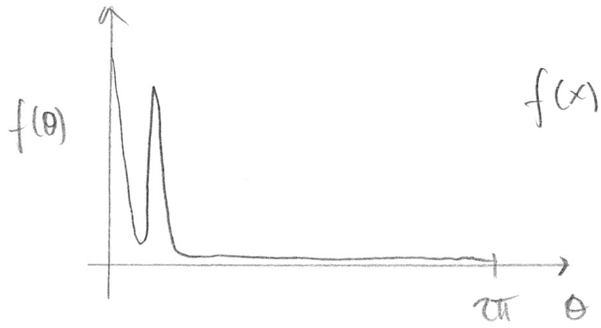
x^2

Harmonisk oscillator

$$\frac{1}{h^2}\left(-\Delta\Psi + |x|^2\Psi\right) = E\Psi$$

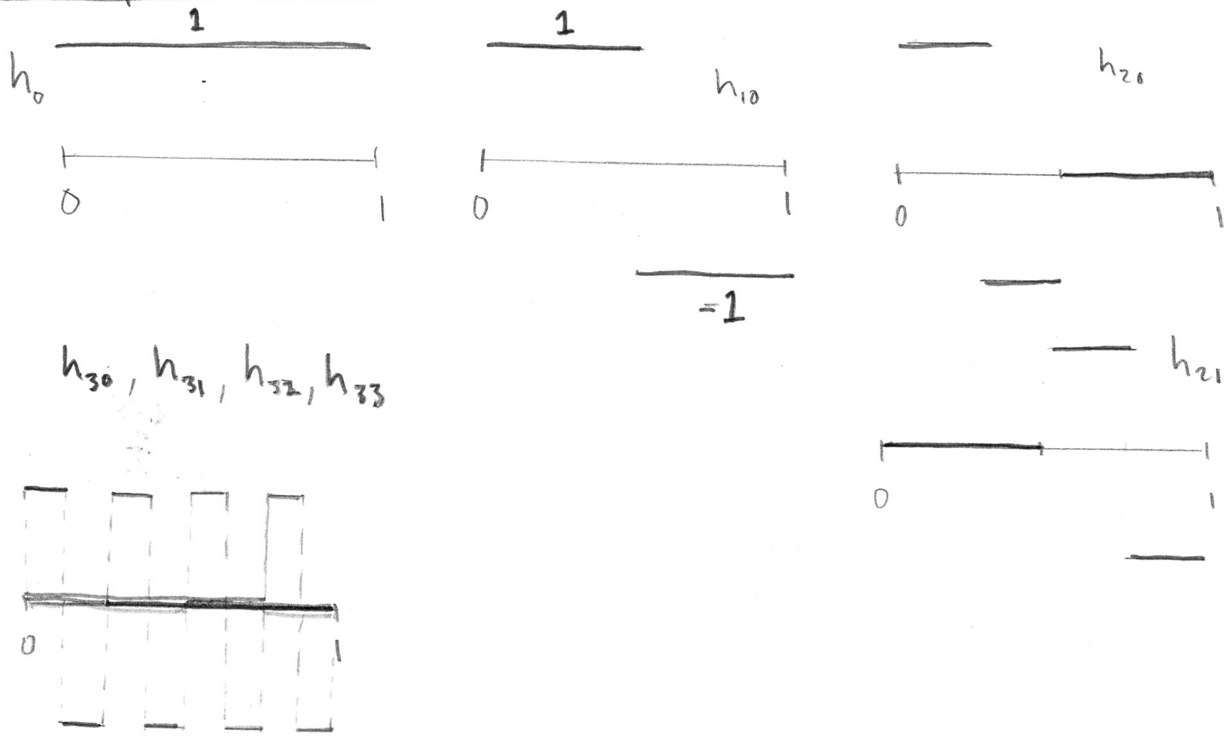
? ?

$$-\underset{?}{h^2}\Delta\Psi - \underset{?}{\frac{Z}{|x|^2}}\Psi = F\Psi$$



$$f(x) = \sum c_k e^{ik\theta}$$

Haarfunktioner



$h_{10} = \varphi(x) \longleftarrow$ mother function

$h_{20} = \varphi(2x)$

$h_{21} = \varphi(2x-1)$

$h_{30} = \varphi(4x)$

$h_{31} = \varphi(4x-1)$

$h_{32} = \varphi(4x-2)$

$h_{33} = \varphi(4x-3)$

Allmänt:

$$h_{nk} = \varphi(2^{n-1}x - k), \quad k = 0, \dots, 2^{n-1}$$

Haarfunktioner av nivå n.

I Daubechies:

$\varphi(x)$ glatt; $\varphi_{nk}(x) = \varphi(2^n x - k)$, $k = 0, 1, \dots, 2^{n-1}$

φ_{nk} ortogonala Wavelets

$u''_{xx} - \frac{1}{x} u + \lambda u = 0$ (väte)

$x \in]0, \infty[$, $u(0)$ ändlig

1. $\lambda > 0$; $\lambda = \mu^2$, $0 < \mu < \infty$; $\varphi_\mu(x)$

2. $\lambda < 0$, $\lambda = -\mu^2$, $\mu = \mu_n$, $n = 1, 2, \ldots$; φ_n

$$f(x) = \int_0^\infty c(\mu)\, \varphi_\mu(x)\, d\mu + \sum c_n \varphi_n(x)$$

Radial lösn $u(r,t)$

$$u'_t = \Delta u - u, \quad r<2$$

$$\begin{cases} u(r,0) = 4-r^2, & 1 \le r \le 2 \\ u(r,0) = 3, & r<1 \\ u(2,t) = 0 \end{cases}$$

$u(r,t) = R(r)T(t)$:

$$RT' = R''T + \frac{1}{r}R'T - RT$$

$$\Rightarrow \frac{T'}{T} + 1 = \frac{R'' + \frac{1}{r}R'}{R} = -\mu^2$$

$$T' + (1+\mu^2)T = 0$$

$$R'' + \frac{1}{r}R' + \mu^2 R = 0, \quad \nu = 0$$

$$\Rightarrow R_k(r) = J_0\left(\frac{\lambda_k r}{2}\right), \quad \mu_k = \frac{\lambda_k}{2}, \quad \lambda_k = \text{pos. nollst till } J_0$$

a) b=2

$$u(r,t) = \sum J_0\left(\frac{\lambda_k r}{2}\right) T_k(t)$$

Enkla fallet:

$$T_k' + (1+\mu_k^2)T_k = 0; \quad T_k(t) = T_k(0)\, e^{-(\mu_k^2+1)t}$$

$$u(r,0) = \sum J_0\left(\frac{\lambda_k r}{2}\right) T_k(0) = f(r) = \begin{cases} 4-r^2, & 1 \le r \le 2 \\ 3, & r<1 \end{cases}$$

Skal-mult. med $J_0\left(\frac{\lambda_n r}{2}\right)$ med vikt r:

$$T_n(0)\, \left\| J_0\left(\frac{\lambda_n r}{2}\right) \right\|^2_{w=r} = \left\langle f(r), J_0\left(\frac{\lambda_n r}{2}\right) \right\rangle_{w=r}$$