MVE 290/030 PROOFS OF THE THEORY QUESTIONS

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ABSTRACT. This is completely optional, non-required, supplementary material. Caveat emptor (although you are not buying anything because this document is free). These proofs and explanations may be helpful for studying for the exam, which is the intention behind their creation. However, the proofs of all theory items are also contained in the wonderful textbook by Folland. The statements and proofs in Folland are perfectly acceptable on the exam. Moreover, if you are able to come up with your own proof of any of these theory items, as long as it is correct and complete, that is awesome and shall be happily accepted as well.

(1) Proof of pointwise convergence of Fourier series (Theorem 2.1 of Folland).
(2) Proof of the formula for the relationship between the Fourier coefficients for a function and its derivative (Theorem 2.2 of Folland).
(3) Proof of Theorem 7.3 in Folland.
(4) The Fourier inversion formula.
(5) Plancharel's Theorem.
(6) Proof of the Sampling Theorem.
(7) Proof of Theorem 3.4.
(8) Proof of Theorem 3.8 on the best approximation.
(9) Proof of Theorem 3.9 (a) and (b).
(10) Proof of the Generating Function for $J_n(x)$, formula (5.20) in Folland.
(11) Proof of the orthogonality of the Hermite polynomials (this is part of the proof of Theorem 6.11 in Folland).
(12) Proof of Theorem 6.13, that is to derive the generating formula for the Hermite polynomials (6.35).

1. How do we learn proofs?

In mathematical research, we have to come up with proofs of theorems which have never been proven before! There is no handy "proofs compendium" like this one here. Moreover, many such proofs end up being really, really, really long (like over 50 pages). So, how do we do it? We all start out the same way: we begin by studying proofs written by other people.

1.1. **Step one: line-by-line.** First, read the proof carefully, line-by-line. It is okay if you cannot really see the whole proof in your head all at once. No worries! Just read line-by-line. If you can understand each line and how it leads to the next line, that is totes adorbs! (= great) This is the way you should begin studying the proofs.

1.2. **Step two: red thread.** Next step: learn the "red thread." (Isn't this like a Swedish saying? Hope I am getting it right...) This is a sequence of key ideas in the proof. It is like street lamps lighting a dark Swedish night in winter. They provide enough light to find your way in the dark. In this document, I've tried to collect what seems to me to be enough lamps to guide your way through the proof. This is called the "red thread," and these points are listed after each complete proof. If you come up with your own red thread which is different, in that it contains more or less points, or different points altogether, that is fine! Good for you! The goal with the "red thread" is that it is a smaller amount to memorize than the whole proof. So, you memorize the red thread, *after* you have read through the proof many times line-by-line. You first need to read through the proof line-by-line many times, so that you feel you understand it. Then, you memorize the red thread (either my red thread or your own). Then you'll be ready for step three...

1.3. **Step three: fill in the details.** Once you have (1) studied the proof carefully line-by-line and (2) memorized the red thread, you can proceed to the third (3) phase of proof-practice. Phase three is: with your red thread steps as a guide, try to do the proof yourself! The idea is that you fill in all the details in between the points on the red thread. Give yourself plenty of time, because it is not easy. However, I promise that while not easy, it is SUPER rewarding. I mean, look at the first proof below. It is a BEAST. (It's like Marshawn Lynch!) Looks impossible to master, right? That's the beauty of it. If you follow this process, you *can* master it, and then that feeling of being able to do the proof yourself is super awesome. (Unlike the end of the Superbowl in 2015).

2. Pointwise convergence of Fourier series for continuous, piecewise $\mathcal{C}^1$ functions

This is a big theorem. The statement we shall prove is the following

Theorem 2.1. *Let f be a 2π periodic function. Assume that f is piecewise continuous on $\mathbb{R}$, and that for every $x \in \mathbb{R}$, the left and right limits of both f and f' exist at x, and these are finite. Let*

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Then

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_-) + f(x_+)\right), \quad \forall x \in \mathbb{R}.$$

Proof: The result should hold for each and every point $x \in \mathbb{R}$. So, first, we fix a point $x \in \mathbb{R}$. Next, as usual, we should use the definitions, so we expand the series using its definition. So, we write

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx}.$$

Now, let's move that lonely e^{inx} inside the integral so it can get close to its friend, e^{-iny}. Then,

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny+inx} dy.$$

OBS! that f on the right is not involved with x, but in the theorem we are trying to prove, we want to relate $S_N(x)$ to $f(x)$. How can we get an x inside the f? Simple, we change the variable. Let $t = y - x$. Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt.$$

Remember that very first fact we proved for periodic functions? It said that the integral of a periodic function of period P from any point a to $a+P$ is the same, no matter what a is. Here $P = 2\pi$. Hence

$$\int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt = \int_{-\pi}^{\pi} f(t+x) e^{-int} dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) e^{-int} dt = \int_{-\pi}^{\pi} f(t+x) \frac{1}{2\pi} \sum_{-N}^{N} e^{int} dt.$$

This is how we get to the N^{th} Dirichlet kernel. Let

$$D_N(t) = \frac{1}{2\pi} \sum_{-N}^{N} e^{int}.$$

Proposition 2.2. *The N^{th} Dirichlet kernel satisfies:*

$$\int_{-\pi}^{0} D_N(t)dt = \frac{1}{2} = \int_0^{\pi} D_N(t)dt. \tag{2.1}$$

$$D_N(t) = \frac{1}{2\pi}e^{-iNt}\frac{1-e^{i(2N+1)t}}{1-e^{it}} = \frac{e^{-iNt}-e^{i(N+1)t}}{2\pi(1-e^{it})}. \tag{2.2}$$

The proof of this proposition shall be given after continuing with the proof of pointwise convergence of Fourier series. The reason for this is because you may use this proposition without proving it! We have:

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x)D_N(t)dt.$$

We want to show that $S_N(x)$ converges to the average of the right and left hand limits of f. In other words, this is equivalent to showing that

$$\lim_{N\to\infty}\left|S_N(x) - \frac{1}{2}\left(f(x_-)+f(x_+)\right)\right| = 0.$$

The S_N has an integral, but the $f(x_\pm)$ don't. They have got a convenient factor of one half, so we use (2.3) to exploit this

$$\frac{1}{2}f(x_-) = \int_{-\pi}^{0} D_N(t)dtf(x_-), \quad \frac{1}{2}f(x_+) = \int_0^{\pi} D_N(t)dtf(x_+).$$

Hence we are bound to prove that

$$\lim_{N\to\infty}\left|S_N(x) - \int_{-\pi}^{0} D_N(t)f(x_-)dt - \int_0^{\pi} D_N(t)f(x_+)dt\right| = 0.$$

Now, we use that

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x)D_N(t)dt.$$

Hence, we want to show that

$$\left|\int_{-\pi}^{\pi} f(t+x)D_N(t)dt - \int_{-\pi}^{0} D_N(t)f(x_-)dt - \int_0^{\pi} D_N(t)f(x_+)dt\right| \to 0, \text{ as } N\to\infty.$$

It is quite natural to split things into two parts

$$\left|\int_{-\pi}^{0} D_N(t)(f(t+x)-f(x_-))dt + \int_0^{\pi} D_N(t)(f(t+x)-f(x_+))dt\right|.$$

Now, we know we've got to use the second expression for $D_N(t)$, and here's where it will come in handy. Let's insert it

$$\left|\int_{-\pi}^{0} \frac{e^{-iNt}-e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_-))dt + \int_0^{\pi} \frac{e^{-iNt}-e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_+))dt\right|.$$

Now, we know that if we take a function which is bounded, then its Fourier coefficients tend to 0, meaning $c_n \to 0$ as $|n| \to \infty$. We've got those e^{-iNt} and $e^{i(N+1)t}$ which look a lot like part of the definition of Fourier coefficient c_n for $|n|$ large... However, we've got this integrand defined two different ways on either side of zero. So, let's just make a try for something and define a new function

$$g(t) = \frac{f(t+x)-f(x_-)}{1-e^{it}}, \quad \text{for } t<0,$$

$$g(t) = \frac{f(t+x)-f(x_+)}{1-e^{it}}, \quad \text{for } t>0.$$

How to define this function at zero? Let's look at the limit

$$\lim_{t\to 0_-}\frac{f(t+x)-f(x_-)}{1-e^{it}} = \lim_{t\to 0_-}\frac{t(f(t+x)-f(x_-))}{t(1-e^{it})} = \frac{f'(x_-)}{-ie^{i0}} = if'(x_-).$$

For the other side, a similar argument shows that

$$\lim_{t\to 0_+} \frac{f(t+x) - f(x_-)}{1-e^{it}} = if'(x_+).$$

So, depending upon whether $f'(x_-) = f'(x_+)$ or not, the function g will be continuous at 0, or not. However, even if it's not continuous, it is at least piecewise continuous, as well as piecewise differentiable, just like the original function f is. To see this, we see that for all other points $t \in [-\pi, \pi]$, the denominator of g is non-zero, and the numerator has the same properties as f. Therefore the above shows that g is indeed quite a lovely function on $[-\pi, \pi]$. The most important fact is that it is bounded on the closed interval $[-\pi, \pi]$, and hence its Fourier coefficients tend to zero by Bessel's inequality. This follows from the fact that any bounded function on a bounded interval, like $[-\pi, \pi]$, is in L^2 on that interval, i.e. in $L^2([-\pi, \pi])$.

Hence, we are looking at

$$\lim_{N\to\infty} \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-iNt}dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(-N-1)t}g(t)dt \right| = \lim_{N\to\infty} \left| c_N(g) - c_{-N-1}(g) \right|,$$

where above, $c_N(g)$ is the N^{th} Fourier coefficient of g,

$$c_N(g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-iNt}dt,$$

and similarly, $c_{-N-1}(g)$ is the $-N-1^{st}$ Fourier coefficient of g,

$$c_{-N-1}(g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(-N-1)t}g(t)dt.$$

By Bessel's inequality,

$$c_N(g) \to 0 \text{ as } N \to \infty, \quad \text{and } c_{-N-1}(g) \to 0 \text{ as } N \to \infty.$$

Hence

$$\lim_{N\to\infty} \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-iNt}dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(-N-1)t}g(t)dt \right| = |0+0| = 0.$$

♡

Proof of the facts about the Dirichlet kernel: Recall that

$$n \in \mathbb{N} \implies e^{int} + e^{-int} = 2\cos(nt), n > 0.$$

Hence, we can pair up all the terms ± 1, ± 2, etc, and write

$$D_N(t) = \frac{1}{2\pi} + \sum_{n=1}^{N} \frac{1}{\pi}\cos(nt).$$

Integrating, we obtain

$$\int_{-\pi}^{\pi} D_N(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi}\cos(nt)dt.$$

The integrals of the cosines all vanish, so we obtain that

$$\int_{-\pi}^{\pi} D_N(t)dt = 1.$$

Moreover, since cosines are all even functions, as is a constant, $D_N(t)$ is an even function, hence

$$\int_{-\pi}^{0} D_N(t)dt = \frac{1}{2} = \int_{0}^{\pi} D_N(t)dt. \tag{2.3}$$

The second observation is that $D_N(t)$ looks almost like a geometric series, the problem is that it goes from minus exponents to positive ones. We can fix that right up by factoring out the largest negative exponent, so

$$D_N(t) = \frac{1}{2\pi}e^{-iNt}\sum_{n=0}^{2N} e^{int}.$$

We know how to sum a partial geometric series, don't we? This gives

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \frac{1 - e^{i(2N+1)t}}{1 - e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})}.$$

♡

2.1. **The red thread.**

(1) Fix the point $x \in \mathbb{R}$.

(2) Write down the definition of

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx}.$$

(3) Make a substitution in the integral defining the Fourier coefficients: let $t = y - x$. Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt.$$

(4) Use the periodicity to move the integral:

$$\int_{-\pi-x}^{\pi-x} f(t+x) e^{-int} dt = \int_{-\pi}^{\pi} f(t+x) e^{-int} dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) e^{-int} dt.$$

(5) Define the N^{th} Dirichlet kernel:

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \sum_{n=0}^{2N} e^{int}.$$

(6) Remember (or if you forgot, show) two things about the Dirichlet kernel:

$$\int_{-\pi}^{0} D_N(t) dt = \frac{1}{2} = \int_0^{\pi} D_N(t) dt$$

and

$$D_N(t) = \frac{1}{2\pi} e^{-iNt} \frac{1 - e^{i(2N+1)t}}{1 - e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})}.$$

(7) Write

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt,$$

so the goal is to prove:

$$\lim_{N\to\infty} \left| S_N(x) - \frac{1}{2}\left(f(x_-) + f(x_+)\right) \right| = 0.$$

(8) Use the integration fact about the Dirichlet kernel to re-write:

$$\frac{1}{2} f(x_-) = \int_{-\pi}^{0} D_N(t) dt f(x_-), \quad \frac{1}{2} f(x_+) = \frac{1}{2} = \int_0^{\pi} D_N(t) dt f(x_+).$$

(9) Show that it now suffices to estimate:

$$\left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_0^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right| \to 0$$

as $N \to \infty$. Pick one of these. I pick the first one.

(10) Use the second expression for the N^{th} Dirichlet kernel. Based on this, define a new function

$$g(t) = \frac{f(t+x) - f(x_-)}{1 - e^{it}}, \quad \text{for } t < 0,$$

$$g(t) = \frac{f(t+x) - f(x_+)}{1 - e^{it}}, \quad \text{for } t > 0.$$

(11) Show that g is piecewise continuous and piecewise differentiable. Show that g is bounded.
(12) Show that one is in fact estimating $c_N(g)$, the N^{th} Fourier coefficient of g minus $c_{-N-1}(g)$, the $-N-1$ Fourier coefficient of g.
(13) Use Bessel's inequality to prove that these coefficients both tend to zero as $N \to \infty$.

3. The Fourier coefficients of a function and its derivative

The nickname for this theory item is do NOT differentiate the series termwise!!! Sure, there is a result in the text later on which says that a function satisfying these hypotheses has a Fourier series which converges absolutely and uniformly, but do you know how to prove that? You use this result. Hence, if you try to use that result to prove this one, you've just run around in a circle and proven nothing. If you wanted to go down that road - correctly - using termwise differentiation of the Fourier series of f, you'd need to prove the absolute, uniform convergence by some independent means. I do not recommend this. This looks hard. As you will see, the proof below is pleasantly elementary. So, why make things hard and complicated?

Theorem 3.1. *This time in Swedish for fun! Låt f vara en 2π-periodisk funktion med $f \in C^1(\mathbb{R})$. Sedan Fourierkoefficienterna c_n av f och Fourierkoefficienterna c_n' av f' uppfyller*

$$c_n' = inc_n.$$

Proof: We use the definition of the fourier coefficient of f',

$$c_n' := \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx.$$

Now we integrate by parts:

$$= \frac{1}{2\pi} \left(f(x)e^{-inx}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-in)e^{-inx} dx \right).$$

The first term vanishes because by periodicity

$$f(-\pi) = f(\pi), \quad e^{-in\pi} = e^{in\pi}.$$

So, we end up with the second term only which is

$$\frac{in}{2\pi} \int_{\pi}^{\pi} f(x) e^{-inx} dx = inc_n.$$

♡

3.1. Red thread.

(1) Use the definition of the Fourier coefficient of f', c_n'. Write it down.
(2) Integrate by parts: move the derivative from f' to the e^{-inx}.
(3) Use the fact that f, f', and e^{inx} are 2π periodic to kill off the boundary terms. The result should be $c_n' = inc_n$.

4. Proof of the 3 equivalent conditions to be an ONB in a Hilbert space

This seems to be a fun one for some reason. It is rather nicely straightforward. Perhaps what makes it so nice is the pleasant setting of a Hilbert space, or translated directly from German, a Hilbert room. Hilbert rooms are cozy.

Theorem 4.1. *Låt $\{\phi_n\}_{n\in\mathbb{N}}$ vara ortonormala i ett Hilbert-rum, H. Följande tre är ekvivalenta:*

$$(1) \quad f \in H \text{ och } \langle f, \phi_n \rangle = 0 \forall n \in \mathbb{N} \implies f = 0.$$

$$(2) \quad f \in H \implies f = \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n.$$

$$(3) \quad ||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2.$$

Proof. We shall proceed in order prove (1) $\implies$ (2), then (2) $\implies$ (3), and finally (3) $\implies$ (1). Just stay calm and carry on. So we begin by assuming (1) holds, and then we shall show that (2) must hold as well. First, we note that by Bessel's inequality, the series

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n \rangle|^2 \le ||f||^2 < \infty.$$

Hence, if we know anything about convergent series, then we sure better know that the tail of the series tends to zero. The tail of the series is

$$\sum_{n\ge N} |\langle f, \phi_n \rangle|^2 \to 0 \text{ as } N \to \infty.$$

Now, let us define some elements in our Hilbert space, which we shall show comprise a Cauchy sequence. Let

$$g_N := \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n.$$

For $M \ge N$, we have, using the Pythagorean Theorem and the orthonormality of the $\{\phi_n\}$,

$$||g_M - g_N||^2 = ||\sum_{n=N+1}^{M} \langle f, \phi_n \rangle \phi_n||^2 = \sum_{n=N+1}^{M} |\langle f, \phi_n \rangle|^2 \le \sum_{n=N+1}^{\infty} |\langle f, \phi_n \rangle|^2 \to 0 \text{ as } N \to \infty.$$

Hence, by definition of Cauchy sequence (which one really should know at this point!), $\{g_N\}_{N\ge1}$ is a Cauchy sequence in our Hilbert space. By definition of Hilbert space, every Hilbert space is complete. Thus every Cauchy sequence converges to a unique limit. Let us now call the limit of our Cauchy sequence, which is by definition,

$$\lim_{N\to\infty} g_N = \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n = \sum_{n\in\mathbb{N}} \langle f, \phi_n \rangle \phi_n = g.$$

We will now show that $f - g$ satisfies

$$\langle f - g, \phi_n \rangle = 0 \forall n \in \mathbb{N}.$$

Then, because we are assuming (1) holds, this implies that $f - g = 0$, ergo $f = g$. So, we compute this inner product,

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle.$$

We insert the definition of g as the series,

$$\langle g, \phi_n \rangle = \langle \sum_{m\ge1} \langle f, \phi_m \rangle \phi_m, \phi_n \rangle = \sum_{m\ge1} \langle f, \phi_m \rangle \langle \phi_m, \phi_n \rangle = \langle f, \phi_n \rangle.$$

Above, we have used in the second equality the linearity of the inner product and the continuity of the inner product. In the third equality, we have used that $\langle \phi_m, \phi_n \rangle$ is 0 if $m \ne n$, and is 1 if $m = n$. Hence, only the term with $m = n$ survives in the sum. Thus,

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle f, \phi_n \rangle = 0, \quad \forall n \in \mathbb{N}.$$

By (1), this shows that $f - g = 0 \implies f = g$.

Next, we shall assume that (2) holds, and we shall use this to demonstrate (3). Well, note that

$$f = \lim_{N\to\infty} g_N \implies ||f - g_N||^2 \to 0, \quad \text{as } N \to \infty.$$

Then, by the triangle inequality,

$$||f||^2 = ||f - g_N + g_N||^2 \le ||f - g_N||^2 + ||g_N||^2 = ||f - g_N||^2 + \sum_{n=1}^N |\langle f, \phi_n\rangle|^2 \le |f - g_N||^2 + \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

On the other hand, by Bessel's Inequality,

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \le ||f||^2.$$

So, we have a little sandwich, en smörgås, if you will, with $||f||^2$ right in the middle of our sandwich,

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \le ||f||^2 \le ||f - g_N||^2 + \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

Letting $N \to \infty$ on the right side, the term $||f - g_N|| \to 0$, and so we indeed have

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \le ||f||^2 \le \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2.$$

This of course means that all three terms are equal, because the terms all the way on the left and right side are the same!

Finally, we assume (3) holds and use it to show that (1) must also hold. This is pleasantly straightforward. We assume that for some f in our Hilbert space, $\langle f, \phi_n\rangle = 0$ for all n. Using (3), we compute

$$||f||^2 = \sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 = \sum_{n\in N} 0 = 0.$$

The only element in a Hilbert space with norm equal to zero is the 0 element. Thus $f = 0$. □

4.1. **Red thread.**

(1) Assume that (1) is true and use it to prove (2). First, prove that

$$g_N := \sum_{n=1}^N \langle f, \phi_n\rangle \phi_n$$

is a Cauchy sequence in your Hilbert space. Use this together with the completeness of Hilbert spaces to conclude that

$$\lim_{N\to\infty} g_N = \sum_{n\ge 1} \langle f, \phi_n\rangle\phi_n = g \in H.$$

(2) Show that

$$\langle g - f, \phi_n\rangle = 0 \quad \forall n \in \mathbb{N}.$$

By the assumption that (1) is true, this shows that

$$g - f = 0 \implies g = f,$$

thereby proving (2).

(3) Assume now that (2) is true and use it to prove (3). To do this, use the Pythagorean theorem and the fact that $\{\phi_n\}$ are orthonormal.

(4) Assume now that (3) is true and use it to prove (1). Since (3) is true, if $f \in H$ and $\langle f, \phi_n\rangle = 0$ for all n, then because (3) is true

$$||f||^2 = 0,$$

which shows that $f = 0$ because the only element in a Hilbert space with norm zero is zero.

5. The Best Approximation Theorem

This is another fun and cozy Hilbert room theory item.

Theorem 5.1. *Låt* $\{\phi_n\}_{n\in\mathbb{N}}$ *vara en ortonormal mängd i ett Hilbert-rum,* H*. Om* $f \in H$*,*

$$||f - \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n|| \leq ||f - \sum_{n\in\mathbb{N}} c_n \phi_n||, \quad \forall \{c_n\}_{n\in\mathbb{N}} \in \ell^2,$$

och $=$ *gäller* $\iff$ $c_n = \langle f, \phi_n\rangle$ *gäller* $\forall n \in \mathbb{N}$*.*

Proof. We make a few definitions: let

$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$

and

$$\varphi := \sum c_n \phi_n.$$

Then we compute

$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 + 2\Re\langle f - g, g - \varphi\rangle.$$

I claim that

$$\langle f - g, g - \varphi\rangle = 0.$$

Just write it out (stay calm and carry on):

$$\langle f, g\rangle - \langle f, \varphi\rangle - \langle g, g\rangle + \langle g, \varphi\rangle$$
$$= \sum \overline{\widehat{f_n}}\langle f, \phi_n\rangle - \sum \overline{c_n}\langle f, \phi_n\rangle - \sum \widehat{f_n}\langle \phi_n, \sum \widehat{f_m}\phi_m\rangle + \sum \widehat{f_n}\langle \phi_n, \sum c_m\phi_m\rangle$$
$$= \sum |\widehat{f_n}|^2 - \sum \overline{c_n}\widehat{f_n} - \sum |\widehat{f_n}|^2 + \sum \widehat{f_n}\overline{c_n} = 0,$$

where above we have used the fact that ϕ_n are an orthonormal set. Then, we have

$$||f - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 \geq ||f - g||^2,$$

with equality iff

$$||g - \varphi||^2 = 0.$$

Let us now write out what this norm is, using the definitions of g and φ. By their definitions,

$$g - \varphi = \sum (\widehat{f_n} - c_n)\phi_n.$$

By the Pythagorean theorem, due to the fact that the ϕ_n are an orthonormal set, and hence multiplying them by the scalars, $\widehat{f_n} - c_n$, they remain orthogonal, we have

$$||g - \varphi||^2 = \sum \left|\widehat{f_n} - c_n\right|^2.$$

This is a sum of non-negative terms. Hence, the sum is only zero if all of the terms in the sum are zero. The terms in the sum are all zero iff

$$\left|\widehat{f_n} - c_n\right| = 0 \forall n \iff c_n = \widehat{f_n} \forall n \in \mathbb{N}.$$

□

5.1. Red thread.

(1) Define

$$g := \sum \widehat{f_n}\phi_n, \quad \widehat{f_n} = \langle f, \phi_n\rangle,$$

and

$$\varphi := \sum c_n \phi_n.$$

(2) A clever trick:

$$||f - \varphi||^2 = ||f - g + g - \varphi||^2 = ||f - g||^2 + ||g - \varphi||^2 + 2\Re\langle f - g, g - \varphi\rangle.$$

(3) Prove that

$$\langle f - g, g - \varphi\rangle = 0.$$

To do this, just pop in the definitions of g and φ and use the properties about scalar products (which you MUST MEMORIZE!!).

(4) After this calculation we get
$$||f-\varphi||^2 = ||f-g+g-\varphi||^2 = ||f-g||^2 + ||g-\varphi||^2 \geq ||f-g||^2,$$
with equality if and only if
$$||g-\varphi||^2 = 0.$$
(5) Use the Pythagorean Theorem to conclude that
$$||g-\varphi||^2 = 0 \iff \widehat{f_n} = c_n \quad \forall n \in \mathbb{N}.$$

6. Cute properties of SLPs

This is a rather nice, follow-your-nose, theory problem. Of course, the really amazing and magical part of this theorem is the third statement, which is one of the gems of functional analysis. We shall not include that third statement here, however, because its proof is beyond the scope of this humble course.

Theorem 6.1 (Cute facts about SLPs)**.** *Let f and g be eigenfunctions for a regular SLP in an interval $[a,b]$ with weight function $w(x) > 0$. Let λ be the eigenvalue for f and μ the eigenvalue for g. Then:*

(1) $\lambda \in \mathbb{R}$ och $\mu \in \mathbb{R}$;
(2) If $\lambda \neq \mu$, then:
$$\int_a^b f(x)\overline{g(x)}w(x)dx = 0.$$

Proof: By definition we have $Lf + \lambda w f = 0$. Moreover, L is self-adjoint, so we have
$$\langle Lf, f\rangle = \langle f, Lf\rangle.$$
By being an eigenfunction,
$$Lf = -\lambda w f.$$
So combining these facts:
$$\langle Lf, f\rangle = \langle -\lambda w f, f\rangle = -\lambda\langle wf, f\rangle$$
$$= \langle f, Lf\rangle = \langle f, -\lambda w f\rangle = -\overline{\lambda}\langle f, wf\rangle.$$
Since w is real valued,
$$\langle wf, f\rangle = \int_a^b w(x)f(x)\overline{f(x)}dx = \int_a^b |f(x)|^2 w(x)dx,$$
$$\langle f, wf\rangle = \int_a^b f(x)\overline{w(x)f(x)}dx = \int_a^b |f(x)|^2 w(x)dx.$$
Since $w > 0$ and f is an eigenfunction,
$$\int_a^b |f(x)|^2 w(x)dx > 0.$$
So, the equation
$$-\lambda\langle wf, f\rangle = -\lambda\int_a^b |f(x)|^2 w(x)dx = -\overline{\lambda}\langle f, wf\rangle = -\overline{\lambda}\int_a^b |f(x)|^2 w(x)dx$$
implies
$$\lambda = \overline{\lambda}.$$

For the second part, we use basically the same argument based on self-adjointness:
$$\langle Lf, g\rangle = \langle f, Lg\rangle.$$
By assumption
$$\langle Lf, g\rangle = -\lambda\langle wf, g\rangle = -\lambda\int_a^b w(x)f(x)\overline{g(x)}dx.$$
Similarly,
$$\langle f, Lg\rangle = \langle f, -\mu w g\rangle = -\overline{\mu}\langle f, wg\rangle = -\mu\langle f, wg\rangle = -\mu\int_a^b f(x)\overline{g(x)}w(x)dx,$$

since $\mu \in \mathbb{R}$ and $w(x)$ is real. So we have

$$-\lambda \int_a^b w(x) f(x) \overline{g(x)} dx = -\mu \int_a^b f(x) \overline{g(x)} w(x) dx.$$

If the integral is non-zero, then it forces $\lambda = \mu$ which is false. Thus the integral must be zero.

6.1. **Red thread.**

(1) Use the fact that L is self-adjoint so

$$\langle Lf, f \rangle = \langle f, Lf \rangle.$$

(2) Use the fact that $Lf = -\lambda wf$ and the properties of scalar products (which you have memorized!!!) in the above equality to change the left and right sides to:

$$-\lambda \langle wf, f \rangle = -\overline{\lambda} \langle wf, f \rangle.$$

Remember that w is real valued so it can be on either side of the scalar product and it does the same thing

(3) Recognize (since you have so thoroughly memorized the properties of scalar products!!!)

$$\langle f, wf \rangle = \int_a^b |f(x)|^2 w(x) dx > 0$$

since eigenfunctions cannot be the zero function and $w > 0$. Consequently

$$-\lambda = -\overline{\lambda} \iff \lambda = \overline{\lambda} \iff \lambda \in \mathbb{R}.$$

(4) For the next part, similar idea. Assume $\lambda \neq \mu$. By self-adjointness

$$\langle Lf, g \rangle = \langle f, Lg \rangle.$$

(5) By definition of f and g

$$\langle Lf, g \rangle = \langle -\lambda wf, g \rangle = -\lambda \langle wf, g \rangle$$
$$= \langle f, Lg \rangle = \langle f, -\mu wg \rangle = -\overline{\mu} \langle f, wg \rangle.$$

(6) Since $\lambda \neq \mu$ this necessitates

$$\langle f, g \rangle_w = 0.$$

7. The big bad convolution approximation theorem

This theory item is Theorem 7.3, regarding approximation of a function by convoluting it with a so-called "approximate identity." This theorem and its proof are both rather long. The proof relies very heavily on knowing the definition of limits and how to work with those definitions, so if you're not comfortable limits, it is strongly advised to brush up on them. Remember, you are always welcome to ask for help and/or explanations!

Theorem 7.1. *Let $g \in L^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} |g(x)| dx = 1.$$

Define

$$\alpha = \int_{-\infty}^0 g(x) dx, \quad \beta = \int_0^\infty g(x) dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\epsilon > 0$,

$$g_\epsilon(x) = \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof: We would like to show that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(x-y)g_\epsilon(y)dy = \alpha f(x+) + \beta f(x-)$$

which is equivalent to showing that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(x-y)g_\epsilon(y)dy - \alpha f(x+) - \beta f(x-) = 0.$$

We now insert the definitions of α and β, so we want to show that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(x-y)g_\epsilon(y)dy - \int_{-\infty}^0 f(x+)g(y)dy - \int_0^\infty f(x-)g(y)dy = 0.$$

We can prove this if we show that

$$\lim_{\epsilon\to 0}\int_{-\infty}^0 f(x-y)g_\epsilon(y)dy - \int_{-\infty}^0 f(x+)g(y)dy = 0$$

and also

$$\lim_{\epsilon\to 0}\int_0^\infty f(x-y)g_\epsilon(y)dy - \int_0^\infty f(x-)g(y)dy = 0.$$

In the textbook, Folland proves that the second of these holds. So, for the sake of diversity, we prove that the first of these holds. The argument is the same for both, so proving one of them is sufficient.

Hence, we would like to show that by choosing ϵ sufficiently small, we can make

$$\int_{-\infty}^0 f(x-y)g_\epsilon(y)dy - \int_{-\infty}^0 f(x+)g(y)dy$$

as small as we like. We would really like to smash the two integrals together. To achieve this, let $z = \epsilon y$, so $y = z/\epsilon$, and $dz/\epsilon = dy$. The limits of integration don't change, so

$$\int_{-\infty}^0 g(y)dy = \int_{-\infty}^0 g(z/\epsilon)\frac{dz}{\epsilon} = \int_{-\infty}^0 g_\epsilon(z)dz$$

By this calculation,

$$\int_{-\infty}^0 f(x+)g(y)dy = \int_{-\infty}^0 f(x+)g_\epsilon(y)dy.$$

(Above the integration variable was called z, but what's in a name? The name of the integration variable doesn't matter!). Note that $f(x+)$ is a constant, so it's just sitting there doing nothing. Hence, we have computed that

$$\int_{-\infty}^0 \left(f(x-y)g_\epsilon(y) - f(x+)g(y)\right)dy = \int_{-\infty}^0 g_\epsilon(y)\left(f(x-y) - f(x+)\right)dy.$$

Remember that $y \le 0$ where we are integrating Therefore, $x - y \ge x$. By definition

$$\lim_{y\uparrow 0} f(x-y) = f(x+) \implies \lim_{y\uparrow 0} f(x-y) - f(x+) = 0.$$

By definition of limit there exists $\delta > 0$ such that for all $y \in (-\delta, 0)$

$$|f(x-y) - f(x+)| \text{ is as small as we would like it to be.}$$

Consequently, we can estimate

$$\left|\int_{-\delta}^0 (f(x-y) - f(x+))g_\epsilon(y)dy\right| \le \int_{-\delta}^0 |f(x-y) - f(x+)||g_\epsilon(y)|dy$$

$$\le (\text{as small as we like})\int_{-\delta}^0 |g_\epsilon(y)|dy \le (\text{as small as we like})\int_{\mathbb{R}} |g_\epsilon(y)|dy$$

$$= (\text{as small as we like})\int_{\mathbb{R}} |g(y)|dy = (\text{as small as we like!})$$

We have used the same substitution trick to see that

$$\int_{\mathbb{R}} |g_\epsilon(y)|dy = \int_{\mathbb{R}} |g(z)|dz = 1.$$

So, we have shown that we can make

$$\left| \int_{-\delta}^{0} (f(x-y) - f(x+)) g_\epsilon(y) dy \right| \leq \text{(as small as we like)}.$$

To complete the proof, we just need to estimate the other part of the integral, from $-\infty$ to $-\delta$, because

$$\left| \int_{-\infty}^{0} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy \right| \leq \left| \int_{-\infty}^{-\delta} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy \right| + \left| \int_{-\delta}^{0} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy \right|.$$

We can make the second term as small as we like.

So, we wish to estimate

$$\left| \int_{-\infty}^{-\delta} (f(x-y) - f(x+)) g_\epsilon(y) dy \right|.$$

Here we need to consider the two possible cases given in the statement of the theorem separately. First, let us assume that f is bounded, which means that there exists $M > 0$ such that $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$. Hence

$$|f(x-y) - f(x+)| \leq |f(x-y)| + |f(x+)| \leq 2M.$$

So, we have the estimate

$$\left| \int_{-\infty}^{-\delta} (f(x-y) - f(x+)) g_\epsilon(y) dy \right| \leq \int_{-\infty}^{-\delta} |f(x-y) - f(x+)| |g_\epsilon(y)| dy \leq 2M \int_{-\infty}^{-\delta} |g_\epsilon(y)| dy.$$

We shall do a substitution now, letting $z = y/\epsilon$. Then, as we have computed before,

$$\int_{-\infty}^{-\delta} |g_\epsilon(y)| dy = \int_{-\infty}^{-\delta/\epsilon} |g(z)| dz.$$

Here the limits of integration do change, because $-\delta$ is neither zero nor infinity. If we let ϵ get very small, then $-\delta/\epsilon$ becomes very large and negative. We know that

$$\int_{-\infty}^{\infty} |g(z)| dz = 1.$$

So, the "ends" of the integral (the so-called "tails") out near infinity must be small (similar to when a series converges, because integrals are like the continuous version of series). This means that we can choose ϵ small and thereby make

$$\int_{-\infty}^{-\delta/\epsilon} |g(z)| dz \text{ as small as we like.}$$

Therefore, we can estimate

$$\left| \int_{-\infty}^{-\delta} (f(x-y) - f(x+)) g_\epsilon(y) dy \right| \leq (2M)(\text{as small as we like}) = \text{still very small}.$$

Hence, we can make this part as small as we like. This completes the proof in this case!

Finally, we consider the other case in the theorem, which is that g vanishes outside a bounded interval. By assumption, there exists some $R > 0$ such that

$$g(x) = 0 \forall x \in \mathbb{R} \text{ with } |x| > R.$$

Hence, we may choose ϵ small to guarantee that

$$-\frac{\delta}{\epsilon} < -R.$$

Specifically, let

$$\epsilon_0 = \frac{\delta}{R} > 0.$$

Then, we compute as before using the substitution $z = y/\epsilon$,

$$\int_{-\infty}^{-\delta} |f(x-y) - f(x+)| |g_\epsilon(y)| dy = \int_{-\infty}^{-\delta/\epsilon} |f(x - \epsilon z) - f(x+)| |g(z)| dz = 0,$$

because $g(z) = 0 \forall z \in (-\infty, -\delta/\epsilon)$. So, the proof is done in this case as well!

♡

7.1. Red thread. This theorem and the first theorem (pointwise convergence of Fourier series) are by far the most challenging. Don't let that discourage you!

(1) Show that it is enough to prove that
$$\lim_{\epsilon \to 0} \int_{-\infty}^{0} f(x-y) g_\epsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy = 0$$
and also
$$\lim_{\epsilon \to 0} \int_{0}^{\infty} f(x-y) g_\epsilon(y) dy - \int_{0}^{\infty} f(x-) g(y) dy = 0.$$
The argument is same for both, so choose one. I choose the first one.

(2) Our mission is now to prove that if we take ϵ small, we can make the quantity
$$\int_{-\infty}^{0} f(x-y) g_\epsilon(y) dy - \int_{-\infty}^{0} f(x+) g(y) dy$$
small. We would like to smash the two integrals together. To achieve this, do a substitution in the second integral, setting $z = \epsilon y$, so $y = z/\epsilon$, and $dz/\epsilon = dy$. This shows that:
$$\int_{-\infty}^{0} \left(f(x-y) g_\epsilon(y) - f(x+) g(y) \right) dy = \int_{-\infty}^{0} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy.$$

(3) Now, to estimate
$$\int_{-\infty}^{0} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy,$$
split the integral into $\int_{-\infty}^{-\delta} + \int_{-\delta}^{0}$.

(4) Estimate
$$\int_{-\delta}^{0} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy.$$
To do this, use the fact that the integral is over negative values of y, so $x - y > x$, together with the definition of $f(x+)$ as the right-hand-limit. In this way make $|f(x-y) - f(x+)|$ super small by choosing $\delta > 0$ but small. Then you can pull out a factor of "super small" and estimate
$$(\text{super small}) \int_{-\delta}^{0} |g_\epsilon(y)| dy \leq (\text{super small}) \int_{-\infty}^{0} |g_\epsilon(y)| dy \leq (\text{super small}).$$

(5) Observe that
$$\left| \int_{-\infty}^{0} \right| \leq \left| \int_{-\infty}^{-\delta} \right| + \left| \int_{-\delta}^{0} \right|.$$
So, we just need to estimate now:
$$\int_{-\infty}^{-\delta} g_\epsilon(y) \left(f(x-y) - f(x+) \right) dy.$$

(6) In case f is bounded, note that $|f(x-y) - f(x+)| \leq 2$(the number that bounds f). So you pull this out. Change variables to make the integral go from $-\infty$ to $-\delta/\epsilon$. Use the fact that the tail of a convergent integral can be made small to make this small.

(7) In case g vanishes outside a compact set, choose ϵ small so that $g = 0$ on the set $(-\infty, -\delta/\epsilon)$. Then the integrand is zero over here, hence sufficiently small.

8. The Fourier inversion formula

This theory item is really a julklapp. All one must know is the Fourier inversion formula.

Theorem 8.1 (FIT)**.** *Assume that $f \in L^2(\mathbb{R})$. Define the Fourier transform to be:*

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(y) e^{-iy\xi} dy.$$

Then (as an equality in $L^2(\mathbb{R})$ we have

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{ixy} dy.$$

♡

8.1. **Red thread.** Just memorize the statement! Simple as that!

9. Plancharel's Theorem

This one is also on the light side.

Theorem 9.1. *Assume $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. With the Fourier transform defined by*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,$$

then we have

$$\langle \hat{f}, \hat{g} \rangle = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = 2\pi \langle f, g \rangle = 2\pi \int_{\mathbb{R}} f(x) \overline{g(x)} dx,$$

and

$$\int_{\mathbb{R}} |\hat{f}(x)|^2 dx = ||\hat{f}||^2 = 2\pi ||f||^2 = 2\pi \int_{\mathbb{R}} |f(x)|^2 dx.$$

Proof: There is one idea which is key here, and that is to start on the right side. Why? Because it is easier, at least it is easier for me. When I try starting on the left side, it gets very messy very quickly. So, better not to do that.

Start on the right side:

$$\int_{\mathbb{R}} f(y) \overline{g(y)} dy.$$

Next, we use the FIT to write

$$f(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyx} \hat{f}(x) dx.$$

Substituting,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iyx} \hat{f}(x) \overline{g(y)} dy dx.$$

Next, we observe that we've got something very close to the Fourier transform of g sitting there,

$$\int_{\mathbb{R}} e^{iyx} \overline{g(y)} dy.$$

This isn't quite the Fourier transform, because the sign of the exponential is wrong. However, observe that

$$\overline{e^{-iyx}} = e^{iyx},$$

so

$$\int_{\mathbb{R}} e^{iyx} \overline{g(y)} dy = \int_{\mathbb{R}} \overline{e^{-iyx} g(y)} dy = \overline{\int_{\mathbb{R}} e^{-iyx} g(y) dy} = \overline{\hat{g}(x)}.$$

Thus, we have computed that

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x) \overline{\hat{g}(x)} dx = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle.$$

Moving the 2π around gives us

$$2\pi \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle.$$

Setting $f = g$ immediately also gives

$$2\pi ||f||^2 = ||\tilde{f}||^2.$$

♡

9.1. **Red thread.**

(1) Start on the RIGHT SIDE! Note that if this item appears on an exam, it is going to always be written as above. So, I'm not gonna swap the left and right sides ever because I don't think that is a nice thing to do.

(2) Use the FIT to write
$$f(y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iyx}\hat{f}(x)dx.$$
Stick this integral expression as well as the $\frac{1}{2\pi}$ factor inside the right side in place of $f(y)$.

(3) Use the magic of complex conjugation to show that
$$\int_{\mathbb{R}} e^{iyx}\overline{g(y)}dy = \overline{\hat{g}(x)}.$$

(4) Substitute this inside the right side again.

(5) Move the 2π factor as needed.

(6) Finally, for the statement relating $||f||^2$ and $||\hat{f}||^2$, just set $f = g$.

10. The Sampling Theorem

Theorem 10.1. *Let $f \in L^2(\mathbb{R})$. Assume that there is $L > 0$ so that $\hat{f}(\xi) = 0$ $\forall \xi \in \mathbb{R}$ with $|\xi| > L$, then:*
$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{L}\right)\frac{\sin(n\pi - tL)}{n\pi - tL}.$$

Proof. This theorem is all about the interaction between Fourier series and Fourier coefficients and how to work with both simultaneously. Since the Fourier transform $\hat{f}$ has compact support (meaning that it vanishes outside of a closed, bounded interval), the following equality holds as elements of $L^2([-L, L])$,
$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx.$$

We use the Fourier inversion theorem (FIT) to write
$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt}\hat{f}(x)dx = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\hat{f}(x)dx.$$
On the right side we have used the fact that $\hat{f}$ is supported in the interval $[-L, L]$, thus the integrand is zero outside of this interval, so we can throw that part of the integral away.

We next substitute the Fourier expansion of $\hat{f}$ into this integral,
$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty} c_n e^{in\pi x/L}dx.$$
Let us take a closer look at the coefficients
$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx = \frac{1}{2L}\int_{\mathbb{R}} e^{ix(-n\pi/L)}\hat{f}(x)dx = \frac{2\pi}{2L}f\left(\frac{-n\pi}{L}\right).$$
In the second equality we have used the fact that $\hat{f}(x) = 0$ for $|x| > L$, so by including that part we don't change the integral. In the third equality we have used the FIT!!! So, we now substitute this into our formula above for
$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty}\frac{\pi}{L}f\left(\frac{-n\pi}{L}\right)e^{in\pi x/L}dx$$
This is approaching the form we wish to have in the theorem, but the argument of the function f has a pesky negative sign. That can be remedied by switching the order of summation, which does not change the sum, so
$$f(t) = \frac{1}{2L}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)e^{-in\pi x/L}dx.$$

We may also interchange the summation with the integral[i]

$$f(t) = \frac{1}{2L}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\int_{-L}^{L} e^{x(it-in\pi/L)}dx.$$

We then compute

$$\int_{-L}^{L} e^{x(it-in\pi/L)}dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)}\sin(Lt-n\pi).$$

Substituting,

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\frac{\sin(Lt-n\pi)}{Lt-n\pi}.$$

□

10.1. **Red thread.**

(1) Expand $\hat{f}(x)$ in a Fourier series on the interval $[-L, L]$

$$\hat{f}(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx.$$

(2) Use the FIT to write

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt}\hat{f}(x)dx = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\hat{f}(x)dx.$$

(3) Substitute the Fourier expansion of $\hat{f}$ into this integral,

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty} c_n e^{in\pi x/L}dx.$$

(4) Compute the Fourier coefficients

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L}\hat{f}(x)dx = \frac{1}{2L}\int_{\mathbb{R}} e^{ix(-n\pi/L)}\hat{f}(x)dx = \frac{2\pi}{2L}f\left(\frac{-n\pi}{L}\right).$$

(5) Substitute back into $f(t)$,

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} e^{ixt}\sum_{-\infty}^{\infty}\frac{\pi}{L}f\left(\frac{-n\pi}{L}\right)e^{in\pi x/L}dx.$$

(6) Swap the sum and the integral

$$f(t) = \frac{1}{2L}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\int_{-L}^{L} e^{x(it-in\pi/L)}dx.$$

(7) Compute:

$$\int_{-L}^{L} e^{x(it-in\pi/L)}dx = \frac{e^{L(it-in\pi/L)}}{i(t-n\pi/L)} - \frac{e^{-L(it-in\pi/L)}}{i(t-n\pi/L)} = \frac{2i}{i(t-n\pi/L)}\sin(Lt-n\pi).$$

(8) Substitute back inside.

[i] None of this makes sense pointwise; we are working over L^2. The key property which allows interchange of limits, integrals, sums, derivatives, etc is *absolute convergence*. This is the case here because elements of L^2 have $\int |f|^2 < \infty$. That is precisely the type of absolute convergence required.

11. The generating function for the Bessel functions

This is a lovely, follow your nose and use the definitions type of proof.

Theorem 11.1. *For all x and for all $z \neq 0$, the Bessel functions, J_n satisfy*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}.$$

Proof. We begin by writing out the familiar Taylor series expansion for the exponential functions

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

These converge beautifully, absolutely and uniformly for z in compact subsets of $\mathbb{C} \setminus \{0\}$. So, since we presume that $z \neq 0$, we can multiply these series and fool around with them to try to make the Bessel functions pop out... Thus, we write

$$e^{xz/2} e^{-x/(2z)} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k \geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}. \tag{11.1}$$

Here is where the one and only clever idea enters into this proof, but it's rather straightforward to come up with it. We would like a sum with $n = -\infty$ to ∞. So we look around into the above expression on the right, hunting for something which ranges from $-\infty$ to ∞. The only part which does this is $j - k$, because each of j and k range over 0 to ∞. Thus, we keep k as it is, and we let $n = j - k$. Then $j + k = n + 2k$, and $j = n + k$. However, now, we have $j! = (n + k)!$, but this is problematic if $n + k < 0$. There were no negative factorials in our original expression! So, to remedy this, we use the equivalent definition via the Gamma function,

$$j! = \Gamma(j + 1), \quad k! = \Gamma(k + 1).$$

Moreover, we observe that in (11.1), $j!$ and $k!$ are for j and k non-negative. We also observe that

$$\frac{1}{\Gamma(m)} = 0, \quad m \in \mathbb{Z}, \quad m \leq 0.$$

Hence, we can write

$$e^{xz/2} e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n+k+1)k!}.$$

This is because for all the terms with $n + k + 1 \leq 0$, which would correspond to $(n + k)!$ with $n + k < 0$, those terms ought not to be there, but indeed, the $\frac{1}{\Gamma(n+k+1)}$ causes those terms to vanish!

Now, by definition,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k! \Gamma(k + n + 1)}.$$

Hence, we have indeed see that

$$e^{xz/2} e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} J_n(x) z^n.$$

□

11.1. **Red thread.**

(1) Write out the Taylor series expansion for the exponential functions:
$$e^{xz/2} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$
and
$$e^{-x/(2z)} = \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}.$$

(2) Multiply these together:
$$e^{xz/2}e^{-x/(2z)} = \sum_{j\geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!} \sum_{k\geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!} = \sum_{j,k\geq 0} (-1)^k \left(\frac{x}{2}\right)^{j+k} \frac{z^{j-k}}{j!k!}.$$

(3) We need a sum over $\mathbb{Z}$ but we just have two sums over $j, k \geq 0$. To get this, define the variable
$$n = j - k.$$
Write everything in terms of n and k, which gives
$$e^{xz/2}e^{-x/(2z)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{z^n}{\Gamma(n+k+1)k!}.$$

(4) OBS! For the Γ function part, recall that $k! = \Gamma(k+1$ and
$$\frac{1}{\Gamma(m)} = 0 \quad \forall 0 \geq m \in \mathbb{Z}.$$
So the terms looking like $(n+k+1)!$ with $n+k+1 > 0$ which are not in the original sum are then all zero, so we have not introduced any problems.

(5) Recall the definition of
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(k+n+1)}.$$
Pop it into the series to complete the proof!

12. Orthogonality of the Hermite polynomials

This is a fun application of integration by parts many times.

Theorem 12.1. *The Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Recall here that*
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$
and so the statement is that
$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}dx = 0, \quad n \neq m.$$

Proof. We are showing that the weighted inner product of H_n and H_m vanishes if $n \neq m$. Hence, we may assume without loss of generality that $n > m$. Due to the fact that H_n begin with $n = 0$, this means that we must have $m \geq 0$ and $n > m$ so $n \geq 1$. Next, we insert the definition of H_n into the inner product, so we look at
$$\int_{\mathbb{R}} (-1)^n e^{x^2} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x)e^{-x^2}dx = \int_{\mathbb{R}} (-1)^n \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x)dx$$
$$= (-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x)dx.$$
Let us do integration by parts one time, since we know that $n \geq 1$. Then, we have
$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_m(x)dx = (-1)^n \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_m(x)\Big|_{x=-\infty}^{\infty}$$

$$+(-1)^{n+1}\int_{\mathbb{R}}\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)H_m'(x)dx.$$

The first, second, and higher order derivatives of e^{-x^2} are all of the form

$$\frac{d^n}{dx^n}e^{-x^2}=p_n(x)e^{-x^2},$$

where $p_n(x)$ is a polynomial. This follows from the chain rule. If you really want to, you can prove this by induction, but you do not need to do that on the exam. For the sake of completeness, however, I'll just go ahead and prove it. For the base case, $n=0$, we haven't taken any derivatives, so $p_0(x)=1$, the constant polynomial of order 0. For the first derivative, $(e^{-x^2})'=-2xe^{-x^2}$, so $p_1(x)=-2x$. Proceeding by induction, assuming $\frac{d^n}{dx^n}e^{-x^2}=p_n(x)e^{-x^2}$, then

$$\frac{d^{n+1}}{dx^{n+1}}e^{-x^2}=\frac{d}{dx}\left(p_n(x)e^{-x^2}\right)=p_n'(x)e^{-x^2}-2xp_n(x)e^{-x^2}=\left(p_n'(x)-2xp_n(x)\right)e^{-x^2}.$$

The derivative of a polynomial is a polynomial, hence we have $p_{n+1}(x)=p_n'(x)-2xp_n(x)$ is also a polynomial which proves this small fact.

Thus,

$$(-1)^n\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)H_m(x)\Big|_{x=-\infty}^{\infty}=(-1)^np_{n-1}(x)e^{-x^2}H_m(x)\Big|_{x=-\infty}^{\infty}=0,$$

due to the fact that $e^{-x^2}\to 0$ as $x\to\pm\infty$ much, much faster than any polynomial tends to $\pm\infty$ as $x\to\pm\infty$. It's like Godzilla, e^{-x^2}, versus an ant, the polynomial part. Godzilla wins.

So, we have

$$(-1)^n\int_{\mathbb{R}}\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)dx=(-1)^{n+1}\int_{\mathbb{R}}\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)H_m'(x)dx.$$

We can repeat this using the same argument, until we run out of derivatives. We've got n derivatives, so we repeat this argument n times, arriving at

$$(-1)^n\int_{\mathbb{R}}\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)dx=(-1)^{n+n}\int_{\mathbb{R}}e^{-x^2}\left(\frac{d^n}{dx^n}H_m(x)\right)dx.$$

Now, we just need to pause and think for a moment. H_m is a polynomial of degree $m<n$. If you differentiate a polynomial of degree m more than m times, you end up with nothing! Zero! So, we actually know that, because $n>m$,

$$\left(\frac{d^n}{dx^n}H_m(x)\right)=0.$$

Hence,

$$(-1)^n\int_{\mathbb{R}}\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)dx=(-1)^{n+n}\int_{\mathbb{R}}e^{-x^2}\left(\frac{d^n}{dx^n}H_m(x)\right)dx=(-1)^{2n}\int_{\mathbb{R}}e^{-x^2}0dx=0.$$

□

12.1. **Red thread.**

(1) Assume that

$$n>m\geq 0.$$

Write out the thing you want to show vanishes:

$$\int_{\mathbb{R}}(-1)^ne^{x^2}\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)e^{-x^2}dx=\int_{\mathbb{R}}(-1)^n\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)dx$$

$$=(-1)^n\int_{\mathbb{R}}\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)dx.$$

(2) Do integration by parts

$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2} \right) H_m(x) dx = (-1)^n \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H_m(x) \Big|_{x=-\infty}^{\infty}$$

$$+(-1)^{n+1} \int_{\mathbb{R}} \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H_m'(x) dx.$$

(3) Use the fact that

$$\frac{d^{n-1}}{dx^{n-1}} e^{-x^2} = \text{ polynomial times } e^{-x^2}$$

and the fact that e^{-x^2} goes to zero faster as $|x| \to \infty$ than any polynomial to conclude that

$$(-1)^n \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H_m(x) \Big|_{x=-\infty}^{\infty} = 0.$$

(4) Show inductively that you can do this n times to get

$$(-1)^n \int_{\mathbb{R}} \left(\frac{d^n}{dx^n} e^{-x^2} \right) H_m(x) dx = (-1)^{n+n} \int_{\mathbb{R}} e^{-x^2} \left(\frac{d^n}{dx^n} H_m(x) \right) dx.$$

(5) If one differentiates H_m, a polynomial of degree $m < n$, n *times,* the result is zero. So the integral on the right is just zero.

13. The generating function for the Hermite polynomials

This is similar to the analogous result for the Bessel functions, but with a bit of a twist.

Theorem 13.1. *For any $x \in \mathbb{R}$ and $z \in \mathbb{C}$, the Hermite polynomials,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

satisfy

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Proof. The key idea with which to begin is to consider instead

$$e^{-(x-z)^2} = e^{-x^2 + 2xz - z^2}.$$

We consider the Taylor series expansion of this guy, with respect to z, viewing x as a parameter. By definition, the Taylor series expansion for

$$e^{-(x-z)^2} = \sum_{n \geq 0} a_n z^n,$$

where

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} e^{-(x-z)^2}, \quad \text{evaluated at } z = 0.$$

To compute these coefficients, we use the chain rule, introducing a new variable $u = x - z$. Then,

$$\frac{d}{dz} e^{-(x-z)^2} = -\frac{d}{du} e^{-u^2},$$

and more generally, each time we differentiate, we get a -1 popping out, so

$$\frac{d^n}{dz^n} e^{-(x-z)^2} = (-1)^n \frac{d^n}{du^n} e^{-u^2},$$

Hence, evaluating with $z = 0$, we have

$$a_n = \frac{1}{n!} (-1)^n \frac{d^n}{du^n} e^{-u^2}, \quad \text{evaluated at } u = x.$$

The reason it's evaluated at $u = x$ is because in our original expression we're expanding in a Taylor series around $z = 0$ and $z = 0 \iff u = x$ since $u = x - z$. Now, of course, we have

$$\frac{d^n}{du^n} e^{-u^2}, \quad \text{evaluated at } u = x = \frac{d^n}{dx^n} e^{-x^2}.$$

Hence, we have the Taylor series expansion

$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Now, we multiply both sides by e^{x^2} to obtain

$$e^{2xz-z^2} = e^{x^2} \sum_{n\geq 0} \frac{z^n}{n!}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

We can bring e^{x^2} inside because everything converges beautifully. Then, we have

$$e^{2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!} e^{x^2}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Voilà! The definition of the Hermite polynomials is staring us straight in the face! Hence, we have computed

$$e^{2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!} H_n(x).$$

□

13.1. **Red thread.**

(1) Start with
$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2}.$$

(2) Compute the Taylor series expansion of this guy, with respect to z, viewing x as a parameter. By definition, it is
$$e^{-(x-z)^2} = \sum_{n\geq 0} a_n z^n,$$
where
$$a_n = \frac{1}{n!}\frac{d^n}{dz^n} e^{-(x-z)^2}, \quad \text{evaluated at } z = 0.$$

(3) Compute the coefficients using the chain rule with the variable $u = x - z$, so
$$\frac{d^n}{dz^n} e^{-(x-z)^2} = (-1)^n \frac{d^n}{du^n} e^{-u^2},$$

(4) Evaluate at $z = 0 \implies u = x$:
$$a_n = \frac{1}{n!}(-1)^n \frac{d^n}{du^n} e^{-u^2}, \quad \text{evaluated at } u = x = \frac{d^n}{dx^n} e^{-x^2}.$$

(5) Pop it back into the Taylor series expansion:
$$e^{-(x-z)^2} = e^{-x^2+2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

(6) Multiply both sides by e^{x^2}:
$$e^{2xz-z^2} = e^{x^2} \sum_{n\geq 0} \frac{z^n}{n!}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

(7) Bring e^{x^2} inside the sum on the right (convergence is beautiful here).
$$e^{2xz-z^2} = \sum_{n\geq 0} \frac{z^n}{n!} e^{x^2}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

(8) Recognize your friends, the Hermite polynomials sitting inside the right side!

14. Please please do not try to fake it!

It is very unfortunate that some people will attempt to fake it. A colleague of mine calls this "vomiting symbols onto the page." It pretty much never works. If you memorize some of the symbols from the proof and try to arrange them as you memorized them, this will be immediately apparent to me. One cannot fake math. Some small detail which you have overlooked will SCREAM out to me POPPYCOCK! RUBBISH! HOGWASH! GOBBELDYGOOK! To give you an example of how sensitive we mathematicians are, have you ever noticed that sometimes we will say (..."assume that such-and-such is non-negative" or ..."assume that such-and-such is positive"). We make this distinction, because we are so persnickety that we think very carefully about what are the absolute most minimal assumptions needed. This is how we mathematicians can so easily spot it when people try to fake it. Some tiny detail to you is a giant glaring monster to me.

Now, this does not mean that you can't get some credit on a proof or problem which you cannot solve up to the finest detail! By no means! It is totally fine if you don't completely master the proofs, but you give them a genuine honest effort. In this sense, if you present your solutions and proofs and acknowledge their shortcomings, that is great! For example, if you have studied and understood some part(s) of the proof(s), and you can explain the parts you understand as well as the parts you don't understand (or are forgetting), that is really nice. In this way, we can see how far your understanding goes, and we can determine a fair and reasonable score. On the other hand, if you try to pretend like you understood the whole proof, but you really don't, and a small detail betrays you, then the deduction will be higher, because it appears that your level of mathematical understanding is zilch and even worse, you're trying to fake it.