Föreläsning 11/10-13

Laplacetransformer

$u(t) = 0, \quad t < 0$

$|u(t)| \leq Me^{at}$

$\mathcal{L}(u)(s) = \tilde{u}(s) = \int_0^\infty u(t)e^{-st}\,dt \quad , \text{Re}(s) > a$

Anm: om $s = iz$ så $\tilde{u}(s) = \tilde{u}(iz) = \hat{u}(z)$

där $\hat{u}(z) = \int_{-\infty}^\infty u(t)e^{-itz}\,dt$ är den

komplexa fouriertransformen.

Egenskaper:

(i) $\mathcal{L}(u')(s) = s\tilde{u} - u(0)$

(ii) $\mathcal{L}(u^{(k)})(s) = s^k\tilde{u} - (s^{k-1}u(0) + \ldots + u^{(k-1)}(0))$

(iii) $\frac{d}{ds}\tilde{u}(s) = -\mathcal{L}(tu)(s)$

(iv) $\mathcal{L}(u(t-a))(s) = e^{-sa}\tilde{u}(s) \quad , a > 0$

(v) $\mathcal{L}(e^{ct}u)(s) = \tilde{u}(s-c)$

$$\left(\begin{aligned} &\mathcal{L}(e^{ct}u(t))(s) = \int_0^\infty e^{ct}u(t)e^{-st}\,dt = \\ &= \int_0^\infty u(t)e^{-(s-c)t}\,dt = \tilde{u}(s-c) \end{aligned} \right)$$

(vi) $\mathcal{L}(u(bt)) = \frac{1}{b}\tilde{u}(s/b) \quad , b > 0.$

Standardtransformer

(1) $\mathcal{L}(1)(s) = 1/s$

(2) $\mathcal{L}(t^k)(s) = k!/s^{k+1}$

$$\left(\begin{aligned} &\mathcal{L}(t^k) = \mathcal{L}(t\,t^{k-1}) = -\frac{d}{ds}\mathcal{L}(t^{k-1}) \text{ enligt (iii)} \\ &\mathcal{L}(1) = 1/s \quad , \mathcal{L}(t) = 1/s^2 \quad , \mathcal{L}(t^2) = 2/s^3 \ldots \end{aligned} \right)$$

(3) $\mathcal{L}(e^{ct}) = 1/(s-c) \quad , c \in \mathbb{C}$

(4) $\mathcal{L}(\sin At) = \dfrac{A}{A^2+s^2}$

(5) $\mathcal{L}(\cos At) = \dfrac{s}{A^2+s^2}$

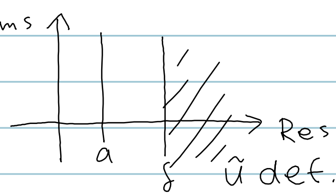
(6) $\mathcal{L}(u*v)(s) = \tilde{u}\,\tilde{v}$

Faltning:

$$u*v(t) = \int_{-\infty}^{\infty} u(t-y)v(y)\,dy = \int_0^t u(t-y)v(y)\,dy$$

Inversionsformel

$$u(t) = \frac{1}{2\pi i}\int_{\operatorname{Re}s=\delta} \hat{u}(s)e^{ts}\,ds$$



där $\delta > a$, där $|u(t)| \le Me^{at}$

ex Bestäm u så att

$$\tilde{u} = \frac{1}{s^2-2s+3}$$

Två sätt: a) inversionsformeln

b) använd räkneregler + kända transformer.

vi kör på b)-sättet (a finns i boken).

$$\hat{u}(s) = \frac{1}{(s-1)^2+2}$$

sätt $\tilde{v} = \dfrac{1}{s^2+2} = \dfrac{\sqrt{2}}{s^2+2}\dfrac{1}{\sqrt{2}}$, $\therefore$ $v = \dfrac{\sin t\sqrt{2}}{\sqrt{2}}$

$\hat{u} = \hat{v}(s-1)$ (V) $\Rightarrow$ $u = e^t v = \dfrac{e^t \sin\sqrt{2}\cdot t}{\sqrt{2}}$

Laplacetransf. och diff. ekv. (ordinära)

ex $u' = u^{(*)}$, $u(0) = 1$ $[u = e^t]$

$(*) \Rightarrow s\hat{u}(s) - u(0) = \hat{u}$

$(s-1)\hat{u} = 1$

$\hat{u} = \frac{1}{s-1} \Rightarrow u = e^t$

jfr. Four. transf.

$u' = u$

$ix\hat{u} = \hat{u}$

$(ix-1)\hat{u} = 0$

$\hat{u} = 0$

$\therefore u = 0 \leftarrow$ enda lösningen som har en four. trans. (= är integrabel).

ex $u' + u = f$, $u(0) = 1$, $t > 0$

$s\hat{u} - u(0) + \hat{u} = \hat{f}$

$(s+1)\hat{u} = 1 + \tilde{f}$

$\hat{u} = \frac{1}{s+1} + \tilde{f}\frac{1}{s+1}$

$u = e^{-t} + f * e^{-t}$

$f * e^{-t} = \int_0^t f(t-y)e^{-y}\,dy$

ex från tenta 17/1-13.

$u'' + 2u' + u = \sin t$, $t > 0$

$u(0) = 1$, $u'(0) = 0$

$\mathcal{L}(u') = s\tilde{u} - u(0) = s\tilde{u} - 1$

$\mathcal{L}(u'') = s^2\tilde{u} - (su(0) + u'(0)) = s^2\tilde{u} - s$

Får $s^2\tilde{u} - s + 2s\tilde{u} - 2 + \tilde{u} = \dfrac{1}{1+s^2}$

$(s^2+2s+1)\hat{u}(s) = s + 2 + \dfrac{1}{1+s^2}$

$\hat{u}(s) = \dfrac{s+2}{(s+1)^2} + \dfrac{1}{(s+1)^2(s^2+1)}$

en möjlighet nu:

$\hat{u} = \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{(s+1)^2(s^2+1)}$ (har partialbråksuppdelat $(s+2)/(s+1)^2$)

$\updownarrow \quad \updownarrow \quad \updownarrow$

$e^{-t} \quad te^{-t} \quad (te^{-t}) * \sin t$

$\therefore u(t) = e^{-t} + te^{-t} + (te^{-t}) * \sin t$ (men får inte svara med faltning)

kan räkna ut faltningen...

en annan möjlighet:

Partialbråksuppdela $\dfrac{1}{(s+1)^2(s^2+1)} =$

$= \dfrac{As+B}{(s^2+1)} + \dfrac{C}{(s+1)} + \dfrac{D}{(s+1)^2}$

$D = 1/2$, $Ai + B = -i/2 \Rightarrow B = 0$, $A = -1/2$, $C = 1/2$

$\Rightarrow \dfrac{1}{(s+1)^2(s^2+1)} = -\dfrac{1}{2}\dfrac{s}{s^2+1} + \dfrac{1}{2}\dfrac{1}{s+1} + \dfrac{1}{2}\dfrac{1}{(s+1)^2}$

$\hat{u}(s) = \dfrac{1}{2}\left[\dfrac{-s}{s^2+1} + \dfrac{3}{s+1} + \dfrac{3}{(s+1)^2}\right]$

$\therefore u(t) = \dfrac{1}{2}\left[-\cos t + 3e^{-t} + 3te^{-t}\right]$

ex $\tilde{u}(s) = \frac{e^{-2s}}{(s-2)(s-3)}$, bestäm u.

Säg $\tilde{v} = \frac{1}{(s-2)(s-3)} = \frac{1}{s-3} - \frac{1}{s-2}$

$\therefore v(t) = e^{3t} - e^{2t}$, $t > 0$

$\tilde{u}(s) = e^{-2s}\tilde{v}(s)$

Regel (iv) $\Longrightarrow e^{-2s}\tilde{v}(s) = \mathcal{L}(v(t-2))$

$\therefore u \begin{cases} = v(t-2) = e^{3(t-2)} - e^{2(t-2)}, & t > 2 \\ = 0, & t < 2 \end{cases}$

Skiss av bevis för inversionsformeln

Säg $u(t) = 0$, $t < 0$. Då är

$\hat{u}(z) = \int_0^\infty u(t)\underbrace{e^{-itz}}_{e^{-itx+ty}}\,dt$ def för $y << 0$.

Detta är den komplexa four. transf. Om $z = x + iy$ så $\hat{u}(x+iy) = \int_{-\infty}^{\infty} (u(t)e^{ty})e^{-itx}\,dt =$

$= \mathcal{F}(ue^{ty})(x)$

Inversionsformeln för vanlig four. transf. $\Longrightarrow$

$\Longrightarrow u(t)e^{ty} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(x+iy)e^{itx}\,dx$

$\therefore u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\underbrace{x+iy}_{z})e^{itx-ty}\,dx =$

$= \frac{1}{2\pi}\int_{\operatorname{Im} z} \hat{u}(z)e^{itz}\,dz$