Föreläsning 25/9-13

$f(z)$ holo i D

Hur många lösn. har ekv. $f(z)=0$ i D?

Residysatsen

Låt D vara enkelt sammanhängande.

f holo i $D - \{z_1, \dots, z_n\}$

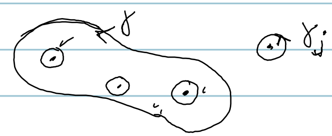
Låt γ vara en enkel sluten kurva i D.

Då
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_j \in \text{inre}(\gamma)} \text{Res}(f, z_j)$$

Bevis

Låt $\gamma_j = \{|z - z_j| = \delta\}$, $0 < \delta << 1$

f holo i området som begränsas av γ och $\cup \gamma_j$.



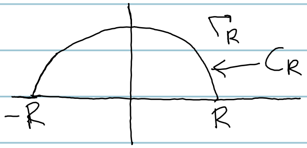
Greens formel $\Rightarrow$ $\displaystyle\int_\gamma f\,dz - \sum \int_{\gamma_j} f\,dz = 0$

$$\therefore \int_\gamma f\,dz = \sum_j \int_{\gamma_j} f\,dz = 2\pi i \sum \text{Res}(f, z_j) \quad \blacksquare$$

ex

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2}$$

Def. Γ_R:



$\Gamma_R = [-R, R] \cup C_R$

forts. →

Residysatsen $\Rightarrow$

$$I_R = \int_{\Gamma_R} \frac{dz}{1+z^2} = 2\pi i \sum_{\substack{z_j \\ \text{sing. innanför } \Gamma_R}} \operatorname{Res}\left(\frac{1}{1+z^2}, z_j\right)$$

Sing. pkter: $1+z^2=0 \Rightarrow z = \pm i$

$$I_R = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, i\right) = \frac{1}{2i} \cdot 2\pi i$$

$$\therefore \int_{\Gamma_R} \frac{dz}{1+z^2} = \pi$$

$$\|$$

$$\int_{-R}^{R} \frac{dx}{1+x^2} + \int_{C_R} \frac{dz}{1+z^2}$$

Claim: $\underbrace{\int_{C_R} \frac{dz}{1+z^2} \xrightarrow[R\to\infty]{} 0}_{(*)}$

Bevis av $(*)$: $\left|\int_{C_R} \frac{dz}{1+z^2}\right| \leq |C_R| \max_{C_R} \left|\frac{1}{1+z^2}\right| \leq$

$$\leq \pi R \max \frac{1}{|z^2|-1} = \frac{\pi R}{R^2-1} \longrightarrow 0$$

$$\therefore \pi = \int_{-R}^{R} + \int_{C_R} \xrightarrow[R\to\infty]{} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} + 0$$

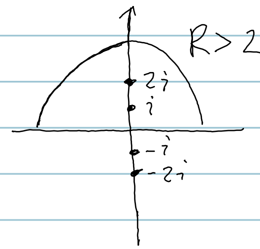
ex

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)(4+x^2)}$$ Def. Γ_R som förut.

forts. $\rightarrow$

$\underline{\text{Residysatsen}} \Rightarrow \quad \displaystyle\int_{\Gamma_R} \frac{z^2\,dz}{(z^2+1)(4+z^2)} = 2\pi i \sum \operatorname{Res}(f, z_j)$

där $f(z) = \dfrac{z^2}{(1+z^2)(4+z^2)}$



$\underline{\text{Sing. pkter:}}$ $1+z^2=0$ el. $4+z^2=0$

$z = \pm i \quad , \quad z = \pm 2i$

$$\therefore \int_{\Gamma_R} \frac{z^2\,dz}{(1+z^2)(4+z^2)} = 2\pi i \left(\operatorname{Res}_i + \operatorname{Res}_{2i}\right) =$$

$$= \left[\frac{i^2}{(4+i^2)\,2i} + \frac{(2i)^2}{(1+(2i)^2)\,4i}\right] 2\pi i = \frac{\pi}{3}$$ Som förut

$$\underset{\pi/3}{\overset{\|}{\int_{\Gamma_R}}} = \int_{-R}^{R} + \underbrace{\int_{C_R}}_{\text{ska på tenta kolla att } I_{C_R} \xrightarrow{R\to\infty} 0} \longrightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)(4+x^2)}\,dx$$

$\underline{\text{När funkar Residysatsen?}}$

Betrakta $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$ $\qquad$ $\operatorname{grad} P = p$, $\operatorname{grad} Q = q$

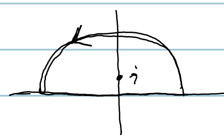
OK om $\operatorname{grad} P \le \operatorname{grad} Q - 2$

$\underline{\text{Koll:}}$ $\displaystyle\left|\int_{C_R}\right| \le \pi R \max_{|z|=R} \left|\frac{P}{Q}\right| \approx \pi R \cdot \frac{|z|^p}{|z|^q} = \frac{\pi R^{p+1}}{R^q} \longrightarrow 0$

om $q \ge p+2$

(ex)

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx \qquad \text{Def } \Gamma_R$$



forts. →

$$\int_{\Gamma_R} \frac{\cos z}{1+z^2}\,dz = 2\pi i\,\frac{\cos i}{2i} \qquad \text{sing. } \pm i,\ i \in \text{inre}(\Gamma_R)$$

Betrakta $$\left|\int_{C_R} \frac{\cos z}{1+z^2}\right| \le \frac{\pi R}{R^2-1}\,\max\frac{\left(|e^{iz}|+|e^{-iz}|\right)}{2} \le$$

$$\le \frac{\pi R}{R^2-1}\,\max\left(e^{-y}+e^{y}\right) \not\longrightarrow 0$$

Istället: använd

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx = \text{Re}\int_{-\infty}^{\infty}\frac{e^{ix}}{1+x^2}\,dx$$

+ favoritmetoden:

$$\int_{\Gamma_R}\frac{e^{iz}}{1+z^2} = 2\pi i\,\text{Res}\left(\frac{e^{iz}}{1+z^2},\ i\right) = 2\pi i\,\frac{e^{i^2}}{2i} = \frac{\pi}{e}$$

$$\left|\int_{C_R}\frac{e^{iz}}{1+z^2}\,dz\right| \le \frac{\pi R}{K^2-1}\,\max\underbrace{|e^{iz}|}_{=\,e^{-y}\,\le\,1} \longrightarrow 0$$

Som förut

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e}$$

$$\parallel$$

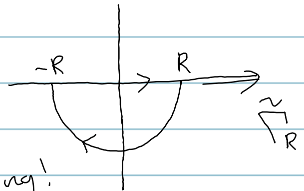
$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx$$

Variant

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{1+x^2}\,dx, \quad \text{ger} \quad \int_{\Gamma_R} \frac{e^{-iz}}{1+z^2}\,d =$$

$$\left| \int_{C_R} \frac{e^{iz}}{1+z^2} \right| \le \frac{\pi R}{R^2-1} \max e^{y} \longrightarrow \infty$$

Välj istället



orientering!

$$\int_{\tilde{\Gamma}_R} \frac{e^{iz}}{1+z^2}\,dz = -2\pi i \,\mathrm{Re}\left(\frac{e^{-iz}}{1+z^2}, -i \right) = -\frac{2\pi i\, e^{i^2}}{-2i} = \frac{\pi}{e} \,!$$

ex

$$\int_{-\infty}^{\infty} \frac{\cos \alpha x}{1+x^2}\,dx, \quad \alpha \in \mathbb{R}$$

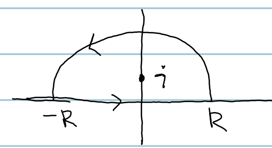
$$= \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\,dx$$

dela upp i olika fall för $\alpha > 0$ och $\alpha < 0$.

ex

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$$

Γ_R:



$$\int_{\Gamma_R} \frac{dz}{(1+z^2)^2} = 2\pi i \,\mathrm{Res}\left(\frac{1}{(1+z^2)^2}, i \right) = \Bigg\{$$ dubbelt nollst. i nämnaren,

forts. →

$$\frac{1}{(1+z^2)^2} = \frac{1}{(z+i)^2(z-i)^2} = \frac{H(z)}{(z-i)^2}$$

om $H = \dfrac{1}{(z+i)^2}$

$$\therefore \ \mathrm{Res}_i = \frac{H'(i)}{1!} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\int_{\Gamma_R} \frac{dz}{(1+z^2)^2} = \frac{2\pi i}{4i} = \frac{\pi}{2}$$

som förut: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$

Teoretisk anv. av residysatsen

Säg f holo i D.

Antag $f(z_0) = 0$.

Betrakta $\dfrac{f'(z)}{f(z)}$ är singulär i z_0.

Hur ser sing. ut?

Vet att $f(z) = (z-z_0)^m g(z)$

g holo och $g(z_0) \neq 0$

$$f' = (z-z_0)^m g' + m(z-z_0)^{m-1} g$$

$$\frac{f'}{f} = \frac{g'}{g} + \frac{m}{(z-z_0)} \quad \text{pol av ordn.} = 1$$

$$\therefore \ \mathrm{Res}\left(\frac{f'}{f}, z_0\right) = m$$

Sats

Antag f holo i D enkelt sammanh.

γ enkel sluten kurva, $f \neq 0$ på γ.

forts. →

Då gäller:

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i \cdot N$$ där N = antalet lösn. till $f(z)=0$ innanför γ, räknade med multiplicitet.

<u>Bevis</u>

Säg att nollst. innanför γ är $z_1, \dots, z_n$ med multiplicitet $m_1, \dots, m_n$.

Då $$\int_\gamma \frac{f'}{f}\,dz = 2\pi i \sum_1^n \operatorname{Res}\left(\frac{f'}{f}, z_j\right) = 2\pi i \sum_1^n m_j =$$

$$= 2\pi i N.$$

$\blacksquare$

<u>Precisering av satsen</u>

Säg att f också har poler i $w_1, \dots, w_q$ av mult. p_j

∴ Nära w_j i $f = \dfrac{g(z)}{(z-w_j)^{p_j}}$

$$f'(z) = \frac{g'}{(z-w_j)^{p_j}} - p_j \frac{g}{(z-w_j)^{p_j+1}}$$

$$\frac{f'}{f} = \frac{g'}{g} - \frac{p_j}{(z-w_j)}$$ pol av ordn. $=1$

$$\operatorname{Res}\left(\frac{f'}{f}, w_j\right) = -p_j$$

Samma bevis som förut ger

<u>Sats</u>: Om f har poler så $\displaystyle\int_\gamma \frac{f'}{f}\,dz = 2\pi i\,(N-P)$ där

$N = \#$ nollst., $P = \#$ poler, räknade med mult.