Storgruppsövning 6/9-13

1.2.21 a) $\{z;\ \underbrace{|z-\alpha| < |1-\bar{\alpha}z|}_{(*)}\} = \{z;\ |z|<1\}$

där $|\alpha|<1$

$(*) \Leftrightarrow |z-\alpha|^2 < |1-\bar{\alpha}z|^2$

$|z|^2 + |\alpha|^2 - \cancel{2\operatorname{Re}\bar{\alpha}z} < 1 + |\bar{\alpha}z|^2 - \cancel{2\operatorname{Re}\bar{\alpha}z}$

$|z|^2 - |\bar{\alpha}z|^2 < 1 - |\alpha|^2$

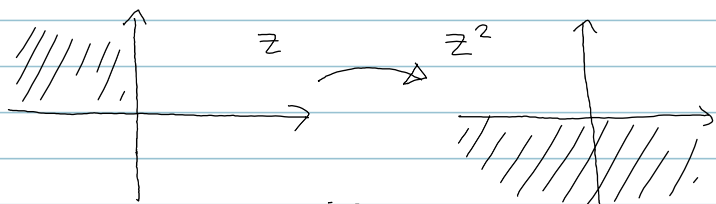
$|z|^2 - |\alpha|^2|z|^2 < 1 - |\alpha|^2$

$|z|^2(1-|\alpha|^2) < 1 - |\alpha|^2$

$|z|^2 < 1$

b) och c) p.s.s.

1.3.10



Polär form: $z = re^{i\theta}$, $\pi/2 < \theta < \pi$

$z^2 = r^2 e^{2i\theta}$, $\pi < 2\theta < 2\pi$

1.5.16 $\cos(x+iy) = \dfrac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} =$

$= \dfrac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2} = \dfrac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2}$

$= \cos x\left(\dfrac{e^{y}+e^{-y}}{2}\right) + i\sin x\left(\dfrac{e^{-y}-e^{y}}{2}\right) =$

$= \cos x\cosh y - i\sin x\sinh y$

$\boxed{\text{Spec: } \cos iy = \cosh y}$

1.5.21 iv) $|\cosh z|^2 = \left| \frac{e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin}{2} \right.$

$= (\cos y \cosh \cancel{y})^2 + (\sin y \sinh x)^2 =$

$= \cos^2 y \cosh^2 x + \sin^2 y \sinh^2 x =$

$= \{\cosh^2 x = \sinh^2 x + 1\} =$

$= \cos^2 y \sinh^2 x + \cos^2 y + \sin^2 y \sinh^2 x =$

$= \sinh^2 x (\underbrace{\cos^2 y + \sin^2 y}_{=1}) + \cos^2 y =$

$= \sinh^2 x + \cos^2 y$

1.6.2

$\cdot\, z_0$

$0 \cdot \quad \gamma = [0, z_0]$

$\int_{[0,z_0]} e^z \, dz \;\; ?$

Parametrisera γ:

$\gamma(t) = t z_0 \qquad 0 \le t \le 1$

$\dot\gamma(t) = z_0$

$I = \int_{[0,z_0]} e^z \, dz = \int_0^1 e^{tz_0} z_0 \, dt$

Vore bra om $\frac{d}{dt} e^{tz_0} = z_0 e^{tz_0} \quad (*)$

I så fall $I = \int_0^1 \frac{d}{dt} e^{tz_0} \, dt = e^{z_0} - 1$

Bevis av $(*)$: $z_0 = a + ib$

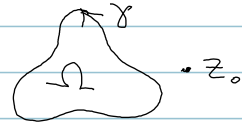
$e^{tz_0} = e^{ta} e^{tib} = e^{ta}(\cos tb + i \sin tb)$

$\frac{d}{dt} e^{tz_0} = a e^{tz_0} + e^{ta}(-b \sin tb + ib \cos tb) =$

$= a e^{tz_0} + ib\, e^{ta} (\cos tb + i \sin tb) = a e^{tz_0} + ib e^{tz_0} =$

$= z_0 e^{tz_0}$. $\blacksquare$

6.16 γ enkel och sluten
z_0 utanför γ



$$\int_\gamma \frac{dz}{(z-z_0)^m} = 0 \quad , \quad m \in \mathbb{Z}$$

$f(z) = 1/(z-z_0)^m$ glatt innanför γ

Green: $$\int_\gamma \frac{dz}{(z-z_0)^m} = i\iint_\Omega \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{1}{(z-z_0)^m}\,dx\,dy = \star$$

$$\frac{\partial}{\partial x}\frac{1}{(z-z_0)^m} = \frac{\partial}{\partial x}\frac{1}{(x+iy-z_0)^m} = \frac{-m}{(x+iy-z_0)^{m+1}}$$

$$\frac{\partial}{\partial y}\frac{1}{(z-z_0)^m} = \frac{-im}{(x+iy-z_0)^{m+1}}$$

tar ut varandra

$$\star = \iint_\Omega \left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right)\left(\frac{1}{(z-z_0)^2}\right)dx\,dy = 0$$