MTF053 - Fluid Mechanics

2023-01-05 08.30 – 13.30

Approved aids:

- The formula sheet handed out with the exam (attached as an appendix)
- *Beta - Mathematics Handbook for Science and Engineering*
- *Physics Handbook : for Science and Engineering*
- Graph drawing calculator with cleared memory

Exam Outline:

- In total 6 problems each worth 10p

Grading:

number of points on exam (including bonus points)	24-35	36-47	48-60
grade	3	4	5

Problem 1 - Boundary-Layer Flow (10 p.)

As a fluid flows past a flat plate, a boundary layer is developed where the flow velocity is gradually decelerated from the freestream velocity (U) a bit away from the flat plate to zero at the surface (if the flat plate is stationary). A very simple approximation of the velocity distribution $u(y)$ from the surface of the flat plate ($y = 0$) to the outer part of the boundary layer ($y = \delta$) is given by the linear relation below.

$$u(y) = U\frac{y}{\delta} \quad \text{for } y \text{ in the range } 0 \leq y \leq \delta$$

$$u = U \quad \text{for } y > \delta$$

(a) Using the linear velocity distribution given above, determine the wall shear stress $\tau_w(x)$ using the momentum thickness equation

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

(6p)

(b) Compare your result with the shear stress calculated using the Blasius velocity profile. How off is the shear stress calculated using the very simple velocity profile given above? (2p)

(c) What is the wall boundary condition used in this problem called and what does it imply? (0.5p)

(d) What assumptions are made in the derivation of the boundary-layer equations, i.e. the boundary-layer formulation of the Navier-Stokes equations? (0.5p)

(e) For laminar flow over a flat plate, the velocity profile is self-similar - what does that mean? (1p)

Problem 2 - Flow Acceleration and Continuity (10 p.)

Steady-state flow through a converging nozzle can be approximated by a one-dimensional velocity distribution $u = u(x)$ where x is the coordinate direction aligned with the nozzle centerline. Let's assume that the velocity varies linearly from $u = u_o$ at the entrance ($x = 0$) to $u = 3u_o$ at the exit ($x = L$).

(a) Derive an expression for the flow acceleration through the nozzle as a function of x (4p)

(b) What is the flow acceleration at the nozzle entrance (0.5p)

(c) What is the flow acceleration at the nozzle exit (0.5p)

(d) Explain the physical meaning of the local acceleration term and the convective acceleration term (1p)

(e) Derive the continuity equation on differential form starting from the integral form for a fixed control volume

$$\int_{cv} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

and let the size of the control volume reduce to infinitesimal size (3p)

(f) How can we simplify the continuity equation on differential form under the following circumstances? (1p)

- steady-state flow
- incompressible flow

Problem 3 - Pipe Flows (10 p.)

A venturi meter is a device used for tube flow rate measurements. It is a simple construction where the flow is passing through a tube section with locally reduced cross-section area. The pressure difference between a location upstream of the contraction and the location in the contraction where the tube diameter is the smallest is measured. The measured pressure difference can then be used to calculate the flow rate.

(a) Derive a general expression for the flow rate (Q) through a venturi meter as a function of fluid density (ρ), pressure and cross-section area upstream of the contraction (p_1 and A_1), and pressure and cross-section area in the contraction (p_2 and A_2). (6p.)

(b) For the range of flow rates given in the venturi meter specification below, determine the range of pressure differences that any connected pressure measurement device needs to be able to handle. (2p.)

Venturi meter specification:

Fluid	Kerosene
Flow rate range	$0.005\ m^3/s \leq Q \leq 0.050 m^3/s$
Tube diameter before and after contraction	$0.1\ m$
Tube diameter in the contraction	$0.06\ m$

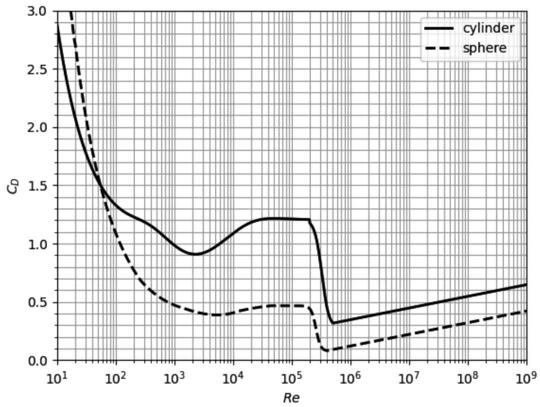
(c) The Moody diagram can be used for estimation of pressure losses in tubes. Why does the Moody chart not give reliable values in the Reynolds number range $2000 < Re < 4000$? (0.5p)

(d) What does fully developed pipe flow mean? (0.5p)

(e) Is the wall shear stress, τ_w, in general higher or lower in a fully developed laminar pipe flow than in a fully developed turbulent pipe flow? Explain why. (1p)

Problem 4 - Water Tower (10 p.)

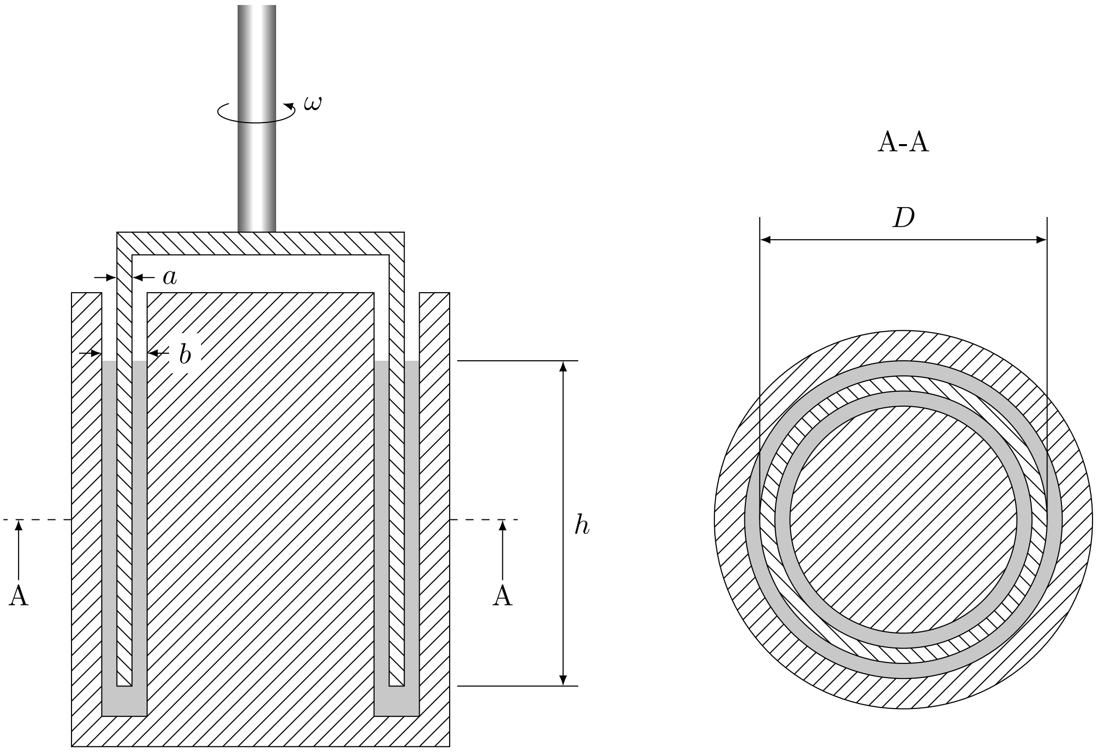
(a) The union water tower (see figure below) can be assumed to be a combination of a spherical water tank (diameter 12.0 m) and a cylinder (diameter 4.5 m, height 30.0m). Estimate the bending moment at the base of the water tower if the wind speed is 27 m/s. The velocity can be assumed to be constant, i.e. you don't have to account for the fact that in a real situation there would be a boundary layer. (6p)



(b) Explain the concepts favorable pressure gradient and adverse pressure gradient. What is the implication of pressure gradients on separation for an external flow (describe differences in separation tendency for flows with favorable pressure gradient, adverse pressure gradient, and no pressure gradient) (4p)

PROBLEM 5 - VISCOSITY (10 P.)

The construction illustrated below is used for measurement of the viscosity of oils. The device consists of a rotating cylinder and a stationary container. The gaps between the container walls and the cylinder are filled with oil and thus there is a thin oil layer on both the inside and outside of the cylinder. The upper part of the cylinder is not in contact with the oil. The material in the rotating cylinder can be assumed to be thin and thus the friction contribution from the end surface can be neglected.



(a) Derive an expression for the moment required to turn the cylinder based on the geometrical dimensions given in the figure above. (6p)

(b) Calculate h in the figure above such that the viscosity meter can be used to measure the viscosity of oils of the type SAE 10W-30 in the temperature range 20 $°C$ - 80 $°C$ if the turning moment must not exceed 1.5×10^{-2} Nm and if all other parameters are defined according to the table below (3p)

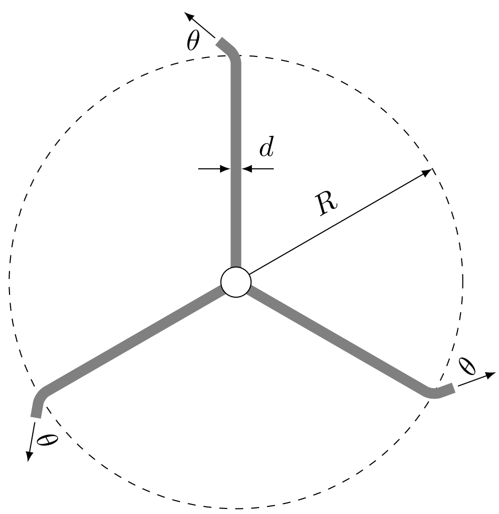
ω	one revolution per second
a	1.5 mm
b	3.5 mm
D	50 mm

(c) What is the viscosity of a fluid? (0.5p)

(d) What does it mean that a fluid is Newtonian? (0.5p)

Problem 6 - Sprinkler (10 p.)

The figure below sows a three-armed lawn sprinkler from above. Water enters the sprinkler from below (normal to the paper) at a flow rate of $Q = 8.0 \times 10^{-4}\ m^3/s$. The radius of the sprinkler is $R = 0.175\ m$ and the inner diameter of the sprinkler arms is $d = 7.0\ mm$. Friction between rotating and stationary parts of the sprinkler can be neglected.



(a) Derive an expression for calculation of the rotational velocity of the sprinkler as function of the angle θ (and other relevant parameters) (6p)

(b) Calculate the rotational velocity if the angle θ is $40°$ (1p)

(c) For what angle θ will we get the maximum rotation velocity? (1p)

(d) Make a schematic representation of the non-dimensional velocity u^+ as a function of the non-dimensional wall distance y^+ for a turbulent boundary layer. The velocity profile can be divided into different regions. Name these regions. (2p)

Fluid Mechanics MTF053

Formulas, Tables & Graphs

Division of Fluid Dynamics
Department of Mechanics and Maritime Sciences
Chalmers University of Technology

Contents

Non-dimensional Numbers

parameter	definition	interpretation	importance
Reynolds number	$Re = \frac{\rho U L}{\mu}$	$\frac{\text{inertia}}{\text{viscosity}}$	almost always
Mach number	$M = \frac{U}{a}$	$\frac{\text{flow speed}}{\text{speed of sound}}$	compressible flow
Froude number	$Fr = \frac{U^2}{gL}$	$\frac{\text{inertia}}{\text{gravity}}$	free-surface flow
Weber number	$We = \frac{\rho U^2 L}{\Upsilon}$	$\frac{\text{inertia}}{\text{surface tension}}$	free-surface flow
Prandtl number	$Pr = \frac{\mu C_p}{k}$	$\frac{\text{dissipation}}{\text{conduction}}$	heat convection
specific heat ratio	$\gamma = \frac{C_p}{C_v}$	$\frac{\text{enthalpy}}{\text{internal energy}}$	compressible flow
Strouhal number	$St = \frac{\omega L}{U}$	$\frac{\text{oscillation}}{\text{mean speed}}$	oscillating flow
roughness ratio	$\frac{\varepsilon}{L}$	$\frac{\text{wall roughness}}{\text{body length}}$	turbulent flow
pressure coefficient	$C_p = \frac{p - p_\infty}{0.5\rho U^2}$	$\frac{\text{static pressure}}{\text{dynamic pressure}}$	aerodynamics
lift coefficient	$C_L = \frac{F_L}{0.5\rho U^2 A}$	$\frac{\text{lift force}}{\text{dynamic force}}$	aerodynamics
drag coefficient	$C_D = \frac{F_D}{0.5\rho U^2 A}$	$\frac{\text{drag force}}{\text{dynamic force}}$	aerodynamics
skin friction coefficient	$c_f = \frac{\tau_{wall}}{0.5\rho U^2}$	$\frac{\text{wall shear stress}}{\text{dynamic pressure}}$	boundary layers

Conversion Factors

It is recommended to only use SI units when doing the calculations. To convert from non-SI to SI units the following may help:

1 foot = 0.3048 m

1 inch = 0.0254 m

1 slug = 14.593902937 kg

1 pound (force) = 4.448221615 N

1 atm = 101325 Pa

1 psi = 6894.757293178 Pa

1 degree R = 0.555556 degree K

1 lb/ft^2 = 47.88025898 Pa

1 Introduction

1.6 Thermodynamic Properties of a Fluid

Perfect-gas law

$$p = \rho R T \tag{1.10}$$

Gas constant R and specific heats

$$R = C_p - C_v$$

Internal energy $\hat{u}$ and specific heat at constant volume C_v

$$C_v = \left(\frac{\partial \hat{u}}{\partial T}\right)_\rho = \frac{d\hat{u}}{dT} = C_v(T)$$

$$d\hat{u} = C_v(T) dT \tag{1.14}$$

Enthalpy h and specific heat at constant pressure C_p

$$h = \hat{u} + \frac{p}{\rho} = \hat{u} + RT = h(T)$$

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p = \frac{dh}{dT} = C_p(T)$$

$$dh = C_p(T) dT \tag{1.15}$$

Ratio of specific heats γ

$$\gamma = \frac{C_p}{C_v} \tag{1.16}$$

$$C_v = \frac{R}{\gamma - 1}, \quad C_p = \frac{\gamma R}{\gamma - 1}$$

1.7 Viscosity and Other Secondary Properties

Shear stress for Newtonian fluids:

$$\tau = \mu \frac{du}{dy} \tag{1.23}$$

The Reynolds number:

$$Re = \frac{\rho V L}{\mu} = \frac{V L}{\nu} \tag{1.24}$$

Dynamic and kinematic viscosity:

$$\nu = \frac{\mu}{\rho} \tag{1.25}$$

Sutherland's law:

$$\frac{\mu}{\mu_0} \approx \frac{(T/T_0)^{3/2}(T_0 + S)}{T + S} \tag{1.27}$$

where μ_0 is a known viscosity at the temperature T_0 (usually 273 K) and S is a constant.

2 Pressure Distribution in a Fluid

2.3 Hydrostatic Pressure Distribution

$$\sum \mathbf{f} = \mathbf{f}_{press} + \mathbf{f}_{grav} + \mathbf{f}_{visc} = -\nabla p + \rho\mathbf{g} + \mathbf{f}_{visc} = \rho\mathbf{a} \tag{2.8}$$

For a fluid at rest Eqn. 2.8 reduces to

$$\nabla p = \rho\mathbf{g} \tag{2.9}$$

and with $\mathbf{g} = -g\mathbf{e}_z$

$$\frac{dp}{dz} = -\rho g \Leftrightarrow p_2 - p_1 = -\int_1^2 \rho g dz \tag{2.12}$$

$$p_2 - p_1 = -\rho g(z_2 - z_1) \tag{2.14}$$

Hydrostatic Pressure in Gases

$$\ln\frac{p_2}{p_1} = -\frac{g}{R}\int_1^2 \frac{dz}{T} \tag{2.17}$$

$$p_2 = p_1 \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] \tag{2.18}$$

$$T \approx T_0 - Bz \tag{2.19}$$

$T_0 = 288.16\ K,\ B = 0.00650\ K/m$ can be used for air and altitudes from 0 to 11000 m.

$$p = p_a\left(1 - \frac{Bz}{T_0}\right)^{g/(RB)},\ \rho = \rho_o\left(1 - \frac{Bz}{T_0}\right)^{g/(RB)-1} \tag{2.20}$$

where $\rho_o = 1.2255\ kg/m^3$ and $p_a = 101350\ Pa$ for air.

2.8 Buoyancy and Stability

$$(F_B)_{LF} = \sum \rho_i g(displaced\ volume)_i \tag{2.35}$$

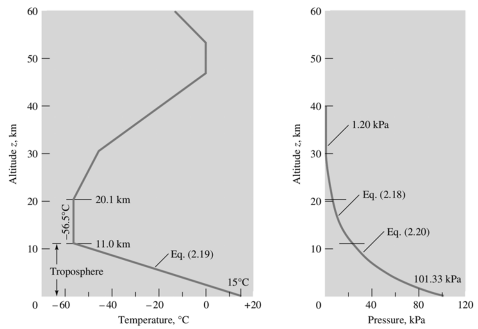


Figure 2.7: Temperature and pressure distribution in standard atmosphere (Table A.6)

3 Integral Relations for a Control Volume

3.1 Basic Physical Laws of Fluid Mechanics

Volume flow through surface S:

$$Q = \int_s (\mathbf{V} \cdot \mathbf{n}) dA \tag{3.7}$$

Mass flow through surface S:

$$\dot{m} = \int_s \rho(\mathbf{V} \cdot \mathbf{n}) dA$$

3.2 The Reynolds Transport Theorem

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{cv} \beta\rho d\mathcal{V}\right) + \int_{cs} \beta\rho(\mathbf{V}_r \cdot \mathbf{n}) dA \tag{3.16}$$

where B is an extensive property of the fluid and β the corresponding intensive property (the amount of B per unit mass) $\beta = dB/dm$.

3.3 Conservation of Mass

General form:

$$\frac{d}{dt}\left(\int_{cv} \rho d\mathcal{V}\right) + \int_{cs} \rho(\mathbf{V}_r \cdot \mathbf{n}) dA = 0 \tag{3.20}$$

Fixed control volume:

$$\int_{cv} \frac{\partial\rho}{\partial t} d\mathcal{V} + \int_{cs} \rho\,(\mathbf{V} \cdot \mathbf{n})\, dA = 0 \tag{3.21}$$

Fixed control volume and a finite number of inlets and outlets with one-dimensional flow:

$$\int_{cv} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0 \tag{3.22}$$

3.4 Conservation of Linear Momentum

General form:

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{cv} \mathbf{V}\rho d\mathcal{V}\right) + \int_{cs} \mathbf{V}\rho(\mathbf{V}_r \cdot \mathbf{n}) dA \tag{3.35}$$

Finite number of inlets and outlets:

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{cv} \mathbf{V}\rho d\mathcal{V}\right) + \sum_i (\dot{m}_i \mathbf{V}_i)_{out} - \sum_i (\dot{m}_i \mathbf{V}_i)_{in} \tag{3.40}$$

3.5 Frictionless Flow: The Bernoulli Equation

Unsteady frictionless flow along a streamline:

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0 \tag{3.53}$$

Steady-state, incompressible flow

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 = const \tag{3.54}$$

3.6 The Angular Momentum Theorem

General form:

$$\sum \mathbf{M}_o = \frac{d}{dt}\left(\int_{cv} (\mathbf{r} \times \mathbf{V})\rho d\mathcal{V}\right) + \int_{cs} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n}) dA \tag{3.59}$$

3.7 The Energy Equation

General form:

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt} = \frac{d}{dt}\left(\int_{cv} e\rho d\mathcal{V}\right) + \int_{cs} e\rho(\mathbf{V} \cdot \mathbf{n}) dA \tag{3.61}$$

where

$$e = \hat{u} + \frac{1}{2}V^2 + gz \tag{3.62}$$

Work divided into viscous work, shaft work, and surface pressure work (surface integral term):

$$\frac{dQ}{dt} - \frac{dW_s}{dt} - \frac{dW_\nu}{dt} = \frac{d}{dt}\left(\int_{cv} e\rho d\mathcal{V}\right) + \int_{cs} \left(e + \frac{p}{\rho}\right)\rho(\mathbf{V} \cdot \mathbf{n}) dA \tag{3.66}$$

Modified surface integral term (enthalpy-form):

$$\hat{h} = \hat{u} + \frac{p}{\rho}$$

$$\frac{dQ}{dt} - \frac{dW_s}{dt} - \frac{dW_\nu}{dt} = \frac{d}{dt}\left(\int_{cv} (\hat{u} + \frac{1}{2}V^2 + gz)\rho d\mathcal{V}\right) + \int_{cs} (\hat{h} + \frac{1}{2}V^2 + gz)\rho(\mathbf{V} \cdot \mathbf{n}) dA \tag{3.67}$$

Steady flow with one inlet and one outlet, both one-dimensional:

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu \tag{3.70}$$

Steady-state, incompressible flow with shaft and friction work:

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_{in} = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_{out} + h_{turbine} - h_{pump} + h_{friction} \tag{3.73}$$

Kinetic energy correction factor (α):

$$\left(\frac{p}{\rho g} + \frac{\alpha V_{av}^2}{2g} + z\right)_{in} = \left(\frac{p}{\rho g} + \frac{\alpha V_{av}^2}{2g} + z\right)_{out} + h_{turbine} - h_{pump} + h_{friction} \tag{3.75}$$

$$\int_{in/out} \left(\frac{1}{2}V^2\right)\rho(\mathbf{V} \cdot \mathbf{n}) dA = \alpha\left(\frac{1}{2}V_{av}^2\right)\dot{m}, \quad \text{where} \quad V_{av} = \frac{1}{A}\int u dA$$

4 Differential Relations for Fluid Flow

4.1 The Acceleration Field of a Fluid

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} \tag{4.2}$$

where $D\mathbf{V}/Dt$ is the substantial derivative, an operator that can be applied on any variable φ

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + (\mathbf{V} \cdot \nabla)\varphi = \frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} + v\frac{\partial \varphi}{\partial y} + w\frac{\partial \varphi}{\partial z} \tag{4.3}$$

4.2 The Differential Equation of Mass Conservation

Continuity (for cylindrical coordinates see Eqn. D.2):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{4.4}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{4.6}$$

4.3 The Differential Equation of Linear Momentum

The viscous stress tensor τ_{ij}:

$$\tau_{ij} = \begin{vmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{vmatrix} \tag{4.31}$$

Conservation of linear momentum on vector form ($D\mathbf{V}/Dt$ from Eqn. 4.2):

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij} \tag{4.32}$$

Viscous stress components for Newtonian fluids:

$$\begin{aligned}
\tau_{xx} &= \mu \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right) = 2\mu \frac{\partial u}{\partial x} \\
\tau_{yx} = \tau_{xy} &= \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\
\tau_{zx} = \tau_{xz} &= \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\
\tau_{yy} &= \mu \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \right) = 2\mu \frac{\partial v}{\partial y} \\
\tau_{yz} = \tau_{zy} &= \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
\tau_{zz} &= \mu \left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} \right) = 2\mu \frac{\partial w}{\partial z}
\end{aligned} \tag{4.37}$$

Conservation of linear momentum (Du/Dt, Dv/Dt, and Dw/Dt from Eqn. 4.3, for cylindrical coordinates see Eqns. D.5-D.7):

$$\begin{aligned}
\rho \frac{Du}{Dt} &= \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\rho \frac{Dv}{Dt} &= \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\rho \frac{Dw}{Dt} &= \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{aligned} \tag{4.38}$$

4.5 The Differential Equation of Energy

The internal energy ($\hat{u}$) formulation of the differential energy equation ($D\hat{u}/Dt$ from Eqn. 4.3):

$$\rho \frac{D\hat{u}}{Dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k \nabla T) + \Phi \tag{4.51}$$

where k is the thermal conductivity and Φ is the viscous dissipation function defined as:

$$\begin{aligned}
\Phi = \mu \Bigg[& 2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 \\
& + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \Bigg]
\end{aligned} \tag{4.50}$$

for moderate temperatures $\hat{u} = C_p T$ and thus

$$\rho \frac{D\hat{u}}{Dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k \nabla T) + \Phi \tag{4.51}$$

4.7 The Stream Function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{4.85}$$

$$\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x}(\nabla^2 \psi) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x}(\nabla^2 \psi) = \nu \nabla^2 (\nabla^2 \psi) \tag{4.87}$$

(ψ is constant along a streamline)

4.8 Vorticity and Irrotationality

Flow rotation (ω):

$$\omega = \frac{1}{2}(curl \mathbf{V}) = \frac{1}{2} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{4.110}$$

$$
\begin{aligned}
\omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\
\omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\
\omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned} \tag{4.109}
$$

Flow vorticity (ζ):

$$\zeta = 2\omega = curl\mathbf{V} \tag{4.111}$$

5 Dimensional Analysis and Similarity

5.4 Nondimensionalization of the Basic Equations

Continuity and momentum equations on non-dimensional form:

$$\nabla^* \cdot \mathbf{V}^* = 0 \tag{5.24a}$$

$$\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho UL}\nabla^{*^2}(\mathbf{V}^*) \tag{5.24b}$$

Drag coefficients for cylinders and spheres:

$$C_{D_{cylinder}} = \frac{F_D}{\frac{1}{2}\rho U^2 Ld} \tag{5.26}$$

$$C_{D_{sphere}} = \frac{F_D}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2} \tag{5.26}$$

6 Viscous Flow in Ducts

6.3 Head Loss – The Friction Factor

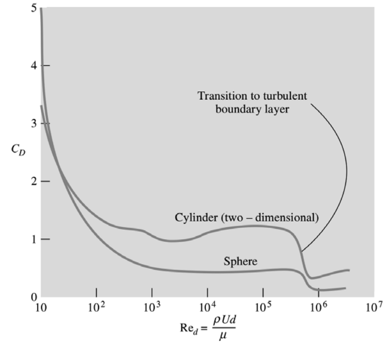
Head loss (h_f) and the Darcy friction factor (f):

$$h_f = f\frac{L}{d}\frac{V^2}{2g}, \text{ where } f = fcn(Re_d, \epsilon/d, \text{duct shape}) \tag{6.10}$$

$$f = \frac{8\tau_w}{\rho V^2} \tag{6.11}$$

6.4 Laminar Fully Developed Pipe Flow

$$f_{lam} = \frac{64}{Re_d} \tag{6.13}$$



Cylinder length effect

$(10^4 < Re < 10^5)$

L/d	C_D
∞	1.20
40	0.98
20	0.91
10	0.82
5	0.74
3	0.72
2	0.68
1	0.64

Figure 5.3: C_D for cylinders and spheres as a function of Reynolds number

6.5 Turbulence Modeling

Reynolds decomposition of a flow variable ϕ

$$\phi = \overline{\phi} + \phi'$$

$$\overline{\phi} = \frac{1}{T}\int_0^T \phi dt$$

$$\overline{\phi'} = \frac{1}{T}\int_0^T (\phi - \overline{\phi})dt = \overline{\phi} - \overline{\phi} = 0$$

$$\overline{\phi'^2} = \frac{1}{T}\int_0^T \phi'^2 dt \neq 0$$

The Reynolds-Averaged Navier-Stokes (RANS) equations ($D\overline{u}/Dt$ from Eqn. 4.3):

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{6.20}$$

$$\rho\frac{D\overline{u}}{Dt} = -\frac{\partial \overline{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x}\left(\mu\frac{\partial \overline{u}}{\partial x} - \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial \overline{u}}{\partial z} - \rho\overline{u'w'}\right) \tag{6.21}$$

RANS equations for duct and boundary layer flows ($D\overline{u}/Dt$ from Eqn. 4.3):

$$\rho\frac{D\overline{u}}{Dt} \approx -\frac{\partial \overline{p}}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y} \tag{6.22}$$

where

$$\tau = \tau_{lam} + \tau_{turb} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} \tag{6.23}$$

The friction velocity u^* is defined as

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$

Velocity distribution in the log-region (see Fig. 6.10):

$$\frac{u}{u^*} = \frac{1}{\kappa} \ln \frac{y u^*}{\nu} + B \tag{6.28}$$

Note that the constant B in Eqn. 6.28 is case dependent but unless another value is suggested, B can be assumed to be 5.0

Velocity distribution in the viscous sublayer (see Fig. 6.10):

$$u^+ = \frac{u}{u^*} = \frac{y u^*}{\nu} = y^+ \tag{6.29}$$

Prandtl's mixing length concept

Eddy viscosity μ_t (turbulent viscosity):

$$-\rho \overline{u'v'} \approx \mu_t \frac{du}{dy} \quad \text{where } \mu_t \approx \rho l^2 \left| \frac{du}{dy} \right| \tag{6.30}$$

$$l \approx \kappa y \quad \text{where } \kappa \text{ is von Kármán's constant } \kappa \approx 0.41 \tag{6.31}$$

6.6 Turbulent Pipe Flow

$$\frac{u(r)}{u^*} \approx \frac{1}{\kappa} \ln \frac{(R-r)u^*}{\nu} + B \tag{6.32}$$

Average velocity:

$$V = \frac{Q}{A} = \frac{1}{\pi R^2} \int_0^R u^* \left(\frac{1}{\kappa} \ln \frac{(R-r)u^*}{\nu} + B \right) 2\pi r dr = \frac{1}{2} u^* \left(\frac{2}{\kappa} \ln \frac{R u^*}{\nu} + 2B - \frac{3}{\kappa} \right) \tag{6.33}$$

Friction factor:

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}(Re_d \sqrt{f}) - 0.8 \tag{6.38}$$

Effect of rough walls

Non-dimensional wall roughness ε^+ (compare with y^+, Eqn. 6.29):

$$\varepsilon^+ = \frac{\varepsilon u^*}{\nu}$$

$\varepsilon^+ < 5$	hydraulically smooth
$5 \geq \varepsilon^+ \leq 70$	transitional
$\varepsilon^+ > 70$	fully rough

Log-law downshift (added to Eqn. 6.28, see Fig. 6.10):

$$\Delta B \approx \frac{1}{\kappa} \ln \varepsilon^+ - 3.5, \text{(for fully rough surfaces)} \tag{6.45}$$

$$u^+ = \frac{1}{\kappa} \ln y^+ B - \Delta B \tag{6.46}$$

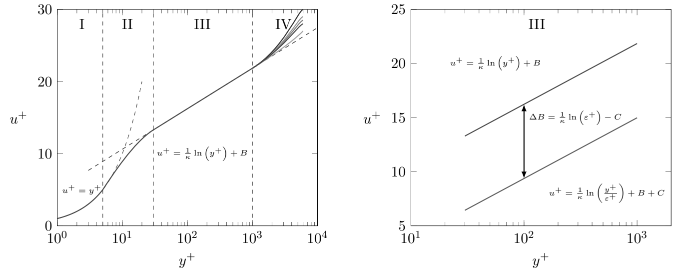


Figure 6.10: Velocity in a turbulent boundary layer (left). Log-law shift ΔB due to surface roughness (right). Boundary layer regions: I – the viscous sublayer, II – the buffer layer, III – the log-law region, and IV – the outer layer.

Friction factor

Colebrook/Moody implicit friction factor formula (f, see Fig. 6.13):

$$\frac{1}{\sqrt{f}} = -2.0 \log_{10} \left(\frac{\varepsilon/d}{3.7} + \frac{2.51}{Re_d \sqrt{f}} \right) \tag{6.48}$$

Haaland's explicit friction factor formula (f):

$$\frac{1}{\sqrt{f}} \approx -1.8 \log_{10} \left(\frac{6.9}{Re_d} + \left(\frac{\varepsilon/d}{3.7} \right)^{1.11} \right) \tag{6.49}$$

Table 6.1: Recommended roughness values (ε) for commercial ducts

Material	Condition	ε [mm]	Uncertainty [%]
Steel	Sheet metal (new)	0.05	±60
	Stainless (new)	0.002	±50
	Commercial (new)	0.046	±30
	Riveted	3.0	±70
	Rusted	2.0	±50
Iron	Cast,new	0.26	±50
	Wrought (new)	0.046	±20
	Galvanized (new)	0.15	±40
	Asphalted cast	0.12	±50
Concrete	Smoothed	0.04	±60
	Rough	2.0	±50
Brass	Drawn (new)	0.002	±50
Plastic	Drawn tubing	0.0015	±60
Glass	–	Smooth	–
Rubber	Smoothed	0.01	±60
Wood	Stave	0.5	±40

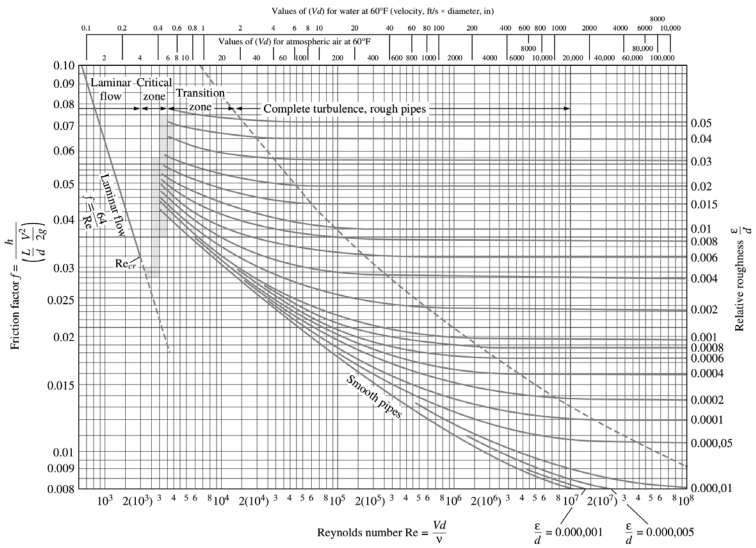


Figure 6.13: The Moody chart for pipe flow friction

6.8 Flow in Noncircular Ducts

For non-circular cross section ducts, the diameter D is replaced with the hydraulic diameter D_h calculated as

$$D_h = \frac{4A}{\mathcal{P}} \tag{6.56}$$

The Reynolds number based on the hydraulic diameter Re_{D_h} is obtained as

$$Re_{D_h} = \frac{V D_h}{\nu}$$

where A is the cross-section area and $\mathcal{P}$ is the wetted perimeter.

Turbulent Flow

Use the same formulas (or the Moody chart) as for flow in circular pipes but replace the diameter with the hydraulic diameter D_h

$$\Delta p_f = f \frac{L}{D_h} \frac{\rho V^2}{2}$$

Non-dimensional surface roughness is calculated as

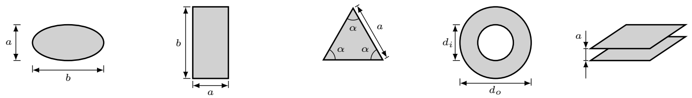
$$\frac{\varepsilon}{D_h}$$

Laminar Flow

Calculate the friction factor as

$$f = \frac{C}{Re_{D_h}}$$

where Re_{D_h} is the Reynolds number based on the hydraulic diameter and C is a constant that depends on duct shape (for circular cross sections $C = 64$ and $D_h = D$). The table below gives values of C for a selection of cross sections.



a/b	D_h	C
0.7	$1.17a$	65.0
0.5	$1.30a$	68.0
0.3	$1.44a$	73.0
0.2	$1.50a$	78.0
0.1	$1.55a$	79.0

b/a	D_h	C
1.0	$1.00a$	57.0
1.25	$1.11a$	57.6
2.0	$1.33a$	62.0
3.0	$1.50a$	69.0
4.0	$1.60a$	73.0
5.0	$1.67a$	78.0
8.0	$1.78a$	83.0
10.0	$1.82a$	85.0

D_h	C
$0.58a$	53.0

d_i/d_o	C
$\frac{d_i}{d_o} = 0.10$	89.2
$\frac{d_i}{d_o} = 0.25$	94.0
$0.5 < \frac{d_i}{d_o} < 1.0$	96.0

$D_h = d_o - d_i$

D_h	C
$2.0a$	96.0

7 Flow Past Immersed Bodies

7.1 Reynolds Number and Geometry Effects

Boundary-layer thickness:

$$\frac{\delta}{x} \approx \begin{cases} \dfrac{5.0}{Re_x^{1/2}} \text{ laminar } 10^3 < Re_x < 10^6 \\ \\ \dfrac{0.16}{Re_x^{1/7}} \text{ turbulent } 10^6 < Re_x \end{cases} \tag{7.1}$$

7.2 Momentum Integral Estimates

Drag force (D), momentum thickness (θ) and wall-shear stress (τ_w):

$$D(x) = \rho b \int_0^{\delta(x)} u(U-u)dy \tag{7.2}$$

$$D(x) = \rho b U^2 \theta, \text{ where } \theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \tag{7.3}$$

$$D(x) = b \int_0^x \tau_w(x) dx \tag{7.4}$$

$$\frac{dD}{dx} = \rho b U^2 \frac{d\theta}{dx}$$

$$\tau_w = \rho U^2 \frac{d\theta}{dx} \tag{7.5}$$

Displacement thickness (δ^*):

$$\delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy \tag{7.12}$$

7.3 The Boundary Layer Equations

Governing equations (continuity and momentum) for boundary layer flows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7.19a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \approx U\frac{dU}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{7.19b}$$

where

$$\tau = \begin{cases} \mu\dfrac{\partial u}{\partial y}, \text{ for laminar flows} \\ \\ \mu\dfrac{\partial u}{\partial y} - \rho\overline{u'v'}, \text{ for turbulent flows} \end{cases}$$

7.4 The Flat-Plate Boundary Layer

Laminar Flow - Blasius

Boundary layer thickness (δ):

$$\frac{\delta}{x} \approx \frac{5.0}{Re_x^{1/2}} \tag{7.24}$$

Skin friction coefficient (c_f), displacement thickness (δ^*):

$$c_f = \frac{2\tau_w}{\rho U^2} \approx \frac{0.664}{Re_x^{1/2}}, \quad \frac{\delta^*}{x} \approx \frac{1.721}{Re_x^{1/2}} \tag{7.25}$$

Wall-shear stress (τ_w):

$$\tau_w(x) \approx \frac{0.332\rho^{1/2}\mu^{1/2}U^{3/2}}{x^{1/2}}$$

Drag force (D):

$$D(x) = b\int_0^x \tau_w(x)dx \approx 0.664 b \rho^{1/2}\mu^{1/2}U^{3/2}x^{1/2} \tag{7.26}$$

Drag coefficient (C_D):

$$C_D = \frac{2D(L)}{\rho U^2 b L} = 2c_f(L) \approx \frac{1.328}{Re_L^{1/2}} \tag{7.27}$$

Momentum thickness (θ):

$$\frac{\theta}{x} \approx \frac{0.664}{Re_x^{1/2}} \tag{7.30}$$

Shape factor (H):

$$H = \frac{\delta^*}{\theta} \approx 2.59 \tag{7.31}$$

Table 7.1: The Blasius velocity profile

$y[U/(\nu x)]^{1/2}$	u/U	$y[U/(\nu x)]^{1/2}$	u/U
0.0	0.00000	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	∞	1.00000
2.6	0.77246		

Turbulent Flow

Prandtl's one-seventh-power law:

$$\left(\frac{u}{U}\right)_{turb} \approx \left(\frac{y}{\delta}\right)^{1/7} \tag{7.39}$$

Momentum thickness (θ):

$$\theta \approx \int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left(1-\left(\frac{y}{\delta}\right)^{1/7}\right)dy = \frac{7}{72}\delta \tag{7.40}$$

Boundary layer thickness (δ):

$$Re_\delta \approx 0.16 Re_x^{6/7}, \text{ or } \frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}} \tag{7.42}$$

Skin friction coefficient (c_f):

$$c_f = \frac{2\tau_w}{\rho U^2} \approx \frac{0.027}{Re_x^{1/7}} \tag{7.43}$$

Wall-shear stress (τ_w):

$$\tau_w \approx \frac{0.0135\mu^{1/7}\rho^{6/7}U^{13/7}}{x^{1/7}} \tag{7.44}$$

Drag coefficient (C_D):

$$C_D = \frac{7}{6}c_f(L) \approx \frac{0.031}{Re_L^{1/7}} \tag{7.45}$$

Displacement thickness (δ^*):

$$\delta^* \approx \int_0^\delta \left(1-\left(\frac{y}{\delta}\right)^{1/7}\right)dy = \frac{1}{8}\delta \tag{7.46}$$

Shape factor (H):

$$H = \frac{\delta^*}{\theta} \approx 1.3 \tag{7.47}$$

Boundary layer drag for flat plate with transition (Eqns. 7.43 and 7.25 combined):

$$D \approx b\frac{1}{2}\rho U^2\left[\int_{>0}^{x_{cr}} \frac{0.664}{Re_x^{1/2}}dx + \int_{x_{cr}}^{L} \frac{0.027}{Re_x^{1/7}}dx\right]$$

where b is the width of the flat plate and x_{cr} is the transition location.

Skin friction coefficient (c_f) and drag coefficient (C_D) for rough surfaces with the surface roughness ε:

$$c_f \approx \left(2.87 + 1.58\log_{10}\frac{x}{\varepsilon}\right)^{-2.5} \tag{7.48a}$$

$$C_D \approx \left(1.89 + 1.62\log_{10}\frac{L}{\varepsilon}\right)^{-2.5} \tag{7.48b}$$

Drag coefficient in transition region from Schlichting for two transition point Reynolds numbers (see Fig. 7.6):

$$C_D = \begin{cases} \dfrac{0.031}{Re_L^{1/7}} - \dfrac{1400}{Re_L} & Re_{trans} = 5\times 10^5 \\[2ex] \dfrac{0.031}{Re_L^{1/7}} - \dfrac{8700}{Re_L} & Re_{trans} = 3\times 10^6 \end{cases} \tag{7.49}$$

7.5 Experimental External Flows

Lift (C_L) and drag (C_D):

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A_p} \tag{7.66a}$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A_p} \tag{7.66b}$$

Lift (C_L) of finite span wings:

$$C_L = \frac{2\pi\sin(\alpha + 2h/c)}{1+2/AR}, \quad \text{where } AR = \frac{b^2}{A_p} = \frac{b}{c} \tag{7.70}$$

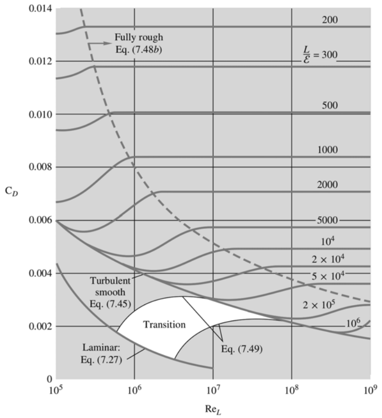


Figure 7.6: Drag coefficient of laminar and turbulent boundary layers

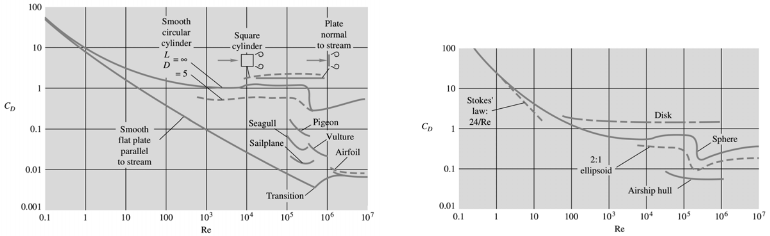


Figure 7.16: Drag coefficient for smooth bodies at low Mach numbers: (left) 2D bodies, (right) 3D bodies.

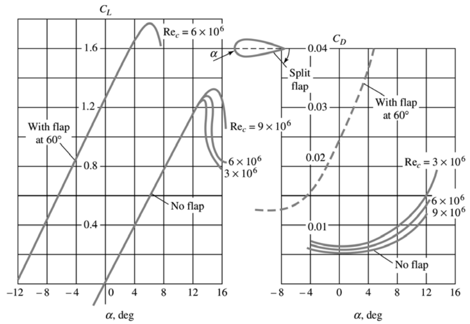


Figure 7.25: Lift and drag of a symmetric NACA 0009 airfoil

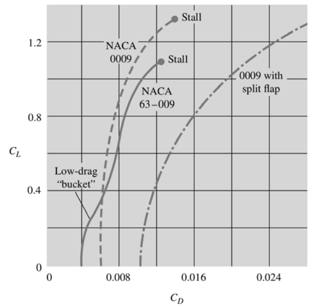


Figure 7.26: Lift-drag polar plot for standard (0009) and laminar flow (63-009) NACA airfoil

Shape	C_D based on frontal area	Shape	C_D based on frontal area	Shape	C_D based on frontal area
Square cylinder:	2.1	Half-cylinder:	1.2	Plate:	2.0
	1.6		1.7	Thin plate normal to a wall:	1.4
Half tube:	1.2	Equilateral triangle:	1.6		
	2.3		2.0	Hexagon:	→1.0 ↑0.7

Shape	C_D based on frontal area							
Rounded nose section:	L/H:	0.5	1.0	2.0	4.0	6.0		
	C_D:	1.16	0.90	0.70	0.68	0.64		
Flat nose section	L/H:	0.1	0.4	0.7	1.2	2.0	2.5	3.0	6.0
	C_D:	1.9	2.3	2.7	2.1	1.8	1.4	1.3	0.9

Elliptical cylinder:	Laminar	Turbulent
1:1	1.2	0.3
2:1	0.6	0.2
4:1	0.35	0.15
8:1	0.25	0.1

Body	C_D based on frontal area
Cube:	1.07
	0.81
Cup:	1.4
	0.4
Disk:	1.17
Parachute (Low porosity):	1.2

Body	C_D based on frontal area							
Cone:	θ:	10°	20°	30°	40°	60°	75°	90°
	C_D:	0.30	0.40	0.55	0.65	0.80	1.05	1.15
Short cylinder, laminar flow:	L/D:	1	2	3	5	10	20	40	∞
	C_D:	0.64	0.68	0.72	0.74	0.82	0.91	0.98	1.20
Porous parabolic dish [23]:	Porosity:	0	0.1	0.2	0.3	0.4	0.5
	← C_D:	1.42	1.33	1.20	1.05	0.95	0.82
	→ C_D:	0.95	0.92	0.90	0.86	0.83	0.80
Average person:	→ $C_DA \approx 9\ ft^2$	↑ $C_DA \approx 1.2\ ft^2$						
Pine and spruce trees [24]:	U, m/s:	10	20	30	40			
	C_D:	1.2 ± 0.2	1.0 ± 0.2	0.7 ± 0.2	0.5 ± 0.2			

Body	Ratio	C_D based on frontal area	Body	Ratio	C_D based on frontal area
Rectangular plate:	b/h 1	1.18	Flat-faced cylinder:	L/d 0.5	1.15
	5	1.2		1	0.90
	10	1.3		2	0.85
	20	1.5		4	0.87
	∞	2.0		8	0.99

Ellipsoid:	Ratio	Laminar	Turbulent
	L/d 0.75	0.5	0.2
	1	0.47	0.2
	2	0.27	0.13
	4	0.25	0.1
	8	0.2	0.08

9 Compressible Flow

9.1 Introduction: Review of Thermodynamics

Mach number (M):

$$M = \frac{V}{a}$$

Ratio of specific heats (γ):

$$\gamma = \frac{C_p}{C_v} \tag{9.2}$$

Eqns. 1.14 and 1.15 and constant specific heats gives:

$$\hat{u}_2 - \hat{u}_1 = C_v(T_2 - T_1), \quad \hat{h}_2 - \hat{h}_1 = C_p(T_2 - T_1) \tag{9.5}$$

Entropy change (from the first and second law of thermodynamics):

$$T ds = dh - \frac{dp}{\rho} \tag{9.6}$$

$$\int_1^2 ds = \int_1^2 C_p \frac{dT}{T} - R \int_1^2 \frac{dp}{p} \tag{9.7}$$

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} \tag{9.8}$$

Isentropic relations:

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \tag{9.9}$$

9.2 The Speed of Sound

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{9.15}$$

$$a = \sqrt{\gamma R T} = \sqrt{\frac{\gamma p}{\rho}} \tag{9.16}$$

9.3 Adiabatic and Isentropic Steady Flow

Adiabatic energy equation without viscous work or shaft work (Eqn. 3.70):

$$h + \frac{1}{2}V^2 = h_o = const \tag{9.21}$$

Eqn. 9.21 with $h = C_p T$ gives:

$$C_p T + \frac{1}{2}V^2 = C_p T_o \tag{9.23}$$

Isentropic flow relations (Table B.1)

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{9.26}$$

$$\frac{a_o}{a} = \left(\frac{T_o}{T}\right)^{1/2} \tag{9.27}$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\gamma/(\gamma-1)} \tag{9.28a}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{1/(\gamma-1)} \tag{9.28b}$$

Critical Values at the Sonic Point

$$\begin{aligned} \frac{T^*}{T_o} &= \left(\frac{2}{\gamma + 1}\right) \\ \frac{p^*}{p_o} &= \left(\frac{T^*}{T_o}\right)^{\gamma/(\gamma-1)} \\ \frac{\rho^*}{\rho_o} &= \left(\frac{T^*}{T_o}\right)^{1/(\gamma-1)} \\ \frac{a^*}{a_o} &= \left(\frac{T^*}{T_o}\right)^{1/2} \end{aligned} \tag{9.32}$$

9.4 Isentropic Flow with Area Changes

The area-velocity relation:

$$\frac{dV}{V}(M^2 - 1) = \frac{dA}{A} \tag{9.40}$$

The area-Mach-number relation (Table B.1):

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1}\right]^{(\gamma+1)/(\gamma-1)} \tag{9.44}$$

Choked mass flow:

$$\dot{m} = \frac{p_o A^*}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

9.5 The Normal Shock Wave

Governing equations (continuity, momentum and energy):

$$\rho_1 V_1 = \rho_2 V_2 \tag{9.49a}$$

$$p_1 - p_2 = \rho_2 V_2^2 - \rho_1 V_1^2 \tag{9.49b}$$

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 = h_o = \text{constant} \tag{9.49c}$$

The Rankine-Hugoniot relation:

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) \tag{9.50}$$

The normal shock relations (Table B.2):

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \tag{9.55}$$

$$M_2^2 = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)} \tag{9.57}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} \tag{9.58}$$

$$\frac{T_2}{T_1} = \left(2 + (\gamma-1)M_1^2\right)\frac{2\gamma M_1^2 - (\gamma-1)}{(\gamma+1)M_1^2} \tag{9.58}$$

$$T_{o_1} = T_{o_2} \tag{9.58}$$

$$\frac{p_{o_2}}{p_{o_1}} = \frac{\rho_{o_2}}{\rho_{o_1}} = \left[\frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)}\right]^{1/(\gamma-1)} \tag{9.58}$$

$$\left(\frac{A_2^*}{A_1^*}\right)^2 = \left(\frac{M_2}{M_1}\right)^2\left[\frac{2 + (\gamma-1)M_1^2}{2 + (\gamma-1)M_2^2}\right]^{(\gamma+1)/(\gamma-1)} \tag{9.59}$$

9.9 Mach Waves and Oblique Shock Waves

Mach Waves

Mach wave angle (μ):

$$\mu = \sin^{-1}\frac{1}{M} \tag{9.79}$$

The Oblique Shock Wave

Governing equations (continuity, momentum in the shock normal and shock tangential directions, and energy):

$$\rho_1 V_{n_1} = \rho_2 V_{n_2} \tag{9.80a}$$

$$p_1 - p_2 = \rho_2 V_{n_2}^2 - \rho_1 V_{n_1}^2 \tag{9.80b}$$

$$0 = \rho_1 V_{n_1}(V_{t_2} - V_{t_1}) \tag{9.80c}$$

$$h_1 + \frac{1}{2}V_{n_1}^2 + \frac{1}{2}V_{t_1}^2 = h_2 + \frac{1}{2}V_{n_2}^2 + \frac{1}{2}V_{t_2}^2 = h_o \tag{9.80d}$$

$$V_{t_2} = V_{t_1} = V_t = \text{const} \tag{9.81}$$

Shock-normal Mach numbers (M_{n_1} and M_{n_2}):

$$\begin{aligned} M_{n_1} &= M_1 \sin\beta \\ M_{n_2} &= M_2 \sin(\beta - \theta) \end{aligned} \tag{9.82}$$

The θ-β-Mach relation (see Fig. 9.1):

$$\tan\theta = \frac{2\cot\beta(M_1^2\sin^2\beta - 1)}{M_1^2(\gamma + \cos(2\beta)) + 2} \tag{9.86}$$

9.10 Prandtl-Meyer Expansion Waves

The Prandtl-Meyer supersonic expansion function (Table B.5):

$$\omega(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \tag{9.99}$$

$$\Delta\theta = \omega(M_2) - \omega(M_1) \tag{9.101}$$

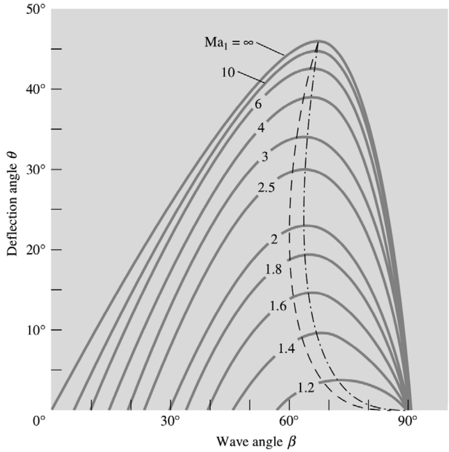


Figure 9.1: Oblique shock deflection versus wave angle for various Mach numbers ($\gamma = 1.4$). Dashed line indicates division of weak and strong solutions and the dash-dotted line connects the maximum deflection agngles θ_{max} for each Mach number.

A Physical Properties of Fluids

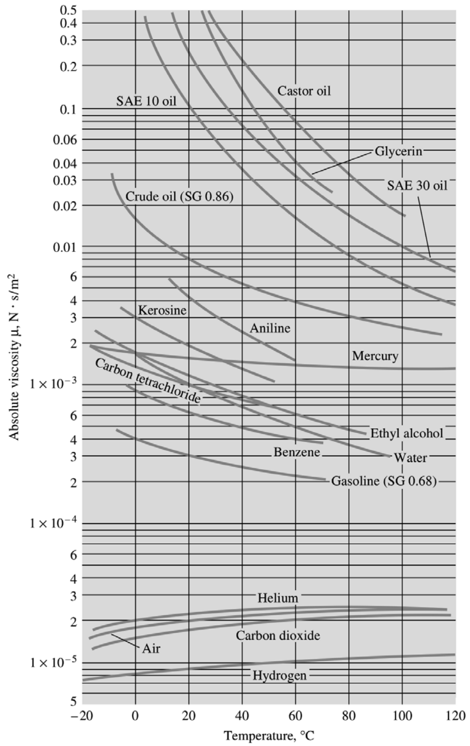


Figure A.1: Absolute viscosity (μ) of common fluids at 1 atm

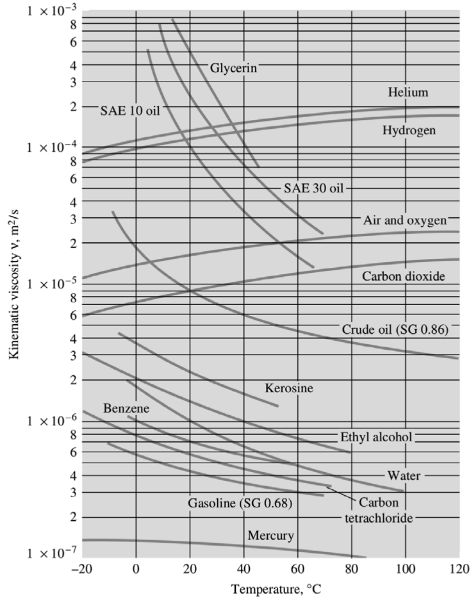


Figure A.2: Kinematic viscosity (ν) of common fluids at 1 atm

Table A.1: Viscosity and density of water at 1 atm

T $[K]$	T $[°C]$	ρ $[kg/m^3]$	μ $[kg/(ms)]$	ν $[m^2/s]$
273	0	1000	1.788 E-03	1.788 E-06
283	10	1000	1.307 E-03	1.307 E-06
293	20	998	1.003 E-03	1.005 E-06
303	30	996	0.799 E-03	0.802 E-06
313	40	992	0.657 E-03	0.662 E-06
323	50	988	0.548 E-03	0.555 E-06
333	60	983	0.467 E-03	0.475 E-06
343	70	978	0.405 E-03	0.414 E-06
353	80	972	0.355 E-03	0.365 E-06
363	90	965	0.316 E-03	0.327 E-06
373	100	958	0.283 E-03	0.295 E-06

Suggested curve fits for water in the range $273K \leq T \leq 373K$:

$$\rho = 1000.0 - 0.0178|T - 277|^{1.7} \pm 0.2\%$$

$$\ln\left(\frac{\mu}{\mu_0}\right) \approx -1.704 - 5.306z + 7.003z^2$$

$$z = \frac{273}{T}, \ \mu_0 = 1.788 \text{ E-03 kg/ms}$$

Table A.2: Viscosity and density of air at 1 atm

T $[K]$	T $[°C]$	ρ $[kg/m^3]$	μ $[kg/(ms)]$	ν $[m^2/s]$
233	-40	1.520	1.51 E-05	0.99 E-05
273	0	1.290	1.71 E-05	1.33 E-05
293	20	1.200	1.80 E-05	1.50 E-05
323	50	1.090	1.95 E-05	1.79 E-05
373	100	0.946	2.17 E-05	2.30 E-05
423	150	0.835	2.38 E-05	2.85 E-05
473	200	0.746	2.57 E-05	3.45 E-05
523	250	0.675	2.75 E-05	4.08 E-05
573	300	0.616	2.93 E-05	4.75 E-05
673	400	0.525	3.25 E-05	6.20 E-05
773	500	0.457	3.55 E-05	7.77 E-05

Suggested curve fits for air:

$$\rho = \frac{p}{RT}$$

Power law: $\frac{\mu}{\mu_0} \approx \left(\frac{T}{T_0}\right)^{0.7}$

Sutherland: $\frac{\mu}{\mu_0} \approx \left(\frac{T}{T_0}\right)^{3/2}\left(\frac{T_0+S}{T+S}\right)$

with R=287 J/(kg K), T_0=273 K, μ_0=1.71 E-05 kg/ms, and S=110.4 K

Table A.3: Properties of common liquids at 1 atm and 20°C

Liquid	ρ $[kg/m^3]$	μ $[kg/(ms)]$	Υ [a] $[N/m]$	p_v $[N/m^2]$	Bulk modulus $[N/m^2]$	Viscosity parameter C[b]
Ammonia	608	2.20 E-04	2.13 E-02	9.10 E+05	1.82 E+09	1.05
Benzene	881	6.51 E-04	2.88 E-02	1.01 E+04	1.47 E+09	4.34
Carbon tetrachloride	1590	9.67 E-04	2.70 E-02	1.20 E+04	1.32 E+09	4.45
Ethanol	789	1.20 E-03	2.28 E-02	5.70 E+03	1.09 E+09	5.72
Ethylene glycol	1117	2.14 E-02	4.84 E-02	1.20 E+01	3.05 E+09	11.7
Freon 12	1327	2.62 E-04	–	–	7.95 E+08	1.76
Gasoline	680	2.92 E-04	2.16 E-02	5.51 E+04	1.30 E+09	3.68
Glycerin	1260	1.49	6.33 E-02	1.40 E-02	4.35 E+09	28.0
Kerosene	804	1.92 E-03	2.80 E-02	3.11 E+03	1.41 E+09	5.56
Mercury	13,550	1.56 E-03	4.84 E-01	1.10 E-03	2.85 E+10	1.07
Methanol	791	5.98 E-04	2.25 E-02	1.34 E+04	1.03 E+09	4.63
SAE 10W oil	870	1.04 E-01[c]	3.60 E-02	–	1.31 E+09	15.7
SAE 10W30 oil	876	1.70 E-01[c]	–	–	–	14.0
SAE 30W oil	891	2.90 E-01[c]	3.50 E-02	–	1.38 E+09	18.3
SAE 50W oil	902	8.60 E-01[c]	–	–	–	20.2
Water	998	1.00 E-03	7.28 E-02	2.34 E+03	2.19 E+09	Table A.1
Seawater (30‰)	1025	1.07 E-03	7.28 E-02	2.34 E+03	2.33 E+09	7.28

[a] In contact with air

[b] The viscosity temperature variation of these liquids may be fitted to the empirical expression
$$\frac{\mu}{\mu_{293K}} = exp\left[C\left(\frac{293}{T} - 1\right)\right]$$
with an accuracy of ±6 percent in the range $273K \leq T \leq 373K$

[c] Representative values. The SAE oil classification allow a viscosity variation of up to ±50 percent, especially at lower temperatures.

Table A.4: Properties of common gases at 1 atm and 20°C

Gas	Molecular weight	R $[m^2/(s^2K)]$	ρg $[N/m^3]$	μ $[kg/(ms)]$	ν $[m^2/s]$	Specific-heat ratio	Power-law exponent n^a
H_2	2.016	4124	0.822	9.05 E-06	1.08 E-04	1.41	0.68
He	4.003	2077	1.63	1.97 E-05	1.18 E-04	1.66	0.67
H_2O	18.02	461	7.35	1.02 E-05	1.36 E-05	1.33	1.15
Ar	39.944	208	16.3	2.24 E-05	1.35 E-05	1.67	0.72
Dry air	28.96	287	11.8	1.80 E-05	1.49 E-05	1.40	0.67
CO_2	44.01	189	17.9	1.48 E-05	8.09 E-06	1.30	0.79
CO	28.01	297	11.4	1.82 E-05	1.56 E-05	1.40	0.71
N_2	28.02	297	11.4	1.76 E-05	1.51 E-05	1.40	0.67
O_2	32.00	260	13.1	2.00 E-05	1.50 E-05	1.40	0.69
NO	30.01	277	12.1	1.90 E-05	1.52 E-05	1.40	0.78
N_2O	44.02	189	17.9	1.45 E-05	7.93 E-06	1.31	0.89
Cl_2	70.91	117	28.9	1.03 E-05	3.49 E-06	1.34	1.00
CH_4	16.04	518	6.54	1.34 E-05	2.01 E-05	1.32	0.87

[a] The power-law curve $\mu/\mu_{293K} \approx (T/293)^n$ fits these gases to within ±4 percent in the range $250K \leq T \leq 1000K$

Table A.5: Surface tension, vapor pressure, and sound speed of water

T $[°C]$	Υ $[N/m]$	p_v $[Pa]$	a $[m/s]$
0	0.0756	6.1100 E02	1402
10	0.0742	1.2270 E03	1447
20	0.0728	2.3370 E03	1482
30	0.0712	4.2420 E03	1509
40	0.0696	7.3750 E03	1529
50	0.0679	1.2340 E04	1542
60	0.0662	1.9920 E04	1551
70	0.0644	3.1160 E04	1553
80	0.0626	4.7350 E04	1554
90	0.0608	7.0110 E04	1550
100	0.0589	1.0130 E05	1543
120	0.0550	1.9850 E05	1518
140	0.0509	3.6130 E05	1483
160	0.0466	6.1780 E05	1440
180	0.0422	1.0020 E06	1389
200	0.0377	1.5540 E06	1334
220	0.0331	2.3180 E06	1268
240	0.0284	3.3440 E06	1192
260	0.0237	4.6880 E06	1110
280	0.0190	6.4120 E06	1022
300	0.0144	8.5810 E06	920
320	0.0099	1.1274 E07	800
340	0.0056	1.4586 E07	630
360	0.0019	1.8651 E07	370
* 374	0.0	2.2090 E07	0

* critical point

Table A.6: Properties of the Standard Atmosphere

z $[m]$	T $[K]$	p $[Pa]$	ρ $[kg/m^3]$	a $[m/s]$	z $[m]$	T $[K]$	p $[Pa]$	ρ $[kg/m^3]$	a $[m/s]$
-500	291.41	107,508	1.2854	342.2	12,500	216.66	17,847	0.2870	295.1
0	288.16	101,350	1.2255	340.3	13,000	216.66	16,494	0.2652	295.1
500	284.91	95,480	1.1677	338.4	13,500	216.66	15,243	0.2451	295.1
1000	281.66	89,889	1.1120	336.5	14,000	216.66	14,087	0.2265	295.1
1500	278.41	84,565	1.0583	334.5	14,500	216.66	13,018	0.2094	295.1
2000	275.16	79,500	1.0067	332.6	15,000	216.66	12,031	0.1935	295.1
2500	271.91	74,684	0.9570	330.6	15,500	216.66	11,118	0.1788	295.1
3000	268.66	70,107	0.9092	328.6	16,000	216.66	10,275	0.1652	295.1
3500	265.41	65,759	0.8633	326.6	16,500	216.66	9496	0.1527	295.1
4000	262.16	61,633	0.8191	324.6	17,000	216.66	8775	0.1411	295.1
4500	258.91	57,718	0.7768	322.6	17,500	216.66	8110	0.1304	295.1
5000	255.66	54,008	0.7361	320.6	18,000	216.66	7495	0.1205	295.1
5500	252.41	50,493	0.6970	318.5	18,500	216.66	6926	0.1114	295.1
6000	249.16	47,166	0.6596	316.5	19,000	216.66	6401	0.1029	295.1
6500	245.91	44,018	0.6237	314.4	19,500	216.66	5915	0.0951	295.1
7000	242.66	41,043	0.5893	312.3	20,000	216.66	5467	0.0879	295.1
7500	239.41	38,233	0.5564	310.2	22,000	218.60	4048	0.0645	296.4
8000	236.16	35,581	0.5250	308.1	24,000	220.60	2972	0.0469	297.8
8500	232.91	33,080	0.4949	306.0	26,000	222.50	2189	0.0343	299.1
9000	229.66	30,723	0.4661	303.8	28,000	224.50	1616	0.0251	300.4
9500	226.41	28,504	0.4387	301.7	30,000	226.50	1197	0.0184	301.7
10,000	223.16	26,416	0.4125	299.5	40,000	250.40	287	0.0040	317.2
10,500	219.91	24,455	0.3875	297.3	50,000	270.70	80	0.0010	329.9
11,000	216.66	22,612	0.3637	295.1	60,000	255.70	22	0.0003	320.6
11,500	216.66	20,897	0.3361	295.1	70,000	219.70	6	0.0001	297.2
12,000	216.66	19,312	0.3106	295.1					

B Compressible Flow Tables

Table B.1: Isentropic flow of a perfect gas $\gamma = 1.4$

M	p/p_o	ρ/ρ_o	T/T_o	A/A^*	M	p/p_o	ρ/ρ_o	T/T_o	A/A^*
0.00	1.0000	1.0000	1.0000	∞	2.10	0.1094	0.2058	0.5313	1.8369
0.10	0.9930	0.9950	0.9980	5.8218	2.20	0.0935	0.1841	0.5081	2.0050
0.20	0.9725	0.9803	0.9921	2.9635	2.30	0.0800	0.1646	0.4859	2.1931
0.30	0.9395	0.9564	0.9823	2.0351	2.40	0.0684	0.1472	0.4647	2.4031
0.40	0.8956	0.9243	0.9690	1.5901	2.50	0.0585	0.1317	0.4444	2.6367
0.50	0.8430	0.8852	0.9524	1.3398	2.60	0.0501	0.1179	0.4252	2.8960
0.60	0.7840	0.8405	0.9328	1.1882	2.70	0.0430	0.1056	0.4068	3.1830
0.70	0.7209	0.7916	0.9107	1.0944	2.80	0.0368	0.0946	0.3894	3.5001
0.80	0.6560	0.7400	0.8865	1.0382	2.90	0.0317	0.0849	0.3729	3.8498
0.90	0.5913	0.6870	0.8606	1.0089	3.00	0.0272	0.0762	0.3571	4.2346
1.00	0.5283	0.6339	0.8333	1.0000	3.10	0.0234	0.0685	0.3422	4.6573
1.10	0.4684	0.5817	0.8052	1.0079	3.20	0.0202	0.0617	0.3281	5.1210
1.20	0.4124	0.5311	0.7764	1.0304	3.30	0.0175	0.0555	0.3147	5.6286
1.30	0.3609	0.4829	0.7474	1.0663	3.40	0.0151	0.0501	0.3019	6.1837
1.40	0.3142	0.4374	0.7184	1.1149	3.50	0.0131	0.0452	0.2899	6.7896
1.50	0.2724	0.3950	0.6897	1.1762	3.60	0.0114	0.0409	0.2784	7.4501
1.60	0.2353	0.3557	0.6614	1.2502	3.70	0.0099	0.0370	0.2675	8.1691
1.70	0.2026	0.3197	0.6337	1.3376	3.80	0.0086	0.0335	0.2572	8.9506
1.80	0.1740	0.2868	0.6068	1.4390	3.90	0.0075	0.0304	0.2474	9.7990
1.90	0.1492	0.2570	0.5807	1.5553	4.00	0.0066	0.0277	0.2381	10.7188
2.00	0.1278	0.2300	0.5556	1.6875					

Tabulated data obtained using Eqns. 9.26, 9.28, and 9.44.

Table B.5: Prandtl-Meyer supersonic expansion function for $\gamma = 1.4$

M	$\omega[deg]$	M	$\omega[deg]$	M	$\omega[deg]$	M	$\omega[deg]$
1.00	00.00	3.10	51.65	5.10	77.84	7.10	91.49
1.10	01.34	3.20	53.47	5.20	78.73	7.20	92.00
1.20	03.56	3.30	55.22	5.30	79.60	7.30	92.49
1.30	06.17	3.40	56.91	5.40	80.43	7.40	92.97
1.40	08.99	3.50	58.53	5.50	81.24	7.50	93.44
1.50	11.91	3.60	60.09	5.60	82.03	7.60	93.90
1.60	14.86	3.70	61.60	5.70	82.80	7.70	94.34
1.70	17.81	3.80	63.04	5.80	83.54	7.80	94.78
1.80	20.73	3.90	64.44	5.90	84.26	7.90	95.21
1.90	23.59	4.00	65.78	6.00	84.96	8.00	95.62
2.00	26.38	4.10	67.08	6.10	85.63	8.10	96.03
2.10	29.10	4.20	68.33	6.20	86.29	8.20	96.43
2.20	31.73	4.30	69.54	6.30	86.94	8.30	96.82
2.30	34.28	4.40	70.71	6.40	87.56	8.40	97.20
2.40	36.75	4.50	71.83	6.50	88.17	8.50	97.57
2.50	39.12	4.60	72.92	6.60	88.76	8.60	97.94
2.60	41.41	4.70	73.97	6.70	89.33	8.70	98.29
2.70	43.62	4.80	74.99	6.80	89.89	8.80	98.64
2.80	45.75	4.90	75.97	6.90	90.44	8.90	98.98
2.90	47.79	5.00	76.92	7.00	90.97	9.00	99.32
3.00	49.76						

Tabulated data obtained using Eqn. 9.99.

Table B.2: Normal shock relations for a perfect gas $\gamma = 1.4$

M_{n_1}	M_{n_2}	p_2/p_1	$V_1/V_2 = \rho_2/\rho_1$	T_2/T_1	p_{o_2}/p_{o_1}	A_2^*/A_1^*
1.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.10	0.9118	1.2450	1.1691	1.0649	0.9989	1.0011
1.20	0.8422	1.5133	1.3416	1.1280	0.9928	1.0073
1.30	0.7860	1.8050	1.5157	1.1909	0.9794	1.0211
1.40	0.7397	2.1200	1.6897	1.2547	0.9582	1.0436
1.50	0.7011	2.4583	1.8621	1.3202	0.9298	1.0755
1.60	0.6684	2.8200	2.0317	1.3880	0.8952	1.1171
1.70	0.6405	3.2050	2.1977	1.4583	0.8557	1.1686
1.80	0.6165	3.6133	2.3592	1.5316	0.8127	1.2305
1.90	0.5956	4.0450	2.5157	1.6079	0.7674	1.3032
2.00	0.5774	4.5000	2.6667	1.6875	0.7209	1.3872
2.10	0.5613	4.9783	2.8119	1.7705	0.6742	1.4832
2.20	0.5471	5.4800	2.9512	1.8569	0.6281	1.5920
2.30	0.5344	6.0050	3.0845	1.9468	0.5833	1.7144
2.40	0.5231	6.5533	3.2119	2.0403	0.5401	1.8514
2.50	0.5130	7.1250	3.3333	2.1375	0.4990	2.0039
2.60	0.5039	7.7200	3.4490	2.2383	0.4601	2.1733
2.70	0.4956	8.3383	3.5590	2.3429	0.4236	2.3608
2.80	0.4882	8.9800	3.6636	2.4512	0.3895	2.5676
2.90	0.4814	9.6450	3.7629	2.5632	0.3577	2.7954
3.00	0.4752	10.3333	3.8571	2.6790	0.3283	3.0456
3.10	0.4695	11.0450	3.9466	2.7986	0.3012	3.3199
3.20	0.4643	11.7800	4.0315	2.9220	0.2762	3.6202
3.30	0.4596	12.5383	4.1120	3.0492	0.2533	3.9483
3.40	0.4552	13.3200	4.1884	3.1802	0.2322	4.3062
3.50	0.4512	14.1250	4.2609	3.3151	0.2129	4.6960
3.60	0.4474	14.9533	4.3296	3.4537	0.1953	5.1200
3.70	0.4439	15.8050	4.3949	3.5962	0.1792	5.5806
3.80	0.4407	16.6800	4.4568	3.7426	0.1645	6.0801
3.90	0.4377	17.5783	4.5156	3.8928	0.1510	6.6213
4.00	0.4350	18.5000	4.5714	4.0469	0.1388	7.2069
4.10	0.4324	19.4450	4.6245	4.2048	0.1276	7.8397
4.20	0.4299	20.4133	4.6749	4.3666	0.1173	8.5227
4.30	0.4277	21.4050	4.7229	4.5322	0.1080	9.2591
4.40	0.4255	22.4200	4.7685	4.7017	0.0995	10.0522
4.50	0.4236	23.4583	4.8119	4.8751	0.0917	10.9054
4.60	0.4217	24.5200	4.8532	5.0523	0.0846	11.8222
4.70	0.4199	25.6050	4.8926	5.2334	0.0781	12.8065
4.80	0.4183	26.7133	4.9301	5.4184	0.0721	13.8620
4.90	0.4167	27.8450	4.9659	5.6073	0.0667	14.9928
5.00	0.4152	29.0000	5.0000	5.8000	0.0617	16.2032

Tabulated data obtained using Eqns. 9.57, 9.55, 9.58, and 9.59.

D Equations of Motion in Cylindrical Coordinates

The equations of motion of an incompressible newtonian fluid with constant μ, k, and C_p are given here in cylindrical coordinates (r, θ, z), which are related to cartesian coordinates (x, y, z) as follows

$$z = r\cos\theta, \quad y = r\sin\theta \tag{D.1}$$

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0 \tag{D.2}$$

Convective derivative:

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{1}{r}v_\theta\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z} \tag{D.3}$$

Laplacian operator:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{D.4}$$

The r-momentum equation:

$$\frac{\partial v_r}{\partial t} + (\mathbf{V}\cdot\nabla)v_r - \frac{1}{r}v_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) \tag{D.5}$$

The θ-momentum equation:

$$\frac{\partial v_\theta}{\partial t} + (\mathbf{V}\cdot\nabla)v_\theta - \frac{1}{r}v_r v_\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) \tag{D.6}$$

The z-momentum equation:

$$\frac{\partial v_z}{\partial t} + (\mathbf{V}\cdot\nabla)v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\nabla^2 v_z \tag{D.7}$$

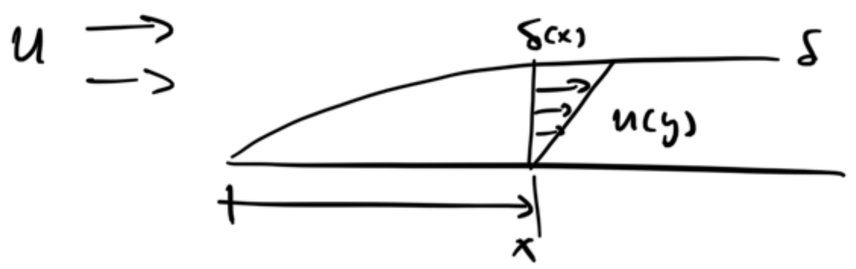
P1

GIVEN :

VELOCITY PROFILE $u(y) = U\frac{y}{\delta}$

WHERE U IS THE FREE STREAM VELOCITY AND δ IS THE BOUNDARY-LAYER THICKNESS

NOTE! $\delta = \delta(x)$



$$\boxed{\begin{array}{ll} u(y) = U\frac{y}{\delta} & 0 \le y \le \delta \\ u = U & y > \delta \end{array}}$$

DERIVE $\tau_w(x)$ USING THE MOMENTUM THICKNESS RELATION GIVEN BY

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$u(y) = U\frac{y}{\delta} \Rightarrow \theta = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy$$

$$\theta = \int_0^\delta \frac{y}{\delta} - \frac{y^2}{\delta^2}\, dy = \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta =$$

$$= \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

$$\therefore \boxed{\theta = \frac{\delta}{6}, \quad \delta = \delta(x)}$$

Eqn. 7.5: $\tau_w = \rho U^2 \frac{d\theta}{dx}$

$$\left.\begin{array}{l} \tau_w = \rho U^2 \frac{d\theta}{dx} \\ \tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \left\{\frac{\partial u}{\partial y} = \frac{U}{\delta}\right\} = \mu\frac{U}{\delta} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mu\frac{U}{\delta} = \rho U^2 \frac{d}{dx}\left(\frac{\delta(x)}{6}\right)$$

$$\mu\frac{U}{\delta} = \rho U^2 \frac{1}{6}\frac{d\delta}{dx} \Rightarrow \frac{6\mu}{\rho U} dx = \delta\, d\delta$$

$$\frac{1}{2}\delta^2 = \frac{6\mu x}{\rho U}$$

$$\left(\Rightarrow \frac{\delta}{x} = \sqrt{12}\left(\frac{\mu}{\rho U x}\right)^{1/2}\right)$$

$$\tau_w(x) = \frac{\mu U}{\sqrt{12}\, x \left(\frac{\mu}{\rho U x}\right)^{1/2}}$$

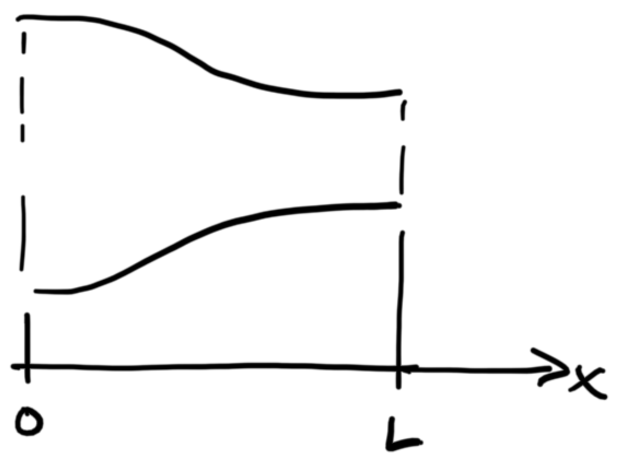
$$\Rightarrow \tau_w(x) = \frac{\mu^{1/2} U^{3/2} \rho^{1/2}}{\sqrt{12}\, x^{1/2}}$$

BLASIUS LAMINAR PROFILE $\Rightarrow$

$$\tau_{w_B}(x) = \frac{0.332\, \rho^{1/2} \mu^{1/2} U^{3/2}}{x^{1/2}}$$

$$\frac{\tau_w(x)}{\tau_{w_B}(x)} = \frac{(1/\sqrt{12})}{0.332} = 0.87$$

P2



GIVEN: LINEAR VELOCITY DISTRIBUTION

$$\left.\begin{aligned} u(0) &= U_0 \\ u(L) &= 3U_0 \end{aligned}\right\} \Rightarrow u(x) = U_0\left(1 + \frac{2}{L}x\right)$$

GET FLOW ACCELERATION THROUGH THE CONVERGENT NOZZLE.

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}$$

STEADY-STATE: $\frac{\partial u}{\partial t} = 0$

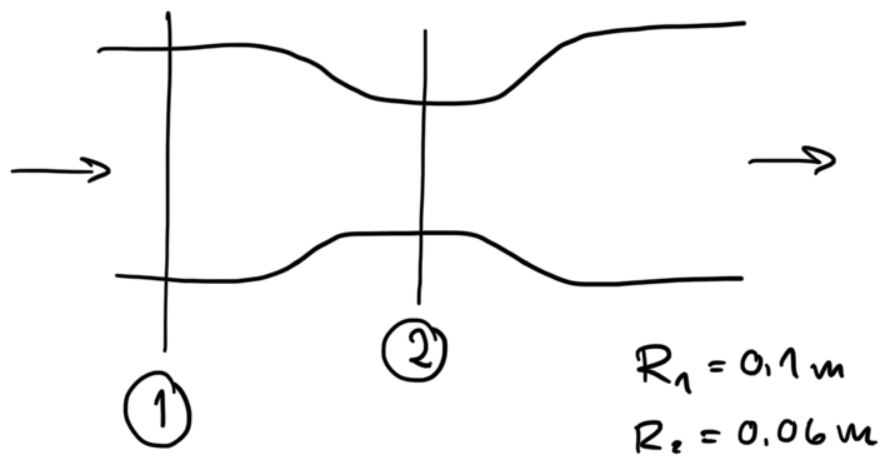
ONE-DIMENSIONAL FLOW $\Rightarrow$ $v=0$, $w=0$

$$\Rightarrow \frac{Du}{Dt} = u\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{2u_0}{L} \Rightarrow \frac{Du}{Dt} = \frac{2u_0^2}{L}\left(1 + \frac{2}{L}x\right)$$

$$\boxed{\begin{aligned} \left.\frac{Du}{Dt}\right|_{x=0} &= \frac{2u_0^2}{L} \\ \left.\frac{Du}{Dt}\right|_{x=L} &= \frac{6u_0^2}{L} \end{aligned}}$$

P3]



Assume incompressible, steady-state flow.

continuity: $u_1 A_1 = u_2 A_2 = Q \quad (1)$

energy eqn:

$$\frac{P_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{u_2^2}{2g} + z_2 + h_L$$

$$\left.\begin{matrix} z_1 = z_2 \\ h_L \simeq 0. \end{matrix}\right\} \Rightarrow P_1 + \frac{1}{2}\rho u_1^2 = P_2 + \frac{1}{2}\rho u_2^2 \quad (2)$$

(1) in (2) $\Rightarrow$

$$P_1 + \frac{1}{2}\rho \frac{Q^2}{A_1^2} = P_2 + \frac{1}{2}\rho \frac{Q^2}{A_2^2}$$

$$\frac{2}{\rho}(P_1 - P_2) = Q^2\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)$$

$$\frac{2}{\rho}(P_1 - P_2) = Q^2\left(\frac{A_1^2 - A_2^2}{A_1^2 A_2^2}\right)$$

$$Q^2 = \left(\frac{A_1^2 A_2^2}{A_1^2 - A_2^2}\right)\frac{2}{\rho}(P_1 - P_2)$$

$$\boxed{Q = A_1 A_2 \sqrt{\frac{2}{\rho}\,\frac{P_1 - P_2}{A_1^2 - A_2^2}}}$$

$Q_{min} = 0.005\ m^3/s$

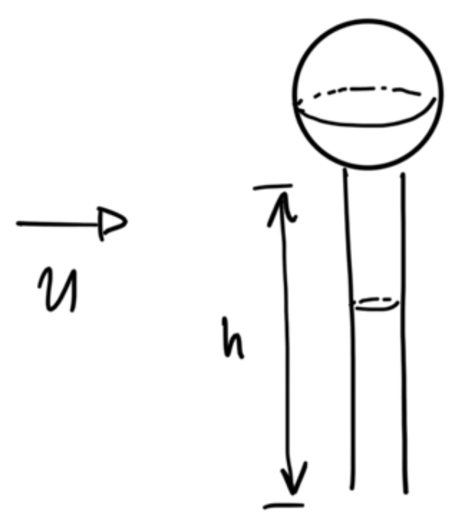
$Q_{max} = 0.050\ m^3/s$

$$Q^2 = \left(\frac{A_1^2 A_2^2}{A_1^2 - A_2^2}\right)\frac{2}{\rho}\Delta p$$

$$\Delta p = \frac{\rho}{2}\left(\frac{A_1^2 - A_2^2}{A_1^2 A_2^2}\right) Q^2 \Rightarrow$$

$$\begin{cases} \Delta P_{min} = 1.11 \cdot 10^3\ Pa \\ \Delta P_{max} = 1.11 \cdot 10^5\ Pa \end{cases}$$

$\boxed{P_4}$



$D_s = 12.0\,m$

$D_c = 4.5\,m$

$h = 30.0\,m$

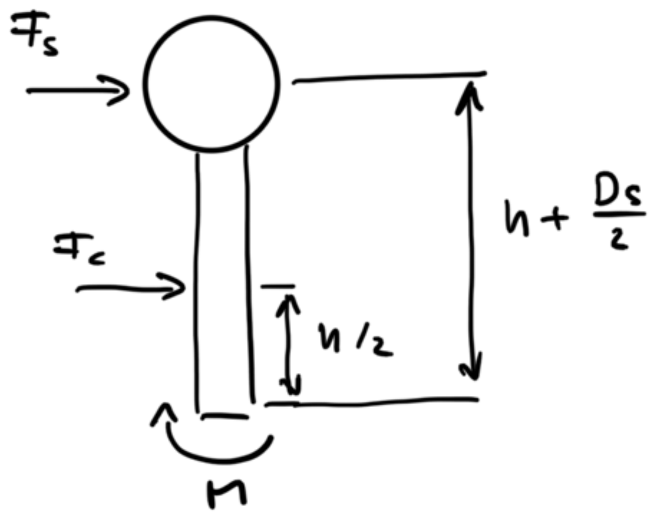
$U = 27.0\,m/s$

air @ 20°C:

$\rho = 1.2\ kg/m^3$

$\mu = 1.81 \cdot 10^{-5}\ kg/m \cdot s$

CALCULATE THE BENDING MOMENT AT THE BASE OF THE WATER TOWER.



$$M = F_s \cdot \left(h + \frac{D_s}{2}\right) + F_c \frac{h}{2} \qquad (1)$$

PROJECTED AREA:

CYLINDER: $D_c \cdot h$

SPHERE: $\frac{\pi D_s^2}{4}$

FORCES:

$$F_s = \frac{1}{2} \rho U^2 \frac{\pi D_s^2}{4} C_{D_s} \qquad (2)$$

$$F_c = \frac{1}{2} \rho U^2 h D_c C_{D_c} \qquad (3)$$

DRAG COEFFICIENTS:

$$Re_s = \frac{\rho U D_s}{\mu} \approx 2.15 \cdot 10^7 \Rightarrow$$

$$\Rightarrow C_{D_s} \approx 0.25 \qquad (4)$$

$$Re_c = \frac{\rho U D_c}{\mu} \approx 8.05 \cdot 10^6 \Rightarrow$$

$$\Rightarrow C_{D_c} \approx 0{,}44 \qquad (5)$$

(4) in (2) $\Rightarrow$

$$F_s \approx 1.26 \cdot 10^4\ N \qquad (6)$$

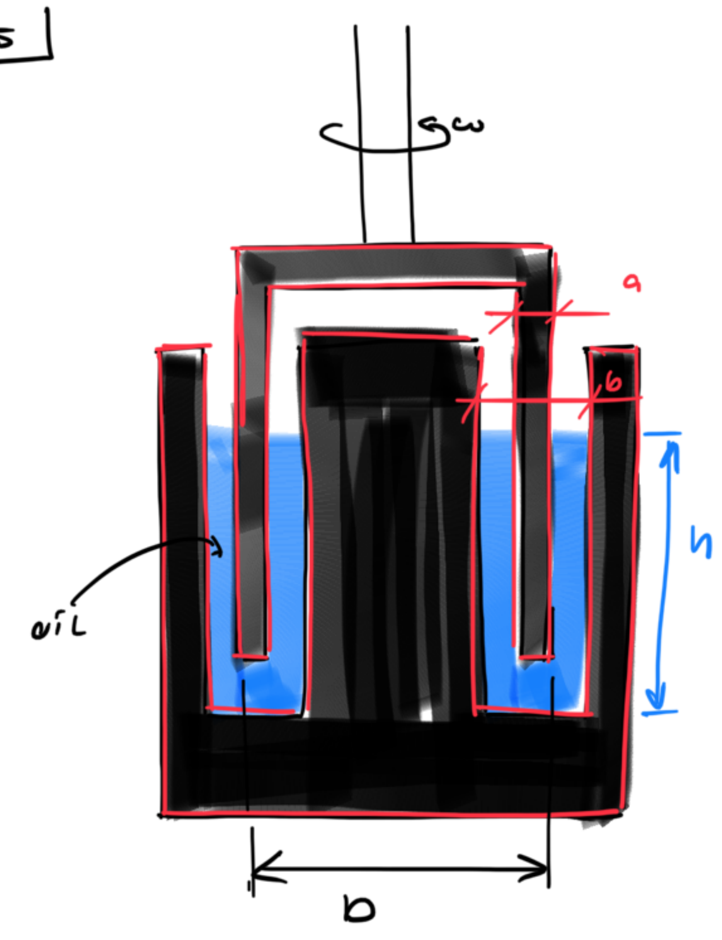
(5) in (3) $\Rightarrow$

$$F_c = 2{,}58 \cdot 10^4\ N \qquad (7)$$

(6) & (7) in (1) $\Rightarrow$

$$\boxed{M \approx 8.39 \cdot 10^5\ Nm}$$

P5



FIND AN EXPRESSION FOR THE MOMENT REQUIRED TO TURN THE CYLINDER

THE ROTATING CYLINDER IS THIN AND THE CONTRIBUTION FROM THE END SURFACE CAN BE NEGLECTED. THE INNER AND OUTER SURFACES OF THE ROTATING CYLINDER WILL CONTRIBUTE TO THE RESISTING MOMENT.

OUTSIDE:

$$\tau_o = \mu \frac{\partial u}{\partial y} = \mu \frac{\omega D}{2} \frac{2}{b-a} = \mu \frac{\omega D}{b-a} \quad (1)$$

$$F_o = \tau_o \cdot 2\pi \frac{D}{2} h = \mu \pi h \frac{\omega D^2}{b-a} \quad (2)$$

$$M_o = F_o \frac{D}{2} = \frac{1}{2} \mu \pi h \frac{\omega D^3}{b-a} \quad (3)$$

INSIDE:

$$\tau_i = \mu \frac{\partial u}{\partial y} = \mu \frac{\omega (D-2a)}{2} \frac{2}{b-a} = \mu \frac{\omega (D-2a)}{b-a} \quad (4)$$

$$F_i = \tau_i \, 2\pi \frac{(D-2a)}{2} h = \mu \pi h \frac{\omega (D-2a)^2}{b-a} \quad (5)$$

$$M_i = F_i \frac{(D-2a)}{2} = \frac{1}{2} \mu \pi h \frac{\omega (D-2a)^3}{b-a} \quad (6)$$

TURNING MOMENT:

$$\boxed{M = M_i + M_o = \frac{\mu \pi h \omega}{2(b-a)} \left[D^3 + (D-2a)^3 \right]} \quad (7)$$

b)

$\omega = 2\pi / s$

$a = 1.5$ mm

$b = 3.5$ mm

$D = 50.0$ mm

SAE 10W30 oil @ $20^\circ C \leq T \leq 80^\circ C$

$M \leq 1.5 \cdot 10^{-2}$ Nm

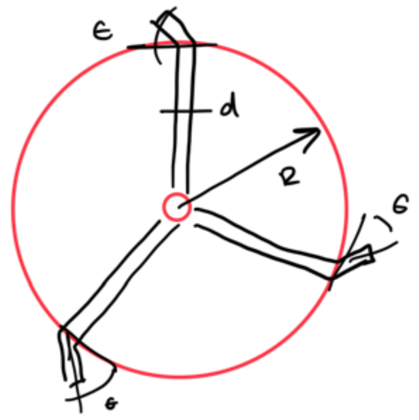
$T = 20^\circ C \Rightarrow \mu = 1.7 \cdot 10^{-1}$ kg/ms

INCREASING THE TEMPERATURE WILL REDUCE THE VISCOSITY OF THE OIL AND THUS THE LOWEST TEMPERATURE IN THE GIVEN RANGE WILL BE THE ONE THAT SETS THE OIL LEVEL (h)

($T = 80^\circ C \Rightarrow \mu = 1.57 \cdot 10^{-2}$ kg/ms)

(7) $\Rightarrow$ $\underline{h = 78 \text{ mm}}$

P6



$Q = 8.0 \cdot 10^{-4}\ m^3/s$
$R = 0,175\ m$
$d = 7.0\ mm$

FRICTION CAN BE NEGLECTED

DERIVE AN EXPRESSION THAT RELATES THE ROTATIONAL VELOCITY (Ω) TO THE ANGLE ϵ.

STARTING POINT: THE ANGULAR MOMENTUM EQUATION ON INTEGRAL FORM FOR A FINITE NUMBER OF INLETS AND OUTLETS.

$$\sum M = \frac{d}{dt}\int_{cv} (r \times u)\rho\, dV + \sum_{cs} r_{out} \times \dot{m}_{out} u_{out} - \sum_{cs} r_{in} \times \dot{m}_{in} u_{in}$$

STEADY STATE: $\frac{d}{dt}\int_{cv} (r \times u)\rho\, dV = 0$

NO TANGENTIAL VELOCITY COMPONENT AT THE INLET

$$\sum_{cs} r_{in} \times \dot{m}_{in} u_{in} = 0$$

$\Rightarrow$

$$\sum M = \sum_{cs} r_{out} \times \dot{m}_{out} u_{out}$$

$$u = \frac{Q}{3A}\cos\epsilon - \Omega R$$

$$A = \frac{\pi d^2}{4}$$

Ω : ROTATING VELOCITY OF THE SPRINKLER.

$$\sum M = \dot{m}\begin{bmatrix} 0 \\ R \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ \frac{Q}{3A}\sin\epsilon \\ \frac{Q}{3A}\cos\epsilon - \Omega R \end{bmatrix}$$

$$\Rightarrow \sum M = \dot{m} R\left(\frac{Q}{3A}\cos\epsilon - \Omega R\right)$$

NO FRICTION $\Rightarrow \sum M = 0 \Rightarrow$

$$\frac{Q}{3A}\cos\epsilon - \Omega R = 0$$

$$\Rightarrow \Omega = \frac{Q}{3AR}\cos\epsilon$$

$$\boxed{\Omega = \frac{4Q\cos\epsilon}{3\pi d^2 R}}$$

$$\Omega = \frac{4Q\cos\epsilon}{6\pi^2 d^2 R} \quad \text{varv}/s$$

$$\Omega = \frac{40Q\cos\epsilon}{\pi^2 d^2 R} \quad rpm$$

$$\Omega_{max} = \Omega(\epsilon = 0) = \frac{40Q}{\pi^2 d^2 R} \quad rpm$$

$$\Omega(\epsilon = 40^\circ) = 290\ rpm \quad (30\ rad/s)$$