Examination

MHA021 Finite Element Method

Date and time:	05-04-2024, 08.30-12.30
Instructors:	Jim Brouzoulis (phone 2253). An instructor will visit the exam around 09:30 and 11:30.
Solutions:	Example solutions will be posted within a few days after the exam on the course homepage.
Grading:	The grades will be reported to the registration office on 26 April the latest.
Review:	For a review of the exam corrections, please make an appointment with your examiner.
Permissible aids:	Chalmers type approved pocket calculator. **Note**: A formula sheet is available as a pdf-file alongside with this exam thesis.

Exam instructions

All exam problems require a hand-in on paper. For some of the problems, it may be convenient to also use MATLAB including CALFEM. If you use MATLAB and CALFEM as part of your solutions, you must make sure to also hand in any MATLAB code you have written yourself. You do this by saving your files under `C:\__Exam__\Assignments\` in the appropriate sub-directories created for each problem. When doing so, it is strongly recommended that you write your anonymous exam code in any MATLAB files that you want to hand in. Finally, **it is also absolutely necessary that you write the name of your computer on the cover page for the exam!**

Note that most CALFEM files (but not all) are provided for your convenience. In addition, please note that the CALFEM function `extract.m` also exists in the CALFEM directory as `extract_dofs.m` (to avoid a conflict with a built-in MATLAB function). These CALFEM finite element files can be found under the directory `C:\__Exam__\Assignments`. You can utilize these files by copying appropriate files into the sub-directories for the problem where they are needed. Should you need to refer to the CALFEM manual, you can find this also (excluding the examples section) under `C:\__Exam__\Assignments`.

Finally, remember to close MATLAB and log-out from the computer when you are finished with the exam.

Problem 1

Consider the plane truss in Figure 1(a), subjected to three point forces. All the members have the same cross-sectional area A and Young's modulus E. A finite element model of the structure has been created and consists of 7 bar elements, with nodal numbering and element orientation according to Figure 1(b).

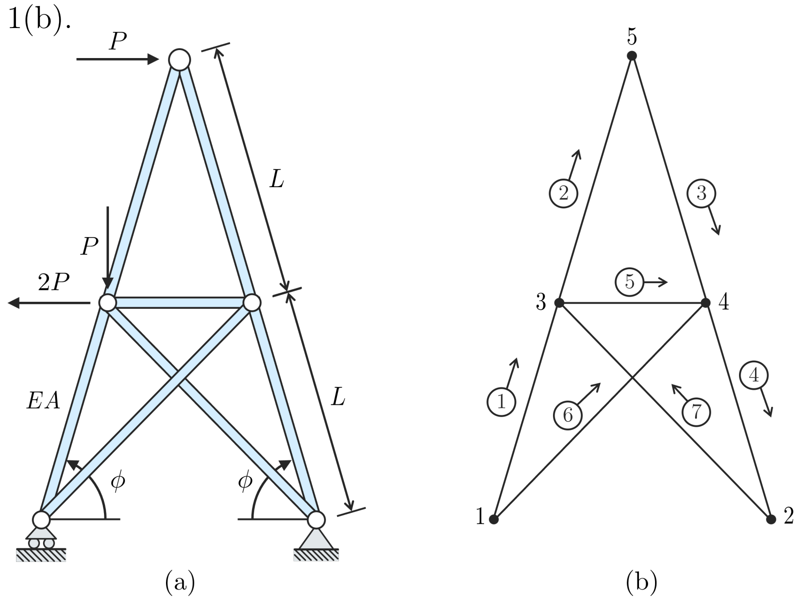

Figure 1: Plane struss structure to be analysed in Problem 1.

Tasks:

a) **Determine the horisontal displacement at node 5.** Start from the provided file `problem1.m` (see the subdirectory for Problem 1 under `C:\__Exam__`), containing the coordinates for the elements (the `Ex` and `Ey` matrices), and write a script that establishes the system of equations $\mathbf{K}\mathbf{a} = \mathbf{f}$ for the structure, and solves this. It is important that you use the provided node and element numbering.

Please write your answer to the problem on the hand-in paper. **(4.0p)**

b) Extend the script from subtask (a) and **determine the magnitude of the force** $P = P_{\mathbf{y}}$ which causes first yielding in any of the members (i.e. $|\sigma_{\max}| = \sigma_{\mathrm{y}}$.).

Please write your answer to the problem on the hand-in paper **(2.0p)**

Use the following numerical values: $\phi = 60°$, $L = 1.8$ m, $P = 13$ kN, $E = 210$ GPa, $A = 2 \cdot 10^{-4}$ m^2 and $\sigma_{\mathrm{y}} = 250$ MPa (also given in the provided MATLAB file `Problem_1.m`).

Problem 2

Consider the water divider as shown in Figure 2. As a result of the separation of water flow, a normal pressure and a shear traction act on the outer surface as indicated in the figure. Both the normal pressure and the shear traction are constant (with magnitudes equal to $\overline{p} = 1$ MPa and $\overline{\tau} = 2$ MPa, respectively) on the inclined surfaces (from node 4 to 8 and from node 4 to 1, respectively), and decreases from left to right along the planar edges (from node 8 to 10 and from node 1 to 3 respectively).

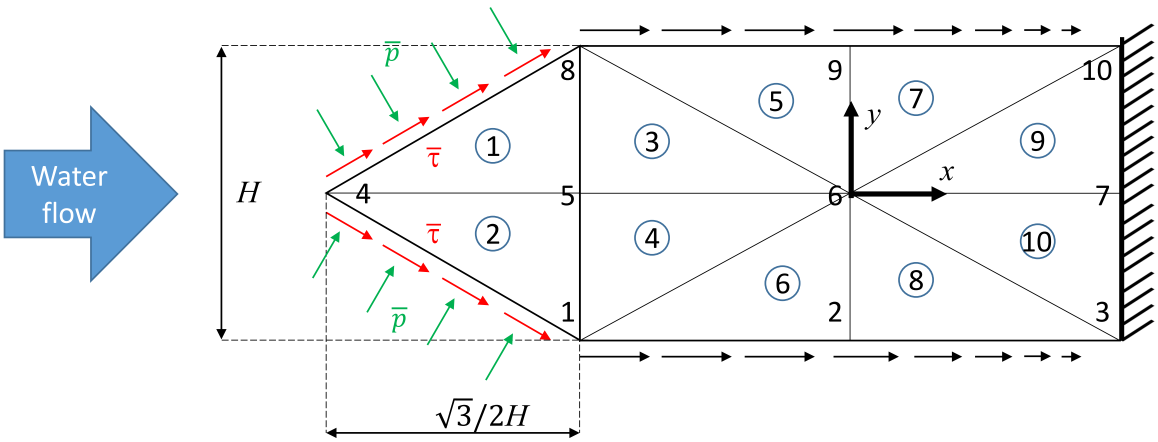

Figure 2: Water divider considered in Problem 2.

As no variations are considered perpendicular to the plane shown in Figure 2, the problem can be considered as a 2D plane strain problem. The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\mathbf{v}\right)^{\mathrm{T}} \mathbf{D}\tilde{\nabla}\mathbf{u}\, t \, \mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\mathbf{b}\, t\, \mathrm{d}A + \int_{\mathcal{L}_g} \mathbf{v}^{\mathrm{T}}\mathbf{t}\, t\, \mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \mathbf{v}^{\mathrm{T}}\mathbf{h}\, t\, \mathrm{d}\mathcal{L}$$

where A denotes the area of the specimen, t its thickness, $\mathbf{v} = [\mathbf{v_x}, \mathbf{v_y}]^{\mathbf{T}}$ a vector arbitrary weight function, $\mathbf{u} = [\mathbf{u_x}, \mathbf{u_y}]^{\mathbf{T}}$ the displacement field (x- and y-component), $\mathcal{L}_g$ the part of the boundary with prescribed degrees of freedom ($\mathbf{g}$), $\mathcal{L}_h$ the part of the boundary with prescribed tractions ($\mathbf{h}$) and where $\mathbf{D}$ is the constitutive matrix relating stresses (σ) and strains ($\boldsymbol{\varepsilon} = \tilde{\nabla}\mathbf{u}$) on Voigt form such that

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}.$$

As the problem is under the state of plane strain, the $\mathbf{D}-$matrix becomes

$$\mathbf{D} = \frac{\mathbf{E}}{(\mathbf{1}+\nu)(\mathbf{1}-\mathbf{2}\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}.$$

Tasks:

a) Introduce suitable FE approximations for **v** and **u** and then **derive and state the FE formulation of the current problem**. Be careful to clearly indicate the contents of any matrices you introduce (or you will not be able to get full points for this subtask). (**2.0p**)

Please note that there is no need at this point to introduce the specific form of the traction boundary conditions.

b) **Do the following:**

Define an appropriate numbering scheme for the degrees-of-freedom associated with the displacement field.

Define the topology matrix `Edof` (or similar) corresponding to your numbering scheme which links degrees-of-freedom to the element numbering. It is enough to write the first two lines of that matrix.

Finally, **write down**, with pen and paper, **the MATLAB code necessary to assemble the element stiffness matrix** (you may call it `Ke`) into the global stiffness matrix (`K`). (**2.0p**)

c) Consider specifically element 2 and the edge between nodes 4 and 1. For this edge, **define the traction vector expressed in the global coordinates that is acting on this edge**. Then **use this to calculate the traction contribution to the global load vector** given $H = 1.0\,\mathrm{m}$ and $t = 2\,\mathrm{m}$. For full points, both the values and how these are assembled needs to be correctly explained. (**2.0p**)

Problem 3

A sealant with rectangular cross-section Ω containing a cylindrical hole centrally placed between two insulating walls to minimise the heat outflow from a heat chamber, cf. Figure 3. The temperature of the air in the heat chamber is T_{hot} and the ambient temperature of the air on the outside is T_{cold}. Furthermore, inside the sealant flows cooling water with the temperature T_w.

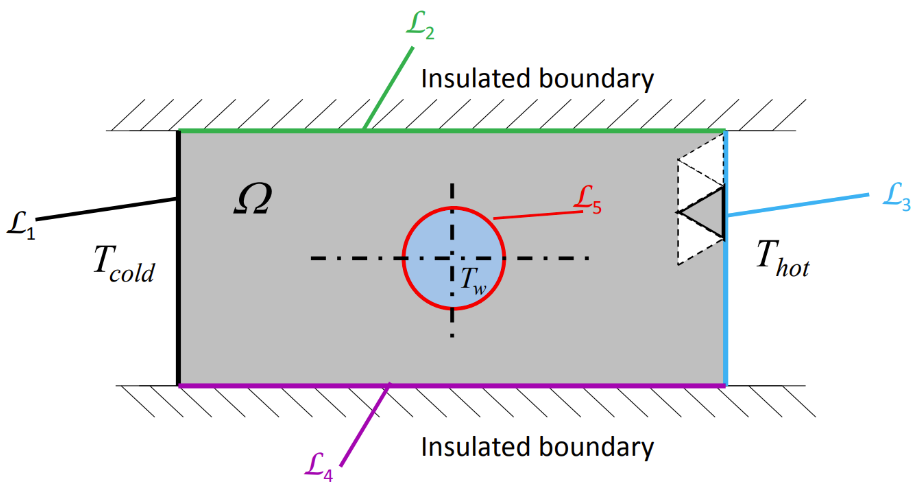

Figure 3: Sketch of the 2D heat flow problem.

The sealant material is assumed to be isotropic (w.r.t heat flow) and obey Fourier's law $\boldsymbol{q} = -k\boldsymbol{\nabla}T$. Furthermore, the material in the sealant is such that the heat conductivity is k and that the heat transfer coefficient between the sealant and air is α_{air} . Between the sealant and water it is α_w.

No heat is assumed to flow out of the plane shown in Figure 3. Thus, the problem can be considered as a 2D heat flow problem and the governing partial differential equation for the heat flow problem thereby becomes

$$\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{q} = 0 \text{ in } \Omega$$

Tasks are on the next page!

Tasks:

a) Motivate whether or not symmetry can be utilised for a more efficient solution of the current problem. Then state the full strong form (in terms of T) for the smallest part of the cross section that can be analysed by FE for this particular problem. (**2.0p**)

b) Derive the weak form corresponding to the strong form in the general format without suppressing boundary reactions. Be specific in how the boundary conditions enter in the weak form. (**2.0p**)

c) By introducing the finite element approximation on the form $T(x, y) \approx \boldsymbol{N}(x, y)\boldsymbol{a}$ and using Galerkin's method, the discrete form of the problem can be obtained as:

$$[\boldsymbol{K} + \boldsymbol{K}_c]\,\boldsymbol{a} = \boldsymbol{f}_b$$

Based on this, derive the expressions for $\boldsymbol{K}, \boldsymbol{K}_c$ and $\boldsymbol{f}_b$ for the particular problem at hand. Please note that $\boldsymbol{f}_b$ may have different contributions so be sure to state which of these will yield non-zero contributions and why. (**2.0p**)