Problem 1:

$$\begin{cases} -\dfrac{d}{dx}\left[E\,A\,\dfrac{du}{dx}\right] = q_x \quad 0 < x < L \\ u(0) = 0 \\ N(L) = 0 \end{cases} \qquad q_x(x) = q_0\left(1 - 0.2\left(\frac{x}{L}\right)\right) \qquad A(x) = \frac{A_0}{2}\left(2 - \left(\frac{x}{L}\right)^2\right)$$

where $N = EA\dfrac{du}{dx}$

a) Multiply with a test function $v(x)$ and integrate over the bar length $x \in [0,L]$ $\Rightarrow$

$$-\int_0^L v \cdot N'\,dx \int_0^L v \cdot q_x\,dx \Rightarrow \{\text{I.B.P}\} \Rightarrow \int_0^L v' \cdot N\,dx - [v \cdot N]_0^L = \int_0^L v \cdot q_x\,dx$$

$$\Rightarrow \int_0^L v' EAu'\,dx = \underbrace{vN\Big|_L}_{=0} - vN\Big|_0 + \int_0^L vq_x\,dx \Rightarrow \int_0^L v' EAu'\,dx = -v(0)N(0) + \int_0^L vq_x\,dx$$

Weak form: Find $u(x)$ such that

$$\begin{cases} \int_0^L v' EAu'\,dx = -v(0)N(0) + \int_0^L vq_x\,dx \quad x \in [0,L] \\ u(0) = 0 \end{cases} \qquad \text{(W.F)}$$

for arbitrary $v(x)$

FE-form:

Introduce approximation: degrees of freedom, shape functions

$$u(x) \approx u_h(x) = N_1 a_1 + \dots N_n a_n = [N_1, \dots, N_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbb{N}\mathbb{a}$$

Galerkin method: use the same shape functions $N_1, \dots, N_n$ to describe $v(x)$

$$\Rightarrow v(x) = [N_1, \dots, N_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbb{N}\mathbb{c}$$, where $c_1, \dots, c_n$ are arbitrary coefficients.

$$\Rightarrow u'(x) = \mathbb{N}'\mathbb{a} = \mathbb{B}\mathbb{a} \quad \& \quad v'(x) = \mathbb{N}'\mathbb{c} = \mathbb{B}\mathbb{c} \quad \text{with } \mathbb{B} = \mathbb{N}'$$

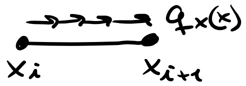
Insert into (W.F) $\Rightarrow$ normal force

$$\mathbb{c}^T \underbrace{\int_0^L \mathbb{B}^T EA\mathbb{B}\,dx}_{\mathbb{K}}\,\mathbb{a} + \mathbb{c}^T \underbrace{\mathbb{N}^T(0)N(0)}_{-\mathbb{f}_b} - \mathbb{c}^T \underbrace{\int_0^L \mathbb{N}^T q_x\,dx}_{\mathbb{f}_l} = 0$$

$$\Rightarrow \mathbb{c}^T\left[\mathbb{K}\mathbb{a} - \mathbb{f}_b - \mathbb{f}_l\right] = 0 \Rightarrow \left\{\begin{matrix}\text{since } \mathbb{c} \text{ is an} \\ \text{arbitrary vector}\end{matrix}\right\} \Rightarrow \begin{cases} \mathbb{K}\mathbb{a} = \mathbb{f}_b + \mathbb{f}_l \\ u(0) = 0 \end{cases} \qquad \text{(FE-form)}$$

b) from (a)-part we have $\mathbb{f}_e = \int_0^L \mathbb{N}^T q_x \, dx$

consider an element with two nodes ⇒



$$\mathbb{f}_e^e = \int_{x_i}^{x_{i+1}} (\mathbb{N}^e)^T q_x \, dx \Rightarrow \left\{ \begin{array}{l} \text{using Linear shape functions:} \\ \mathbb{N}^e = [N_1^e(x), N_2^e(x)] \\ N_1^e(x) = \dfrac{x_{i+1} - x}{L^e}, \quad N_2^e = \dfrac{x - x_i}{L^e} \\ L^e = x_{i+1} - x_i \\ \mathbb{B}^e = \left[-\dfrac{1}{L^e} \quad \dfrac{1}{L^e} \right] \end{array} \right\} = \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} q_0 \left(1 - 0.2\left(\frac{x}{L}\right)\right) dx$$

$$= \{\text{integration in Matlab}\}$$

$$\Rightarrow \mathbb{f}_e^e = \frac{q_0 h}{30L} \begin{bmatrix} 15L - 2x_i - x_{i+1} \\ 15L - x_i - 2x_{i+1} \end{bmatrix}$$

(where $h = L/3$ for discretization with three elements)

c) To establish $\mathbb{K}\mathbb{a} = \mathbb{f}$, we need to compute $\mathbb{K} = \int_0^L \mathbb{B}^T EA \, \mathbb{B} \, dx \Rightarrow$ assembly of $\mathbb{K}^e = \int_{x_i}^{x_{i+1}} (\mathbb{B}^e)^T EA \, \mathbb{B}^e \, dx$

$$\mathbb{K}^e = \{\text{constant } \mathbb{B}^e\} = \frac{E}{(L^e)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_i}^{x_{i+1}} A(x) \, dx = \frac{E}{(L^e)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_i}^{x_{i+1}} \frac{A_0}{2} \left(2 - \left(\frac{x}{L}\right)^2\right) dx = \left\{ \begin{array}{c} \text{integration} \\ \text{in} \\ \text{Matlab} \end{array} \right\}$$

$$= \frac{EA_0}{L^e} \left(\frac{6L^2 - x_i^2 - x_i x_{i+1} - x_{i+1}^2}{6L^2} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

with numerical values: $\mathbb{K}^1 = 4.71 \cdot 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ N/m, $\mathbb{K}^2 = 4.18 \cdot 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ N/m, $\mathbb{K}^3 = 3.11 \cdot 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ N/m

$\mathbb{f}_e^1 = \begin{bmatrix} 1.63 \\ 1.59 \end{bmatrix} \cdot 10^4$ N, $\mathbb{f}_e^2 = \begin{bmatrix} 1.52 \\ 1.48 \end{bmatrix} \cdot 10^4$ N, $\mathbb{f}_e^3 = \begin{bmatrix} 1.40 \\ 1.37 \end{bmatrix} \cdot 10^4$ N

$$\Rightarrow \begin{bmatrix} 4.71 & -4.71 & 0 & 0 \\ & 8.89 & -4.18 & 0 \\ & & 7.29 & -3.11 \\ \text{sym.} & & & 3.11 \end{bmatrix} 10^8 \cdot \begin{bmatrix} 0 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} f_{b1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.63 \\ 3.11 \\ 2.89 \\ 1.37 \end{bmatrix} \cdot 10^4 \text{ N} \Rightarrow$$

solve in Matlab

$\Rightarrow a_4 = 0.3$ mm