Examination

MHA021 Finite Element Method
VSM167 Finite Element Method – basics

Date and time:	11-01-2024, 08.30-12.30
Instructors:	Martin Fagerström (phone 1300) and Jim Brouzoulis (phone 2253). An instructor will visit the exam around 09:30 and 11:30.
Solutions:	Example solutions will be posted within a few days after the exam on the course homepage.
Grading:	The grades will be reported to the registration office on 1 February the latest.
Review:	For a review of the exam corrections, please make an appointment with your examiner.
Permissible aids:	Chalmers type approved pocket calculator. **Note**: A formula sheet is available as a pdf-file alongside with this exam thesis.

Exam instructions

All exam problems require a hand-in on paper. For some of the problems, it may be convenient to also use MATLAB including CALFEM. If you use MATLAB and CALFEM as part of your solutions, you must make sure to also hand in any MATLAB code you have written yourself. You do this by saving your files under `C:\__Exam__\Assignments\` in the appropriate sub-directories created for each problem. When doing so, it is strongly recommended that you write your anonymous exam code in any MATLAB files that you want to hand in. Finally, **it is also absolutely necessary that you write the name of your computer on the cover page for the exam!**

Note that most CALFEM files (but not all) are provided for your convenience. In addition, please note that the CALFEM function `extract.m` also exists in the CALFEM directory as `extract_dofs.m` (to avoid a conflict with a built-in MATLAB function). These CALFEM finite element files can be found under the directory `C:\__Exam__\Assignments`. You can utilize these files by copying appropriate files into the sub-directories for the problem where they are needed. Should you need to refer to the CALFEM manual, you can find this also (excluding the examples section) under `C:\__Exam__\Assignments`.

Finally, remember to close MATLAB and log-out from the computer when you are finished with the exam.

Problem 1

Consider the bar in Figure 1 that has a varying cross-sectional area A [m^2] and is subjected to a varying axially distributed load $q(x)$ [N/m]. The axial displacement $u(x)$ [m] of the bar is given as the solution to the boundary value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[E\,A\,\dfrac{\mathrm{d}u}{\mathrm{d}x}\right] = q_x \quad 0 < x < L \\ u(0) = 0 \\ N(L) = 0 \end{cases}$$

where E [Pa] is Young's modulus and $N = E\,A\,\dfrac{\mathrm{d}u}{\mathrm{d}x}$ is the normal force.

For this problem, E is a constant, $A(x) = \dfrac{A_0}{2}\left(2 - \left(\dfrac{x}{L}\right)^2\right)$ and $q_x(x) = q_0\left(1 - 0.2\left(\frac{x}{L}\right)\right)$.

Tasks:

a) **Derive the weak form and global FE-form of the problem**. **(2.0p)**

b) Consider the bar discretized into three equally long linear elements. **Derive the explicit expression for the element load vector contribution for an element with coordinates x_i and x_{i+1} resulting from the distributed load q_x**. For full point you need to perform the integration and can use either analytical or numerical integration. **(2.0p)**

c) For the same discretzation, **determine the explicit system of equations $\mathbf{K}\,\mathbf{a} = \mathbf{f}$ and determine the axial displacement at $x = L$**. Use the following numerical values: $L = 1.0\,\mathrm{m}$, $E = 80\ \mathrm{GPa}$, $A_0 = 20\,\mathrm{cm}^2$ and $q_0 = 100\ \mathrm{kN/m}$. **(2.0p)**

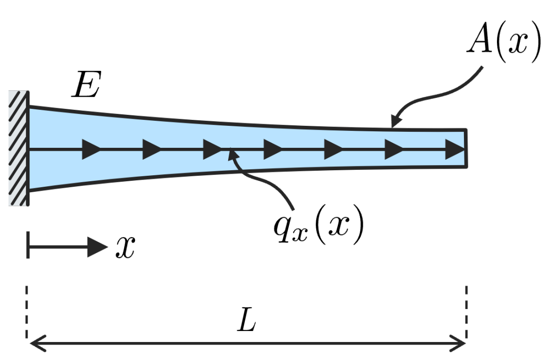

Figure 1: Axially loaded bar considered in Problem 1.

Problem 2

Consider the solid roof of a long building with a triangular cross-section and which is insulated at the bottom, see Figure 2. The roof has a height $H = 5\,\mathrm{m}$ and width $W = 10\,\mathrm{m}$ and is made of an isotropic material which is considered to obey Fourier's law $\mathbf{q} = -k\,\nabla T$ with heat conductivity k.

On a sunny winter day, the air temperature is $T_{\mathrm{air}} = -5\,°\mathrm{C}$ and the sun is shining on one of the roof sides (in blue). The heat from the sun gives rise to a measured constant heat *influx* $h = 10\,\mathrm{W/m^2}$. In addition, the temperature at the bottom right corner is measured to be $T_0 = -4\,°\mathrm{C}$. Finally, the heat transfer coefficient between the roof material and air is α_{air}.

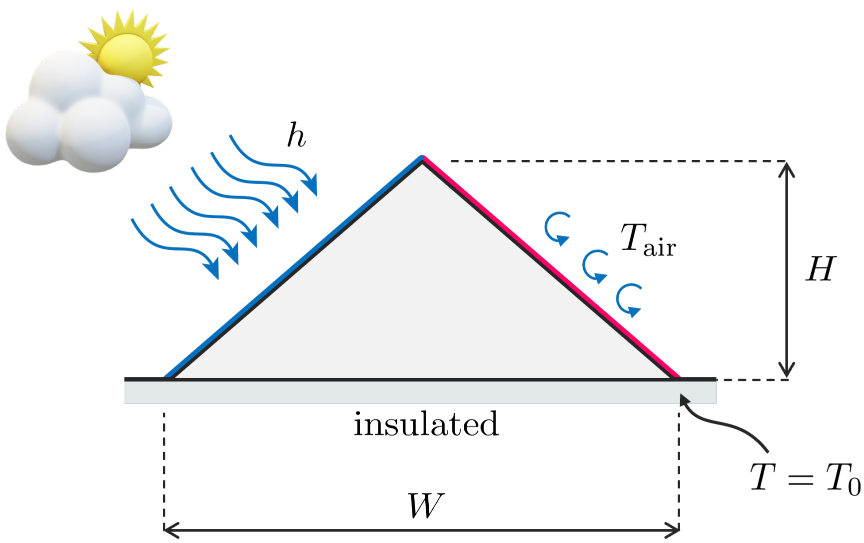

Figure 2: Sketch of the roof for Problem 2.

As the roof is long, the problem can be considered as two dimensional and the governing partial differential equation for the heat flow problem thereby becomes

$$\nabla^{\mathrm{T}}\,\mathbf{q} = 0 \quad \text{in } \Omega$$

Tasks:

a) Derive the weak form of the particular problem. Be specific in how the boundary conditions enter in the weak form. **(2.0p)**

b) By discretizing the domain into four elements (see Figure 3) and introducing the linear finite element approximation on the form $T(x, y) \approx \mathbf{N}(x, y)\,\mathbf{a}$ and using Galerkin's method, the discrete form of the problem can be obtained as:

$$[\mathbf{K} + \mathbf{K}_c]\ \mathbf{a} = \mathbf{f}_b$$

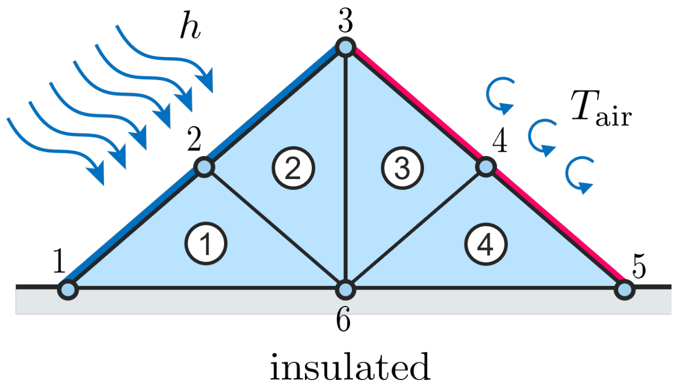

Figure 3: The roof domain in Problem 2 discretized into four elements.

Based on this, **calculate the element boundary load contribution $\mathbf{f}_b^e$ from element no 2** if the temperature is approximated with linear shape functions. Also **show how this contribution is assembled into the global boundary load $\mathbf{f}_b$. (2.0p)**

c) For the current problem discretization (with four elements) the resulting discretized system of FE-equations will be of the form:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

Partition the system into free and prescribed degrees of freedom and show how the problem should be solved when, as in this case, $T_5 = T_0$. **(2.0p)**

Problem 3

Consider a rotating blade that has been discretized using 4-node isoparametric bilinear elements, as shown in Figure 4.

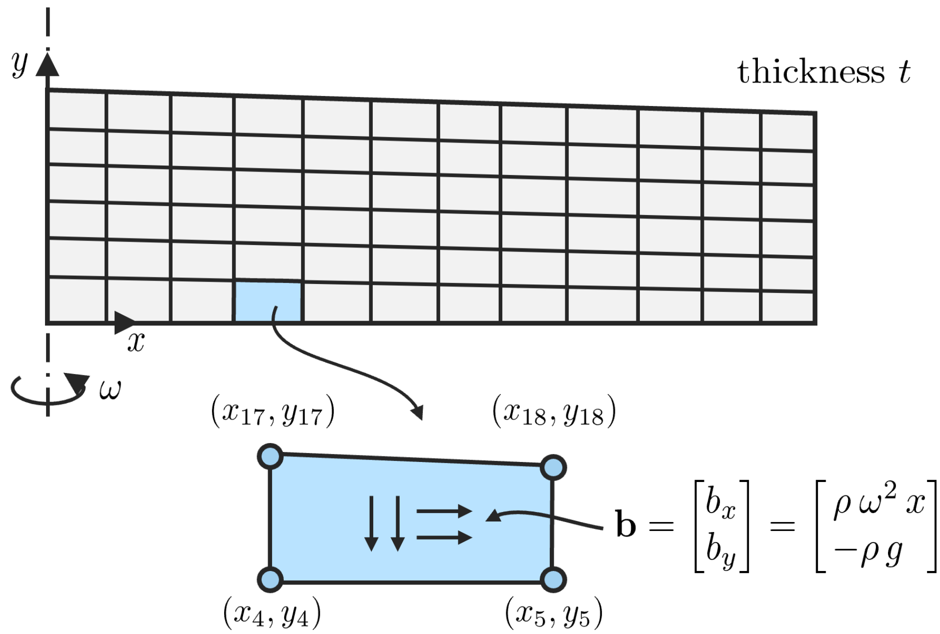

Figure 4

The 2D elasticity equation on weak form is written as:

$$\int_A \left(\tilde{\nabla}\mathbf{v}\right)^{\mathrm{T}} \mathbf{D}\,\tilde{\nabla}\mathbf{u}\,t\;\mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\,\mathbf{b}\,t\;\mathrm{d}A + \int_{\mathcal{L}_g} \mathbf{v}^{\mathrm{T}}\,\mathbf{t}\,t\;\mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \mathbf{v}^{\mathrm{T}}\,\mathbf{h}\,t\;\mathrm{d}\mathcal{L},$$

for any domain with prescribed displacements $\mathbf{u} = \mathbf{g}$ along $\mathcal{L}_g$ and prescribed tractions $\mathbf{t} = \mathbf{h}$ along $\mathcal{L}_h$. Here, the left boundary (the axis around which the blade rotates) is treated equally to a symmetry boundary of a 2D problem.

Gravity is assumed to act in the negative y-direction which gives a loading component $\rho\,g$. The rotation of the blade around the y-axis gives rise to a centrifugal loading in the positive x-axis with magnitude $\rho\,\omega^2\,x$. In summary, the volume load vector is stated as

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} \rho\,\omega^2\,x \\ -\rho\,g \end{bmatrix}.$$

Tasks on the next page

Tasks:

a) **Derive the FE form of the given problem and provide a sketch of the domain and explain any regions (domains, surfaces, edges etc.) you introduce**. Make sure to specify (in general terms) the contents of any vectors or matrices you introduce. **(2.0p)**

b) **Compute the element volume load vector $\mathbf{f}_l^e$** for the indicated element. For this task, use the following numerical values: $\rho = 7,800\,\text{kg/m}^3$, $\omega = 20$ rad/s, $g = 9.81$ m/s^2 and the following nodal coordinates (expressed in centimeters):

$$(x_4, y_4) = (3, 0), \quad (x_5, y_5) = (4, 0) \quad (x_{17}, y_{17}) = (3, 1), \quad (x_{18}, y_{18}) = (4, 0.8)$$

(1.5p)

c) **Show how $\mathbf{f}_l^e$ can be assembled into the global load vector $\mathbf{f}$** with proper consideration of a global degree of freedom numbering of your choice. **(0.5p)**

d) **Use numerical integration to compute the element area**. Please note that no points will be given to an alternative solution of the problem. **(2.0p)**