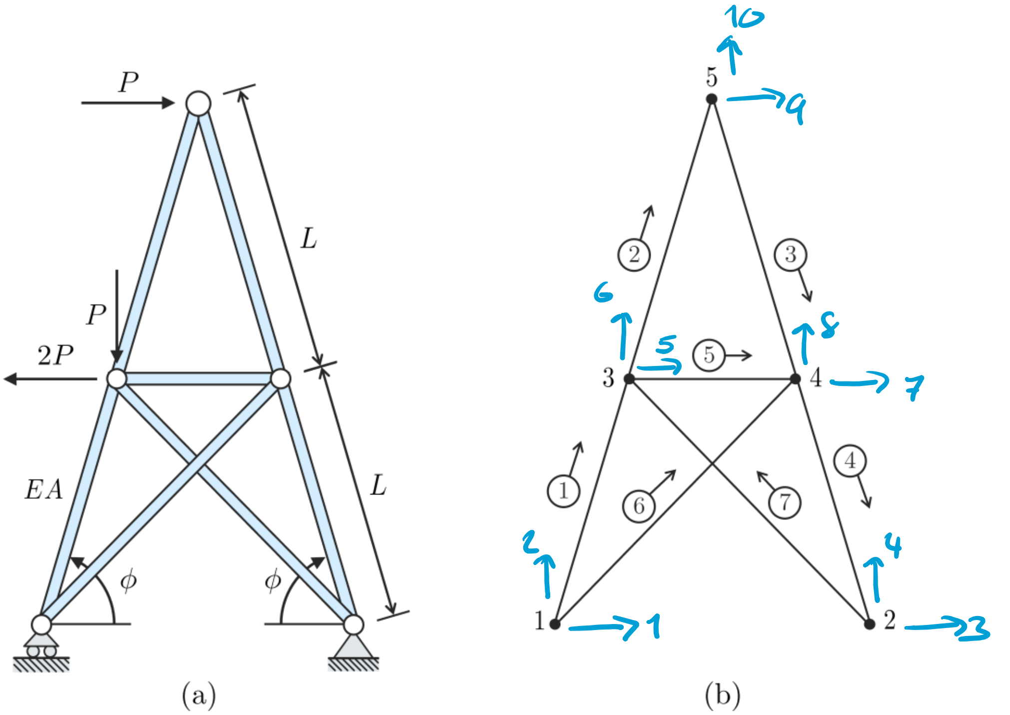
Problem 1



Task a) • dof 1, 2, 4 are prescribed to zero due to the supports

• $f_{b5} = -2P$, $f_{b6} = -P$, $f_{b9} = P$

See code for implementation

Answer: Horisontal displacement at node 5 $= a_9 \approx -0{,}0023\,m$ meaning a movement to the left.

Task b)

• Determine the normal force in each bar.

$\Rightarrow N_1, \dots, N_7$

• Compute stresses $\Rightarrow \sigma_1, \dots, \sigma_7$

$\Rightarrow |\text{max stress}| = \sigma_{max}$ (largest tensile or compressive stress)

• Scale the load such that $\sigma_{max} = \sigma_Y$

$$\Rightarrow P_Y = \frac{\sigma_Y}{\sigma_{max}(P)} \cdot P \approx 25{,}2\ kN$$

Problem 1

```
% Jim Brouzoulis, 10-04-2024
clc, clear variables, close all

phi = 60; % [degrees]
L = 1.8; % [m]

c = cosd(phi); s = sind(phi); % helper variables

% Element cooridnates
Ex = [0 L*c
L*c 2*L*c
2*L*c 3*L*c
3*L*c 4*L*c
L*c 3*L*c
0 3*L*c
4*L*c L*c];

Ey = [ 0 L*s
L*s 2*L*s
2*L*s L*s
L*s 0
L*s L*s
0 L*s
0 L*s ];

eldraw2(Ex, Ey) % Draw the truss structure

E = 210e9; % Young's modulus [Pa]
A = 2.0e-4; % cross-sectional area [m^2]
P = 13e3; % force magnitude [N]
sig_y = 250e6; % material yield limit [Pa]

%% Write you implementation below
% Anonumous code:
% a)

% Degrees of freedom numbered starting from node 1, gives the following
% topology
Edof = [1 1 2 5 6
2 5 6 9 10
3 9 10 7 8
4 7 8 3 4
5 5 6 7 8
6 1 2 7 8
7 3 4 5 6]

ep = [E A];
nel = 7;
ndofs = 10;
K = zeros(ndofs, ndofs);
f = zeros(ndofs, 1);
```

```
% Assemble stiffness matrix
for el = 1:nel
Ke = bar2e(Ex(el,:), Ey(el,:), ep);
K = assem(Edof(el, :), K, Ke);
end

% Loads and support condiions
f(5) = -2*P;
f(6) = -P;
f(9) = P;

bc = [
2 0
3 0
4 0];
% Solve equation system
[a, r] = solveq(K, f, bc);

Ed = extract_dofs(Edof, a);
hold on
eldisp2(Ex, Ey, Ed);

% Horisontal displacement in node 5 -> degree of freedom a5 & a6
hor_disp = a(9) % a6 = -0.0023 m meaning it moves to the left


%%
% b)

% Post-processing
N = zeros(nel,1);
for el = 1:nel
N(el) = bar2s(Ex(el,:), Ey(el,:), ep, Ed(el, :));
end
sigma = N/A
% maximum magnitude of the stresses among all elements
max_sig = max(abs(sigma))

n = sig_y / max_sig % factor of safety wrt yielding
P_y = n * P % Load at first yielding
```

Point distribution on Problem 1:

a) If answer is correct & code is also ⇒ 4p

Partial points:

Edof matrix: 1p

$\mathbb{K}^e$ - matrices: 0.5p

assembly of $\mathbb{K}$: 0.5p

boundary conditions: 1p

solve equation sys: 0.5p

correct answer: 0.5p // 4p

b)

If answer is correct & code is also ⇒ 2p

Partial points:

- Determine normal forces/stresses in all elements: 1p
- Compute $|\sigma_{max}|$: 0.5p
- Correctly determine P_Y: 0.5p // 2p

Problem 2

P2/ $\mathbf{u} \simeq \mathbf{u}_h = \sum_{i=1}^{n} N_i \mathbf{a}_i^e = \mathbb{N}\mathbf{a}$ with

$$\mathbf{a}_i^e = \begin{bmatrix} u_{x,i} \\ u_{y,i} \end{bmatrix}$$

$$\mathbb{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix}$$

where $N_i(x,y)$ is a 2D shape function associated with node i

$$\& \quad \mathbf{a} = \begin{bmatrix} u_{x,1} \\ u_{y,1} \\ u_{x,2} \\ u_{y,2} \\ \vdots \\ u_{x,n} \\ u_{y,n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\boldsymbol{\varepsilon} = \tilde{\nabla}\mathbf{u} \simeq \tilde{\nabla}\mathbf{u}_h = \tilde{\nabla}(\mathbb{N}\mathbf{a}) = \underbrace{(\tilde{\nabla}\mathbb{N})}_{\mathbb{B}}\mathbf{a} = \mathbb{B}\mathbf{a} \quad \text{with}$$

$$\mathbb{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

Using Galerkins method to approxi

v we obtain:

$$v \approx v_h = N c \Rightarrow v^T = c^T N^T$$

$$\tilde{\nabla} v \approx \tilde{\nabla} v_h = B c \Rightarrow (\tilde{\nabla} v_h)^T = c^T B^T$$

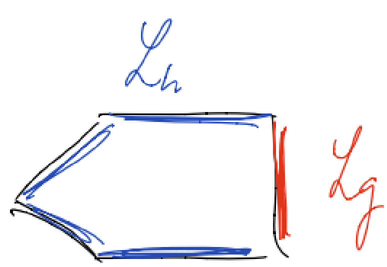
where c contains arbitrary coefficients.

Inserted in the weak form yields:

$$c^T \Big[\underbrace{\int_A B^T D B \, t \, dA \, a}_{K} - \underbrace{\int_A N^T b \, t \, dA}_{f_l} - \underbrace{\int_{L_g} N^T t \, t \, dL}_{f_b^g} - \underbrace{\int_{L_h} N^T h \, t \, dL}_{f_b^h} \Big] = 0$$

c - arbitrary yields

$$\begin{cases} K a = f_l + f_b^g + f_b^h \\ u = 0 \text{ on } L_g \\ t = h \text{ on } L_h \end{cases}$$ with



P2b) To number the degrees of freedom we use that for node k then

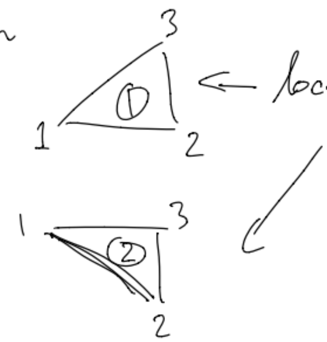
$$u_{x,k} = a_{2k-1}$$

$$u_{y,k} = a_{2k}$$

A possible form of Edof is

$$Edof = \begin{array}{c} \text{el no} \\ \left[\begin{array}{ccccccc} 1 & 7 & 8 & 9 & 10 & 15 & 16 \\ 2 & 7 & 8 & 1 & 2 & 9 & 10 \\ \vdots & & & & & & \\ \vdots & & & & & & \end{array}\right] \end{array}$$

with



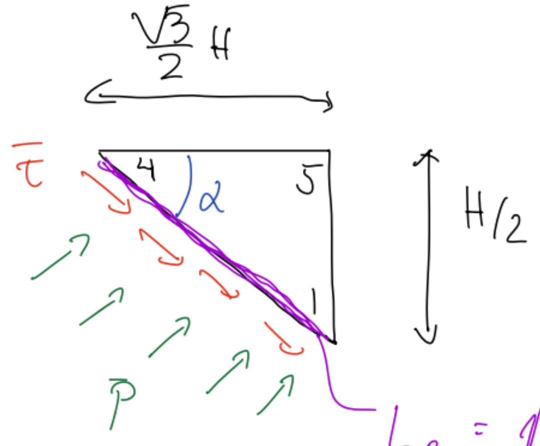
← local node numbering

Given the element stiffness matrix Ke for element i, it can be assembled into the global stiffness matrix as:

$$K(Edof(i,2:end),Edof(i,2:end)) =$$

$$K(Edof(i,2:end),Edof(i,2:end)) + Ke$$

P2c)



$\bar{\tau} = 2\ MPa$

$\bar{p} = 1\ MPa$

$$L_e = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 H^2 + \frac{H^2}{4}} = H$$

$$\alpha = 30^\circ$$

<u>Traction:</u> $\mathbb{h} = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = \begin{bmatrix} \bar{\tau}\cos 30^\circ + \bar{p}\sin 30^\circ \\ -\bar{\tau}\sin 30^\circ + \bar{p}\cos 30^\circ \end{bmatrix} =$

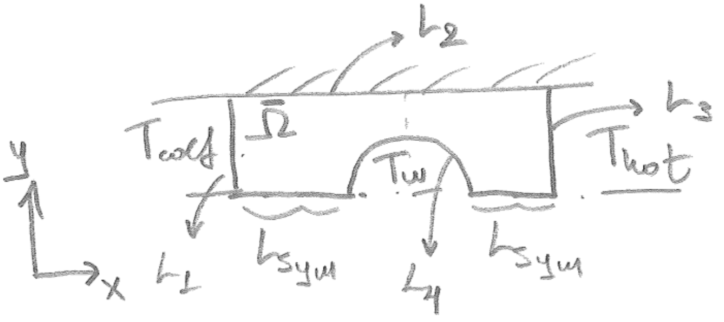
$$f_e^{l} = \int_{L^e} \mathbb{N}^T \mathbb{h}\, t\, dL = \int_{L^e} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix} t\, dL$$

assembled into

$$= \frac{L_e}{2} \begin{bmatrix} h_x \\ h_y \\ h_x \\ h_y \end{bmatrix} t = \ldots = \begin{bmatrix} 2.23 \cdot 10^6 \\ -1.34 \cdot 10^5 \\ 2.23 \cdot 10^6 \\ -1.34 \cdot 10^5 \end{bmatrix} N \begin{matrix} \longrightarrow & 1 \\ \longrightarrow & 2 \\ \longrightarrow & 7 \\ \longrightarrow & 8 \end{matrix}$$

Problem 3

a) Symmetry can be utilized, and the smallest part that can be studied is the following:



0.5 p for symmetry and defining all boundaries

Strong form:

$$
\begin{cases}
\nabla^T q = 0 \text{ in } \bar{\Omega}, \quad q = -k(\nabla T) \\
q_n = \alpha_{air}(T - T_{cold}) \text{ on } L_1 \quad \text{(convective BC)} \\
q_n = 0 \ (= q^T n) \text{ on } L_{sym} \quad \text{(symmetry nat. BC)} \\
q_n = 0 \text{ on } L_2 \quad \text{(natural BC)} \\
q_n = \alpha_w (T - T_w) \text{ on } L_4 \quad \text{(convective BC)} \\
q_n = \alpha_{air}(T - T_{hot}) \text{ on } L_3 \quad \text{(convective BC)}
\end{cases}
$$

1.5 p for all bnd.c

$\sum 2.0$ p

b) Multiply dif. eq. by test function $v(x,y)$ and integrate over the domain:

$$\int_{\Omega} v(x,y)\, \operatorname{div}(-k\nabla T)\, dA = 0 \Rightarrow \{\text{Green-Gauss}\}$$

$$\oint_L v(x,y)\underbrace{(-k\nabla T)}_{q}^T n\, dL - \int_{\Omega} (\nabla v)^T(-k\nabla T)\, dA = 0 \Rightarrow$$

$$\int_{\Omega} (\nabla v)^T k \nabla T\, dA = -\oint_L v(x,y)\, \underbrace{q^T n}_{q_n}\, dL$$

$$= -\int_{L_1} v(x,y)\, q_n\, dL - \int_{L_2} v(x,y)\, q_n\, dL +$$

$$-\int_{L_3} v(x,y)\, q_n\, dL - \int_{L_4} v(x,y)\, q_n\, dL - \int_{L_{sym}} v(x,y)\, q_n\, dL$$

<u>weak form</u>, where the boundary conditions enter into the boundary terms (see strong form).

0.5p

0.5p expand boundary terms

(c) Replacing the known (natural) BCs into the weak form, we get:

$$\int_{\Omega} (\nabla v)^T k \nabla T\, dA = -\int_{L_1} v(x,y)\, \alpha_{air}(T - T_{cold})\, dy +$$

$$-\int_{L_2} v(x,y)\, \cancelto{0}{q_n}\, dx - \int_{L_3} v(x,y)\, \alpha_{air}(T - T_{hot})\, dy +$$

$$-\int_{L_4} v(x,y)\, \alpha_w(T - T_w)\, dL - \int_{L_{sym}} v(x,y)\, \cancelto{0}{q_n}\, dx$$

1.0p combine and insert bndc

$\sum 2.0$ p

c)

FE-approximation on T:

$T(x,y) = \underline{N}(x,y)\,\underline{a}$, $\underline{a}$ are nodal temperatures

where $\underline{N}(x,y) = [N_1(x,y)\ N_2(x,y) \cdots N_n(x,y)]$

$\underline{\nabla} T = \underline{\nabla}(\underline{N}\,\underline{a}) = (\underline{\nabla}\,\underline{N})\,\underline{a} = \underline{\underline{B}}(x,y)\,\underline{a}$

where $\underline{\underline{B}}(x,y) = \underline{\nabla}\,\underline{N}$ //

0.25 p

Galerkin's method:

$v(x,y) = \underline{N}(x,y)\,\underline{c}$, $v = v^T = \underline{c}^T \underline{N}^T$

$\underline{\nabla} v = (\underline{\nabla}\,\underline{N})\,\underline{c}$, $(\underline{\nabla} v)^T = \underline{c}^T \underline{\underline{B}}^T_{(x,y)}$

0.25 p

FE-form: Replacing everything in the weak form, we get

$$\underline{c}^T\left[\int_\Omega \underline{\underline{B}}^T(x,y)\,k\,\underline{\underline{B}}(x,y)\,dA\,\underline{a}\right] =$$

$$\underline{c}^T\left[-\int_{L_1} \underline{N}^T(x,y)\,\alpha_{air}\left(\underline{N}(x,y)\,\underline{a} - T_{cold}\right)dy\ + \right.$$

$$-\int_{L_3} \underline{N}^T(x,y)\,\alpha_{air}\left(\underline{N}(x,y)\,\underline{a} - T_{hot}\right)dy\ +$$

$$\left.-\int_{L_4} \underline{N}^T(x,y)\,\alpha_w\left(\underline{N}(x,y)\cdot\underline{a} - T_w\right)dL\right] \Rightarrow$$

1.0 p if correctly inserts into the weak form

$$\{ \underline{c} \text{ is arbitrary} \}$$

$$[\underline{\underline{k}} + \underline{\underline{k}}_c]\, \underline{a} = \underline{f}_b$$

where $\underline{\underline{k}} = \int_{\Omega} \underline{\underline{B}}^T(x,y)\, k\, \underline{\underline{B}}(x,y)\, dA$

$$\left.\begin{aligned}\underline{\underline{k}}_c = &\int_{L_1} \alpha_{air} \underline{\underline{N}}^T(x,y) \underline{\underline{N}}(x,y)\, dy + \int_{L_3} \alpha_{air} \underline{\underline{N}}^T(x,y) \underline{\underline{N}}(x,y)\, dy + \\ &+ \int_{L_4} \alpha_w \underline{\underline{N}}^T(x,y) \underline{\underline{N}}(x,y)\, dL\end{aligned}\right\} \text{0.25 p}$$

$$\left.\begin{aligned}\underline{f}_b = &\int_{L_1} \alpha_{air} \underline{\underline{N}}^T(x,y) T_{cold}\, dy + \int_{L_3} \alpha_{air} \underline{\underline{N}}^T(x,y) T_{hot}\, dy + \\ &\int_{L_4} \alpha_w \underline{\underline{N}}^T(x,y) T_w\, dL\end{aligned}\right\} \text{0.25 p}$$

$\sum 2.0$ p