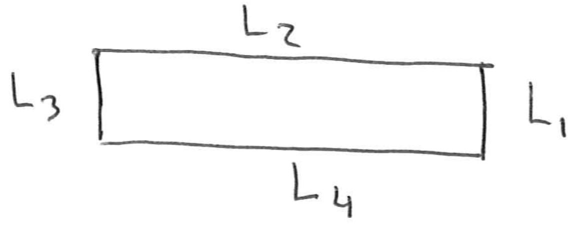
P3a) Derive the FE-form



$\mathbf{u} = \mathbf{0}$ along L_1

$\mathbf{t} = \mathbf{0}$ along L_3 & L_4

$\mathbf{t} = \begin{pmatrix} 0 \\ -h_0\left(1-\left(\frac{x}{b}\right)\right) \end{pmatrix}$ along L_2

Weak form:

$$\int_A (\tilde{\nabla}\mathbf{N})^T \mathbf{D}\,\tilde{\nabla}\mathbf{u}\; t\, dA = \int_A \mathbf{N}^T \overset{\mathbf{0}}{\mathbf{b}}\, t\, dA + \int_{\mathcal{L}_1} \mathbf{N}^T \mathbf{t}\, t\, d\mathcal{L} + \int_{\mathcal{L}_4} \mathbf{N}^T \begin{pmatrix} 0 \\ h_y \end{pmatrix} t\, d\mathcal{L}$$

FE-approx

$\mathbf{u} = \mathbf{N}\mathbf{a}$, $\mathbf{N} = \begin{bmatrix} N_1 & 0 & - & N_{15} & 0 \\ 0 & N_1 & - & 0 & N_{15} \end{bmatrix}$

$\mathbf{a} = \begin{bmatrix} u_{x,1} \\ u_{y,1} \\ | \\ u_{x,15} \\ u_{y,15} \end{bmatrix}$

$\Rightarrow$

$\tilde{\nabla}\mathbf{u} = \mathbf{B}\mathbf{a}$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & - - & \frac{\partial N_{15}}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & & 0 & \frac{\partial N_{15}}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & & \frac{\partial N_{15}}{\partial y} & \frac{\partial N_{15}}{\partial x} \end{bmatrix}$$

Galerkin's method

$$\mathbb{v} = \mathbb{N}\mathbb{c} \quad , \ \mathbb{c} - \text{arbitrary}$$

$$\tilde{\nabla}\mathbb{v} = \mathbb{B}\mathbb{c} \Rightarrow (\tilde{\nabla}\mathbb{v})^T = \mathbb{c}^T\mathbb{B}^T$$

Insert in weak form.

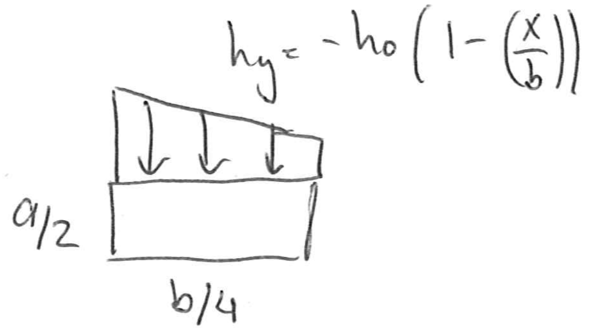
$$\mathbb{c}^T \underbrace{\int_A \mathbb{B}^T \mathbb{D}\mathbb{B}\, t\, dA}_{\mathbb{K}}\, \mathbb{a} = \mathbb{c}^T\left[\underbrace{\int_{L_1} \mathbb{N}^T \mathbb{t}\, t\, dL}_{\mathbb{f}_g} + \underbrace{\int_{L_4} \mathbb{N}^T \begin{pmatrix} 0 \\ h_y \end{pmatrix} t\, dL}_{\mathbb{f}_h}\right]$$

$$\mathbb{c}^T(\mathbb{K}\mathbb{a} - \mathbb{f}_g - \mathbb{f}_h) = 0$$

$$\Rightarrow \mathbb{K}\mathbb{a} = \mathbb{f}_g + \mathbb{f}_h$$

reaction force

3b)



Determine f_b^e for element (5) and explain how this is assembled into the global vector.

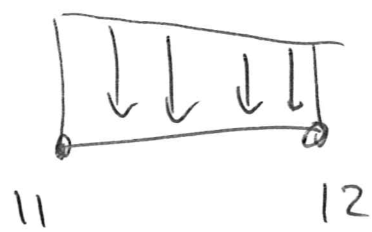
<u>Numbering scheme</u>

$$u_{x,i} = a_{2i-1}$$

$$u_{y,i} = a_{2i}$$

<u>Compute f_b^e</u>

Here, we consider only the dofs that will have non-zero force contributions:



$$f_b^e = \int_0^{b/4} \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{12} & 0 \\ 0 & N_{12} \end{bmatrix} \begin{pmatrix} 0 \\ h_y \end{pmatrix} t\, dx$$

$$= \int_0^{b/4} \begin{bmatrix} 0 \\ N_{11} h_y \\ 0 \\ N_{12} h_y \end{bmatrix} t\, dx$$

$$\int_0^{b/4} N_{11} h_y t \, dx = -h_0 t \int_0^{b/4} \left(1 - \frac{x}{b/4}\right)\left(1 - \frac{x}{b}\right) dx$$

$$= -h_0 t \int_0^{b/4} \left(1 - \frac{x}{b} - \frac{4x}{b} + \frac{4x^2}{b^2}\right) dx$$

$$= -h_0 t \left[x - \frac{5x^2}{2b} + \frac{4x^3}{3b^2} \right]_0^{b/4}$$

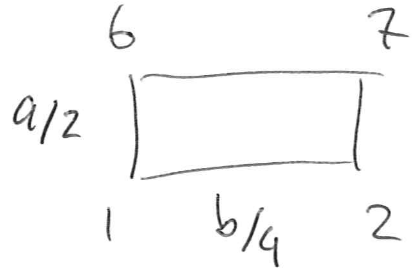
$$= -h_0 t \left[\frac{b}{4} - \frac{5b^2}{2 \cdot 16 b} + \frac{\cancel{4} b^3}{3 \cdot \cancel{4} \cdot 16 \cdot b^2} \right]$$

$$= -h_0 t \left[\frac{(16 - 10 + 2)\, b}{64} \right]$$

$$= \underline{\underline{-\frac{h_0 t b}{8}}} \quad \text{assembled into } f_{22}$$

$$\int_0^{b/4} N_{12} h_y t \, dx = \ldots = \underline{\underline{-\frac{7 h_0 t b}{64}}} \quad \text{assembled into } f_{24}$$

3c



Compute the Jacobian matrix & its determinant in the element midpoint

$$\mathbb{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$x = \mathbb{N}\hat{x} \Rightarrow \frac{\partial x}{\partial \xi} = \frac{\partial \mathbb{N}}{\partial \xi}\hat{x}, \quad \frac{\partial x}{\partial \eta} = \frac{\partial \mathbb{N}}{\partial \eta}\hat{x}$$

$$y = \mathbb{N}\hat{y} \Rightarrow \frac{\partial y}{\partial \xi} = \frac{\partial \mathbb{N}}{\partial \xi}\hat{y}, \quad \frac{\partial y}{\partial \eta} = \frac{\partial \mathbb{N}}{\partial \eta}\hat{y}$$

$$\hat{x} = \begin{bmatrix} 0 \\ b/4 \\ b/4 \\ 0 \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} 0 \\ 0 \\ a/2 \\ a/2 \end{bmatrix}$$

$$\frac{\partial N_1^e}{\partial \xi} = \frac{1}{4}(\eta - 1), \quad \frac{\partial N_1^e}{\partial \eta} = \frac{1}{4}(\xi - 1)$$

$$\frac{\partial N_2^e}{\partial \xi} = -\frac{1}{4}(\eta - 1) \qquad \frac{\partial N_2^e}{\partial \eta} = -\frac{1}{4}(\xi + 1)$$

$$\frac{\partial N_3^e}{\partial \xi} = \frac{1}{4}(\eta + 1) \qquad \frac{\partial N_3^e}{\partial \eta} = \frac{1}{4}(\xi + 1)$$

$$\frac{\partial N_4^e}{\partial \xi} = -\frac{1}{4}(\eta + 1) \qquad \frac{\partial N_4^e}{\partial \eta} = -\frac{1}{4}(\xi - 1)$$

$$\frac{\partial N}{\partial \xi}(0,0) = \left[-\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad -\frac{1}{4}\right]$$

$$\frac{\partial N}{\partial \eta}(0,0) = \left[-\frac{1}{4} \quad -\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}\right]$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N(0,0)}{\partial \xi} \hat{x} = \frac{1}{4}\frac{b}{4} + \frac{1}{4}\frac{b}{4} = \frac{2b}{16} = \frac{b}{8}$$

$$\frac{\partial x}{\partial \eta} = \sim = -\frac{1}{4}\frac{b}{4} + \frac{1}{4}\frac{b}{4} = 0$$

$$\frac{\partial y}{\partial \xi} = \sim = \frac{1}{4}\frac{a}{2} - \frac{1}{4}\frac{a}{2} = 0$$

$$\frac{\partial y}{\partial \eta} = \sim = \frac{1}{4}\frac{a}{2} + \frac{1}{4}\frac{a}{2} = \frac{2a}{8} = \frac{a}{4}$$

$$\mathbb{J} = \begin{pmatrix} \frac{b}{8} & 0 \\ 0 & \frac{a}{4} \end{pmatrix} \qquad \det(\mathbb{J}) = \frac{b}{8} \cdot \frac{a}{4} = \frac{ab}{8 \cdot 4}$$

area scaling

true element area

$$\frac{b}{4} \cdot \frac{a}{2} = \frac{ab}{8}$$

isoparametric elem. area

$$4$$

area scaling $\frac{ab}{8 \cdot 4}$ ok!