MHA021 Finite element method
Exam 2023-04-05, 14:00-18:00

Instructor: Martin Fagerström (phone 070-2248731) and Jim Brouzoulis (phone 070-2615494). Instructor will visit the exam around 15:00 and 17:00.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on Monday 24 April the latest.

Review: For a review of the exam corrections, please make an appointment with Jim Brouzoulis.

Permissible aids: Chalmers type approved pocket calculator. **Note**: A formula sheet is appended to this exam thesis.

Exam instructions

All exam problems require a hand-in on paper. For some of the problems, it may be convenient to also use MATLAB including CALFEM (see specifically Problem 3). If you use MATLAB and CALFEM as part of your solutions, you must make sure to also hand in any MATLAB code you have written yourself. You do this by saving your files under `C:\__Exam__\Assignments\VSM167` in the appropriate subdirectories created for each problem. When doing so, it is strongly recommended that you write your anonymous exam code in any MATLAB files that you want to hand in. Finally, **it is also absolutely necessary that you write the name of your computer on the cover page for the exam!**

Specifically, note that part of the solution to Problem 1 could be solved by use of MATLAB and CALFEM. Most CALFEM files (but not all) are therefore provided for your convenience. Please note that the CALFEM function `extract.m` also exists in the CALFEM directory as `extract_dofs.m` (to avoid a conflict with a built-in MATLAB function). These CALFEM finite element files can be found under the directory `C:\__Exam__\Assignments`. You can utilise these files by copying appropriate files into the subdirectories for the problem where they are needed. Should you need to refer to the CALFEM manual, you can find this also (excluding the examples section) under `C:\__Exam__\Assignments`.

Finally, remember to close MATLAB and log-out from the computer when you are finished with the exam.

Problem 1

Consider the following boundary value problem:
Find $u(x)$ in $0 \le x \le L$ such that

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[k\,\dfrac{\mathrm{d}u(x)}{\mathrm{d}x}\right] + v_x\dfrac{\mathrm{d}u(x)}{\mathrm{d}x} = f_0 \\ u(0) = u_0 \\ u(L) = 0 \end{cases}$$

where $k > 0$, $v_x \neq 0$, $u_0 \neq 0$ and f_0 are given constants.

Tasks:

(a) Staring from the boundary value problem above, **derive the weak form and the global FE-form.** Note that integration by parts should not be applied to the last term $v_x\dfrac{\mathrm{d}u(x)}{\mathrm{d}x}$. Make sure to be specific on the structure and contents of any matrices you introduce. (**3p**)

(b) Assume the problem is discretized using linear shape functions and consider one element with the nodes $x = x_i$ and $x = x_{i+1}$, where $0 < x_i < x_{i+1} < L$. From the global FE-form, **derive the local FE-form and show that the element stiffness matrix and load vector become**

$$\mathbf{K}^e = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{v_x}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{f}^e = \frac{f_0\,h}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $h = x_{i+1} - x_i$. (**3p**)

Problem 2

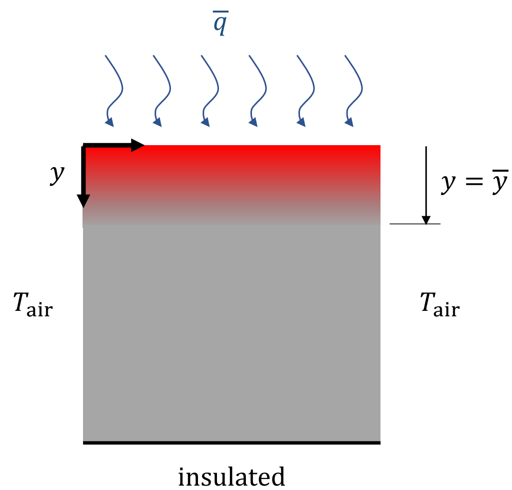

Figure 1: Sketch of the cutting tool considered in Problem 2.

Consider a 2D model of a cutting tool for orthogonal machining shown in Figure 1. To handle the very high loads and temperatures at the tool surface, the tool is made from a so-called Functionally Graded Material (FGM) where the material is changing its properties as illustrated by the colour changes in the image. At the top part of the tool ($y < \overline{y}$), both Young's modulus E and the thermal conductivity k varies with the distance form the surface (coordinate y) as:

$$E(y) = E_0 \left(1 - 0.2 \left(\frac{y}{\overline{y}}\right)^2\right), \quad k(y) = k_0 \left(1 - 0.3 \left(\frac{y}{\overline{y}}\right)^2\right),$$

with $E_0 = 200$ GPa and $k_0 = 200\,\mathrm{W/(m\,°C)}$.

The FGM is considered point-wise isotropic such that the heat flow obeys Fourier's law

$$\mathbf{q} = -k(y)\nabla T,$$

where ∇T is the temperature gradient.

Due to the friction between the cutting tool and the work piece during the cutting, the tool is subjected to a constant net influx of heat ($\overline{q}$) across its upper boundary (see figure). Furthermore, the left and right hand sides can be considered to obey convective condition with heat transfer coefficient α whereas the bottom surface can be considered as insulated.

For a heat transfer analysis, the 2D domain is discretised into rectangular, four-noded quadrilateral elements with side lengths $a = 5\,\mathrm{mm}$ and $b = 6\,\mathrm{mm}$, as shown in Figure 2.

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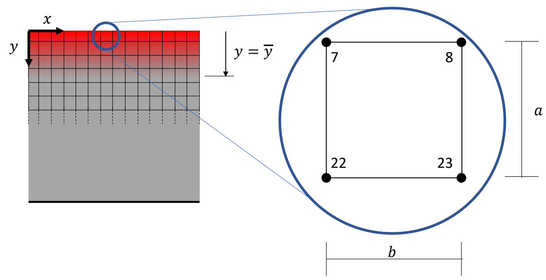

Figure 2: Discretisation of cutting tool considered in Problem 2.

Tasks:

(a) Staring from the heat balance of a two-dimensional body with thickness t, **derive and state the strong form of the heat balance equation that describes the current problem.** (**1.0p**)

(b) The general weak form of 2D heat flow can be written as

$$\int_A (\nabla v)^T \mathbf{D} \nabla T \, t \, \mathrm{d}A = \int_A v Q t \mathrm{d}A - \int_{\mathcal{L}} v q_n t \mathrm{d}\mathcal{L},$$

where v is an arbitrary weight function, $\mathbf{D} = k\mathbf{1}$ is the material heat conductivity matrix, ∇T is the temperature gradient, Q is external heat supplied to the surface and q_n is the heat outflux across the outer boundary $\mathcal{L}$. With this at hand, **specify the explicit weak form for current problem**. (**1.0p**)

(c) Start from the problem specific weak form determined in (b), introduce a global finite element approximation and **derive the corresponding FE form specific to this problem**. Make sure to be specific on the structure and contents of any matrices you introduce. (**1.5p**)

(d) Considering the numerical integration of the element conductivity matrix described using isoparametric mapping as:

$$\mathbf{K}^e = \int_{-1}^{-1} \int_{-1}^{-1} \mathbf{B}^{eT} \mathbf{D} \mathbf{B}^e \det(\mathbf{J}) \, \mathrm{d}\xi \, \mathrm{d}\eta$$

propose and motivate an appropriate integration scheme for numerical integration. Please present the integration scheme in terms of location of integration points (in the parent domain) and their corresponding weights. (**1.0p**)

(e) Calculate the heat flow vector at the midpoint of the edge between nodes 22 and 23 given the following temperatures at the nodes:

$$T_7 = 304 \text{ °C}, \quad T_8 = 306 \text{ °C}, \quad T_{22} = 294 \text{ °C}, \quad T_{23} = 296 \text{ °C}.$$

(**1.5p**)

Problem 3

Consider the 2D elasticity toy problem shown in Figure 3. The simplified domain is considered to be under a state of plane strain.

For this problem, the general finite element form of the equilibrium equation is given by

$$\int_A \mathbf{B}^T \mathbf{D} \mathbf{B} \, t \, \mathrm{d}A = \int_A \mathbf{N}^T \mathbf{b} \, t \, \mathrm{d}A + \int_{\mathcal{L}} \mathbf{N}^T \mathbf{t} \, t \, \mathrm{d}\mathcal{L},$$

where $\mathbf{B}$ is the global 2D elasticity shape function derivative matrix, $\mathbf{D}$ is the plane strain Hooke's elasticity matrix, t is the thickness, $\mathbf{N}$ is the global 2D elasticity shape function matrix, $\mathbf{b}$ are distributed body loads and $\mathbf{t}$ are surface tractions on the outer boundary $\mathcal{L}$.

Fort he current problem, please consider the following data:

$$E = 200\,\mathrm{MPa}, \quad \nu = 0.3, \quad t = 4\,\mathrm{mm}, \quad a = 5\,\mathrm{mm}, \quad \tau = 100\,\mathrm{MPa}.$$

For your convenience, the nodal coordinates have been specified in the MATLAB script file `problem3.m` that is available on the computer.

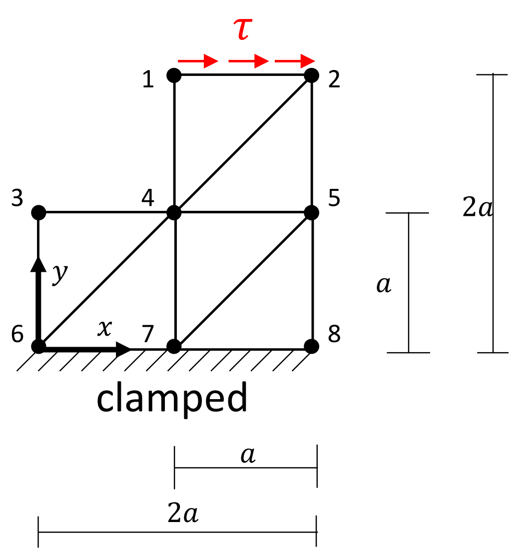

Figure 3: Simple 2D plane strain domain, discretised with linear triangular finite elements, to be considered in Problem 3.

Tasks on the next page

Tasks:

(a) Propose a numbering scheme for the degrees-of-freedom and explain how the numbering of these relate to the numbering of nodes. Please also **provide a sketch of the domain where the degree of freedom numbering is shown**. (**0.5p**)

(b) Explain a procedure for how element stiffness matrices $\mathbf{K}^e$ and element volume load vectors $\mathbf{f}_l^e$ are assembled into the global stiffness matrix $\mathbf{K}$ and the global load vector $\mathbf{f}$ for a 2D elasticity problem. When doing so, please define a suitable matrix representation for how to link global and local degrees of freedom for each element and show how this is used in the assembly. (**1.5p**)

(c) Compute the load vector contribution from the applied shear traction on the top surface of the domain, and explain how this is included in the global load vector. (1.0p)

(d) Write a small finite element programme that solves the problem. Specifically report u_x and u_y for node 2. Here, you may use any CALFEM files you want. (**3.0p**)