

Problem 1

Consider the following boundary value problem:

Find $u(x)$ in $0 \leq x \leq L$ such that

$$\begin{cases} -\frac{d}{dx} \left[k \frac{du(x)}{dx} \right] + v_x \frac{du(x)}{dx} = f_0 \\ u(0) = u_0 \\ u(L) = 0 \end{cases}$$

where $k > 0$, $v_x \neq 0$, $u_0 \neq 0$ and f_0 are given constants.

Tasks:

(a) Starting from the boundary value problem above, **derive the weak form and the global FE-form**. Note that integration by parts should not be applied to the last term $v_x \frac{du(x)}{dx}$. Make sure to be specific on the structure and contents of any matrices you introduce. (3p)

(b) Assume the problem is discretized using linear shape functions and consider one element with the nodes $x = x_i$ and $x = x_{i+1}$, where $0 < x_i < x_{i+1} < L$. From the global FE-form, **derive the local FE-form and show that the element stiffness matrix and load vector become**

$$\mathbf{K}^e = \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{v_x}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{f}^e = \frac{f_0 h}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $h = x_{i+1} - x_i$. (3p)

a) Multiply d.e. with an arbitrary test function $v(x)$ and integrate over the domain $\Rightarrow$

$$-\int_0^L v(x) (k u'(x))' dx + \int_0^L v(x) v_x u'(x) dx = \int_0^L v(x) f_0 dx$$

Use I.B.P. on the first term $\Rightarrow$

$$\begin{aligned} -\int_0^L v (k u')' dx &= \int_0^L v' k u' dx - [v k u']_0^L = \\ &= \int_0^L v' k u' dx - v(L) k u'(L) + v(0) k u'(0) \end{aligned}$$

⇒ Weak form:

Find $u(x)$ such that

$$\begin{cases} \int_0^L v' k u' dx + \int_0^L v \cdot v_x u' dx = v(L) k u'(L) - v(0) k u'(0) + \int_0^L v f_0 dx \\ u(0) = u_0 \\ u(L) = 0 \end{cases}$$

FE-form

Introduce approximations: $u(x) \approx u_h(x) = \sum_{i=1}^n N_i(x) a_i = [N_1, \dots, N_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$$\Rightarrow u_h'(x) = (N a)' = N' a + N a' \underset{=0}{=} N' a = \left[\frac{dN_1}{dx}, \dots, \frac{dN_n}{dx} \right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = B a$$

Use the same shape functions N to describe the test function

$$v(x) = \sum_{i=1}^n N_i(x) c_i = [N_1, \dots, N_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = N c, \text{ where } c_i \text{ are arbitrary constants}$$

$$\Rightarrow v'(x) = (N c)' = N' c + N c' \underset{=0}{=} N' c = B c$$

Insert into W.F. ⇒

$$c^T \int_0^L B^T k B dx a + c^T \int_0^L N^T v_x B dx a = c^T N^T(L) k u'(L) - c^T N^T(0) k u'(0) + c^T \int_0^L N^T f_0 dx$$

$$\Rightarrow c^T (\text{FE-form}) = 0$$

must be equal to 0 since c is arbitrary ⇒

$$\underbrace{\left[\int_0^L B^T k B dx + \int_0^L N^T v_x B dx \right]}_{K} a = \underbrace{N^T(L) k u'(L) - N^T(0) k u'(0) + \int_0^L N^T f_0 dx}_f$$

$$\text{or } K a = f$$

b) Use linear approximation $\Rightarrow \{\text{formula sheet}\} \Rightarrow$

$$\begin{cases} N_1^e(x) = -\frac{1}{h}(x - x_{i+1}) \\ N_2^e(x) = \frac{1}{h}(x - x_i) \end{cases} \quad \text{with } h = x_{i+1} - x_i$$

$$\Rightarrow u^e(x) = [N_1^e \ N_2^e] \begin{bmatrix} a_1^e \\ a_2^e \end{bmatrix} = N^e a^e \Rightarrow B^e = \begin{bmatrix} \frac{dN_1^e}{dx} & \frac{dN_2^e}{dx} \end{bmatrix} = \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} \end{bmatrix}$$

Localize the global FE-form over one element $\Rightarrow$

$$k^e = \int_{x_i}^{x_{i+1}} (B^e)^T h B^e dx + \int_{x_i}^{x_{i+1}} (N^e)^T v_x B^e dx$$

$$= \int_{x_i}^{x_{i+1}} \begin{bmatrix} -\frac{1}{h} \\ \frac{1}{h} \end{bmatrix} \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} \end{bmatrix} dx + \int_{x_i}^{x_{i+1}} \begin{bmatrix} -\frac{1}{h}(x - x_{i+1}) \\ \frac{1}{h}(x - x_i) \end{bmatrix} \begin{bmatrix} \frac{1}{h} & \frac{1}{h} \end{bmatrix} dx \cdot v_x = \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{h} \underbrace{\int_{x_i}^{x_{i+1}} \begin{bmatrix} \frac{x - x_{i+1}}{h} \\ \frac{x - x_i}{h} \end{bmatrix} dx}_{(*)} \begin{bmatrix} 1 & 1 \end{bmatrix} v_x$$

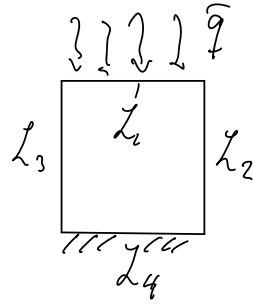
$$= \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{h} \begin{bmatrix} h/2 \\ h/2 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} v_x = \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{v_x}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$f^e = \int_{x_i}^{x_{i+1}} (N^e)^T f_0 dx = \underbrace{\int_{x_i}^{x_{i+1}} \begin{bmatrix} -\frac{(x - x_{i+1})}{h} \\ \frac{(x - x_i)}{h} \end{bmatrix} dx}_{\text{same as } (*)} \cdot f_0 = \begin{bmatrix} h/2 \\ h/2 \end{bmatrix} f_0 = \frac{f_0 h}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 2

g Heat balance of 2D body

$$\int_A Q_t dA = \oint_L q_n t dL$$



$$\left\{ \oint_L q_n t dL = \text{div theorem} = \int_A \text{div}(t q_f) dA \right\}$$

$$\Rightarrow \int_A Q_t dA = \int_A \text{div}(t q_f) dA \Rightarrow$$

$$\int_A -\text{div}(t q_f) + Q_t dA = 0$$

should hold for arbitrary domain $\Rightarrow$

$$-\text{div}(t q_f) + Q_t = 0 \quad \text{0.5p}$$

Add BCs

$$\left. \begin{array}{l} q_n = -\bar{q} \text{ along } L_1 \\ q_n = \alpha(T - T_{\text{air}}) \text{ along } L_2 \cup L_3 \\ q_n = 0 \text{ along } L_4 \end{array} \right\} \text{0.5p}$$

strong form

b)

General weak form:

$$\int_A (\nabla v)^T \mathbb{D} \nabla T \, t \, dA = \int_A v Q \, t \, dA - \int_L v q_n \, t \, dL$$

For this problem:

$$Q = 0$$

$$q_n = -\bar{q} \text{ along } L_1$$

$$q_n = \alpha(T - T_{\infty}) \text{ along } L_2 \cup L_3$$

$$q_n = 0 \text{ along } L_4$$

$\Rightarrow$

$$\int_A (\nabla v)^T \mathbb{D} \nabla T \, t \, dA = \int_{L_1} v \bar{q} \, t \, dL - \int_{L_2 \cup L_3} v \alpha (T - T_{\infty}) \, t \, dL$$

c)

$$T \approx N a_1 \text{ with } N = [N_1 \, N_2 \, \dots \, N_n]$$

$$\Rightarrow \nabla T = (\nabla N) a_1 = B a_1$$

Galerkin's method

$$v = N c$$

$$\nabla v = B c$$

$$a_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ (arbitrary)}$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

Insert in weak form:

$$c^T \int_A B^T D B t dA a = c^T \left[\int_{L_1}^{L_2} N^T \bar{q} t dL - \int_{L_2}^{L_1} \alpha N^T (N a - T_{ax}) t dL \right]$$

$$\Leftrightarrow c^T \left[\underbrace{\left(\int_A B^T D B t dA + \int_{L_2}^{L_1} \alpha N^T N t dL \right)}_{K + K_C} a - \underbrace{\left(\int_{L_2}^{L_1} \alpha N^T T_{ax} t dL + \int_{L_1}^{L_2} N^T \bar{q} t dL \right)}_{f_b} \right] = 0$$

c^T - arbitrary $\Rightarrow$

$$(K + K_C) a - f_b = 0 \Rightarrow$$

$$(K + K_C) a = f_b \quad \text{FE-form}$$

$$d_s \quad K^e = \int_{-1}^1 \int_{-1}^1 B^{eT} D B^e \det(J) d\xi d\eta$$

$B^e \sim \text{linear}$

$D \sim \text{quadratic}$

$\det(J) \sim \text{constant (for this shape)}$

$\Rightarrow$ integral of order $\sim 4 \Rightarrow$ 3 intgr points in each direction
($2 \cdot 3 - 1 = 5$)

Integration points (9 in total) from formula sheet.

$$\xi_1 = -0.7746 \quad \eta_1 = -0.7746$$

$$\text{weight} = (0.5556)^2$$

$$\xi_2 = -0.7746 \quad \eta_2 = 0$$

$$\text{weight} = 0.5556 \times 0.8889$$

$$\xi_3 = -0.7746 \quad \eta_3 = 0.7746$$

$$\text{weight} = (0.5556)^2$$

$$\xi_4 = 0 \quad \eta_4 = -0.7746$$

$$\text{weight} = 0.5556 \times 0.8889$$

$$\xi_5 = 0 \quad \eta_5 = 0$$

$$\text{weight} = 0.8889^2$$

$$\xi_6 = 0 \quad \eta_6 = 0.7746$$

$$\text{weight} = 0.5556 \times 0.8889$$

$$\xi_7 = 0.7746 \quad \eta_7 = -0.7746$$

$$\text{weight} = 0.5556^2$$

$$\xi_8 = 0.7746 \quad \eta_8 = 0$$

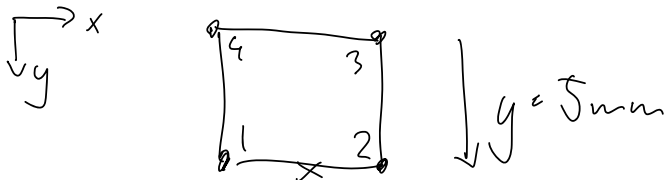
$$\text{weight} = 0.5556 \times 0.8889$$

$$\xi_9 = 0.7746 \quad \eta_9 = 0.7746$$

$$\text{weight} = 0.5556^2$$

c) Heat flow

$$q = -k \nabla T = -k \nabla \bar{\theta}^e$$



$$N_1^e = \frac{-1}{ab} (x - x_2)(y - y_4) \Rightarrow \frac{\partial N_1^e}{\partial x} = -\frac{1}{ab} (y - y_4)$$

$$\frac{\partial N_1^e}{\partial y} = -\frac{1}{ab} (x - x_2)$$

$$N_2^e = \frac{1}{ab} (x-x_1)(y-y_3) \Rightarrow \frac{\partial N_2^e}{\partial x} = \frac{1}{ab} (y-y_3)$$

$$\frac{\partial N_2^e}{\partial y} = \frac{1}{ab} (x-x_1)$$

$$N_3^e = \frac{1}{ab} (x-x_2)(y-y_2) \Rightarrow \frac{\partial N_3^e}{\partial x} = \frac{1}{ab} (y-y_2)$$

$$\frac{\partial N_3^e}{\partial y} = \frac{1}{ab} (x-x_2)$$

$$N_4^e = \frac{1}{ab} (x-x_3)(y-y_1) \Rightarrow \frac{\partial N_4^e}{\partial x} = \frac{1}{ab} (y-y_1)$$

$$\frac{\partial N_4^e}{\partial y} = \frac{1}{ab} (x-x_3)$$

$$B^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial x} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_4^e}{\partial x} & \frac{\partial N_4^e}{\partial y} \end{bmatrix}$$

$$\text{MATLAB} \Rightarrow H = \begin{bmatrix} -6.44 \\ 38.7 \end{bmatrix} \times 10^4 \text{ W/m}^2 \quad (\text{with } \bar{y} = 15 \text{ mm})$$

```
a = 5e-3;
b = 6e-3;
y = a;
x = b/2;
ybar = 3*a; %unfortunately not given in the problem description
k0 = 200;

x7 = 0; y7 = 0;
x8 = b; y8 = 0;
x22 = 0; y22 = a;
x23 = b; y23 = a;

x1 = x22; y1 = y22;
x2 = x23; y2 = y23;
x3 = x8; y3 = y8;
x4 = x7; y4 = y7;

Be = 1/(a*b)*[-(y-y4), (y-y3), -(y-y2), (y-y1);
              -(x-x2), (x-x1), -(x-x4), (x-x3)];
ae = [294;296;306;304];

k = k0*(1-0.3*(y/ybar)^2);

q = -k*Be*ae
```

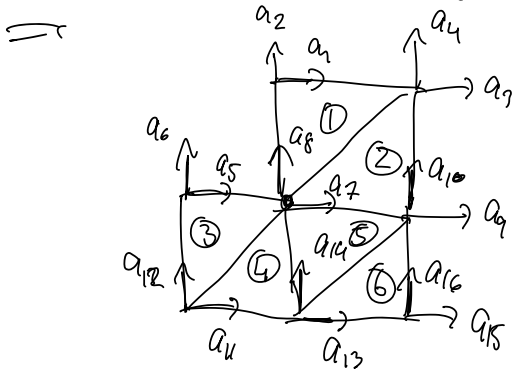
Problem 3

a) Numbering scheme for degrees of freedom.

For node k

$$u_{x,k} = a_{2k-1}$$

$$u_{y,k} = a_{2k}$$



b)

K^e and f^e can be assembled into the global K and f using a topology matrix E_{def} :

$$E_{def} = \begin{matrix} & \text{elno} & \text{def1} & \text{def2} & \text{def3} & \text{def4} & \text{def5} & \text{def6} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 1 & 2 & 7 & 8 & 3 & 4 \\ 2 & 3 & 4 & 7 & 8 & 9 & 10 \\ 3 & 5 & 6 & 11 & 12 & 7 & 8 \\ 4 & 11 & 12 & 13 & 14 & 7 & 8 \\ 5 & 13 & 14 & 9 & 10 & 7 & 8 \\ 6 & 13 & 14 & 15 & 16 & 9 & 10 \end{bmatrix} \end{matrix}$$

For element 1 this can be done in MATLAB:

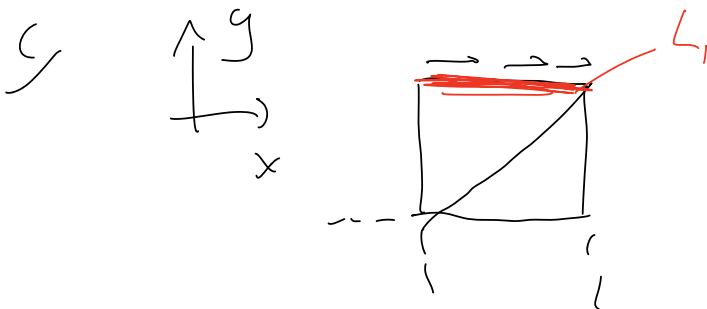
$$K(E_{def}(i, 2:7), E_{def}(i, 2:7)) =$$

$$K(E_{def}(i, 2:7), E_{def}(i, 2:7)) + K^e$$

$$f(E_{def}(i, 2:7)) = f(E_{def}(i, 2:7)) \quad \begin{array}{l} \uparrow \\ \text{element stiffness} \\ \text{for element } i \end{array}$$

$$+ f_b^e$$

$\uparrow$
element load vector
for element i



$$f_b^e = \int_{L_1} N^T t t dL$$

$$t = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

$$t = 4 \text{ mm}$$

$$= \int_{L_1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \tau \\ 0 \end{bmatrix} t dL = \frac{\sigma t}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{added} \\ \text{to} \\ \rightarrow f_1 \\ \rightarrow f_2 \\ \rightarrow f_3 \\ \rightarrow f_4 \end{array}$$

$$= \frac{5.4 \cdot 100}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \varepsilon \cdot 10^3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix}$$

d) see next page

For node 2:

$$u_x \approx 1.73 \text{ mm}$$

$$u_y \approx -5.39 \mu\text{m}$$

```
a = 5e-5;
t = 4e-3;
ptype = 2;
E = 200e9;
nu = 0.3;

Coord = [
    a, 2*a;
    2*a, 2*a;
    0, a;
    a, a;
    2*a, a;
    0, 0;
    a, 0;
    2*a, 0];

Dof = [2*[1:8]'-1 2*[1:8]'];

nen = 3;

Edof = [1 1 2 7 8 3 4;
        2 3 4 7 8 9 10;
        3 5 6 11 12 7 8;
        4 11 12 13 14 7 8;
        5 13 14 9 10 7 8;
        6 13 14 15 16 9 10];

bc = [11 0;
      12 0;
      13 0;
      14 0;
      15 0;
      16 0];

[Ex,Ey] = coordxtr(Edof,Coord,Dof,nen);

D = hooke(ptype,E,nu);
K = zeros(max(max(Edof)));
f = zeros(max(max(Edof)),1);

for i=1:size(Edof,1)
    [Ke,fe] = plante(Ex(i,:),Ey(i,:),[ptype t],D,[0;0]);
    [K,f] = assem(Edof(i,:),K,Ke,f,fe);
end

f(1) = f(1) + 1e3;
f(3) = f(3) + 1e3;

[a,Q]=solveq(K,f,bc);

disp('For node 2:')
```

```
ux = a(3)  
uy = a(4)
```