MHA021 Finite Element Method
Exam 2022-04-13, 14:00-18:00

Instructor: Jim Brouzoulis (phone 070-261 5494). The instructor will visit the exam around 15:00 and 17:00.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on May 3rd 2022 the latest.

Review: It will be possible to review the grading at the Division of Dynamics (floor 3 in M-building) on May 3rd 12:30-13:00.

Permissible aids: Chalmers type approved pocket calculator and MATLAB with the CALFEM toolbox. On the computer, you can find the CALFEM manual (excluding the examples section) and the CALFEM finite element method functions. **Note**: A formula sheet (as a pdf) is also available on the computer.

Problem 1

An igloo, as shown in Figure 1a, has the shape of a half-sphere with inner and outer radius r_{in} and r_{out}, respectively. Let T_{in} denote the temperature on the inside, while T_{out} is the temperature of the outside. It is assumed that the heat transfer between the air and the wall is convective, both on the inside and the outside. The convection constant is denoted α and is assumed to be same on both inside and outside.
The wall temperature $T(r)$ can then be estimated as the solution to the boundary value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r}\left[k\,r^2\,\dfrac{\mathrm{d}T(r)}{\mathrm{d}r}\right] = 0 & r_1 < r < r_2 \\ q(r_{\text{in}}) = -\alpha\,(T(r_{\text{in}}) - T_{\text{in}}) \\ q(r_{\text{out}}) = \alpha\,(T(r_{\text{out}}) - T_{\text{out}}) \end{cases} \tag{1}$$

Here, k is the thermal conductivity and

$$q(r) = -k\,\frac{\mathrm{d}T(r)}{\mathrm{d}r}$$

is the heat flux according to Fourier's law.

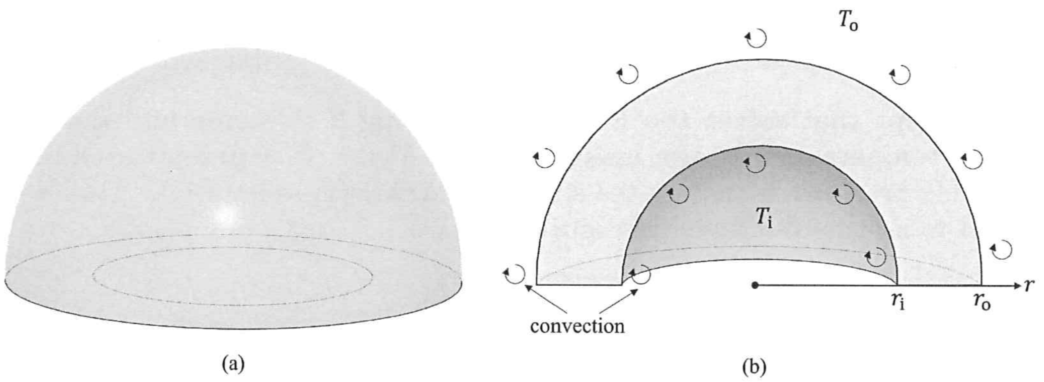

Figure 1: (a) Igloo to be analysed in Problem 1. (b) Section through the igloo.

Tasks on the next page!

Problem 2

According to Euler-Bernoulli theory, beam bending is described by the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E\,I\,\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right) = q(x)$$

where $w(x)$ is the transverse displacement, $q(x)$ is the intensity of the distributed load along the beam, and $E\,I(x)$ is the bending stiffness.
Tasks:

(a) **Derive the weak formulation and make a finite element formulation of the problem.** Assume that the problem is defined on the interval $0 < x < L$, but leave the boundary conditions unspecified. **(2.0p)**

(b) A simple beam element has four basis functions ($0 \le x \le L$):

$$N_1^e = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$
$$N_2^e = x\left(1 - 2\frac{x}{L} + \left(\frac{x}{L}\right)^2\right)$$
$$N_3^e = \left(\frac{x}{L}\right)^2\left(3 - 2\frac{x}{L}\right)$$
$$N_4^e = x\left(\frac{x}{L}\right)\left(\frac{x}{L} - 1\right)$$

Show that the element is complete. (1.0p)

Hint:

The basis functions have been constructed such that

$$N_1^e(0) = 1 \qquad N_2^e(0) = 0 \qquad N_3^e(0) = 0 \qquad N_4^e(0) = 0$$

$$N_1^e(L) = 0 \qquad N_2^e(L) = 0 \qquad N_3^e(L) = 1 \qquad N_4^e(0) = 0$$

And the derivatives

$$\frac{\mathrm{d}N_1^e}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}N_2^e}{\mathrm{d}x} = 1 \qquad \frac{\mathrm{d}N_3^e}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}N_4^e}{\mathrm{d}x} = 0 \quad \text{at } x = 0$$

$$\frac{\mathrm{d}N_1^e}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}N_2^e}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}N_3^e}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}N_4^e}{\mathrm{d}x} = 1 \quad \text{at } x = L$$

Since the constants in $\boldsymbol{c}$ are arbitrary the equation above can only be fulfilled if

$$(\boldsymbol{K}+\boldsymbol{K}_{\mathrm{c}})\,\boldsymbol{a}=\boldsymbol{f}_{\mathrm{c}}$$

where

$$\boldsymbol{K}=\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\boldsymbol{B}^{\mathrm{T}}k\,r^2\,\boldsymbol{B}\,\mathrm{d}r$$
$$\boldsymbol{K}_{\mathrm{c}}=\alpha\,r_{\mathrm{in}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_{\mathrm{in}})\,\boldsymbol{N}(r_{\mathrm{in}})+\alpha\,r_{\mathrm{out}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_{\mathrm{out}})\,\boldsymbol{N}(r_{\mathrm{out}})$$
$$f_{\mathrm{c}}=\alpha\,r_{\mathrm{in}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_1)\,T_{\mathrm{in}}+\alpha\,r_{\mathrm{out}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_{\mathrm{out}})\,T_{\mathrm{out}}$$

These are the global FE-equations.

(c)

The element stiffness matrix is determined by integrating over one element, defined between r_i and r_{i+1}.

$$\boldsymbol{K}^e=\int_{r_i}^{r_{i+1}}\boldsymbol{B}^{\mathrm{T}}k\,r^2\,\boldsymbol{B}\,\mathrm{d}r$$

Using linear shape functions (as given in the problem description), the $\boldsymbol{B}$-matrix becomes

$$\boldsymbol{B}^e=\frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}r}=\begin{bmatrix}\frac{\mathrm{d}N_1^e}{\mathrm{d}r} & \frac{\mathrm{d}N_2^e}{\mathrm{d}r}\end{bmatrix}=\begin{bmatrix}-\frac{1}{h^e} & \frac{1}{h^e}\end{bmatrix}$$

and consequently

$$\boldsymbol{K}^e=\int_{r_i}^{r_{i+1}}r^2\,\mathrm{d}r\,\frac{k}{(h^e)^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\frac{k(r_{i+1}^3-r_i^3)}{3(h^e)^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

(d)

Temperature on the inside wall is 2.30 C. [See separate code for solution] where

$$\boldsymbol{K} = \int_0^L \boldsymbol{B}^{\mathrm{T}} E I \boldsymbol{B} \, \mathrm{d}x$$

$$\boldsymbol{f} = (\boldsymbol{N}'(L))^{\mathrm{T}} M(L) - (\boldsymbol{N}'(0))^{\mathrm{T}} M(0) - (\boldsymbol{N}(L))^{\mathrm{T}} T(L) + (\boldsymbol{N}'(0))^{\mathrm{T}} T(0) + \int_0^L \boldsymbol{N}^{\mathrm{T}} q \, \mathrm{d}x$$

Therefore, non trivial solutions are obtained when $\boldsymbol{K}\,\boldsymbol{a} = \boldsymbol{f}$ – this is the FE-formulation.

(b)

For a beam element to be complete, the element approximation needs to be able to represent a constant deflection w, constant first derivate w', and constant second derivative w''.
Consider an element approximation (superscript e is implied).

$$w(x) = N_1\,a_1 + N_2\,a_2 + N_3\,a_3 + N_4\,a_4$$

Constant w

Select $a_1 = a_3 = c$ and $a_2 = a_4 = 0$

$$w = N_1(x)\,c + N_3(x)\,c = (N_1(x) + N_3(x))\,c = 1 \cdot c = c$$

For this choice of $a_1 - a_4$, w becomes an arbitrary constant function.

Constant w'

$$w'(x) = N_1'\,a_1 + N_2'\,a_2 + N_3'\,a_3 + N_4'\,a_4$$

$$\frac{\mathrm{d}N_1}{\mathrm{d}x} = -\frac{6\,x\,(L-x)}{L^3}$$
$$\frac{\mathrm{d}N_2}{\mathrm{d}x} = \frac{L^2 - 4\,L\,x + 3\,x^2}{L^2}$$
$$\frac{\mathrm{d}N_3}{\mathrm{d}x} = \frac{6\,x\,(L-x)}{L^3}$$
$$\frac{\mathrm{d}N_4}{\mathrm{d}x} = -\frac{x\,(2\,L - 3\,x)}{L^2}$$

Select $a_2 = a_4 = c$, $a_1 = 0$ and $a_3 = c\,L$

$$w' = N_2'(x)\,c + N_3'(x)\,c\,L + N_4'(x)\,c = (N_2'(x) + N_3'(x)\,L + N_4'(x))\,c = 1 \cdot c = c$$

For this choice of $a_1 - a_4$, w becomes an arbitrary constant function.

Problem 3

Consider a hexagonal concrete part of thickness 0.75 m and with a uniform side length $L = 2$ m subjected to an external pressure $p = 0.4$ MPa as shown in Figure 3.

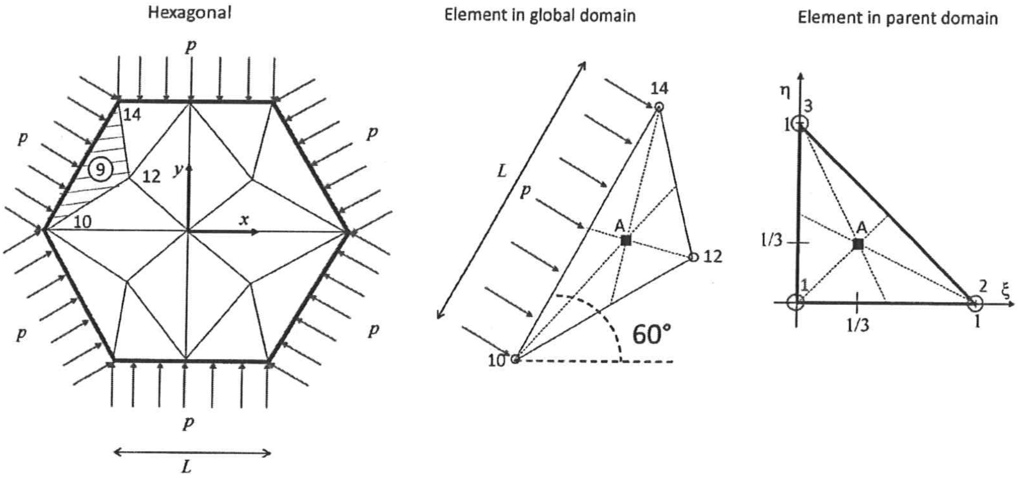

Figure 3: Left: Hexagonal part in Problem 3. Middle: Particular element (no 9) in global domain to be considered in tasks a)-c) where global node numbers are indicated in black. Left: Isoparametric three noded element in the parent domain to be considered in tasks b)-c) where local node numbers are indicated in red.

The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\mathbf{v}\right)^{\mathrm{T}} \mathbf{D}\tilde{\nabla}\mathbf{u}\, t \,\mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\mathbf{b}\, t \,\mathrm{d}A + \int_{\mathcal{L}_g} \mathbf{v}^{\mathrm{T}}\mathbf{t}\, t \,\mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \mathbf{v}^{\mathrm{T}}\mathbf{h}\, t \,\mathrm{d}\mathcal{L}$$

for any domain A with prescribed displacements $\mathbf{u} = \mathbf{g}$ along $\mathcal{L}_g$ and prescribed tractions $\mathbf{t} = \mathbf{h}$ along $\mathcal{L}_h$.

(a) For the resulting FE-problem ($\boldsymbol{Ka} = \boldsymbol{f}$), **calculate the contribution to the global load vector $\boldsymbol{f}$ from the pressure p along the boundary of the dashed element no 9** (also shown in the middle figure) considering the global node numbering in the figure. For full point, both specify the values and the positions of the non-zero contributions in the global load vector. **(2.0p)**

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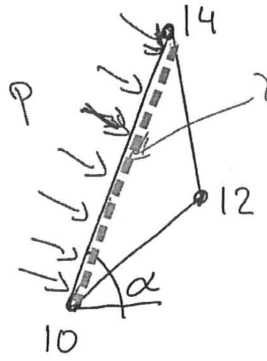
3a)



Calculate the contribution to $\mathbb{f}$ from the pressure p along the edge between nodes 10 & 14.

The contribution comes as a boundary contribution

$$\mathbb{f}_b = \int_{\mathcal{L}} N^T \mathbb{t} \, t \, d\mathcal{L} \quad , t - \text{thickness}$$

For the particular edge γ

0.25p

$$\mathbb{f}_b^{\gamma} = \int_{\mathcal{L}} \begin{bmatrix} N_{10} & 0 \\ 0 & N_{10} \\ N_{14} & 0 \\ 0 & N_{14} \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix} t \, d\mathcal{L} = \begin{Bmatrix} t_x = p \sin\alpha \\ t_y = -p\cos\alpha \end{Bmatrix}$$

0.25p

$$= \int_{\mathcal{L}} \begin{bmatrix} N_{10} & 0 \\ 0 & N_{10} \\ N_{14} & 0 \\ 0 & N_{14} \end{bmatrix} d\mathcal{L} \begin{bmatrix} pt\sin\alpha \\ -pt\cos\alpha \end{bmatrix} = \left\{ \int_{\mathcal{L}} N_{10} d\mathcal{L} = \int_{\mathcal{L}} N_{14} d\mathcal{L} = L_e/2 \text{ with} \right.$$

$$L_e = \sqrt{(x_{14}-x_{10})^2 + (y_{14}-y_{10})^2}$$

$$\mathbb{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad \mathbb{N} = \begin{bmatrix} N_1^e(\xi,\eta) & N_2^e(\xi,\eta) & N_3^e(\xi,\eta) \end{bmatrix}$$

$$N_1^e = 1-\xi-\eta$$

$$N_2^e = \xi$$

$$N_3^e = \eta$$

0.25p

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1^e}{\partial \xi} x_1^e + \frac{\partial N_2^e}{\partial \xi} x_2^e + \frac{\partial N_3^e}{\partial \xi} x_3^e$$

$$= -1 \cdot x_{10} + 1 \cdot x_{12} + 0 \cdot x_{14} =$$

$$\frac{\partial x}{\partial \eta} = \ldots = -x_{10} + x_{14} =$$

$$\frac{\partial y}{\partial \xi} = \ldots = -y_{10} + y_{12} =$$

$$\frac{\partial y}{\partial \eta} = \ldots = -y_{10} + y_{14} =$$

0.5p

c

$$\varepsilon = \tilde{\nabla} u = \tilde{\nabla} \mathbb{N}^e a^e = \mathbb{B}^e a^e$$

$$\mathbb{B}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & 0 & \frac{\partial N_3^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_3^e}{\partial x} \end{bmatrix}$$