Institutionen för tillämpad mekanik, Chalmers tekniska högskola

TENTAMEN I FINIT ELEMENTMETOD — MHA 021

18 DECEMBER 2007

Tid och plats:	$8^{\underline{30}} - 12^{\underline{30}}$ i M–huset
Hjälpmedel:	Ordböcker, lexikon och typgodkänd räknare.
Lärare:	Peter Möller, tel (772) 1505. Besöker sal ca. $9^{\underline{30}}$ samt $11^{\underline{15}}$.
Lösningar:	Anslås på anslagstavlan vid ingången till institutionens lokaler senast 19/12.
Betygsättning:	En fullständig och korrekt lösning på en uppgift ger poäng enligt vad som anges på uppgiftslappen. Smärre fel leder till poängavdrag. Ofullständig lösning (svar på ställt problem saknas) eller omfattande fel ger inte något poäng. Maximal poäng är 20. Det krävs 8 poäng för betyg 3; 12 poäng ger betyg 4; för betyg 5 krävs 16 poäng. **Observera att ovanstående är betygssättning på enbart tentamen; för godkänd examination krävs dessutom godkända inlämningsuppgifter.**
Resultatlista:	Anslås senast 10/1 på samma ställe som lösningarna. Resultaten sänds till betygsexpeditionen senast 16/1 — **för kursdeltagare som inte har alla inlämningsuppgifter godkända vid detta tillfälle inrapporteras betyget U (underkänd).**
Granskning:	fredag 11/1 13–15 samt måndag 21/1 12–15.

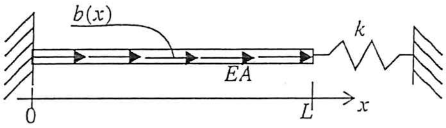
<u>Tänk på:</u>

- Skriv så att den som ska rätta, kan läsa och förstå hur du tänker. Den som rättar tentamen gissar inte eller antar inte vad du menar/tänker — endast vad som verkligen skrivs har betydelse vid poängsättningen.
- Förklara/definiera införda beteckningar.
- Rita tydliga figurer. Ange i förekommande fall vad som är positiva/negativa riktingar (på t.ex förskjutningar och krafter).
- Gör du antaganden utöver de som anges i uppgiftstexten, så ange detta explicit och förklara dessa.

<u>Note for master programme students</u>: English thesis text starts on page 4 — illustrations are found on pages 2 and 3.

1

Betrakta stångproblemet som illustreras i figuren. Stångens axiella förskjutning $u(x)$ ges av lösningen till randvärdesproblemet



$$\begin{cases} -\dfrac{d}{dx}\left[EA\dfrac{du}{dx}\right] = b & 0<x<L \\ u(0) = 0 & \\ \dfrac{du}{dx}(L) + \dfrac{k}{EA}u(L) = \dfrac{P}{EA} & \end{cases}$$

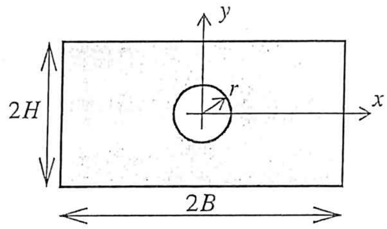
a: Variationsformulera problemet. Det ska framgå hur randvillkoren kommer in i variationsproblemet. Ange också regularitetskrav på ingående funktioner. (2p)

b: Finit–elementformulera problemet med testfunktioner (viktsfunktioner) enligt Galerkin. (2p)

c: Om axialstyvheten EA är konstant fås elementstyvhetsmatrisen $K^e = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ för ett element med längden h och lineära basfunktioner. Antag att fjäderstyvheten är $k = \frac{EA}{L}$ och att $b = \frac{P}{L}$; ställ upp ekvationssystemet $Ka = f$ för fallet att problemet löses med två lika långa sådana element. Det ska framgå hur de två randvillkoren påverkar ekvationssystemet. (3p)

2

Betrakta ett rektangulärt område innehållande ett hål och ett koordinatsystem (x, y), sådant att geometrin är symmetrisk med avseende på såväl x som y. Området är utsatt för en temperaturbelastning $T(x, y)$ som också är symmetrisk med avseende på koordinataxlarna, dvs vi har att $T(x, y) = T(-x, y) = T(x, -y)$. Den svaga formen (variationsproblemet) av detta elasticitetsproblem kan skrivas



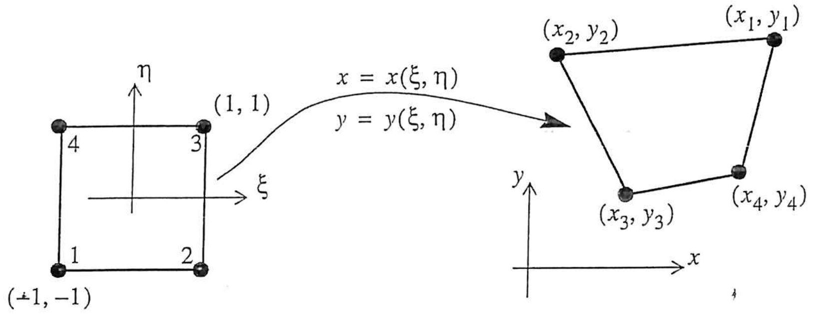
$$\int_A (\tilde{\nabla}v)^T D(\tilde{\nabla}u)dA = \int_A \alpha \cdot \Delta T(\tilde{\nabla}v)^T D\begin{bmatrix}1\\1\\0\end{bmatrix}dA \qquad \tilde{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^T$$

där α är längdutvidgningskoefficienten, D Hooke–matrisen, $\Delta T = T - T_0$ (T_0 är den temperatur vid vilken konstruktionen är spänningsfri), v innehåller testfunktionerna (viktsfunktionerna), och $u = \begin{bmatrix} u_x & u_y \end{bmatrix}^T$ är de obekanta förskjutningarna i respektive koordinatriktning.

a: FE–formulera problemet med testfunktioner enligt Galerkins metod. Det ska framgå hur den obekanta förskjutningsvektorn u approximeras samt hur man får tillräckligt många ekvationer för att bestämma samtliga obekanta nodvariabler. Visa också hur B^e–matrisen ser ur för ett element med n noder. (3p)

b: För att spara beräkningsarbete (eller få bättre nogrannhet för ett givet arbete) kan man utnyttja dubbelsymmetrin. Skissa upp en fjärdedel av området och ange i figuren samtliga randvillkor som gäller för problemet. (2p) c: Antag att vi ska använda isoparametriska bi–lineära element för att diskretisera problemet.

Visa hur de derivator som behövs för att ställa upp B^e beräknas. Du behöver inte ange explicita uttryck för bas– eller formfunktioner, men tänk på att definiera alla beteckningar du inför. (2p)



d: För att den isoparametriska avbildningen ska bli entydig, krävs att $\det J \neq 0$; vilka begränsningar ställer detta på tillåtna elementgeometrier. (1p)

3

Betrakta Poissons ekvation på ett område A med homogena Dirichlet villkor på randen Γ:

$$\begin{cases} -\text{div}(D\nabla u) = f(x, y) & \text{i } A \\ u = 0 & \text{på } \Gamma \end{cases}$$

där den konstitutiva matrisen D är symmetrisk och positivt definit. Om vi definierar $a(u, v) = \int_A (\nabla v)^T D(\nabla u) dA$ och $(v, f) = \int_A vf dA$, så kan variationsproblemet skrivas: Bestäm $u \in V$ så att

$$a(u, v) = (v, f) \qquad \forall v \in V$$

där V är rummet av alla funktioner som är noll på Γ samt tillräckligt reguljära. Finita–elementproblemet kan på samma sätt skrivas: Bestäm $u_h \in V_h$ så att

$$a(u_h, v) = (v, f) \qquad \forall v \in V_h$$

där FE–rummet V_h innehåller alla funktioner som kan skrivas som en lineärkombination av basfunktionerna.

Givet att $V_h \subset V$ visar man enkelt att diskretiseringsfelet $e = u - u_h$ är energiortogonalt mot V_h, dvs att $a(e, v) = 0 \ \forall v \in V_h$ (Galerkin ortogonalitet).

a: Använd Galerkin ortogonaliteten för att visa att energin i felet är lika med felet i energi: $a(e, e) = a(u, u) - a(u_h, u_h)$. (2p)

b: $V_h \subset V$ innebär att FE–approximationen säkert konvergerar — detta kräver att elementen är kompletta och kompatibla. Vad menas med komplett respektive kompatibel? Vilken är den enklast möjliga FE–approximationen u_h på ett element? (3p)

1

Consider the one dimensional elasticity problem depicted in the illustration. The axial deflection $u(x)$ of the bar is obtained as the solution of the boundary value problem

$$\begin{cases} -\dfrac{d}{dx}\left[EA\dfrac{du}{dx}\right] = b & 0 < x < L \\ u(0) = 0 \\ \dfrac{du}{dx}(L) + \dfrac{k}{EA}u(L) = \dfrac{P}{EA} \end{cases}$$

a: Make a variational formulation of the problem; it should be shown how the boundary conditions affect the formulation. Also state regularity conditions of the involved functions. (2p)

b: Derive a FE–formulation of the problem with test (weight) functions according to the Galerkin method. (2p)

c: If the axial stiffness EA is constant, the element stiffness matrix for a linear element with length h becomes $K^e = \dfrac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Assume that the spring stiffness is $k = \dfrac{EA}{L}$ and that $b = \dfrac{P}{L}$; establish the system of equations $Ka = f$ in the case two such elements of equal lengths are used to approximate u. It should be shown how the two boundary conditions affect the equations. (3p)

2

Consider a rectangular domain containing a circular cavity and a Cartesian coordinate system (x, y), such that the geometry is symmetric with respect to x– as well as the y–axis. The domain is subjected to a temperature distribution $T(x, y)$ that also is symmetric with respect to the coordinate axes, i.e. we have that $T(x, y) = T(-x, y) = T(x, -y)$. The weak form of this elasticity problem may be written

$$\int_A (\tilde{\nabla}v)^T D(\tilde{\nabla}u)dA = \int_A \alpha \cdot \Delta T(\tilde{\nabla}v)^T D\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} dA \qquad \tilde{\nabla} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}^T$$

where α is the coefficient of thermal expansion, D is the Hooke matrix, $\Delta T = T - T_0$ (where T_0 is the temperature at which the domain is stress free), the vector v contains the test (weight) functions, and $u = \begin{bmatrix} u_x & u_y \end{bmatrix}^T$ are the unknown displacements in the respective coordinate directions.

a: Make a FE–formulation of the problem with test functions according to the Galerkin method. It should be shown how the unknown displacement vector u is approximated and how one obtains a sufficient number of equations to solve for the node variables. Also show what the B^e–matrix looks like for an element with n nodes. (3p)

b: To reduce the computational effort (for a given accuracy), one should make use of the symmetry. Draw a figure of a quarter of the domain and state all boundary conditions for the problem. (2p)

c: Assume that we want to use isoparametric bi–linear elements to discretize the problem. Show how the derivatives needed to establish B^e are calculated. You do not have to give explicit expressions for the basis and shape functions, but define all the notation that you introduce. (2p)

d: To have a unique isoparametric mapping, one needs to have $\det J \neq 0$. What restrictions does this requirement put on allowable element geometries? (1p)

3

Consider the Poisson equation on a domain A and with homogenous Dirichlet conditions on the boundary Γ:

$$\begin{cases} -\mathrm{div}(D\nabla u) = f(x,y) & \text{i } A \\ u = 0 & \text{på } \Gamma \end{cases}$$

where the constitutive matrix D is symmetric and positive definite. Define $a(u,v) = \int_A (\nabla v)^T D(\nabla u) dA$ and $(v,f) = \int_A vf dA$, so that the variational problem may be written: Find $u \in V$ such that

$$a(u,v) = (v,f) \qquad \forall v \in V$$

where V is the space of functions that vanish on Γ ant that are sufficiently regular. The finite element problem may similarly be expressed as: Find $u_h \in V_h$ such that

$$a(u_h, v) = (v,f) \qquad \forall v \in V_h$$

where the finite element space V_h consists of all functions that are linear combinations of the basis function.

Given that $V_h \subset V$, it is easy to show that the discretization error $e = u - u_h$ is energy orthogonal to V_h, i.e. that $a(e,v) = 0 \ \forall v \in V_h$ (a.k.a. Galerkin orthogonality).

a: Use the Galerkin orthogonality to prove that the energy in the error equals the error in energy: $a(e,e) = a(u,u) - a(u_h, u_h)$. (2p)

b: $V_h \subset V$ ensures that the FE approximation converges; this requires that the elements are complete and compatible. What does complete and compatible, respectively, mean? Which is the simplest possible FE approximation u_h on an element? (3p)

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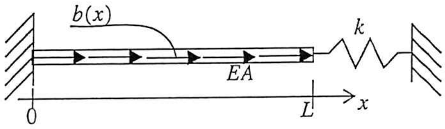

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<u>**Lösning 1a:**</u> Multiplicera differentialekvationen med en testfunktion $v(x)$ och integrera över intervallet

$$-\int_0^L v\frac{d}{dx}\left[EA\frac{du}{dx}\right]dx = \int_0^L vb\,dx$$

Partialintegration av vänsterledet ger

$$\int_0^L EA\frac{dv}{dx}\frac{du}{dx}dx = \int_0^L vb\,dx + v(L)\left[EA\frac{du}{dx}\right]_{x=L} - v(0)\left[EA\frac{du}{dx}\right]_{x=0}$$

I den första randtermen kan vi sätta in randvillkoret vid $x = L$, men $\dfrac{du}{dx}$ är obekant vid $x = 0$ så vi begränsar våra val av testfunktioner till de som uppfyller $v(0) = 0$. Funktionerna måste vidare vara kvadratiskt integrerbara och ha kvadratiskt integrerbar första derivata. Om vi definierar

$$V = \left\{ v\colon\ v(0) = 0 \quad \int_0^L \left(v^2 + \left(\frac{dv}{dx}\right)^2\right)dx < 0 \right\}$$

Så kan variationsproblemet skrivas: Hitta $u \in V$ så att $\displaystyle\int_0^L EA\frac{dv}{dx}\frac{du}{dx}dx + kv(L)u(L) = \int_0^L vb\,dx + Pv(L) \quad \forall v \in V$.

Lösning 1b: Approximera u med en lineärkombination av basfunktioner: $u \approx u_h = \sum_i N_i(x)a_i = Na$,

där $N = [N_1(x)\ N_2(x)\ \ldots\ N_n(x)]$ är en radvektor med basfunktionerna och $a = [a_1\ a_2\ \ldots\ a_n]^T$ innehåller nodvariablerna. Insättning i variationsproblemet ger

$$\left(\int_0^L EA\frac{dv}{dx}\frac{dN}{dx}dx + kv(L)N(L)\right)a - \int_0^L vbdx - Pv(L) = 0$$

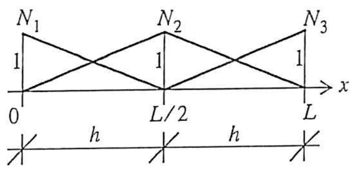
Testfunktionerna väljs nu som basfunktionerna $v = N_1, N_2, \ldots, N_n$, dvs Galerkins metod, vilket ger n ekvationer som kan användas för att beräkna nodvariablerna a; detta är ekvivalent med att sätta $v = c^T N^T$, där c är godtycklig. Insättning ger

$$c^T\left\{\left(\int_0^L EA\left(\frac{dN}{dx}\right)^T\frac{dN}{dx}dx + kN^T(L)N(L)\right)a - \int_0^L N^T bdx - PN^T(L)\right\} = 0$$

För att denna likhet ska vara uppfylld för godtyckligt vald vektor c, så måste uttrycket innanför klammer–parentesen vara en nollvektor; med beteckningen $B = \frac{dN}{dx}$ har vi alltså

$$\left(\int_0^L EAB^T B dx + kN^T(L)N(L)\right)a = \int_0^L N^T bdx + PN^T(L)$$

Lösning 1c: Med $N = [N_1\ N_2\ N_3]$, $a = [a_1\ a_2\ a_3]^T$ och den givna elementindelningen får vi $N(L) = [0\ 0\ 1]$, så med $k = \frac{EA}{L}$ har vi att



$$kN^T(L)N(L) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{EA}{L} \end{bmatrix} \text{ och } PN^T(L) = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix}$$

Vidare, med $b = \frac{P}{L}$, blir

$$\int_0^L N^T bdx = \frac{P}{L}\int_0^L \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dx = \frac{P}{L}\begin{bmatrix} \frac{1}{2}\cdot 1\cdot h \\ \frac{1}{2}\cdot 1\cdot 2h \\ \frac{1}{2}\cdot 1\cdot h \end{bmatrix} = \begin{bmatrix} \frac{P}{4} \\ \frac{P}{2} \\ \frac{P}{4} \end{bmatrix}$$

där vi beräknade integralerna som arean mellan respektive basfunktions graf och x–axeln. Vi har också givet att

$$\int_0^L EAB^{\mathrm{T}}B\,dx = \frac{EA}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{EA}{h}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Vi får alltså ekvationssystemet

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{P}{4}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Randvillkoret $u(0) = 0$ innebär att vi måste sätta $a_1 = 0$ varvid 1a kolumnen i styvhetsmatrisen 'försvinner'; vidare kan vi inte använda 1a ekvationen, eftersom den har fåtts genom att välja testfunktion $v = N_1$ och $N_1(0) \neq 0$, dvs N_1 uppfyller inte randvillkoret vid $x = 0$ vilket vi kräver eftersom vi strukit motsvarande randterm (se uppgift 1a). Ekvationssystemet reduceras alltså till

$$\frac{EA}{L}\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \frac{P}{4}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

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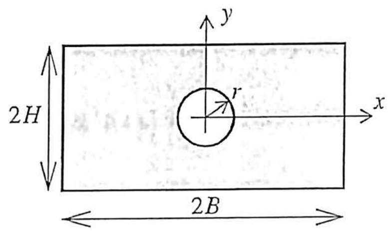

$$\int_A (\tilde{\nabla}v)^{\mathrm{T}} D(\tilde{\nabla}u)\,dA = \int_A \alpha \cdot \Delta T (\tilde{\nabla}v)^{\mathrm{T}} D \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} dA \qquad \tilde{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^{\mathrm{T}}$$

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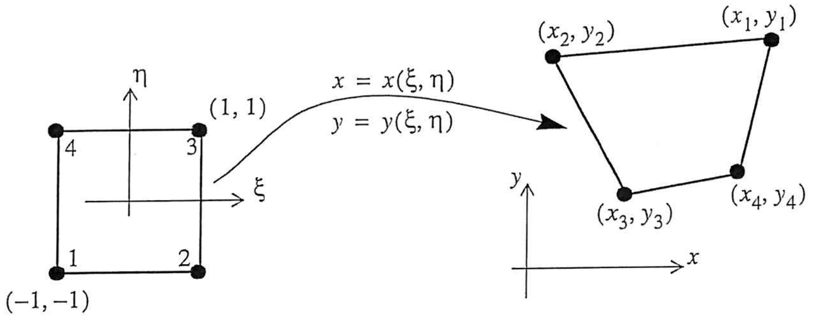

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Lösning 2a: Approximera $u_x \approx u_{xh} = a_{x1}N_1 + a_{x2}N_2 + \ldots + a_{xn}N_n$ och $u_y \approx u_{yh} = a_{y1}N_1 + a_{y2}N_2 + \ldots + a_{yn}N_n$. Om vi samlar de $2n$ nodvariablerna i en vektor a och basfunktionerna i en matris N enligt

$$a = \begin{bmatrix} a_{x1} & a_{y1} & a_{x2} & a_{y2} & \ldots & a_{xn} & a_{yn} \end{bmatrix}^T \qquad N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \ldots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \ldots & 0 & N_n \end{bmatrix}$$

kan vi skriva $u \approx u_h = \begin{bmatrix} u_{xh} \\ u_{yh} \end{bmatrix} = Na$. Approximationen sätts in i variationsproblemet och vi väljer sedan $v = \begin{bmatrix} v_x & v_y \end{bmatrix}^T$ på $2n$ olika sätt, så att vi får lika många ekvationer som obekanta nodvariabler. Galerkins metod: välj $v = \begin{bmatrix} \mathrm{N}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{N}_1 \end{bmatrix}, \begin{bmatrix} \mathrm{N}_2 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{N}_n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{N}_n \end{bmatrix}$. Insättning ger nu

$$\int_A (\tilde{\nabla}N)^T D(\tilde{\nabla}N)dA a = \int_A \alpha \cdot \Delta T(\tilde{\nabla}N)^T D \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} dA$$

För ett element med m noder, samlar vi de $2m$ nodvariablerna i en vektor

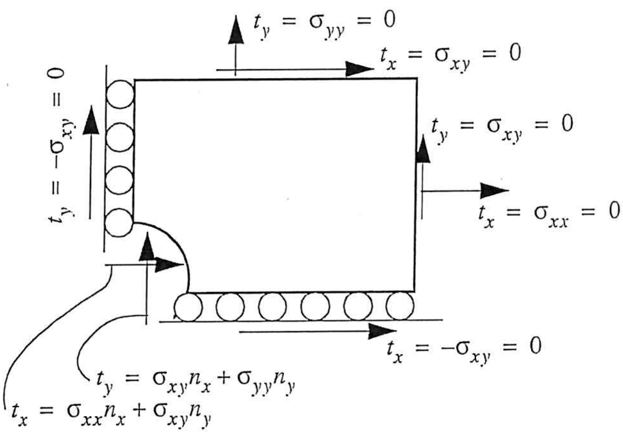
$$a^e = \begin{bmatrix} a^e_{x1} & a^e_{y1} & \ldots & a^e_{xm} & a^e_{ym} \end{bmatrix}^T$$ och motsvarande basfunktioner i matrisen $$N^e = \begin{bmatrix} N^e_1 & 0 & \ldots & N^e_m & 0 \\ 0 & N^e_1 & \ldots & 0 & N^e_m \end{bmatrix}$$, så

att approximationen på elementet kan skrivas $u_h = N^e a^e$. Vi har då $\tilde{\nabla}u_h = \tilde{\nabla}(N^e a^e) = B^e a^e$, där alltså

$$
B^{e} = \tilde{\nabla} N^{e} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \dots & \frac{\partial N_m^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & \dots & 0 & \frac{\partial N_m^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \dots & \frac{\partial N_m^e}{\partial y} & \frac{\partial N_m^e}{\partial x} \end{bmatrix}
$$

Lösning 2b: Längs den horisontella symmetrilinjen har vi $u_y = 0$ och $t_x = 0$ (dvs ingen randlast i x–led); längs den vertikala symmetrilinjen gäller $u_x = 0$ samt $t_y = 0$. Resten av randen är fri och obelastad, så här gäller att normal– och tangentialspänningarna ska vara noll:

$$
t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$



Lösning 2c: Isoparametriska element innebär att basfunktionerna $N_i^e(\xi, \eta)$ används som formfunktioner — avbildningen görs då som

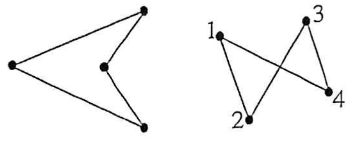
$x(\xi, \eta) = \sum_{i=1}^{4} x_i N_i^e \qquad y(\xi, \eta) = \sum_{i=1}^{4} y_i N_i^e$, där (x_i, y_i) är koordinaten för nod i. Eftersom basfunktionerna är polynom, beräknar vi enkelt derivatorna $\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} x_i \frac{\partial N_i^e}{\partial \xi}$, $\frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} x_i \frac{\partial N_i^e}{\partial \eta}$ och motsvarande för derivatorna av y.

Basfunktionernas derivator med avseende på x och y (som behövs för att ställa upp B^e) fås med kedjeregeln:

$$
\begin{aligned} \frac{\partial N_i^e}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i^e}{\partial y} \\ \frac{\partial N_i^e}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial y} \end{aligned} \Rightarrow \begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix} \text{ där alltså } J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \text{ Med } J^{-1} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix},
$$

$$
\det J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}, \text{ fås då de sökta derivatorna som } \begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix}.
$$

Lösning 2d: Inre öppningsvinklar måste vara mindre än $180°$. Noderna måste numreras i samma ordning i de båda koordinatsystemen (praxis är moturs ordning). Figuren visar exempel på otillåtna elementgeometrier.



3

Betrakta Poissons ekvation på ett område A med homogena Dirichlet villkor på randen Γ:

$$\begin{cases} -\mathrm{div}(D\nabla u) = f(x, y) & \text{i } A \\ u = 0 & \text{på } \Gamma \end{cases}$$

där den konstitutiva matrisen D är symmetrisk och positivt definit. Om vi definierar $a(u, v) = \int_A (\nabla v)^T D(\nabla u) dA$ och $(v, f) = \int_A vf dA$, så kan variationsproblemet skrivas: Bestäm $u \in V$ så att

$$a(u, v) = (v, f) \qquad \forall v \in V$$

där V är rummet av alla funktioner som är noll på Γ samt tillräckligt reguljära. Finita–element-problemet kan på samma sätt skrivas: Bestäm $u_h \in V_h$ så att

$$a(u_h, v) = (v, f) \qquad \forall v \in V_h$$

där FE–rummet V_h innehåller alla funktioner som kan skrivas som en lineärkombination av bas-funktionerna.

Givet att $V_h \subset V$ visar man enkelt att diskretiseringsfelet $e = u - u_h$ är energiortogonalt mot V_h, dvs att $a(e, v) = 0 \;\; \forall v \in V_h$ (Galerkin ortogonalitet).

a: Använd Galerkin ortogonaliteten för att visa att energin i felet är lika med felet i energi: $a(e, e) = a(u, u) - a(u_h, u_h)$. (2p)

b: $V_h \subset V$ innebär att FE–approximationen säkert konvergerar — detta kräver att elementen är kompletta och kompatibla. Vad menas med komplett respektive kompatibel? Vilken är den enklast möjliga FE–approximationen u_h på ett element? (3p)

<u>Lösning 3a:</u>

$$a(e, e) = a(u - u_h, u - u_h) = a(u, u - u_h) - a(u_h, u - u_h) =$$
$$a(u, u) - a(u, u_h) - a(u_h, u) + a(u_h, u_h) = a(u, u) + a(u_h, u_h) - 2a(u, u_h) =$$
$$\{\text{sätt } u = u_h + e \text{ i sista termen}\} = a(u, u) + a(u_h, u_h) - 2a(u_h + e, u_h) =$$
$$a(u, u) + a(u_h, u_h) - 2a(u_h, u_h) - 2a(e, u_h) =$$
$$\{u_h \in V_h \text{ så sista termen är 0 enligt Galerkinortogonalitet}\} = a(u, u) - a(u_h, u_h)$$

Alltså, $a(e, e) = a(u, u) - a(u_h, u_h)$.

<u>Lösning 3b:</u> För att ett element ska vara komplett, måste ansatsen vara sådan att det är möjligt att välja nodvariabelvärden så att approximationen u_h blir konstant och välja (andra) nodvaria-belvärden så att första derivatorna $\frac{\partial u_h}{\partial x}$ och $\frac{\partial u_h}{\partial y}$ blir konstanta på elementet. Enklast möjliga approximation på ett element är alltså $u_h = \alpha_1 + \alpha_2 x + \alpha_3 y$.

Elementen är kompatibla om u_h är C^0-kontinuerlig.

Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION
AUGUST 29 2007

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1 (svensk uppgiftstext nedan)

Formulate the divergence theorem (explain the entities you use) and show how the Green–Gauss theorem, i.e. $\int_{\Omega}\phi\,\mathrm{div}(\boldsymbol{q})d\Omega = \oint_{\Gamma}\phi\boldsymbol{q}^{\mathrm{T}}\boldsymbol{n}d\Gamma - \int_{\Omega}(\nabla\phi)^{\mathrm{T}}\boldsymbol{q}d\Omega$, may be derived from it. (2p)

1 (english text above)

Formulera divergensteoremet och visa hur Gauss–Greens teorem, dvs

$\int_{\Omega}\phi\,\mathrm{div}(\boldsymbol{q})d\Omega = \oint_{\Gamma}\phi\boldsymbol{q}^{\mathrm{T}}\boldsymbol{n}d\Gamma - \int_{\Omega}(\nabla\phi)^{\mathrm{T}}\boldsymbol{q}d\Omega$, kan härledas från detta. Definiera införda beteckningar. (2p)

---

2 (svensk uppgiftstext nedan)

The FE–formulation of a boundary value problem (of the type treated in the course) results in a system of equations $\boldsymbol{Ka} = \boldsymbol{f}$, where the stiffness matrix becomes symmetric if the Galerkin method is used. Also, essential and/or convective boundary conditions ensure that the matrix is positive definite.

a: Show that the FE–approximation minimizes the potential energy $\Pi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Kx} - \boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}$,

i.e. show that $\Pi(\boldsymbol{a}) < \Pi(\boldsymbol{x}) \qquad \forall\boldsymbol{x} \neq \boldsymbol{a}$. (2p)

b: Show that the solution is unique, i.e. show that Π has one minimum only. (1p)

2 (english text above)

FE–formuleringen av de randvärdesproblem som behandlats i kursen leder fram till ett ekvationssystem $\boldsymbol{Ka} = \boldsymbol{f}$, där styvhetsmatrisen blir symmetrisk om testfunktionerna väljs enligt Galerkins metod; väsentliga och/eller konvektiva randvillkor gör att matrisen också blir positivt definit.

a: Visa att FE–approximationen minimerar den potentiella energin $\Pi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Kx} - \boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}$,

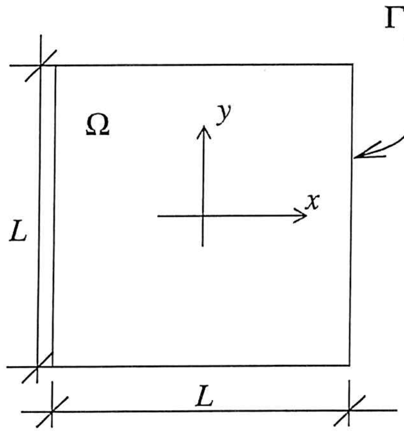
dvs att $\Pi(\boldsymbol{a}) < \Pi(\boldsymbol{x}) \qquad \forall\boldsymbol{x} \neq \boldsymbol{a}$. (2p)

b: Visa att lösningen är unik, dvs att Π bara har ett minimum. (1p)

---

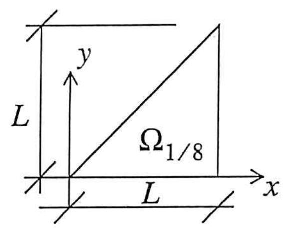
3 (svensk uppgiftstext nedan)

Consider a square membrane with side length L and loaded with a distributed load q in the z-direction; the membrane is pre–tensioned with a force S. The deflection $w(x)$ in z-direction of the membrane, satisfies the boundary value problem



$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_g \end{cases}$$

a: Using the symmetry of the problem, we can settle with considering only $1/8th$ of the domain. State how to formulate the boundary value problem so as to account for the symmetry in this manner. (1p)



b: Use the Green–Gauss theorem to derive the variational formulation of the problem. Regularity conditions of involved functions should be explicitly given, and it should be clear how the boundary conditions affect the variational problem. (2p)

c: Make a finite element formulation of the problem with test (weight) functions according to the Galerkin method. Show what the $\boldsymbol{B}^{e}$ – matrix looks like, for an element with N_e, basis functions. (2p)

3 (english text above)

Betrakta ett kvadratiskt membran med kantlängden L som i z-led belastas med en jämnt fördelad last q. Om membranet är förspänt med kraften S så ges utböjningen $w(x)$ i z-led av lösningen till randvärdesproblemet

$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{i } \Omega \\ w = 0 & \text{på } \Gamma_g \end{cases}$$

a: Genom att utnyttja symmetrin i problemet, räcker det med att betrakta 1/8—del av området. Formulera randvärdesproblemet så att symmetrin beaktas på detta sätt. (1p) b: Använd Green–Gauss teorem för att härleda problemets variationsformuleringen. Regularitetskrav på inblandade funktioner ska anges. Det ska också klart framgå hur randvillkoren beaktas i variationsproblemet. (2p)

c: Finit–elementformulera problemet med test–(vikts–)funktioner enligt Galerkins metod.

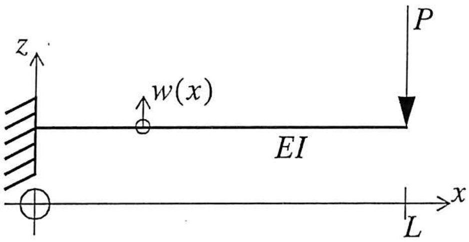
Visa $\boldsymbol{B}^{e}$ – matrisens utseende för ett element med N_e basfunktioner. (2p)

---

4 (svensk uppgiftstext nedan)

Consider a cantilever beam with length L, constant bending stiffness EI and loaded by a force P as depicted in the illustration. The beam displacement $w(x)$ is obtained as the solution of the boundary value problem

$$\begin{cases} EI\dfrac{d^4w}{dx^4} = 0 & \\ w(0) = 0 & \dfrac{dw}{dx}(0) = 0 \\ \dfrac{d^2w}{dx^2}(L) = 0 & \dfrac{d^3w}{dx^3}(L) = \dfrac{P}{EI} \end{cases}$$

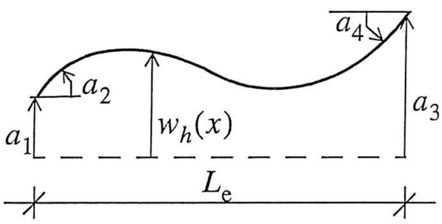


a: Derive the weak form of the problem. It should be shown how the boundary conditions affect the formulation. Also state requirements on involved functions. (2p)

b: Make a finite element formulation of the problem and show what the $\boldsymbol{B}$-matrix looks like. (2p)

c: The most common beam element has 4 cubic basis functions where the node variables have the meaning of displacements and rotations in the 2 nodes. This element formulation yields the element stiffness matrix (with constant bending stiffness)

$$\boldsymbol{K}^e = \frac{EI}{L_e}\begin{bmatrix} \frac{12}{L_e^2} & \frac{6}{L_e} & -\frac{12}{L_e^2} & \frac{6}{L_e} \\ \frac{6}{L_e} & 4 & -\frac{6}{L_e} & 2 \\ -\frac{12}{L_e^2} & -\frac{6}{L_e} & \frac{12}{L_e^2} & -\frac{6}{L_e} \\ \frac{6}{L_e} & 2 & -\frac{6}{L_e} & 4 \end{bmatrix}$$



where L_e is the element length. Solve the problem with 1 element. (2p)

4 (english text above)

Betrakta en konsolbalk med längden L, konstant böjstyvheten EI och belastad med en kraft P enligt figuren. Balkens utböjning $w(x)$ fås som lösningen till randvärdesproblemet

$$\begin{cases} EI\dfrac{d^4w}{dx^4} = 0 & \\ w(0) = 0 & \dfrac{dw}{dx}(0) = 0 \\ \dfrac{d^2w}{dx^2}(L) = 0 & \dfrac{d^3w}{dx^3}(L) = \dfrac{P}{EI} \end{cases}$$

a: Härled den svaga formen av randvärdesproblemet. Det ska framgå hur randvillkoren påverkar variationsproblemet. Ange också krav på ingående funktioner. (2p)

b: Finit-elementformulera problemet med testfunktioner enligt Galerkin och visa hur $\boldsymbol{B}$-matrisen ser ut. (2p)

c: Det vanligaste balkelementet har 4 kubiska basfunktioner och nodvariablerna representerar förskjutningar och rotationer i de 2 noderna. För detta element blir elementstyhetsmatrisen (med konstant böjstyvhet)

$$\boldsymbol{K}^e = \frac{EI}{L_e}\begin{bmatrix} \frac{12}{L_e^2} & \frac{6}{L_e} & -\frac{12}{L_e^2} & \frac{6}{L_e} \\ \frac{6}{L_e} & 4 & -\frac{6}{L_e} & 2 \\ -\frac{12}{L_e^2} & -\frac{6}{L_e} & \frac{12}{L_e^2} & -\frac{6}{L_e} \\ \frac{6}{L_e} & 2 & -\frac{6}{L_e} & 4 \end{bmatrix}$$

där L_e är elementets längd. Lös problemet med 1 element. (2p)

5 (svensk uppgiftstext nedan)

Consider the integral $I = \int_{-1}^{1} f(\xi) d\xi$, where $f(\xi)$ is enough smooth to be expressed by a Taylor series. We may then write $f(\xi) = \sum_{i=1}^{n} f_i l_i^{n-1}(\xi) + P(\xi)(\beta_0 + \beta_1 \xi + \ldots)$, with $f_i = f(\xi_i)$,

ξ_i is an integration point, and

$$l_i^{n-1}(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_{i-1})(\xi - \xi_{i+1})\ldots(\xi - \xi_n)}{(\xi_i - \xi_1)(\xi_i - \xi_2)\ldots(\xi_i - \xi_{i-1})(\xi_i - \xi_{i+1})\ldots(\xi_i - \xi_n)}$$

is the Lagrange polynomial of degree $n-1$ such that $l_i^{n-1}(\xi_j) = 0$ if $\xi_j \neq \xi_i$ and $l_i^{n-1}(\xi_i) = 1$. Further, $P(\xi) = (\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_n)$ is a n:th degree polynomial that is zero in the n points ξ_i, while the coefficients β_i depends of the Taylor series expansion of $f(\xi)$.

Derive the integration point coordinates ξ_i and the integration weights H_i for 3–point Gauss quadrate. You need not utilize the description given above, if you do not want to. (4p)

5 (english text above)

Betrakta integralen $I = \int_{-1}^{1} f(\xi) d\xi$, där $f(\xi)$ är tillräckligt reguljär för att utvecklas i en taylorserie. Man kan då skriva $f(\xi) = \sum_{i=1}^{n} f_i l_i^{n-1}(\xi) + P(\xi)(\beta_0 + \beta_1 \xi + \ldots)$, där $f_i = f(\xi_i)$, ξ_i är en integrationspunkt och

$$l_i^{n-1}(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_{i-1})(\xi - \xi_{i+1})\ldots(\xi - \xi_n)}{(\xi_i - \xi_1)(\xi_i - \xi_2)\ldots(\xi_i - \xi_{i-1})(\xi_i - \xi_{i+1})\ldots(\xi_i - \xi_n)}$$

är ett lagrange polynom av grad $n-1$ sådant att $l_i^{n-1}(\xi_j) = 0$ om $\xi_j \neq \xi_i$ och $l_i^{n-1}(\xi_i) = 1$. Vidare är $P(\xi) = (\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_n)$ ett polynom av grad n som har sina nollställen i de n punkterna ξ_i, medan koefficienterna β_i beror av taylorutvecklingen av $f(\xi)$.

Härled integrationspunkternas ξ_i koordinater och integrationsvikterna H_i för 3–punkts gaussintegration. Du behöver inte utgå från beskrivningen ovan om du inte vill. (4p)

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1

Formulate the divergence theorem (explain the entities you use) and show how the Green–Gauss theorem, i.e. $\int_\Omega \phi \operatorname{div}(q) d\Omega = \oint_\Gamma \phi q^{\mathrm{T}} n d\Gamma - \int_\Omega (\nabla\phi)^{\mathrm{T}} q d\Omega$, may be derived from it. (2p)

Solution 1: Let $n = \begin{bmatrix} n_x & n_y \end{bmatrix}^{\mathrm{T}}$ ($|n| = 1$) be an out-ward normal on the boundary Γ and $q = \begin{bmatrix} q_x(x, y) & q_y(x, y) \end{bmatrix}^{\mathrm{T}}$ be a vector field. Provided that the functions are sufficiently smooth, we have (the divergence theorem):

$$\int_\Omega \operatorname{div}(q) d\Omega = \oint_\Gamma q^{\mathrm{T}} n d\Gamma$$

To obtain the Green–Gauss theorem, we construct the vector field ϕq, where $\phi = \phi(x, y)$ is smooth enough, but otherwise arbitrary. Calculate the divergence of ϕq:

$$\operatorname{div}(\phi q) = \frac{\partial}{\partial x}[\phi q_x] + \frac{\partial}{\partial y}[\phi q_y] = \phi\frac{\partial q_x}{\partial x} + \phi\frac{\partial q_y}{\partial y} + \frac{\partial \phi}{\partial x}q_x + \frac{\partial \phi}{\partial y}q_y = \phi \operatorname{div}(q) + (\nabla\phi)^{\mathrm{T}} q$$

Applying the divergence theorem on ϕq and utilizing the identity $\operatorname{div}(\phi q) = \phi \operatorname{div}(q) + (\nabla\phi)^{\mathrm{T}} q$, we obtain the desired result.

2

The FE–formulation of a boundary value problem (of the type treated in the course) results in a system of equations $Ka = f$, where the stiffness matrix becomes symmetric if the Galerkin method is used. Also, essential and/or convective boundary conditions ensure that the matrix is positive definite.

a: Show that the FE–approximation minimizes the potential energy $\Pi(x) = \frac{1}{2}x^{\mathrm{T}}Kx - x^{\mathrm{T}}f$,

i.e. show that $\Pi(a) < \Pi(x) \quad \forall x \neq a$. (2p)

b: Show that the solution is unique, i.e. show that Π has one minimum only. (1p)

Solution 2a: The potential energy in the FE–approximation is $\Pi(a) = \frac{1}{2}a^{\mathrm{T}}Ka - a^{\mathrm{T}}f$. Let v be an arbitrary vector and $\varepsilon \ll 1$ a real number; we calculate th potential energy subsequent to a small arbitrary perturbation εv of the FE solution:

$$\Pi(a + \varepsilon v) = \frac{1}{2}(a + \varepsilon v)^T K(a + \varepsilon v) - (a + \varepsilon v)^T f =$$

$$= \frac{1}{2}(a^T Ka + \varepsilon v^T Ka + \varepsilon a^T Kv + \varepsilon^2 v^T Kv) - a^T f - \varepsilon v^T f$$

Now, we have that $a^T Kv = (a^T Kv)^T = (Kv)^T a = v^T K^T a = v^T Ka$, since K is symmetric; hence

$$\Pi(a + \varepsilon v) = \frac{1}{2}a^T Ka - a^T f + \frac{\varepsilon^2}{2}v^T Kv + \varepsilon v^T Ka - \varepsilon v^T f = \Pi(a) + \frac{\varepsilon^2}{2}v^T Kv + \varepsilon v^T(Ka - f)$$

The change in potential energy is thus

$\partial\Pi(a) = \Pi(a + \varepsilon v) - \Pi(a) = \frac{\varepsilon^2}{2}v^T Kv + \varepsilon v^T(Ka - f) \approx \varepsilon v^T(Ka - f)$, where the term involving the factor ε^2 is neglected, since $\varepsilon \ll 1$. Because $Ka = f \Rightarrow Ka - f = 0$, we note that $\partial\Pi(a) = 0$, i.e. the FE–approximation makes the potential energy stationary. To show that the stationary point is a minimum, we calculate the second variation:

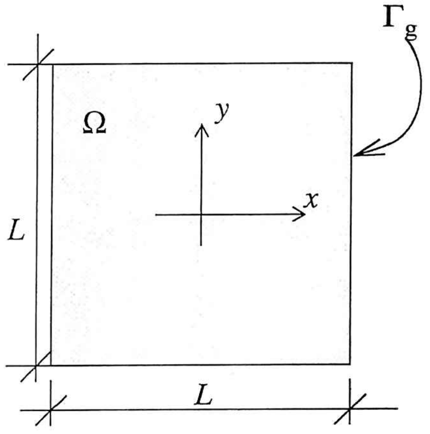
$\partial^2\Pi(a) = \partial\Pi(a + \varepsilon v) - \partial\Pi(a) = \varepsilon v^T(K(a + \varepsilon v) - f) - \varepsilon v^T(Ka - f) = \varepsilon^2 v^T Kv > 0$, where the last inequality steams from the fact that K is positive definite. From $\partial^2\Pi(a) > 0$ it follows that the stationary point is a minimum.

Solution 2b: Assume that a_1 and a_2 minimizes Π. Then we have

$\partial\Pi(a_1) = \varepsilon v^T(Ka_1 - f) = 0 \Rightarrow Ka_1 = f$ and similarly $Ka_2 = f$. Subtraction yields $Ka_1 - Ka_2 = 0$, or $K(a_1 - a_2) = 0$. Since K is positive definite, the only solution is $a_1 - a_2 = 0$, so that $a_1 = a_2$ — i.e. Π has one minimum only.

3

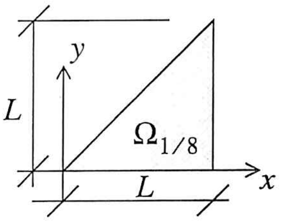
Consider a square membrane with side length L and loaded with a distributed load q in the z–direction; the membrane is pre–tensioned with a force S. The deflection $w(x)$ in z–direction of the membrane, satisfies the boundary value problem



$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_g \end{cases}$$

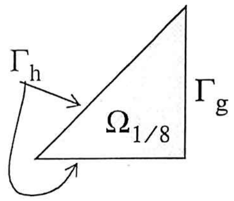
a: Using the symmetry of the problem, we can settle with considering only $1/8th$ of the domain. State how to formulate the boundary value problem so as to account for the symmetry in this manner. (1p)

b: Use the Green–Gauss theorem to derive the variational formulation of the problem. Regularity conditions of involved functions should be explicitly given, and it should be clear how the boundary conditions affect the variational problem. (2p)



c: Make a finite element formulation of the problem with test (weight) functions according to the Galerkin method. Show what the B^e – matrix looks like, for an element with N_e, basis functions. (2p)

Solution 3a: The normal derivative of the deflection has to be zero along the lines of symmetry. Thus, the b.v.p reads



$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega_{1/8} \\ w = 0 & \text{on } \Gamma_g \\ (\nabla w)^{\mathrm{T}} \boldsymbol{n} = 0 & \text{on } \Gamma_h \end{cases}$$

where $\boldsymbol{n}$ is an out–ward unit normal vector of the boundary.

Solution 3b: Multiply both sides of the differential equation by a test function v and integrate over the domain: $-\int_{\Omega_{1/8}} v\,\mathrm{div}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega$. Apply the Green–Green theorem to the left hand side: $\int_{\Omega_{1/8}} (\nabla v)^T(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega + \oint_{\Gamma} v(\nabla w)^T \boldsymbol{n} d\Gamma$. Now study the boundary integral; from the boundary condition on Γ_h, we know that the integrand is zero, but we do not know the value of $(\nabla w)^T n$ on Γ_g. Hence, since it is necessary to be able to evaluate the boundary term, we must enforce $v = 0$ on Γ_g. Thus,

$$\oint_{\Gamma} v(\nabla w)^T \boldsymbol{n} d\Gamma = \int_{\Gamma_g} v(\nabla w)^T \boldsymbol{n} d\Gamma + \int_{\Gamma_h} v(\nabla w)^T \boldsymbol{n} d\Gamma = \int_{\Gamma_g} 0 \cdot (\nabla w)^T \boldsymbol{n} d\Gamma + \int_{\Gamma_h} v \cdot 0 d\Gamma = 0$$

To be able to evaluate the integrals, all involved functions have to be regular enough:

$\int_{\Omega_{1/8}} v^2 d\Omega < \infty$ $\qquad$ $\int_{\Omega_{1/8}} (\nabla v)^T(\nabla v)d\Omega < \infty$. Let V denote the space of functions that are regular enough and that fulfil $v = 0$ on Γ_g. The variational problem may now be expressed as

$$\text{Find } w \in V \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^T(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega \qquad \forall v \in V$$

Solution 3c: Approximate the unknown function by a linear combination of selected basis functions $N_i(x, y)$: $w \approx w_h = \sum_i N_i a_i = Na$ ($N = \begin{bmatrix} N_1 & N_2 & \dots \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots \end{bmatrix}^T$). We substitute this into the variational problem and choose test functions v according to Galerkin, viz. any linear combination of basis functions. If we define the finite element space V_h as the space of functions that can be expressed as a linear combination of the basis functions, the FE–formulation according to Galerkin may be expressed as:

$$\text{Find } w_h \in V_h \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^T(\nabla w_h)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega \qquad \forall v \in V_h$$

If the N_e basis functions that are non-zero on an element are collected into a row vector $\boldsymbol{N}^e = \begin{bmatrix} N_1^e & \dots & N_{N_e}^e \end{bmatrix}$, and the corresponding node variables into a column vector

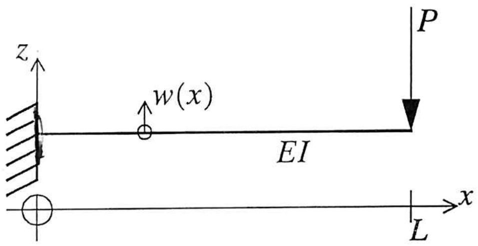
$\boldsymbol{a}^{\mathrm{e}} = \begin{bmatrix} a_1^{\mathrm{e}} & \dots & a_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix}^{\mathrm{T}}$, then the FE–approximation on the element becomes $w_{\mathrm{h}} = \boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$. On the element we thus have $\nabla w_{\mathrm{h}} = \nabla(\boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}) = (\nabla\boldsymbol{N}^{\mathrm{e}})\boldsymbol{a}^{\mathrm{e}} = \boldsymbol{B}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$, where, hence,

$$\boldsymbol{B}^{\mathrm{e}} = \nabla\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} N_1^{\mathrm{e}} & \dots & N_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_1^{\mathrm{e}}}{\partial x} & \dots & \dfrac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} \\ \dfrac{\partial N_1^{\mathrm{e}}}{\partial y} & \dots & \dfrac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} \end{bmatrix}$$

4

Consider a cantilever beam with length L, constant bending stiffness EI and loaded by a force P as depicted in the illustration. The beam displacement $w(x)$ is obtained as the solution of the boundary value problem

$$\begin{cases} EI\dfrac{d^4w}{dx^4} = 0 & 0 < x < L \\ w(0) = 0 & \dfrac{dw}{dx}(0) = 0 \\ \dfrac{d^2w}{dx^2}(L) = 0 & \dfrac{d^3w}{dx^3}(L) = \dfrac{P}{EI} \end{cases}$$

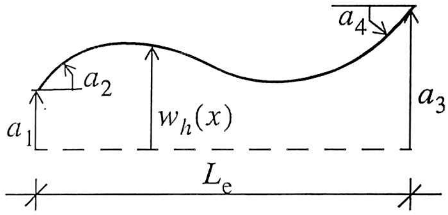


a: Derive the weak form of the problem. It should be shown how the boundary conditions affect the formulation. Also state requirements on involved functions. (2p)

b: Make a finite element formulation of the problem and show what the $\boldsymbol{B}$–matrix looks like. (2p)

c: The most common beam element has 4 cubic basis functions where the node variables have the meaning of displacements and rotations in the 2 nodes. This element formulation yields the element stiffness matrix (with constant bending stiffness)

$$
\mathbf{K}^{e} = \frac{EI}{L_e}\begin{bmatrix} \frac{12}{L_e^2} & \frac{6}{L_e} & -\frac{12}{L_e^2} & \frac{6}{L_e} \\ \frac{6}{L_e} & 4 & -\frac{6}{L_e} & 2 \\ -\frac{12}{L_e^2} & -\frac{6}{L_e} & \frac{12}{L_e^2} & -\frac{6}{L_e} \\ \frac{6}{L_e} & 2 & -\frac{6}{L_e} & 4 \end{bmatrix}
$$



where L_e is the element length. Solve the problem with 1 element. (2p)

Solution 4a: Multiply the differential equation by at test function $v(x)$ and integrate over the domain: $\int_0^L EIv\frac{d^4w}{dx^4}dx = 0$. Integration by parts (twice) gives

$$\int_0^L EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}dx = \left[\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_0^L - \left[vEI\frac{d^3w}{dx^3}\right]_0^L$$

. Now use the boundary conditions to solve for the boundary terms as far as possible:

$$\left[\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_0^L - \left[vEI\frac{d^3w}{dx^3}\right]_0^L = \left(0 - \frac{dv}{dx}EI\frac{d^2w}{dx^2}\bigg|_{x=0}\right) - \left(Pv(L) - vEI\frac{d^3w}{dx^3}\bigg|_{x=0}\right)$$

Now, since we do not know the values of $\frac{d^2w}{dx^2}$ and $\frac{d^3w}{dx^3}$ at $x = 0$, we restrict or choices of test functions to those that fulfil $v(0) = 0$ and $\frac{dv}{dx}(0) = 0$, so as to get rid of the unknown boundary terms. Furthermore, to be able to evaluate the integral in the left hand side, the square of the second derivatives of the functions has to be integrable. Thus, if we define

$$V = \left\{ v(x):\ \int_0^L\left(\frac{d^2v}{dx^2}\right)^2 dx < \infty,\ v(0) = 0,\ \left(\frac{dv}{dx}(0) = 0\right) \right\}$$

the weak form of the problem may be expressed as

$$\text{Find } w \in V \text{ such that} \qquad \int_0^L EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} dx = -Pv(L) \qquad \forall v \in V$$

Solution 4b: Approximate w by a linear combination of n chosen basis functions

$w \approx w_h = \sum_{i=1}^{n} N_i a_i$ and select the basis functions as test functions (Galerkin) to obtain as many equations as unknown node variables a_i. Let V_h be the space of functions that can be expressed as a linear combination of the selected basis function— the FE–formulation may the be expressed as

$$\text{Find } w_h \in V_h \text{ such that} \qquad \int_0^L EI \frac{d^2 v}{dx^2} \frac{d^2 w_h}{dx^2} dx = -Pv(L) \qquad \forall v \in V_h$$

Substituting $w_h = \sum_{i=1}^{n} N_i a_i = Na$, where $N = \begin{bmatrix} N_1 & N_2 & \dots & N_n \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}^T$,

into the integral and collection the equations we obtain by selecting $v = N_i$, $i = 1, 2, \dots, n$,

we obtain $\int_0^L EI \left(\frac{d^2 N}{dx^2} \right)^T \left(\frac{d^2 N}{dx^2} \right) dx\, a = -P N^T(L)$, where $\boldsymbol{B} = \frac{d^2 \boldsymbol{N}}{dx^2} = \begin{bmatrix} \frac{d^2 N_1}{dx^2} & \frac{d^2 N_2}{dx^2} & \dots & \frac{d^2 N_n}{dx^2} \end{bmatrix}$.

Solution 4c: Since we have one element only, the stiffness matrix is simply the element stiffness matrix, $K = K^e$, and the element length is L. Due to the boundary conditions at $x = 0$, we have that $a_1 = a_2 = 0$ so the first 2 columns are to be multiplied by zeros and may thus be omitted. Furthermore, N_1 and N_2 must not be used as test functions, since they do not satisfy the essential boundary conditions at $x = 0$; hence, the first two equations must not be used. We are, thus, left with $\begin{bmatrix} \frac{12}{L^2} & -\frac{6}{L} \\ -\frac{6}{L} & 4 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = -\frac{PL}{EI} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; multiplying by the

inverse of the matrix, we finally obtain $\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = -\frac{L^2}{12} \begin{bmatrix} 4 & 6/L \\ 6/L & 12/L^2 \end{bmatrix} \frac{PL}{EI} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{PL^2}{EI} \begin{bmatrix} L/3 \\ 1/2 \end{bmatrix}$.

5

Consider the integral $I = \int_{-1}^{1} f(\xi)d\xi$, where $f(\xi)$ is enough smooth to be expressed by a Taylor series. We may then write $f(\xi) = \sum_{i=1}^{n} f_i l_i^{n-1}(\xi) + P(\xi)(\beta_0 + \beta_1\xi + \ldots)$, with $f_i = f(\xi_i)$, ξ_i is an integration point, and

$$l_i^{n-1}(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)\ldots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\ldots(\xi-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)\ldots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\ldots(\xi_i-\xi_n)}$$

is the Lagrange polynomial of degree $n-1$ such that $l_i^{n-1}(\xi_j) = 0$ if $\xi_j \neq \xi_i$ and $l_i^{n-1}(\xi_i) = 1$. Further, $P(\xi) = (\xi-\xi_1)(\xi-\xi_2)\ldots(\xi-\xi_n)$ is a n:th degree polynomial that is zero in the n points ξ_i, while the coefficients β_i depends of the Taylor series expansion of $f(\xi)$.

Derive the integration point coordinates ξ_i and the integration weights H_i for 3–point Gauss quadrate. You need not utilize the description given above, if you do not want to. (4p)

<u>**Solution 5:**</u> We have

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n}\left[f_i\int_{-1}^{1} l_i^{n-1}(\xi)d\xi\right] + \int_{-1}^{1} P(\xi)(\beta_0+\beta_1\xi+\ldots)d\xi =$$

$$= \sum_{i=1}^{n} f_i H_i + \int_{-1}^{1} P(\xi)(\beta_0+\beta_1\xi+\ldots)d\xi \approx \sum_{i=1}^{n} f_i H_i$$

where, hence, the integration weights are $H_i = \int_{-1}^{1} l_i^{n-1}(\xi)d\xi$. With n-point Gauss integration, the coordinates ξ_i are chosen so that the integral of the first n terms in the polynomial $P(\xi)(\beta_0+\beta_1\xi+\ldots)$ become zero. Since these polynomial terms, together with the n

Lagrange polynomials, embrace all polynomial terms up to degree $2n-1$, it is obvious that

$$I = \sum_{i=1}^{n} f_i H_i$$ becomes exact if $f(\xi)$ is a polynomial of degree $2n-1$ (or lower).

The condition that the integral of the first n terms of $P(\xi)(\beta_0 + \beta_1\xi + \ldots)$ should vanish, yields the n equations

$$\int_{-1}^{1} \xi^i (\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_n) d\xi = 0 \qquad i = 0, 1, \ldots, n-1$$

from which the n coordinates $\xi_1, \xi_2 \ldots \xi_n$ may be solved.

With $n = 3$ we have, due to symmetry, $\xi_1 = -\xi_3$ and $\xi_2 = 0$, so with $i = 1$ we obtain

$$\int_{-1}^{1} \xi(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3) d\xi = \int_{-1}^{1} \xi(\xi + \xi_3)(\xi)(\xi - \xi_3) d\xi = \int_{-1}^{1} (\xi^4 - \xi^2\xi_3^2) d\xi =$$

$$= \left[\frac{\xi^5}{5} - \frac{\xi^3\xi_3^2}{3}\right]_{-1}^{1} = \frac{2}{5} - \frac{2}{3}\xi_3^2 = 0 \Rightarrow \xi_3 = -\xi_1 = \sqrt{3/5}, \quad \xi_2 = 0$$

Thus, $$l_2^2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{\xi^2 - \xi_3^2}{-\xi_3^2} = \frac{5\left(\frac{3}{5} - \xi^2\right)}{3}$$, so

$$H_2 = \int_{-1}^{1} l_2(\xi) d\xi = \left(\frac{5}{3}\right)\left[\frac{3\xi}{5} - \frac{\xi^3}{3}\right]_{-1}^{1} = \frac{8}{9}$$

Now, due to symmetry we have that $H_1 = H_3$ and since $\sum_i H_i = 2$ we get

$$H_1 = H_3 = \frac{1}{2}(2 - H_2) = \frac{5}{9}.$$

FINITE ELEMENT METHOD (MHA 021)—EXAMINATION

Time: Thursday March 15 2007, 08.30–12.30, in the V-building
Teacher: Peter Hansbo, ext. 1494

Visits by a teacher: approx. 9.30 and 11.30.
Solutions: will be posted on the web homepage after the exam.
Preliminary grading: will be posted at the Department of Applied Mechanics no later than March 29.
Review of the grading: at the Department of Applied Mechanics, Thursday March 29 at 12.00–13.00.

Aids: 'Closed books' examination; only dictionaries and a 'standard' calculator allowed.

Grading: A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. **Note: to pass the course you also have to complete 4 computer assignments.**

Grading limit	Score
3	8–11
4	12–15
5	16–20

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

Problems

1 Consider a square membrane with side length 3 m and loaded with a distributed load $q = 100$ N/m^2 in the z−direction. The membrane is under tension by a force $S = 1000$ N/m in the xy−plane. The deflection $u(x, y)$ in the z−direction of the membrane satisfies the boundary value problem

$$-\text{div}(\nabla u) = \frac{q}{S} \quad \text{in } \Omega, \quad u = 0 \quad \text{on} \quad \Gamma,$$

where Γ is the boundary of Ω. The domain is meshed using 9 equal bilinear finite elements and $u \approx u^h = \sum_{i=1}^{16} N_i(x, y)a_i$. The global and local node numbering is given in Figure 1.

a) To compute the element stiffness matrices, it is sufficient to look at only one element (the matrix will be the same for all elements). Take the one in the lower left-hand corner where $N_1^e = (1 - x)(1 - y)$, $N_2^e = x(1 - y)$, $N_3^e = xy$, and $N_4^e = (1 - x)y$. The element matrix will have the following appearance:

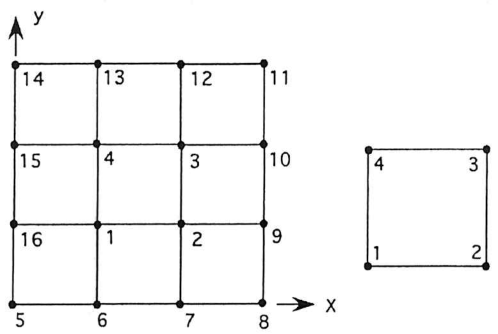
$$\boldsymbol{K}^e = \begin{bmatrix} K & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & K & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & K & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & K \end{bmatrix}$$

Compute the number K. (2p)

b) We only need to consider the unknowns $a_1 - a_4$, the other variables are all zero due to the boundary conditions. Assemble the 4×4 global stiffness matrix $\boldsymbol{K}$ corresponding to these unknowns. (2p)

c) Establish the global load vector $\boldsymbol{f} = [f_1, f_2, f_3, f_4]^{\mathrm{T}}$ and use the symmetry, i.e., that $a_1 = a_2 = a_3 = a_4$ to compute $\boldsymbol{a}$ from $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$. (2p)

d) Compute the approximate deflection at $x = 0.75$, $y = 0.6$. (2p)



Figur 1: Mesh with global numbering and one element with local numbering

2 Consider the weak form of the boundary value problem in the previous task; provided that we have homogeneous essential boundary conditions, it may be expressed as: Find $u \in V$ such that

$$a(u, v) = (f, v) \quad \forall v \in V,$$

where V is the space of all admissible functions, $a(\cdot,\cdot)$ is symmetric and linear in both arguments and $(\cdot,\cdot)$ a scalar product of functions.

Let $u_h \in V_h$ be a conforming FE-approximation of u.

a) Prove Galerkin orthogonality, i.e., show that the discretization error $e = u - u_h$ is a–orthogonal to the FE–space V_h so that $a(e, v_h) = 0$ for all $v_h \in V_h$. (2p)

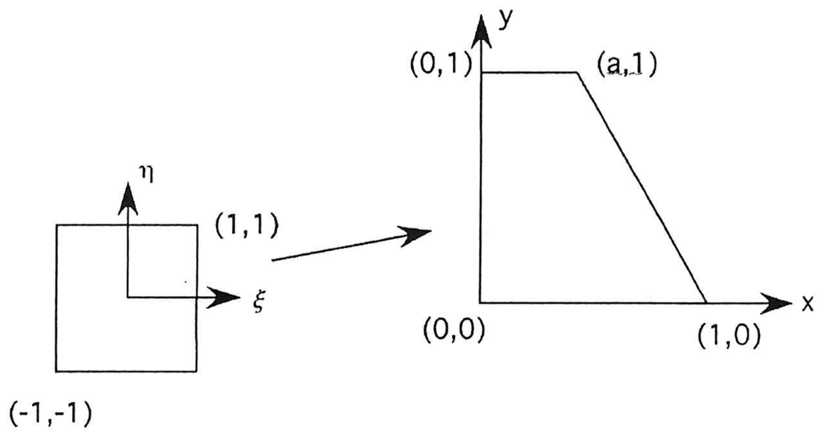
b) Use Galerkin orthogonality to show that the energy of the error equals the error in energy: $a(e, e) = a(u, u) - a(u_h, u_h)$. (2p)

b) What is the engineering interpretation of the fact that $a(u, u) \geq a(u_h, u_h)$ in terms of the apparent stiffness of the FE-model? Give an explanation by use of the potential energy functional $\Pi(v) = \frac{1}{2}a(v, v) - (f, v)$. (2p)

3 When using mapped elements, we have the basis functions defined in terms of some local coordinates (ξ, η), i.e., $N_i^e = N_i^e(\xi, \eta)$, where $N_1^e = \frac{1}{4}(1-\xi)(1-\eta)$, etc.

(a) Derive expressions for the derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ in the case when the mapping $(x, y) = (x(\xi, \eta), y(\xi, \eta))$ to the global coordinates is isoparametric. (2p)

(b) An alternative way to create a triangular element is to move one of the nodes to the same position as another, using $a = 0$ in Figure 2. Calculate the determinant of the Jacobian $\boldsymbol{J}$ of the mapping, as a function of the parameter a. What happens to the determinant of $\boldsymbol{J}$ when $a = 0$? (2p)



Figur 2: Mapping from a reference element

4 Maxwell's equations (of electrodynamics) can in simplified situations be written as

$$\omega^2 \boldsymbol{u} + \operatorname{curl}(\operatorname{curl} \boldsymbol{u}) = \boldsymbol{f}, \text{ in } \Omega \quad \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \Gamma.$$

where Γ is the boundary of Ω, $\boldsymbol{n}$ is the outward pointing normal to Ω, and ω is a number (related to the frequency). In 2D, the curl of a vector is

$$\operatorname{curl} \boldsymbol{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y},$$

the curl of a scalar is

$$\operatorname{curl} v = \begin{bmatrix} \dfrac{\partial v}{\partial y} \\ -\dfrac{\partial v}{\partial x} \end{bmatrix},$$

and $\boldsymbol{a} \times \boldsymbol{b} = a_x b_y - a_y b_x$.

The Green-Gauss formula can be written in 2D as

$$\left.\begin{aligned} \int_\Omega \frac{\partial u}{\partial x} v \, d\Omega &= \int_{\partial\Omega} n_x u \, v \, ds - \int_\Omega u \frac{\partial v}{\partial x} \, d\Omega, \\ \int_\Omega \frac{\partial u}{\partial y} v \, d\Omega &= \int_{\partial\Omega} n_y u \, v \, ds - \int_\Omega u \frac{\partial v}{\partial y} \, d\Omega. \end{aligned}\right\} \tag{1}$$

Use (1) to show that a weak form of Maxwell's equations in 2D is to find $\boldsymbol{u} \in V$ such that

$$\int_\Omega \omega^2 \boldsymbol{v}^{\mathrm{T}} \boldsymbol{u} \, d\Omega + \int_\Omega \operatorname{curl} \boldsymbol{v} \operatorname{curl} \boldsymbol{u} \, d\Omega = \int_\Omega \boldsymbol{v}^{\mathrm{T}} \boldsymbol{f} \, d\Omega$$

for all $\boldsymbol{v} \in V$. (2p)

FINITE ELEMENT METHOD (MHA 021)—EXAMINATION

Time: Thursday March 15 2007, 08.30–12.30, in the V-building
Teacher: Peter Hansbo, ext. 1494

Visits by a teacher: approx. 9.30 and 11.30.
Solutions: will be posted on the web homepage after the exam.
Preliminary grading: will be posted at the Department of Applied Mechanics no later than March 29.
Review of the grading: at the Department of Applied Mechanics, Thursday March 29 at 12.00–13.00.

Aids: 'Closed books' examination; only dictionaries and a 'standard' calculator allowed.

Grading: A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. **Note: to pass the course you also have to complete 4 computer assignments.**

Grading limit	Score
3	8–11
4	12–15
5	16–20

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

Problems

1 Consider a square membrane with side length 3 m and loaded with a distributed load $q = 100$ N/m^2 in the $z-$direction. The membrane is under tension by a force $S = 1000$ N/m in the $xy-$plane. The deflection $u(x, y)$ in the $z-$direction of the membrane satisfies the boundary value problem

$$-\mathrm{div}(\nabla u) = \frac{q}{S} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma,$$

where Γ is the boundary of Ω. The domain is meshed using 9 equal bilinear finite elements and $u \approx u^h = \sum_{i=1}^{16} N_i(x, y)a_i$. The global and local node numbering is given in Figure 1.

a) To compute the element stiffness matrices, it is sufficient to look at only one element (the matrix will be the same for all elements). Take the one in the lower left-hand corner where $N_1^e = (1-x)(1-y)$, $N_2^e = x(1-y)$, $N_3^e = xy$, and $N_4^e = (1-x)y$. The element matrix will have the following appearance:

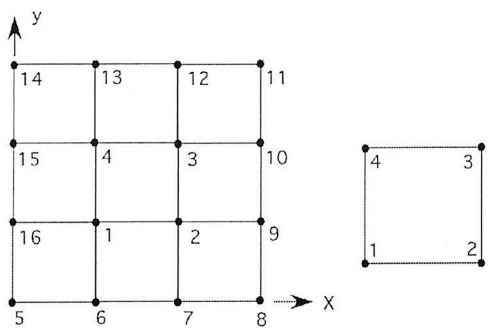
$$\boldsymbol{K}^e = \begin{bmatrix} K & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & K & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & K & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & K \end{bmatrix}$$

Compute the number K. (2p)

b) We only need to consider the unknowns $a_1 - a_4$, the other variables are all zero due to the boundary conditions. Assemble the 4×4 global stiffness matrix $\boldsymbol{K}$ corresponding to these unknowns. (2p)

c) Establish the global load vector $\boldsymbol{f} = [f_1, f_2, f_3, f_4]^{\mathrm{T}}$ and use the symmetry, i.e., that $a_1 = a_2 = a_3 = a_4$ to compute $\boldsymbol{a}$ from $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$. (2p)

d) Compute the approximate deflection at $x = 0.75$, $y = 0.6$. (2p)



Figur 1: Mesh with global numbering and one element with local numbering

Solution:

a) Easiest is to note that the product $\boldsymbol{K}^e \boldsymbol{\varphi} = \mathbf{0}$ if $\boldsymbol{\varphi}$ is a rigid boy mode, in this case a vector with all entries equal (producing a constant). Thus we must have the row-sum $\sum_{j=1}^{4} K_{ij} = 0$ and it follows that

$$K - 1/6 - 1/3 - 1/6 = 0 \Rightarrow K = 2/3.$$

b) Following the local numbering of the matrices, we have that

$$K_{11} = K_{33}^e + K_{44}^e + K_{11}^e + K_{22}^e = \frac{8}{3}, \text{ and in the same way } K_{22} = K_{33} = K_{44} = \frac{8}{3}.$$

$$K_{12} = K_{43}^e + K_{12}^e = -\frac{1}{3}, \; K_{13} = K_{13}^e = -\frac{1}{3},$$

$$K_{14} = K_{23}^e + K_{14}^e = -\frac{1}{3}, \; K_{23} = K_{23}^e + K_{14}^e = -\frac{1}{3}.$$

and thus

$$\boldsymbol{K} = \begin{bmatrix} \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{8}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} \end{bmatrix}$$

c) The element load vector is given by the area of the element multiplied by the intensity of the load and the value of the basis functions in the midpoint (one-point integration is exact in this case). Thus $\boldsymbol{f}^e = (q/S)[1/4, 1/4, 1/4, 1/4]^{\mathrm{T}}$. Each entry in the global load vector is the sum of four contributions and thus $\boldsymbol{f} = [1/10, 1/10, 1/10, 1/10]^{\mathrm{T}}$. Knowing that the a_i are all the same we get

$$\left(\frac{8}{3} - 3\frac{1}{3}\right) a_i = \frac{1}{10} \Rightarrow a_i = \frac{3}{50}, \; i = 1, 2, 3, 4.$$

c) The point $(0.75, 0.6)$ is situated in the elment with nodes $5, 6, 1, 16$ and thus $\boldsymbol{a}^e = [0, 0, 3/50, 0, 0]^{\mathrm{T}}$. We can thus compute

$$u^h(0.75, 0.6) = N_3^e(0.75, 0.6) \times \frac{3}{50} = 0.75 \times 0.6 \times 0.06 = 0.027 \text{ [m]}.$$

2 Consider the weak form of the boundary value problem in the previous task; it may be expressed as: Find $u \in V$ such that

$$a(u, v) = (f, v) \quad \forall v \in V,$$

where V is the space of all admissible functions, $a(\cdot, \cdot)$ is symmetric and linear in both arguments and $(\cdot, \cdot)$ a scalar product of functions.

Let $u_h \in V_h$ be a conforming FE-approximation of u.

a) Prove Galerkin orthogonality, i.e., show that the discretization error $e = u - u_h$ is a–orthogonal to the FE–space V_h so that $a(e, v_h) = 0$ for all $v_h \in V_h$. (2p)

b) Use Galerkin orthogonality to show that the energy of the error equals the error in energy: $a(e, e) = a(u, u) - a(u_h, u_h)$. (2p)

c) What is the engineering interpretation of the fact that $a(u, u) \geq a(u_h, u_h)$ in terms of the apparent stiffness of the FE-model? Give an explanation by use of the potential energy functional $\Pi(v) = \frac{1}{2}a(v, v) - (f, v)$. (2p)

Solution:

a) For all $v_h \in V_h$ there holds $a(u_h, v_h) = (f, v_h)$ and also $a(u, v_h) = (f, v_h)$ since $V_h \subset V$. Thus, using the linearity,

$$0 = (f, v_h) - (f, v_h) = a(u, v_h) - a(u_h, v_h) = a(u - u_h, v_h) = a(e, v_h).$$

b) We have, using the symmetry, linearity, and orthogonality, that

$$\begin{aligned} a(e, e) &= a(e, u) - a(e, u_h) = a(e, u) = a(u, e) = a(u, u) - a(u, u_h) = \\ &= a(u, u) - a(u, u_h) + a(u - u_h, u_h) = a(u, u) - a(u_h, u_h). \end{aligned}$$

c) We know that the exact solution minimizes Π, thus $\Pi(u_h) \geq \Pi(u)$. Now, since

$$a(u_h, u_h) = (f, u_h), \quad \text{there follows} \quad \Pi(u_h) = -\frac{1}{2}a(u_h, u_h),$$

and, by the same argument, $\Pi(u) = -\frac{1}{2}a(u, u)$. Thus $a(u, u) \geq a(u_h, u_h)$ is a consequence of the minimum of potential energy. Since

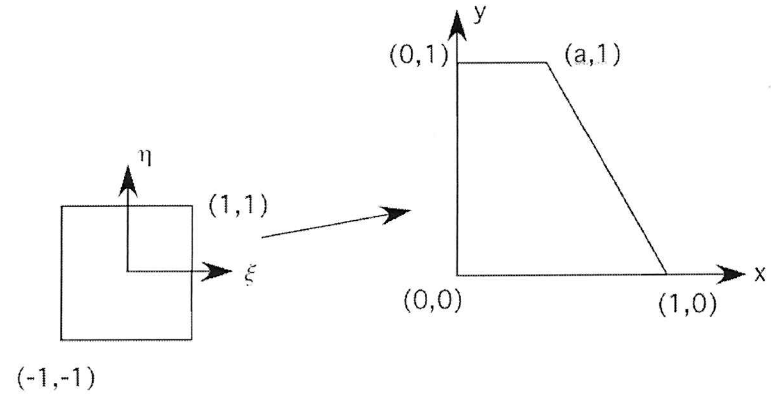
$$a(u, u) = \int_\Omega \nabla u \cdot \nabla u \, d\Omega$$

and ∇u is the strain, the effect of discretization is to give smaller strains than the exact solution. This will result in the discretized model appearing stiffer that the continuous model.

3 When using mapped elements, we have the basis functions defined in terms of some local coordinates (ξ, η), i.e., $N_i^e = N_i^e(\xi, \eta)$, where $N_1^e = \frac{1}{4}(1 - \xi)(1 - \eta)$, etc.

(a) Derive expressions for the derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ in the case when the mapping $(x, y) = (x(\xi, \eta), y(\xi, \eta))$ to the global coordinates is isoparametric. (2p)

(b) An alternative way to create a triangular element is to move one of the nodes to the same position as another, using $a = 0$ in Figure 2. Calculate the determinant of the Jacobian $\boldsymbol{J}$ of the mapping, as a function of the parameter a. What happens to the determinant of $\boldsymbol{J}$ when $a = 0$? (2p)



Figur 2: Mapping from a reference element

Solution:

a) By the chain rule

$$\left.\begin{aligned}\frac{\partial N_i^e}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i^e}{\partial y}\\ \frac{\partial N_i^e}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial y}\end{aligned}\right\} = \boldsymbol{J}\begin{bmatrix}\dfrac{\partial N_i^e}{\partial x}\\ \dfrac{\partial N_i^e}{\partial y}\end{bmatrix}, \quad \text{where} \quad \boldsymbol{J} = \begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix}.$$

Thus

$$\begin{bmatrix}\dfrac{\partial N_i^e}{\partial x}\\ \dfrac{\partial N_i^e}{\partial y}\end{bmatrix} = \boldsymbol{J}^{-1}\begin{bmatrix}\dfrac{\partial N_i^e}{\partial \xi}\\ \dfrac{\partial N_i^e}{\partial \eta}\end{bmatrix}, \quad \text{where} \quad \frac{\partial x}{\partial \xi} = \sum_i x_i \frac{\partial N_i^e}{\partial \xi} \text{ etc.}$$

where (x_i, y_i) are the coordinates of node i.

b) Set $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (1,0)$, $(x_3, y_3) = (a,1)$, $(x_4, y_4) = (0,1)$. Then

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_2^e}{\partial \xi} + a\frac{\partial N_3^e}{\partial \xi} = \frac{1}{4}(1-\eta) + \frac{a}{4}(1+\eta), \quad \frac{\partial x}{\partial \eta} = -\frac{1}{4}(1+\xi) + \frac{a}{4}(1+\xi)$$

$$\frac{\partial y}{\partial \xi} = \frac{\partial N_3^e}{\partial \xi} + \frac{\partial N_4^e}{\partial \xi} = \frac{1}{4}(1+\eta) - \frac{1}{4}(1+\eta), \quad \frac{\partial y}{\partial \eta} = \frac{1}{4}(1+\xi) + \frac{1}{4}(1-\xi)$$

and $\det \boldsymbol{J} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} = (1 + a - \eta + a\eta)/8$. If $a = 0$, then $\det \boldsymbol{J} = 0$ at $\eta = 1$.

4 Maxwell's equations (of electrodynamics) can in simplified situations be written as

$$\omega^2 \boldsymbol{u} + \operatorname{curl}(\operatorname{curl} \boldsymbol{u}) = \boldsymbol{f}, \text{ in } \Omega \quad \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \Gamma.$$

where Γ is the boundary of Ω, $\boldsymbol{n}$ is the outward pointing normal to Ω, and ω is a number (related to the frequency). In 2D, the curl of a vector is

$$\operatorname{curl} \boldsymbol{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y},$$

the curl of a scalar is

$$\operatorname{curl} v = \begin{bmatrix} \dfrac{\partial v}{\partial y} \\ -\dfrac{\partial v}{\partial x} \end{bmatrix},$$

and $\boldsymbol{a} \times \boldsymbol{b} = a_x b_y - a_y b_x$.

The Green-Gauss formula can be written in 2D as

$$\left.\begin{aligned} \int_\Omega \frac{\partial u}{\partial x} v \, d\Omega &= \int_\Gamma n_x u \, v \, ds - \int_\Omega u \frac{\partial v}{\partial x} \, d\Omega, \\ \int_\Omega \frac{\partial u}{\partial y} v \, d\Omega &= \int_\Gamma n_y u \, v \, ds - \int_\Omega u \frac{\partial v}{\partial y} \, d\Omega. \end{aligned}\right\} \tag{1}$$

Use (1) to show that a weak form of Maxwell's equations in 2D is to find $\boldsymbol{u} \in V$ such that

$$\int_\Omega \omega^2 \boldsymbol{v}^{\mathrm{T}} \boldsymbol{u} \, d\Omega + \int_\Omega \operatorname{curl} \boldsymbol{v} \operatorname{curl} \boldsymbol{u} \, d\Omega = \int_\Omega \boldsymbol{v}^{\mathrm{T}} \boldsymbol{f} \, d\Omega$$

for all $\boldsymbol{v} \in V$. (2p)

Solution:

Let $w = \operatorname{curl} \boldsymbol{u}$. We have

$$\int_\Omega \frac{\partial w}{\partial y} v_x \, d\Omega = \int_\Gamma n_y w \, v_x \, ds - \int_\Omega w \frac{\partial v_x}{\partial y} \, d\Omega$$

and

$$-\int_\Omega \frac{\partial w}{\partial x} v_y \, d\Omega = -\int_\Gamma n_x w \, v_y \, ds + \int_\Omega w \frac{\partial v_y}{\partial x} \, d\Omega$$

and thus

$$\int_\Omega \boldsymbol{v}^{\mathrm{T}} \operatorname{curl} w \, d\Omega = -\int_\Gamma w \, \boldsymbol{n} \times \boldsymbol{v} \, ds + \int_\Omega \operatorname{curl} \boldsymbol{v} \, w \, d\Omega.$$

The essential boundary condition $\boldsymbol{u} \times \boldsymbol{n} = 0$ must be fulfilled also by the test function, and thus

$$\int_\Omega \boldsymbol{v}^{\mathrm{T}} \boldsymbol{f} \, d\Omega = \int_\Omega \omega^2 \boldsymbol{v}^{\mathrm{T}} \boldsymbol{u} \, d\Omega + \int_\Omega \boldsymbol{v}^{\mathrm{T}} \operatorname{curl}(\operatorname{curl} \boldsymbol{u}) \, d\Omega = \int_\Omega \omega^2 \boldsymbol{v}^{\mathrm{T}} \boldsymbol{u} \, d\Omega + \int_\Omega \operatorname{curl} \boldsymbol{v} \operatorname{curl} \boldsymbol{u} \, d\Omega.$$