Institutionen för tillämpad mekanik, Chalmers tekniska högskola

TENTAMEN I FINIT ELEMENTMETOD — MHA 021

22 DECEMBER 2006

Lösningar

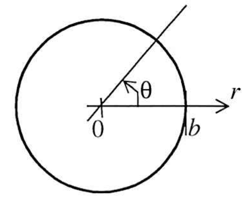
Tid och plats:	$14^{00} - 18^{00}$ i M–huset
Hjälpmedel:	Ordböcker, lexikon och typgodkänd räknare.
Lärare:	Peter Möller, tel (772) 1505. Besöker sal ca. 15^{00} samt 17^{00}.
Lösningar:	Anslås på anslagstavlan vid ingången till institutionens lokaler senast 23/12. Se även kursens www–sidor på http://www.am.chalmers.se/~moller/index_fem_m3_gk.html
Betygsättning:	En fullständig och korrekt lösning på en uppgift ger poäng enligt vad som anges på uppgiftslappen. Smärre fel leder till poängavdrag. Ofullständig lösning (svar på ställt problem saknas) eller omfattande fel ger inte något poäng. Maximal poäng är 20. Det krävs 8 poäng för betyg 3; 12 poäng ger betyg 4; för betyg 5 krävs 16 poäng. **Observera att ovanstående är betygssättning på enbart tentamen; för godkänd examination krävs dessutom godkända inlämningsuppgifter.**
Resultatlista:	Anslås senast 11/1 på samma ställe som lösningarna. Resultaten sänds till betygsexpeditionen senast 16/1 — **för kursdeltagare som inte har alla inlämningsuppgifter godkända vid detta tillfälle rapporteras betyget U (underkänd).**
Granskning:	Torsdag 11/1 13–15 samt måndag 22/11 13–15.

Tänk på:

- Skriv så att den som ska rätta, kan läsa och förstå hur du tänker. Den som rättar tentamen gissar inte eller antar inte vad du menar/tänker — endast vad som verkligen skrivs har betydelse vid poängsättningen.
- Förklara/definiera införda beteckningar.
- Rita tydliga figurer. Ange i förekommande fall vad som är positiva/negativa riktingar (på t.ex förskjutningar och krafter).
- Gör du antaganden utöver de som anges i uppgiftstexten, så ange detta explicit och förklara dessa.

1

Betrakta det rotationssymetriska randvärdesproblemet



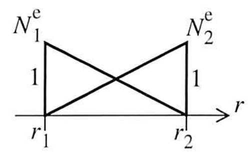
$$\left.\begin{aligned} -\frac{1}{r}\frac{d}{dr}\left[kr\frac{du}{dr}\right] &= Q \qquad 0 < r < b \\ q(b) &= \alpha(u(b) - u_0) \end{aligned}\right\}$$

där k, Q, α och u_0 är givna konstanter, $u = u(r)$ den obekanta funktionen, samt $q = -k\frac{du}{dr}$.

a: Variationsformulera problemet. Du behöver här inte ange regularitetskrav på ingåeende funktioner, med det måste klart framgå hur det konvektiva randvillkoret kommer in i den svaga formen. (Observera att integrationen måste göras över en volym: $\int_V \ldots dV = \iiint \ldots r d\theta dr dz$ (med lämpliga integrationsgränser)). (2p)

b: FE–formulera problemet med testfunktioner (viktsfunktioner) enligt Galerkin. (2p)

c: Tag fram ett uttryck för en lumpad elementlastvektor för ett lineärt element med längden $h = r_2 - r_1$. (2p)



d: Med $b = 8{,}91 \cdot 10^{-3}$, $k = 3{,}5$ och $Q = 63 \cdot 10^6$, så ger integralerna i vänster– respektive högerledet av FE–formuleringen

$$K = \begin{bmatrix} 1{,}75 & -1{,}75 & 0 \\ -1{,}75 & 7{,}00 & -5{,}25 \\ 0 & -5{,}25 & 5{,}25 \end{bmatrix} \qquad f = \begin{bmatrix} 0{,}3126 \\ 1{,}2504 \\ 0{,}9378 \end{bmatrix} \cdot 10^3$$

om problemet diskretiseras med två lineära element. Visa — med förklaring — hur styvhetsmatrisen och lastvektorn ändras av det konvektiva randvillkoret, om $\alpha = 400$ och $u_0 = 150$. (1p)

Lösning 1a: Multiplicera differentialekvationen med en testfunktion $v = v(r)$ och integrera över volymen

$$\int_0^L\int_0^b\int_0^{2\pi} v\left(Q + \frac{1}{r}\frac{d}{dr}\left[kr\frac{du}{dr}\right]\right) r d\theta dr dz = 0 \Rightarrow 2\pi L\int_0^b v\left(Qr + \frac{d}{dr}\left[kr\frac{du}{dr}\right]\right)dr = 0.$$

Efter division med $2\pi L$ och partialintegration av den andra termen i integranden,
$\int_0^b v\left(\frac{d}{dr}\left[kr\frac{du}{dr}\right]\right)dr = \left[vkr\frac{du}{dr}\right]_0^b - \int_0^b kr\frac{dvdu}{drdr}dr$, fås $\int_0^b kr\frac{dvdu}{drdr}dr = \left[vkr\frac{du}{dr}\right]_0^b + \int_0^b vrQdr$. Utveckla nu randtermen och sätt in randvillkoret: $\left[vkr\frac{du}{dr}\right]_0^b = v(b)bk\frac{du}{dr}\bigg|_{r=b} - 0 = v(b)b\alpha u_0 - v(b)b\alpha u(b)$, där sista termen är obekant och flyttas till vänsterledet. Vi har alltså variationsproblemet

$$\int_0^b kr\frac{dv}{dr}\frac{du}{dr}dr + \alpha b v(b)u(b) = \int_0^b vrQdr + \alpha b v(b)u_0$$

<u>**Lösning 1b:**</u> Approximera den obekanta funktionen med en lineärkombination av basfunktioner $u \approx u_h = N_1a_1 + N_2a_2 + \ldots + N_na_n = \boldsymbol{Na}$, där $\boldsymbol{N}$ är en radvektor med basfunktionerna och $\boldsymbol{a}$ är en kolumnvektor med nodvariablerna. Sätt in approximationen i variationsproblemet och låt

$$\frac{du_h}{dr} = \frac{d}{dr}[\boldsymbol{Na}] = \frac{d\boldsymbol{N}}{dr}\boldsymbol{a} = \begin{bmatrix}\frac{dN_1}{dr} & \frac{dN_2}{dr} & \cdots & \frac{dN_n}{dr}\end{bmatrix}\boldsymbol{a} = \boldsymbol{Ba}: \int_0^b kr\frac{dv}{dr}\boldsymbol{B}dr\boldsymbol{a} + \alpha b v(b)\boldsymbol{N}(b)\boldsymbol{a} = \int_0^b vrQdr + \alpha b v(b)u_0.$$

Välj nu basfunktionerna som testfunktioner i tur och ordning (Galerkins metod); varje val ger upphov till en ekvation och vis samlar ekvationerna radvis, så $v \rightarrow \begin{bmatrix}N_1\\N_2\\\ldots\\N_n\end{bmatrix}$ och $\frac{dv}{dr} \rightarrow \begin{bmatrix}\frac{dN_1}{dr}\\\frac{dN_2}{dr}\\\ldots\\\frac{dN_n}{dr}\end{bmatrix} = \boldsymbol{B}^{\mathrm{T}}$. Vi får

alltså $\displaystyle\int_0^b kr\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}dr\boldsymbol{a} + \alpha b\boldsymbol{N}^{\mathrm{T}}(b)\boldsymbol{N}(b)\boldsymbol{a} = \int_0^b \boldsymbol{N}^{\mathrm{T}}rQdr + \alpha b\boldsymbol{N}^{\mathrm{T}}(b)u_0$

<u>**Lösning 1c:**</u> Den konsistenta elementlastvektorn ges av $f^e = \displaystyle\int_{r_1}^{r_2}(N^e)^{\mathrm{T}}rQdr$, där radvektorn N^e innehåller de basfunktioner som antar från noll skilda värden på elementet. Eftersom summan av basfunktionerna är 1, blir lastresultanten

$$\sum_{i=1}^{2} f_i^e = \int_{r_1}^{r_2}(N_1^e + N_2^e)rQdr = Q\int_{r_1}^{r_2}rdr = Q\frac{r_2^2 - r_1^2}{2} = Q\frac{(r_2+r_1)(r_2-r_1)}{2} = \frac{Q(r_2+r_1)h}{2}.$$ En lumpad lastvektor fås om lasten fördelas lika på de två noderna: $f^e \approx \dfrac{Q(r_2+r_1)h}{4}\begin{bmatrix}1\\1\end{bmatrix}$

<u>**Lösning 1d:**</u> Om vi numrerar noderna från vänster ($r = 0$) till höger ($r = b$) är alla basfunktioner utom den sista 0 vid $r = b$, så $\boldsymbol{N}(b) = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$. Bidraget till lastvektorn blir då (se lösning 1b)

$\alpha b\begin{bmatrix}0\\0\\1\end{bmatrix}u_0 = \begin{bmatrix}0\\0\\534{,}6\end{bmatrix}$ och styvhetsmatrisen får bidraget

$$\alpha b\boldsymbol{N}^{\mathrm{T}}(b)\boldsymbol{N}(b) = \alpha b\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix} = \alpha b\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 3{,}564\end{bmatrix}$$

2

FE–formuleringen av de randvärdesproblem som behandlats i kursen leder fram till ett ekvationssystem $Ka = f$, där styvhetsmatrisen blir symmetrisk om testfunktionerna väljs enligt Galerkins metod; väsentliga och/eller konvektiva randvillkor gör att matrisen också blir positivt definit.

a: Visa att FE–approximationen minimerar den potentiella energin $\Pi(x) = \frac{1}{2}x^{T}Kx - x^{T}f$, dvs att $\Pi(a) < \Pi(x) \quad \forall x \neq a$. (2p)

b: Visa att lösningen är unik, dvs att Π bara har ett minimum. (1p)

Lösning 2a: Den potentiella energin i FE–approximationen är $\Pi(a) = \frac{1}{2}a^{T}Ka - a^{T}f$. Låt v vara en godtycklig vektor och $\varepsilon \ll 1$ vara ett litet reellt tal, och beräkna potentiella energin efter en liten godtycklig variation εv:

$$\Pi(a + \varepsilon v) = \frac{1}{2}(a + \varepsilon v)^{T}K(a + \varepsilon v) - (a + \varepsilon v)^{T}f =$$
$$= \frac{1}{2}(a^{T}Ka + \varepsilon v^{T}Ka + \varepsilon a^{T}Kv + \varepsilon^{2}v^{T}Kv) - a^{T}f - \varepsilon v^{T}f$$

Men $a^{T}Kv = (a^{T}Kv)^{T} = (Kv)^{T}a = v^{T}K^{T}a = v^{T}Ka$, eftersom K är symmetrisk; vi har då

$$\Pi(a + \varepsilon v) = \frac{1}{2}a^{T}Ka - a^{T}f + \frac{\varepsilon^{2}}{2}v^{T}Kv + \varepsilon v^{T}Ka - \varepsilon v^{T}f = \Pi(a) + \frac{\varepsilon^{2}}{2}v^{T}Kv + \varepsilon v^{T}(Ka - f)$$

Förändringen i potentiell energi är därmed

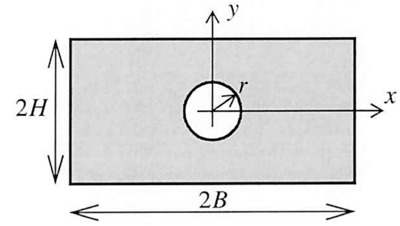
$\partial\Pi(a) = \Pi(a + \varepsilon v) - \Pi(a) = \frac{\varepsilon^{2}}{2}v^{T}Kv + \varepsilon v^{T}(Ka - f) \approx \varepsilon v^{T}(Ka - f)$, där vi försummar ε^{2}–termen eftersom $\varepsilon \ll 1$. Eftersom $Ka = f \Rightarrow Ka - f = \mathbf{0}$, ser vi att $\partial\Pi(a) = 0$, dvs FE–approximationen gör potentiella energin stationär. För att se att den stationära punkten är ett minimum beräknar vi andra–variationen: $\partial^{2}\Pi(a) = \partial\Pi(a + \varepsilon v) - \partial\Pi(a) = \varepsilon v^{T}(K(a + \varepsilon v) - f) - \varepsilon v^{T}(Ka - f) = \varepsilon^{2}v^{T}Kv > 0$, där den sista olikheten följer av att K är positivt definit. Av $\partial^{2}\Pi(a) > 0$ följer att den stationära punkten är ett minimum.

Lösning 2b: Antag att a_1 och a_2 minimerar Π. Vi har då att

$\partial\Pi(a_1) = \varepsilon v^{T}(Ka_1 - f) = 0 \Rightarrow Ka_1 = f$ och på samma sätt att $Ka_2 = f$. Subtraktion ger $Ka_1 - Ka_2 = \mathbf{0}$, eller $K(a_1 - a_2) = \mathbf{0}$. Eftersom K är positivt definit är enda lösningen $a_1 - a_2 = \mathbf{0}$, varvid vi har att $a_1 = a_2$ — dvs Π har bara ett minimum.

3

Betrakta ett rektangulärt område innehållande ett hål och ett koordinatsystem (x, y), sådant att geometrin är symmetrisk med avseende på såväl x som y. Området är utsatt för en temperaturbelastning $T(x, y)$ som också är symmetrisk med avseende på koordinataxlarna, dvs vi har att $T(x, y) = T(-x, y) = T(x, -y)$. Den svaga formen (variationsproblemet) av detta elasticitetsproblem kan skrivas



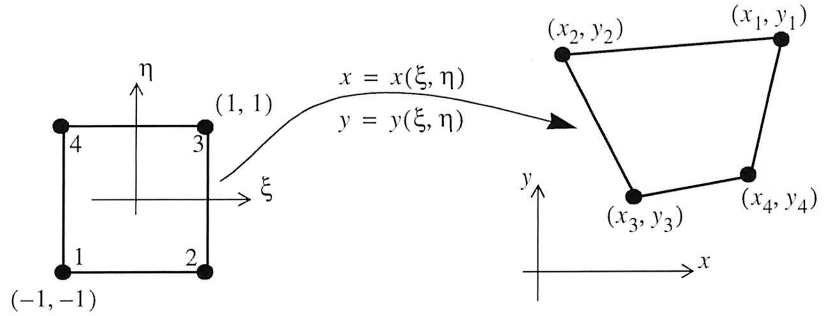
$$\int_A (\tilde{\nabla} \boldsymbol{v})^T \boldsymbol{D}(\tilde{\nabla} \boldsymbol{u}) dA = \int_A \alpha \cdot \Delta T (\tilde{\nabla} \boldsymbol{v})^T \boldsymbol{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} dA \qquad \tilde{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^T$$

där α är längdutvidgningskoefficienten, $\boldsymbol{D}$ Hooke–matrisen, $\Delta T = T - T_0$ (T_0 är den temperatur vid vilken konstruktionen är spänningsfri), $\boldsymbol{v}$ innehåller testfunktionerna (viktsfunktionerna), och $\boldsymbol{u} = \begin{bmatrix} u_x & u_y \end{bmatrix}^T$ är de obekanta förskjutningarna i respektive koordinatriktning.

a: FE–formulera problemet med testfunktioner enligt Galerkins metod. Det ska framgå hur den obekanta förskjutningsvektorn $\boldsymbol{u}$ approximeras samt hur man får tillräckligt många ekvationer för att bestämma samtliga obekanta nodvariabler. Visa också hur $\boldsymbol{B}^e$–matrisen ser ur för ett element med n noder. (3p)

b: För att spara beräkningsarbete (eller få bättre nogrannhet för ett givet arbete) kan man utnyttja dubbelsymmetrin. Skissa upp en fjärdedel av området och ange i figuren samtliga randvillkor som gäller för problemet. (2p)

c: Antag att vi ska använda isoparametriska bi–lineära element för att diskretisera problemet. Visa hur de derivator som behövs för att ställa upp $\boldsymbol{B}^e$ beräknas. Du behöver inte ange explicita uttryck för bas– eller formfunktioner, men tänk på att definiera alla beteckningar du inför. (2p)



d: För att den isoparametriska avbildningen ska bli entydig, krävs att $\det \boldsymbol{J} \neq 0$; vilka begränsningar ställer detta på tillåtna elementgeometrier. (1p)

e: För att integrera fram styvhetsmatrisen för det isoparametriska elementet ger vanligtvis 2×2 gausspunkter tillräcklig nogrannhet, men ibland används så kallad reducerad integration. Beskriv kortfattat varför man skulle kunna vilja använda reducerad integration och vilken risk man löper med detta. (2p)

Lösning 3a: Approximera $u_x \approx u_{xh} = a_{x1}N_1 + a_{x2}N_2 + \ldots + a_{xn}N_n$ och $u_y \approx u_{yh} = a_{y1}N_1 + a_{y2}N_2 + \ldots + a_{yn}N_n$. Om vi samlar de $2n$ nodvariablerna i en vektor $\boldsymbol{a}$ och basfunktionerna i en matris $\boldsymbol{N}$ enligt

$$\boldsymbol{a} = \begin{bmatrix} a_{x1} & a_{y1} & a_{x2} & a_{y2} & \ldots & a_{xn} & a_{yn} \end{bmatrix}^{\mathrm{T}} \qquad \boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \ldots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \ldots & 0 & N_n \end{bmatrix}$$

kan vi skriva $\boldsymbol{u} \approx \boldsymbol{u}_h = \begin{bmatrix} u_{xh} \\ u_{yh} \end{bmatrix} = \boldsymbol{Na}$. Approximationen sätts in i variationsproblemet och vi väljer sedan $\boldsymbol{v} = \begin{bmatrix} v_x & v_y \end{bmatrix}^{\mathrm{T}}$ på $2n$ olika sätt, så att vi får lika många ekvationer som obekanta nodvariabler. Galerkins metod: välj $\boldsymbol{v} = \begin{bmatrix} \mathrm{N}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{N}_1 \end{bmatrix}, \begin{bmatrix} \mathrm{N}_2 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{N}_n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{N}_n \end{bmatrix}$. Insättning ger nu

$$\int_{\mathrm{A}} (\tilde{\nabla}\boldsymbol{N})^{\mathrm{T}} \boldsymbol{D} (\tilde{\nabla}\boldsymbol{N}) dA \boldsymbol{a} = \int_{\mathrm{A}} \alpha \cdot \Delta T (\tilde{\nabla}\boldsymbol{N})^{\mathrm{T}} \boldsymbol{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} dA$$

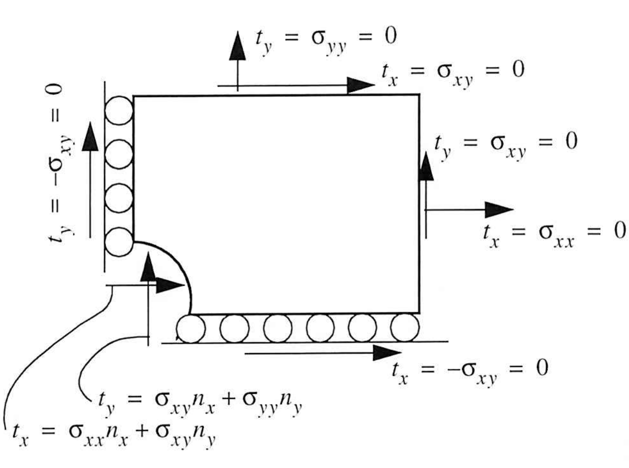
För ett element med m noder, samlar vi de $2m$ nodvariablerna i en vektor

$\boldsymbol{a}^{\mathrm{e}} = \begin{bmatrix} a_{x1}^{\mathrm{e}} & a_{y1}^{\mathrm{e}} & \ldots & a_{xm}^{\mathrm{e}} & a_{ym}^{\mathrm{e}} \end{bmatrix}^{\mathrm{T}}$ och motsvarande basfunktioner i matrisen $\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & 0 & \ldots & N_m^{\mathrm{e}} & 0 \\ 0 & N_1^{\mathrm{e}} & \ldots & 0 & N_m^{\mathrm{e}} \end{bmatrix}$, så att approximationen på elementet kan skrivas $\boldsymbol{u}_h = \boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$. Vi har då $\tilde{\nabla}\boldsymbol{u}_h = \tilde{\nabla}(\boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}) = \boldsymbol{B}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$, där alltså

$$\boldsymbol{B}^{\mathrm{e}} = \tilde{\nabla}\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} \frac{\partial N_1^{\mathrm{e}}}{\partial x} & 0 & \ldots & \frac{\partial N_m^{\mathrm{e}}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{\mathrm{e}}}{\partial y} & \ldots & 0 & \frac{\partial N_m^{\mathrm{e}}}{\partial y} \\ \frac{\partial N_1^{\mathrm{e}}}{\partial y} & \frac{\partial N_1^{\mathrm{e}}}{\partial x} & \ldots & \frac{\partial N_m^{\mathrm{e}}}{\partial y} & \frac{\partial N_m^{\mathrm{e}}}{\partial x} \end{bmatrix}$$

Lösning 3b: Längs den horisontella symmetrilinjen har vi $u_y = 0$ och $t_x = 0$ (dvs ingen randlast i x–led); längs den vertikala symmetrilinjen gäller $u_x = 0$ samt $t_y = 0$. Resten av randen är fri och obelastad, så här gäller att normal– och tangentialspänningarna ska vara noll:

$$\boldsymbol{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



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Lösning 3c: Isoparametriska element innebär att basfunktionerna $N_i^e(\xi, \eta)$ används som form-funktioner — avbildningen görs då som $x(\xi, \eta) = \sum_{i=1}^{4} x_i N_i^e$ $\quad y(\xi, \eta) = \sum_{i=1}^{4} y_i N_i^e$, där (x_i, y_i) är koordinaten för nod i. Eftersom basfunktionerna är polynom, beräknar vi enkelt derivatorna

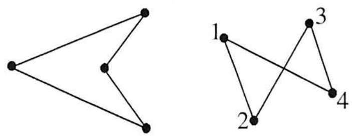
$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} x_i \frac{\partial N_i^e}{\partial \xi}$, $\frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} x_i \frac{\partial N_i^e}{\partial \eta}$ och motsvarande för derivatorna av y.

Basfunktionernas derivator med avseende på x och y (som behövs för att ställa upp B^e) fås med kedjeregeln:

$$\begin{aligned}\frac{\partial N_i^e}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i^e}{\partial y} \\ \frac{\partial N_i^e}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial y}\end{aligned} \Rightarrow \begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix} \text{ där alltså } J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \text{ Med } J^{-1} = \frac{1}{\det J}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix},$$

$$\det J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}, \text{ fås då de sökta derivatorna som } \begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix}.$$

Lösning 3d: Inre öppningsvinklar måste vara mindre än $180°$. Noderna måste numreras i samma ordning i de båda koordinatsystemen (praxis är moturs ordning). Figuren visar exempel på otillåtna elementgeometrier.



Lösning 3e: En konform FE–approximation är för styv, dvs potentiella energin blir större än i den exakta lösningen (alt. det går åt mer energi för att få givna nodförskjutningar). Genom att använda reducerad integration kan man sänka egenvärdena hos elementstyvhetsmatrisen och därmed få en mindre styv lösning. Man riskerar dock att styvhetsmatrisen blir singulär även om man har tillräckligt med randvillkor för att förhindra stelkroppsförskjutningar och stelkroppsrotation.

Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION
AUGUST 30 2006

Time and location:	$14^{\underline{00}}$ – $18^{\underline{00}}$ in the V building. Teacher will be present at about 15 and 17.
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that August 31. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses.*
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments**.
Results:	will be posted at the Department of Applied Mechanics no later than September 11. Results are sent for registration Friday September 15 — **course participants that have not completed the computer assignments by this time will be registered as *not approved.*** You may scrutinize the correction (mark up) of your written examination Monday September 11 $13^{\underline{00}}$–$15^{\underline{00}}$ and Tuesday September 12 $12^{\underline{00}}$–$14^{\underline{00}}$ (at the office space of Department of Applied Mechanics).

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

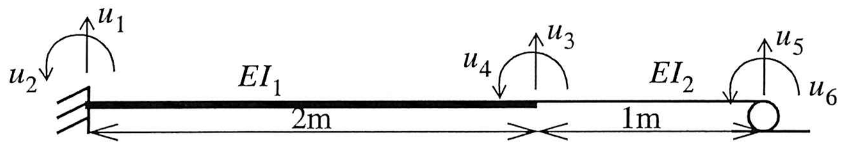
1 (svensk uppgiftstext nedan)

The plane beam shown in the figure is assumed to be fixed at the left–hand end and simply supported at the right–hand end. It is discretized by 2 C^1–continuous beam elements and the bending stiffnesses are $EI_1 = 2000\ \text{Nm}^2$ and $EI_2 = 1000\ \text{Nm}^2$, respectively. The global degree of freedom numbering is shown in the illustration.

a: The structure stiffness matrix becomes $\boldsymbol{K} = 1000\begin{bmatrix} 3 & 3 & -3 & 3 & 0 & 0 \\ 3 & 4 & -3 & 2 & 0 & 0 \\ ? & -3 & ? & 3 & -12 & 6 \\ 3 & 2 & 3 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$. Determine the stiffnesses K_{31} and K_{33}. (2p)

b: Given the prescribed displacement $u_3 = 0.02$ (and no other load acting on the structure), calculate the other degrees of freedom. (2p)

c: Explain the requirements for a beam element to be complete and compatible. (2p)



1 (english text above)

Balken i figuren är fast inspänd i vänster ände och fritt upplagd i den högra. Den diskretiseras med 2 C^1–kontinuerliga element, med böjstyvheterna $EI_1 = 2000\ \text{Nm}^2$ respektive $EI_2 = 1000\ \text{Nm}^2$. Figuren anger den globala numreringen av frihetsgraderna.

a: Strukturstyvhetsmatrisen blir $\boldsymbol{K} = 1000\begin{bmatrix} 3 & 3 & -3 & 3 & 0 & 0 \\ 3 & 4 & -3 & 2 & 0 & 0 \\ ? & -3 & ? & 3 & -12 & 6 \\ 3 & 2 & 3 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$. Bestäm matriselementen K_{31} och K_{33}. (2p)

b: Givet en påtvingad förskjutning $u_3 = 0.02$ (och ingen belastning i övrigt), bestäm övriga frihetsgrader. (2p)

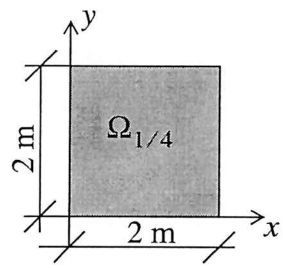
c: Ange kraven för att ett balkelement ska vara komplett och kompatibelt. (2p)

2 (svensk uppgiftstext nedan)

Consider a square membrane with side length 4 m and loaded with a distributed load $q = 150\ \mathrm{N/m^2}$ in the z-direction; the membrane is pre-tensioned with a force $S = 1000\ \mathrm{N/m}$. The deflection $w(x)$ in z-direction of the membrane, satisfies the boundary value problem

$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_g \end{cases}$$

a: Using the symmetry of the problem, we can settle with considering only $1/4th$ of the domain. State how to formulate the boundary value problem so as to account for the symmetry in this manner. (1p)

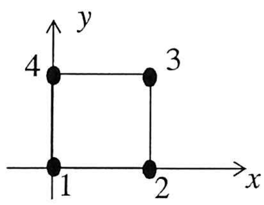


b: Use the Green–Gauss theorem, i.e.

$$\int_{\Omega} \psi\ \mathrm{div}(\boldsymbol{q})d\Omega = \oint_{\Gamma} \psi \boldsymbol{q}^T \boldsymbol{n} d\Gamma - \int_{\Omega} (\nabla\psi)^T \boldsymbol{q} d\Omega$$ (where ψ is a scalar function, $\boldsymbol{q}$ is a vector valued function, and $\boldsymbol{n}$ is an out-ward normal of Ω), to derive the variational formulation of the problem. Regularity conditions of involved functions should be explicitly given, and it should be clear how the boundary conditions affect the variational problem. (2p)

c: Make a finite element formulation of the problem with test (weight) functions according to the Galerkin method. Show what the $\boldsymbol{B}^e$ – matrix looks like, for an element with N_e, basis functions. (2p)

d: Assume that the domain is discretized with a single 4-noded quadrilateral element. With node numbering according to the figure, the basis functions becomes $N_1 = \frac{1}{4}(2-x)(2-y)$,



$N_2 = \frac{1}{4}(2+x)(2-y)$, $N_3 = \frac{1}{4}(2+x)(2+y)$ and

$N_4 = \frac{1}{4}(2-x)(2+y)$. Show that this gives the equation system $\boldsymbol{Ka} = \boldsymbol{f}$, with

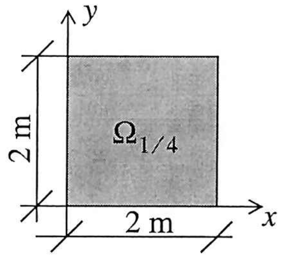
$\boldsymbol{K} = \begin{bmatrix} 2/3 \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 \end{bmatrix}$, and $\boldsymbol{f} = \begin{bmatrix} 0.15 \end{bmatrix}$. (3p)

2 (english text above)

Betrakta ett kvadratiskt membran med kantlängden 4 m som i z-led belastas med en jämnt fördelad last $q = 150\ \mathrm{N/m^2}$. Om membranet är förspänt med kraften $S = 1000\ \mathrm{N/m}$ så ges utböjningen i z-led av lösningen till randvärdesproblemet

$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{i } \Omega \\ w = 0 & \text{på } \Gamma_g \end{cases}$$

a: Genom att utnyttja symmetrin i problemet, räcker det med att betrakta $1/4$—del av området. Formulera randvärdesproblemet så att symmetrin beaktas på detta sätt. (1p)

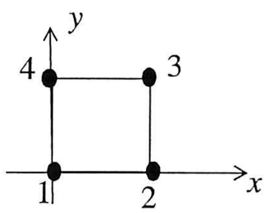


b: Använd Green–Gauss teorem, d v s

$$\int_\Omega \psi\ \mathrm{div}(\boldsymbol{q})d\Omega = \oint_\Gamma \psi \boldsymbol{q}^T\boldsymbol{n}d\Gamma - \int_\Omega (\nabla\psi)^T\boldsymbol{q}d\Omega$$ (där ψ är en skalär funktion, $\boldsymbol{q}$ en vektorvärd funktion och $\boldsymbol{n}$ en utåtriktad normal till Ω), för att härleda problemets variationsformuleringen. Regularitetskrav på inblandade funktioner ska anges. Det ska också klart framgå hur randvillkoren beaktas i variationsproblemet. (2p)

c: Finit–elementformulera problemet med test–(vikts–)funktioner enligt Galerkins metod. Visa $\boldsymbol{B}^e$ – matrisens utseende för ett element med N_e basfunktioner. (2p)

d: Antag att området diskretiseras med en enda 4–nods rektangel. Med nodnumrering enligt vidstående figur blir basfunktionerna



$N_1 = \frac{1}{4}(2-x)(2-y)$, $N_2 = \frac{1}{4}(2+x)(2-y)$,

$N_3 = \frac{1}{4}(2+x)(2+y)$ samt $N_4 = \frac{1}{4}(2-x)(2+y)$. Visa att

detta leder till ekvationssystemet $\boldsymbol{Ka} = \boldsymbol{f}$, med $\boldsymbol{K} = \begin{bmatrix}2/3\end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix}a_1\end{bmatrix}$ och $\boldsymbol{f} = \begin{bmatrix}0.15\end{bmatrix}$.

(3p)

3 (svensk uppgiftstext nedan)

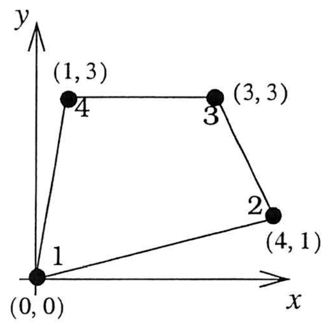
a: The basis functions of a bi–linear element are given by

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

in a local coordinate system $(-1 \le \xi \le 1, -1 \le \eta \le 1)$. Derive an expression for the derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ for the case of an isoparametric mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. (2p)

b: For the depicted element, calculate at least one of the elements in the Jacobian matrix. (2p)



c: Assume that the element i the illustration is to be used to solve a stationary heat transfer problem, where the heat supply is given by the function $Q(x, y) = y + 2x$. Use 1–point Gauss integration to calculate at least one component of the element load vector $f^e = \int_{A^e} (N^e)^T Q dxdy$. **Hint:** for the element geometry we have $\det(J) = \frac{1}{16}[(6-2\eta)(5-\xi) + 2\xi(1-\eta)]$ (2p)

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3 (english text above)

a: Basfunktionerna för ett bi–lineärt element ges av

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

i ett lokalt koordinatsystem $(-1 \le \xi \le 1, -1 \le \eta \le 1)$. Visa (härled) hur derivatorna $\frac{\partial N_i^e}{\partial x}$ och $\frac{\partial N_i^e}{\partial y}$ beräknas då avbildningen $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ är isoparametrisk. (2p)

b: Beräkna minst ett av elementen i jacobianen för elementet som visas i figuren ovan. (2p)

c: Antag att det visade elementet ska användas för att lösa ett stationärt värmeledningsproblem där värmetillförseln ges av funktionen $Q(x, y) = y + 2x$. Använd 1–punkts gaussintegration för att beräkna någon av komponenterna i elementlastvektorn

$f^e = \int_{A^e} (N^e)^T Q\,dx\,dy$. **Ledning**: för den givna elementgeometrin har vi

$$\det(\boldsymbol{J}) = \frac{1}{16}[(6-2\eta)(5-\xi)+2\xi(1-\eta)] \quad \text{(2p)}$$

Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION
AUGUST 30 2006

Solutions

Time and location:	$14^{\underline{00}}$ – $18^{\underline{00}}$ in the V building. Teacher will be present at about 15 and 17.
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that August 31. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses*.
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments**.
Results:	will be posted at the Department of Applied Mechanics no later than September 11. Results are sent for registration Friday September 15 — **course participants that have not completed the computer assignments by this time will be registered as *not approved***. You may scrutinize the correction (mark up) of your written examination Monday September 11 $13^{\underline{00}}$–$15^{\underline{00}}$ and Tuesday September 12 $12^{\underline{00}}$–$14^{\underline{00}}$ (at the office space of Department of Applied Mechanics).

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

1

The plane beam shown in the figure is assumed to be fixed at the left–hand end and simply supported at the right–hand end. It is discretized by 2 C^1–continuous beam elements and the bending stiffnesses are $EI_1 = 2000\ \mathrm{Nm}^2$ and $EI_2 = 1000\ \mathrm{Nm}^2$, respectively. The global degree of freedom numbering is shown in the illustration.

a: The structure stiffness matrix becomes $\boldsymbol{K} = 1000\begin{bmatrix} 3 & 3 & -3 & 3 & 0 & 0 \\ 3 & 4 & -3 & 2 & 0 & 0 \\ ? & -3 & ? & 3 & -12 & 6 \\ 3 & 2 & 3 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$. Determine the stiffnesses K_{31} and K_{33}. (2p)

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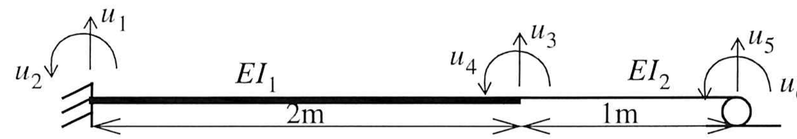

Solution 1a: From symetry we have that $K_{31} = K_{13}$, so $K_{31} = -3000$. To find K_{33}, one may use equilibrium; consider for instance a rigid body translation $u_1 = u_3 = u_5 = 1$, $u_2 = u_4 = u_6 = 0$. Then $\boldsymbol{f} = \boldsymbol{Ku} = \begin{bmatrix} 0 & 0 & (K_{31} + K_{33} - 12000) & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$; however, due to static equilibrium we have that $f_1 + f_3 + f_5 = 0$, so $K_{31} + K_{33} - 12000 = 0$, from which we get $K_{33} = 12000 - K_{31} = 15000$.

Solution 1a: Essential boundary conditions requires that $u_1 = u_2 = u_5 = 0$; since the associated basis functions do not satisfy these boundary conditions, the corresponding rows (equations) in the stiffness matrix may not be used; hence with rows/columns 1, 2 and 5 removed, with $u_3 = 0.02$ and no forces on the structure, we get

$$1000\begin{bmatrix} ? & 3 & 6 \\ 3 & 8 & 2 \\ 6 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0{,}02 \\ u_4 \\ u_6 \end{bmatrix} = \begin{bmatrix} f_3 \\ 0 \\ 0 \end{bmatrix}$$

where f_3 is the force required to obtain $u_3 = 0.02$. Multipying the first column of $\boldsymbol{K}$ by u_3 and moving the result to the right hand side, we get

$$\begin{bmatrix} 3 & 6 \\ 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_4 \\ u_6 \end{bmatrix} = \frac{1}{1000} \begin{bmatrix} f_3 - (0.02 \cdot ?) \\ -60 \\ -120 \end{bmatrix}$$

We can now solve for the unknown node variables from the last 2 equations

$$\begin{bmatrix} u_4 \\ u_6 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 4 & -2 \\ -2 & 8 \end{bmatrix} \cdot \frac{1}{1000} \begin{bmatrix} -60 \\ -120 \end{bmatrix} = \frac{1}{28000} \begin{bmatrix} 0 \\ -840 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.03 \end{bmatrix}$$

Solution 1c: Since the govening equation is a 4:th order equation, a complete element is one the can describe a constant approximation (i.e constant displacement of the beam) and a constant derivative (i.e a constant rotation). A compatible element (for a 4:th order problem) is such that the approximation and its first derivative are continuous across element boundaries — i.e. we do not allow discontinuities in displacements or rotations

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2

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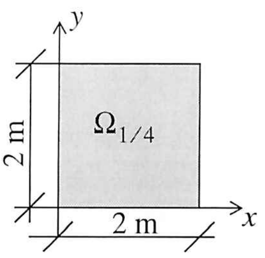

a: Using the symmetry of the problem, we can settle with considering only $1/4th$ of the domain. State how to formulate the boundary value problem so as to account for the symmetry in this manner. (1p)

b: Use the Green–Gauss theorem, i.e.

$$\int_\Omega \psi \ \mathrm{div}(\boldsymbol{q}) d\Omega = \oint_\Gamma \psi \boldsymbol{q}^T \boldsymbol{n} d\Gamma - \int_\Omega (\nabla \psi)^T \boldsymbol{q} d\Omega$$ (where ψ is a scalar function, $\boldsymbol{q}$ is a vector valued function, and $\boldsymbol{n}$ is an out-ward normal of Ω), to derive the variational formulation of the problem. Regularity conditions of involved functions should

be explicitly given, and it should be clear how the boundary conditions affect the variational problem. (2p)

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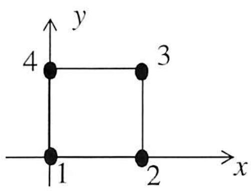
d: Assume that the domain is discretized with a single 4–noded quadrilateral element. With node numbering according to the figure, the basis functions becomes $N_1 = \frac{1}{4}(2-x)(2-y)$,

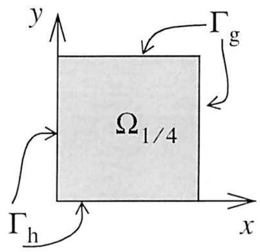


$N_2 = \frac{1}{4}(2+x)(2-y)$, $N_3 = \frac{1}{4}(2+x)(2+y)$ and

$N_4 = \frac{1}{4}(2-x)(2+y)$. Show that this gives the equation system $\boldsymbol{Ka} = \boldsymbol{f}$, with

$\boldsymbol{K} = \begin{bmatrix} 2/3 \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 \end{bmatrix}$, and $\boldsymbol{f} = \begin{bmatrix} 0.15 \end{bmatrix}$. (3p)

Solution 2a: The normal derivative of the deflection has to be zero along the lines of symmetry. Thus, the b.v.p reads



$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega_{1/4} \\ w = 0 & \text{on } \Gamma_g \\ (\nabla w)^{\mathrm{T}}\boldsymbol{n} = 0 & \text{on } \Gamma_h \end{cases}$$

where $\boldsymbol{n}$ is an out–ward unit normal vector of the boundary. The natural boundary condition may alternatively (and maybe a bit more clear) be expressed as $\frac{\partial w}{\partial x} = 0$ on $x = 0$ and $\frac{\partial w}{\partial y} = 0$ on $y = 0$.

Solution 2b: Multiply both sides of the differential equation by a test function v and integrate over the domain: $-\int_{\Omega_{1/4}} v\,\mathrm{div}(\nabla w)d\Omega = \int_{\Omega_{1/4}} v\frac{q}{S}d\Omega$. Apply the Green–Green theorem to the left hand side: $\int_{\Omega_{1/4}} (\nabla v)^{\mathrm{T}}(\nabla w)d\Omega = \int_{\Omega_{1/4}} v\frac{q}{S}d\Omega + \oint_{\Gamma} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma$. Now study the boundary integral; from the boundary condition on Γ_h, we know that the integrand is zero, but we do not know the value of $(\nabla w)^{\mathrm{T}}\boldsymbol{n}$ on Γ_g. Hence, since it is necessary to be able to evaluate the boundary term, we must enforce $v = 0$ on Γ_g. Thus,

$$\oint_{\Gamma} v(\nabla w)^{\mathrm{T}} \boldsymbol{n} d\Gamma = \int_{\Gamma_g} v(\nabla w)^{\mathrm{T}} \boldsymbol{n} d\Gamma + \int_{\Gamma_h} v(\nabla w)^{\mathrm{T}} \boldsymbol{n} d\Gamma = \int_{\Gamma_g} 0 \cdot (\nabla w)^{\mathrm{T}} \boldsymbol{n} d\Gamma + \int_{\Gamma_h} v \cdot 0 d\Gamma = 0$$

To be able to evaluate the integrals, all involved functions have to be regular enough:

$\int_{\Omega_{1/4}} v^2 d\Omega < \infty$ $\quad \int_{\Omega_{1/4}} (\nabla v)^{\mathrm{T}} (\nabla v) d\Omega < \infty$. Let V denote the space of functions that are regular enough and that fulfil $v = 0$ on Γ_g. The variational problem may now be expressed as

$$\text{Find } w \in V \text{ such that } \int_{\Omega_{1/4}} (\nabla v)^{\mathrm{T}} (\nabla w) d\Omega = \int_{\Omega_{1/4}} v \frac{q}{S} d\Omega \qquad \forall v \in V$$

Solution 2c: Approximate the unknown function by a linear combination of selected basis functions $N_i(x, y)$: $w \approx w_{\mathrm{h}} = \sum_i N_i a_i = \boldsymbol{N}\boldsymbol{a}$ ($\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & \dots \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots \end{bmatrix}^{\mathrm{T}}$). We substitute this into the variational problem and choose test functions v according to Galerkin, viz. any linear combination of basis functions. If we define the finite element space V_{h} as the space of functions that can be expressed as a linear combination of the basis functions, the FE–formulation according to Galerkin may be expressed as:

$$\text{Find } w_{\mathrm{h}} \in V_{\mathrm{h}} \text{ such that } \int_{\Omega_{1/4}} (\nabla v)^{\mathrm{T}} (\nabla w_{\mathrm{h}}) d\Omega = \int_{\Omega_{1/4}} v \frac{q}{S} d\Omega \qquad \forall v \in V_{\mathrm{h}}$$

If the N_{e} basis functions that are non–zero on an element are collected into a row vector

$\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & \dots & N_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix}$, and the corresponding node variables into a column vector

$\boldsymbol{a}^{\mathrm{e}} = \begin{bmatrix} a_1^{\mathrm{e}} & \dots & a_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix}^{\mathrm{T}}$, then the FE–approximation on the element becomes $w_{\mathrm{h}} = \boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$. On the element we thus have $\nabla w_{\mathrm{h}} = \nabla(\boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}) = (\nabla \boldsymbol{N}^{\mathrm{e}})\boldsymbol{a}^{\mathrm{e}} = \boldsymbol{B}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$, where, hence,

$$\boldsymbol{B}^{e} = \nabla \boldsymbol{N}^{e} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} N_1^e & \dots & N_{N_e}^e \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \dots & \frac{\partial N_{N_e}^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \dots & \frac{\partial N_{N_e}^e}{\partial y} \end{bmatrix}$$

Solution 2d: From the finite element formulation we have $\int_{\Omega_{1/4}} (\nabla v)^{T}(\nabla \boldsymbol{N}) d\Omega \boldsymbol{a} = \int_{\Omega_{1/4}} v\frac{q}{S} d\Omega$ with $v = N_1$ (the choices $v = N_2$, $v = N_3$ and $v = N_4$ are not valid, since N_2, N_3 and N_4 do not satisfy the essential boundary condition). Furthermore, since the essential boundary condition requires that $a_2 = a_3 = a_4 = 0$, we obtain

$\int_{\Omega_{1/4}} (\nabla N_1)^{T}(\nabla N_1) d\Omega a_1 = \int_{\Omega_{1/4}} N_1 \frac{q}{S} d\Omega$. Now, $\nabla N_1 = \begin{bmatrix} \frac{y-2}{4} \\ \frac{x-2}{4} \end{bmatrix}$, so

$$\int_{\Omega_{1/4}} (\nabla N_1)^{T}(\nabla N_1) d\Omega = \frac{1}{16} \int_{\Omega_{1/4}} [(y-2)^2 + (x-2)^2] d\Omega = \int_0^2 \int_0^2 (y^2 + x^2 + 4y + 4x - 8) dx dy$$

$$= \frac{2}{3}$$

Also, the right hand side becomes $\int_{\Omega_{1/4}} N_1 \frac{q}{S} d\Omega = \frac{q}{4S} \int_0^2 \int_0^2 (2-x)(2-y) dx dy = 0.15$.

Hence, we have $\frac{2}{3} a_1 = 0{,}15$.

3

a: The basis functions of a bi–linear element are given by

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

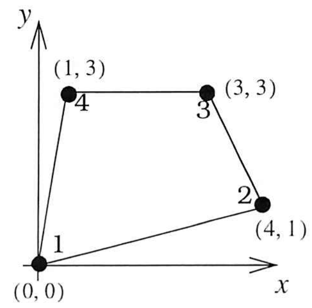
$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

in a local coordinate system $(-1 \le \xi \le 1, -1 \le \eta \le 1)$. Derive an expression for the derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ for the case of an isoparametric mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$.

(2p)

b: For the depicted element, calculate at least one of the elements in the Jacobian matrix. (2p)

c: Assume that the element i the illustration is to be used to solve a stationary heat transfer problem, where the heat supply is given by the function $Q(x, y) = y + 2x$. Use 1–point Gauss integration to calculate at least one component of the element load vector $f^e = \int_{A^e} (N^e)^T Q dxdy$. **Hint**: for the element geometry we have $\det(J) = \frac{1}{16}[(6-2\eta)(5-\xi) + 2\xi(1-\eta)]$ (2p)



Solution 3a: Use the chain rule to calculate the derivatives with respect to ξ and η:

$$\left.\begin{aligned}\frac{\partial N_i^e}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i^e}{\partial y}\\ \frac{\partial N_i^e}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial y}\end{aligned}\right\} = J\begin{bmatrix}\frac{\partial N_i^e}{\partial x}\\ \frac{\partial N_i^e}{\partial y}\end{bmatrix}$$

where, hence, $J = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}$. We then get the desired derivatives by multiplying both sides by J^{-1}: $\begin{bmatrix}\frac{\partial N_i^e}{\partial x}\\ \frac{\partial N_i^e}{\partial y}\end{bmatrix} = J^{-1}\begin{bmatrix}\frac{\partial N_i^e}{\partial \xi}\\ \frac{\partial N_i^e}{\partial \eta}\end{bmatrix}$. The components of J are easily calculate as $\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\sum_i x_i N_i^e = \sum_i x_i \frac{\partial N_i^e}{\partial \xi}$, etc.; (x_i, y_i) is the coordinate of node i on the element.

Solution 3b: We have $(x_1, x_2, x_3, x_4) = (0, 4, 3, 1)$ and $(y_1, y_2, y_3, y_4) = (0, 1, 3, 3)$, while the shape function derivatives are

$$\frac{\partial N_1}{\partial \xi} = \frac{-1}{4}(1-\eta) \quad \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta) \quad \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta) \quad \frac{\partial N_4}{\partial \xi} = \frac{-1}{4}(1+\eta)$$

$$\frac{\partial N_1}{\partial \eta} = \frac{-1}{4}(1-\xi) \quad \frac{\partial N_2}{\partial \eta} = \frac{-1}{4}(1+\xi) \quad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi) \quad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)$$

Hence,

$$\frac{\partial x}{\partial \xi} = \sum_i x_i \frac{\partial N_i^e}{\partial \xi} = \frac{1}{4}[-0\cdot(1-\eta) + 4\cdot(1-\eta) + 3\cdot(1+\eta) - 1\cdot(1+\eta)] = \frac{1}{4}(6-2\eta),$$

$$\frac{\partial y}{\partial \xi} = \sum_i y_i \frac{\partial N_i^e}{\partial \xi} = \frac{1}{4}[-0 \cdot (1-\eta) + 1 \cdot (1-\eta) + 3 \cdot (1+\eta) - 3 \cdot (1+\eta)] = \frac{1}{4}(1-\eta),$$

$$\frac{\partial x}{\partial \eta} = \sum_i x_i \frac{\partial N_i^e}{\partial \eta} = \frac{1}{4}[-0 \cdot (1-\xi) - 4 \cdot (1+\xi) + 3 \cdot (1+\xi) + 1 \cdot (1-\xi)] = \frac{1}{4}(0-2\xi), \text{ and}$$

$$\frac{\partial y}{\partial \eta} = \sum_i y_i \frac{\partial N_i^e}{\partial \eta} = \frac{1}{4}[-0 \cdot (1-\xi) - 1 \cdot (1+\xi) + 3 \cdot (1+\xi) + 3 \cdot (1-\xi)] = \frac{1}{4}(5-\xi), \text{ so}$$

$$\boldsymbol{J} = \frac{1}{4}\begin{bmatrix} (6-2\eta) & (1-\eta) \\ -2\xi & (5-\xi) \end{bmatrix}$$

Solution 3c: With Gauss integration we approximate

$f^e = \int_{A^e} (\boldsymbol{N}^e)^T Q dxdy \approx \sum_i \sum_j H_i H_j (\boldsymbol{N}^e(\xi_i, \eta_j))^T Q(\xi_i, \eta_j) \det\boldsymbol{J}(\xi_i, \eta_j)$. With a single point, the integration weight is $H_1 = 2$, so $H_i H_j = H_1 H_1 = 4$. Furthermore, the integration point is at the centre, i.e $(\xi_i, \eta_j) = (\xi_1, \eta_1) = (0, 0)$. Thus, $\boldsymbol{N}^e(0, 0) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ and $\det\boldsymbol{J}(0, 0) = \frac{15}{8}$. To calculate the value of Q at the integration point, we first determine the (x, y)–coordinate that corresponds to $(\xi, \eta) = (0, 0)$:

$$x(0, 0) = \sum_i x_i N_i^e(0, 0) = \frac{1}{4}(0+4+3+1) = 2$$

$$y(0, 0) = \sum_i y_i N_i^e(0, 0) = \frac{1}{4}(0+1+3+3) = \frac{7}{4}$$

Therefore, $Q = y + 2x = \frac{23}{4}$. So finally we have

$$f^e = \int_{A^e} (\boldsymbol{N}^e)^T Q dxdy \approx 4 \cdot \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{23}{4} \cdot \frac{15}{8} = \frac{345}{32}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION

MARCH 9 2006

Time and location:	$8^{\underline{30}}$ - $12^{\underline{30}}$ in the V building. Teacher will be present at about $9^{\underline{30}}$ and $11^{\underline{30}}$.
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that March 10. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses.*
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments**.
Results:	will be posted at the Department of Applied Mechanics no later than March 20. Results are sent for registration Friday March 24 — **course participants that have not completed the computer assignments by this time will be registered as *not approved*.** You may scrutinize the correction (mark up) of your written examination Tuesday March 21 $13^{\underline{00}}$–$15^{\underline{00}}$ and Thursday March 23 $10^{\underline{00}}$–$12^{\underline{00}}$ (at the office space of Department of Applied Mechanics).

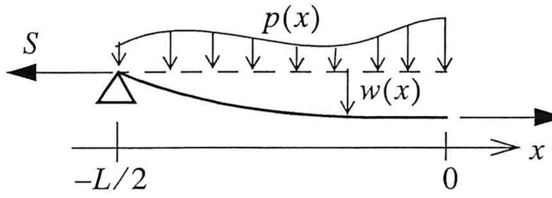
Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

1 (svensk uppgiftstext nedan)

Consider a string of length L that has been pre–tensioned by a force S and that is loaded by a transverse load with intensity $p(x)$. Provided that the problem is symmetric, the deflection $w(x)$ is given by the solution of the boundary value problem

$$(1)\quad \begin{cases} -\dfrac{d^2w}{dx^2} = \dfrac{p}{S} & -\dfrac{L}{2} < x < 0 \\[2ex] w\left(-\dfrac{L}{2}\right) = 0 & \left.\dfrac{dw}{dx}\right|_{x=0} = 0 \end{cases}$$



a: Derive the weak form (variational formulation) of the problem (1) and make a finite element formulation with test (weight) functions according to Galerkin. It shall be clearly shown how the boundary conditions affect the formulation. Also state regularity requirements of involved functions. (3p)

b: The element stiffness matrix for a linear element becomes $\boldsymbol{K}^e = \dfrac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, where h is the element length. Solve the problem with a single linear element. (1p)

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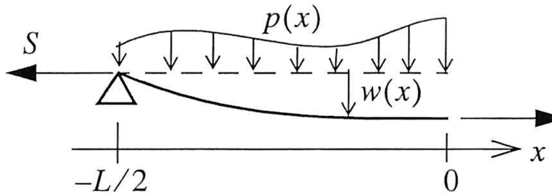
1 (english text above)

Betrakta en lina med längden L som har sträckts med en kraft S och sedan belastas med en transversal last med intensiteten $p(x)$. Om problemet är symmetriskt, så är utböjningen $w(x)$ lösningen till randvärdesproblemet

$$(1)\quad \begin{cases} -\dfrac{d^2w}{dx^2} = \dfrac{p}{S} & -\dfrac{L}{2} < x < 0 \\[2ex] w\left(-\dfrac{L}{2}\right) = 0 & \left.\dfrac{dw}{dx}\right|_{x=0} = 0 \end{cases}$$



a: Härled den svaga formen (variationsformuleringen) av (1), och gör en finit–elementformulering med testfunktioner (viktsfunktioner) enligt Galerkins metod. Det ska klar framgå hur randvillkoren påverkar formuleringen. Ange också regularitetskrav på funktionerna. (3p)

b: Elementstyvhetsmatrisen för ett lineärt element blir $\boldsymbol{K}^e = \dfrac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, där h är elementlängden. Lös problemet med ett lineärt element. (1p)

2 (svensk uppgiftstext nedan)

Consider a finite element approximation of a 2D elasticity problem.

a: Describe briefly what is meant by convergence, and two different methods to achieve convergence. (2p)

b: If there is a singular point present in the exact solution, its strength will govern the rate of convergence. Give examples of possible singular points. (1p)

c: Describe the requirements that the FE–approximation have to fulfil in order for the approximation to converge, and give a physical interpretation of these requirements. (2p)

---

2 (english text above)

Betrakta en finit–elementapproximation av ett 2D elasticitetsproblem.

a: Beskriv kortfattat vad som menas med konvergens samt två olika metoder som ger konvergens. (2p)

b: Om den exakta lösningen har en singulär punkt så kommer singularitetens styrka att styra konvergenshastigheten. Ge exempel på möjliga singulära punkter. (1p)

c: Ange de krav FE–approximationen måste uppfylla för att man säkert ska få konvergens, samt ge en fysikalisk tolkning av kraven. (2p)

---

3 (svensk uppgiftstext nedan)

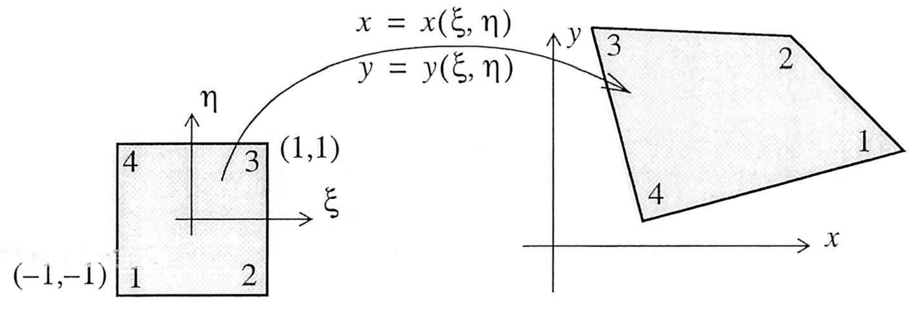
a: The basis functions of a bi–linear element are given by

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

in a local coordinate system (ξ,η). Derive an expression for the derivatives $\dfrac{\partial N_i^e}{\partial x}$ and $\dfrac{\partial N_i^e}{\partial y}$ for the case of an isoparametric mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. (2p)

b: Give a couple of examples of cases where the mapping is not unique, i.e. cases where $\det(J) = 0$ somewhere inside an element. (1p)



3 (english text above)

a: Basfunktionerna för ett bi-lineärt element ges av

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

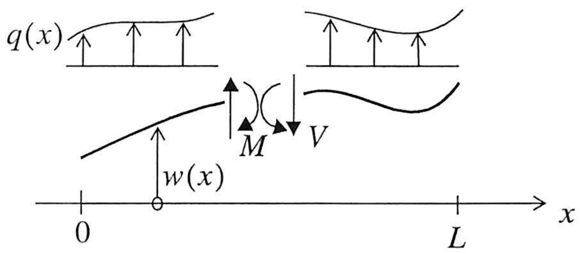
$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

i ett lokalt koordinatsystem (ξ,η). Visa (härled) hur derivatorna $\frac{\partial N_i^e}{\partial x}$ och $\frac{\partial N_i^e}{\partial y}$ beräknas då avbildningen $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ är isoparametrisk (illustration på föregående sida). (2p)

b: Ge ett par exempel på situationer där avbildningen inte är unik, d.v.s då $\det(\boldsymbol{J}) = 0$ någonstans inom ett element. (1p)

4 (svensk uppgiftstext nedan)

According to the Euler–Bernoulli theory, the deflection $w = w(x)$ of a beam is governed by the equation



$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = q \qquad 0 < x < L$$

where EI is the bending stiffness of the beam and $q = q(x)$ is the load intensity (force/length). When the solution is at hand, one may calculate the cross-sectional moment $M(x)$ and shear force $V(x)$ through

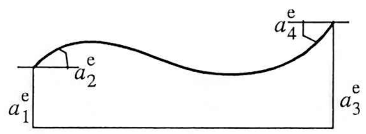
$$M = -EI\frac{d^2w}{dx^2} \qquad V = -\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]$$

a: Derive the weak form of the beam equation. Make a finite-element formulation and show what the $\boldsymbol{B}$-matrix looks like. Leave the boundary conditions unspecified for now. (2p)

b: Specify the requirements for a beam element to be conform (complete an compatible). (2p)

c: The most common beam element has four cubic basis functions, so that

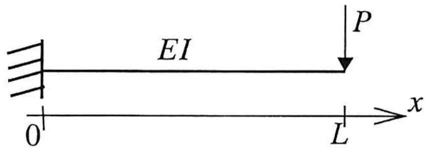
$$w \approx w_h = N_1^e a_1^e + N_2^e a_2^e + N_3^e a_3^e + N_4^e a_4^e$$



on one element; if the bending stiffness is constant and the element length is L, the element stiffness matrix becomes

$$K^{e} = \frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & \frac{-12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & \frac{-6}{L} & 2 \\ \frac{-12}{L^2} & \frac{-6}{L} & \frac{12}{L^2} & \frac{-6}{L} \\ \frac{6}{L} & 2 & \frac{-6}{L} & 4 \end{bmatrix}$$

For the problem depicted in the illustration: establish the FE–equation system $\boldsymbol{Ka} = \boldsymbol{f}$ using a single element — describe how you obtained the system. (2p)



d: Consider the potential energy $\Pi(w_h)$ in the FE–solution of the problem in the previous subtask: do you expect it to be larger, smaller or equal to the potential energy in the exact solution? Motivate your answer. (2p)

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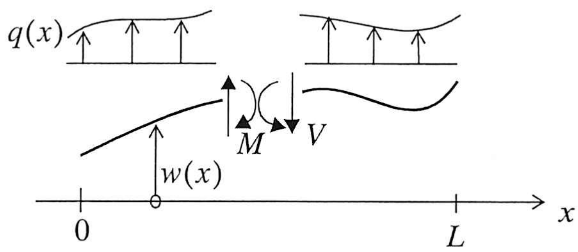
4 (english text above)

Enligt Euler–Bernoullis balkteori, så ges förskjutningen $w = w(x)$ av lösningen till differentialekvationen



$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = q \qquad 0 < x < L$$

där EI är balkens böjstyvhet och $q = q(x)$ är lastintensiteten (kraft/längd). När förskjutningen beräknats, kan snittkrafterna — momentet $M(x)$ och tvärkraften $V(x)$ — beräknas enligt

$$M = -EI\frac{d^2w}{dx^2} \qquad V = -\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]$$

a: Härled svaga formen av balkekvationen. Gör en FE–formulering och visa hur problemets $\boldsymbol{B}$-matris ser ut. (2p) b: Ange villkoren som måste vara uppfyllda för att ett balkelement ska vara konformt (komplett och kompatibelt). (2p)

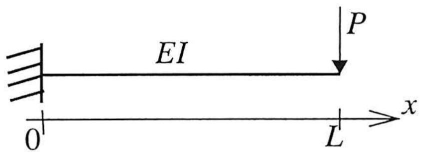
c: Det vanligaste balkelementet har fyra kubiska basfunktioner, så att

$$w \approx w_h = N_1^e a_1^e + N_2^e a_2^e + N_3^e a_3^e + N_4^e a_4^e$$

på ett element. Om elementet har längden L och böjstyvheten är konstant, blir elementstyvhetsmatrisen

$$\boldsymbol{K}^e = \frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & \frac{-12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & \frac{-6}{L} & 2 \\ \frac{-12}{L^2} & \frac{-6}{L} & \frac{12}{L^2} & \frac{-6}{L} \\ \frac{6}{L} & 2 & \frac{-6}{L} & 4 \end{bmatrix}$$

Etablera ekvationssystemet $\boldsymbol{K a} = \boldsymbol{f}$ för problemet som visas i figuren; använd ett element. Förklara hur du får fram ekvationssystemet. (2p)



d: Betrakta den potentiella energin $\Pi(w_h)$ i FE–lösningen av problemet i föregående deluppgift: förväntar du dig att den är större, mindre eller lika stor som potentiella energin i den exakta lösningen? Motivera ditt svar. (2p)

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Solutions

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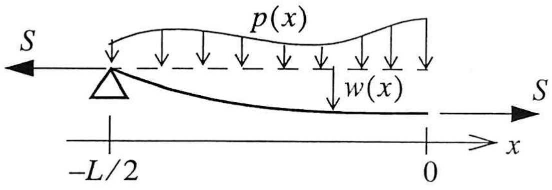
1

Consider a string of length L that has been pre–tensioned by a force S and that is loaded by a transverse load with intensity $p(x)$. Provided that the problem is symmetric, the deflection $w(x)$ is given by the solution of the boundary value problem

$$(1)\quad \begin{cases} -\dfrac{d^2 w}{dx^2} = \dfrac{p}{S} & -\dfrac{L}{2} < x < 0 \\ w\left(-\dfrac{L}{2}\right) = 0 & \left.\dfrac{dw}{dx}\right|_{x=0} = 0 \end{cases}$$



a: Derive the weak form (variational formulation) of the problem (1) and make a finite element formulation with test (weight) functions according to Galerkin. It shall be clearly shown how the boundary conditions affect the formulation. Also state regularity requirements of involved functions. (3p)

b: The element stiffness matrix for a linear element becomes $K^e = \dfrac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, where h is the element length. Solve the problem with a single linear element. (1p)

Solution 1a: Multiply the differential equation by a test function $v(x)$ and integrate over the interval: $-\displaystyle\int_{-L/2}^{0} v(x)\frac{d^2 w}{dx^2}dx = \int_{-L/2}^{0} v(x)\frac{p}{S}dx$. Partial integration of the left hand side yields

$$\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw}{dx}\,dx - \left(v(0)\frac{dw}{dx}(0) - v\left(\frac{-L}{2}\right)\frac{dw}{dx}\left(\frac{-L}{2}\right)\right) = \int_{-L/2}^{0} v(x)\frac{p}{S}dx.$$ The first boundary term vanishes, since $w'(0) = 0$; to get rid of the second boundary term, we restrict the formulation to test functions such that $v(-L/2) = 0$. Thus, defining

$$V = \left\{ v:\ v(-L/2) = 0 \qquad \int_{-L/2}^{0}\left(\frac{dv}{dx}\right)^2 dx < \infty,\quad \int_{-L/2}^{0} v^2 dx < \infty \right\},$$ we are lead to the variational problem: Find $w \in V$ such that $\displaystyle\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw}{dx}\,dx = \int_{-L/2}^{0} v\frac{p}{S}dx \qquad \forall v \in V.$

Finite element formulation: select a set of basis functions N_i and let V_h be the space of functions that can be expressed as a linear combination of the basis functions. Approximate

$w \approx w_h = \sum_i N_i a_i = \boldsymbol{N}\boldsymbol{a}$ (where, hence, $\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & \dots \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots \end{bmatrix}^{\mathrm{T}}$). Now, substitute the approximation into the variational problem and use the basis functions as test functions (i.e. the Galerkin method); we obtain:

Find $w_h \in V_h$ such that $\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw_h}{dx}\,dx = \int_{-L/2}^{0} v\frac{p}{S}\,dx \quad \forall v \in V_h$ (or

$\int_{-L/2}^{0} \left(\frac{d\boldsymbol{N}}{dx}\right)^{\mathrm{T}} \frac{d\boldsymbol{N}}{dx}\,dx\,\boldsymbol{a} = \int_{-L/2}^{0} \boldsymbol{N}^{\mathrm{T}}\frac{p}{S}\,dx$).

Solution 1b: With a single element of length $h = \frac{L}{2}$ we get $\frac{2}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{p}{S}\int_{-L/2}^{0} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$, where we used the fact that p and S are constants; a_1 and a_2 represents the transverse displacements at $x = -\frac{L}{2}$ and $x = 0$, respectively. Hence, by the condition $w\left(-\frac{L}{2}\right) = 0$ we have that $a_1 = 0$ and that the first equation is not valid (since the test function $v = N_1 \neq 0$ at $x = -\frac{L}{2}$. We are thus lead to $2\frac{a_2}{L} = \frac{p}{S}\int_{-L/2}^{0} N_2\,dx$. With $\int_{-L/2}^{0} N_2\,dx = \frac{L}{4}$ we finally obtain $a_2 = \frac{pL^2}{8S}$.

2

Consider a finite element approximation of a 2D elasticity problem.

a: Describe briefly what is meant by convergence, and two different methods to achieve convergence. (2p)

b: If there is a singular point present in the exact solution, its strength will govern the rate of convergence. Give examples of possible singular points. (1p)

c: Describe the requirements that the FE–approximation have to fulfil in order for the approximation to converge, and give a physical interpretation of these requirements. (2p)

Solution 2a: By convergence we mean that the finite element approximation gets better and better, the more degrees of freedom that is used in the discretization. This may be accomplished in two different ways. Using the so called h–method, we use more and more elements, i.e. we reduce the element size h (but retain the element type). In the so-called p–method one keeps the mesh topology, but introduces more an more basis functions on the the elements, viz. by resorting to polynomials of higher and higher degree p.

Solution 2b: In an elasticity problem, typical singular points are re–entrant corners, points of concentrated loads (forces), material interfaces at boundaries, and points where boundary conditions changes abrupt.

Solution 2c: The elements have to be complete, meaning that it should be possible to chose values for the element degrees of freedom such that the approximation becomes constant on the element, and likewise chose values such that the first derivatives become constants. Physically, for the elasticity problem, this means that the elements have to be able to represent rigid body translations (displacements and rotation) as well as a state of constant strain.

In addition, the elements have to be compatible, meaning that the approximation has to be continuous across element edges — i.e. we do not allow discontinuities in the displacements.

3

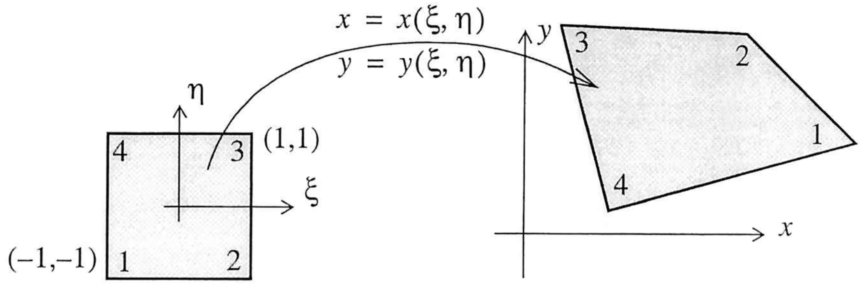
a: The basis functions of a bi–linear element are given by

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

in a local coordinate system (ξ,η). Derive an expression for the derivatives $\dfrac{\partial N_i^e}{\partial x}$ and $\dfrac{\partial N_i^e}{\partial y}$ for the case of an isoparametric mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. (2p)

b: Give a couple of examples of cases where the mapping is not unique, i.e. cases where $\det(\boldsymbol{J}) = 0$ somewhere inside an element. (1p)

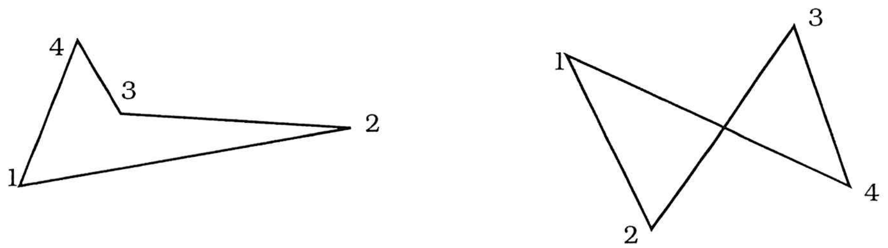


Solution 3a: Isoparametric mapping means that the basis functions are used as shape functions: $x(\xi, \eta) = \sum_{i=1}^{4} x_i N_i(\xi, \eta)$, $y(\xi, \eta) = \sum_{i=1}^{4} y_i N_i(\xi, \eta)$, where (x_i, y_i) is the coordinate

of the i:th node on the element. Hence, the derivatives $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$, and $\frac{\partial y}{\partial \eta}$ are straightforward to calculate; $\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} x_i \frac{\partial N_i}{\partial \xi}$, etc. By the chain rule we have

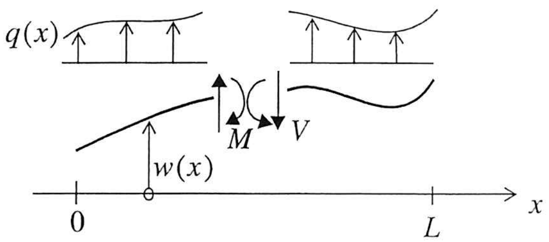
$$\begin{aligned} \frac{\partial N_i}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i}{\partial y} \end{aligned} \quad \text{or} \quad \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}, \text{ where } \boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \text{ Thus, } \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \boldsymbol{J}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

Solution 3b: For the mapping to be unique, the inverse of the Jacobian must exist, i.e. we must have $\det \boldsymbol{J} \neq 0$. This condition is not met if, for instance, the element has an re–entrant vertex or if the nodes in the element definition are numbered in the wrong order:



4

According to the Euler–Bernoulli theory, the deflection $w = w(x)$ of a beam is governed by the equation



$$\frac{d^2}{dx^2}\left[EI\frac{d^2 w}{dx^2}\right] = q \qquad 0 < x < L$$

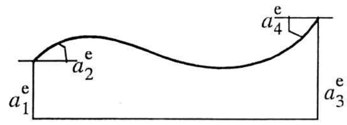
where EI is the bending stiffness of the beam and $q = q(x)$ is the load intensity (force/length). When the solution is at hand, one may calculate the cross–sectional moment $M(x)$ and shear force $V(x)$ through

$$M = -EI\frac{d^2 w}{dx^2} \qquad V = -\frac{d}{dx}\left[EI\frac{d^2 w}{dx^2}\right]$$

a: Derive the weak form of the beam equation. Make a finite–element formulation and sho what the $\boldsymbol{B}$–matrix looks like. Leave the boundary conditions unspecified for now. (2p)

b: Specify the requirements for a beam element to be conform (complete an compatible). (2p)

c: The most common beam element has four cubic basis functions, so that

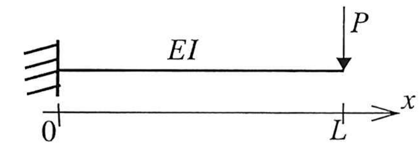


$$w \approx w_h = N_1^e a_1^e + N_2^e a_2^e + N_3^e a_3^e + N_4^e a_4^e$$

on one element; if the bending stiffness is constant and the element length is L, the element stiffness matrix becomes

$$\boldsymbol{K}^e = \frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & \frac{-12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & \frac{-6}{L} & 2 \\ \frac{-12}{L^2} & \frac{-6}{L} & \frac{12}{L^2} & \frac{-6}{L} \\ \frac{6}{L} & 2 & \frac{-6}{L} & 4 \end{bmatrix}$$

For the problem depicted in the illustration: establish the FE–equation system $\boldsymbol{Ka} = \boldsymbol{f}$ using a single element — describe how you obtained the system. (2p)



d: Consider the potential energy $\Pi(w_h)$ in the FE–solution of the problem in the previous subtask: do you expect it to be larger, smaller or equal to the potential energy in the exact solution? Motivate your answer. (2p)

Solution 4a: Multiply the differential equation by a test function $v = v(x)$ and integrate over the domain (interval)

$$\int_0^L v\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right]dx = \int_0^L vq\,dx$$

Now, twice integration by parts of the left hand side yield

$$\int_0^L EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}dx = \left[\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_0^L - \left[v\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]\right]_0^L + \int_0^L vq\,dx$$

Identifying the bending moment and shear force in the boundary terms, we may write

$$\int_0^L EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}dx = \int_0^L vq\,dx + M(0)\frac{dv}{dx}\bigg|_{x=0} - M(L)\frac{dv}{dx}\bigg|_{x=L} + V(L)v(L) - V(0)v(0)$$

Let $w \approx w_h = \begin{bmatrix} N_1 & \dots & N_n \end{bmatrix}\begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^{\mathrm{T}} = \boldsymbol{Na}$, where N_i and a_i are basis functions and node variables, respectively, be a FE–approximation; use the basis functions as test functions. We then obtain

$$\int_0^L EI\frac{d^2N_i}{dx^2}\frac{d^2\boldsymbol{N}}{dx^2}dx\boldsymbol{a} = \int_0^L N_i q\,dx + M(0)\frac{dN_i}{dx}\bigg|_{x=0} - M(L)\frac{dN_i}{dx}\bigg|_{x=L} + V(L)N_i(L) - V(0)N_i(0)$$

$(i = 1, 2, \dots, n)$. Collecting these n equations in matrix form and using

$\boldsymbol{B} = \dfrac{d^2\boldsymbol{N}}{dx^2} = \begin{bmatrix} \dfrac{d^2N_1}{dx^2} & \dots & \dfrac{d^2N_n}{dx^2} \end{bmatrix}$, we obtain

$$\int_0^L EI\,\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}dx\,\boldsymbol{a} = \int_0^L \boldsymbol{N}^{\mathrm{T}}q\,dx + M(0)\frac{d\boldsymbol{N}^{\mathrm{T}}}{dx}\bigg|_{x=0} - M(L)\frac{d\boldsymbol{N}^{\mathrm{T}}}{dx}\bigg|_{x=L} + V(L)\boldsymbol{N}^{\mathrm{T}}(L) - V(0)\boldsymbol{N}^{\mathrm{T}}(0)$$

Solution 4b: The elements have to be complete, meaning that it should be possible to chose values for the element degrees of freedom such that the approximation becomes constant on the element, chose values such that the first derivative becomes constant, and, finally, chose chose values such that the second derivative become constant.

In addition, the elements have to be compatible, meaning that the approximation and its first derivative have to be continuous across the element mesh.

Solution 4c: Since we only have one element, we have $\boldsymbol{K} = \boldsymbol{K}^{\mathrm{e}}$ and, thus

$\boldsymbol{a} = \begin{bmatrix} a_1^{\mathrm{e}} & a_2^{\mathrm{e}} & a_3^{\mathrm{e}} & a_4^{\mathrm{e}} \end{bmatrix}^{\mathrm{T}}$. To evaluate the right hand side of the FE–formulation, we note that

$q = 0$ and

$$N_1(0) = 1 \qquad N_1(L) = \left.\frac{dN_1}{dx}\right|_{x=0} = \left.\frac{dN_1}{dx}\right|_{x=L} = 0$$

$$\left.\frac{dN_2}{dx}\right|_{x=0} = 1 \qquad N_2(0) = N_2(L) = \left.\frac{dN_2}{dx}\right|_{x=L} = 0$$

$$N_3(L) = 1 \qquad N_3(0) = \left.\frac{dN_3}{dx}\right|_{x=0} = \left.\frac{dN_3}{dx}\right|_{x=L} = 0$$

$$\left.\frac{dN_4}{dx}\right|_{x=L} = 1 \qquad N_4(0) = N_4(L) = \left.\frac{dN_4}{dx}\right|_{x=0} = 0$$

Hence we have

$$\frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & \frac{-12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & \frac{-6}{L} & 2 \\ \frac{-12}{L^2} & \frac{-6}{L} & \frac{12}{L^2} & \frac{-6}{L} \\ \frac{6}{L} & 2 & \frac{-6}{L} & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = M(0)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - M(L)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + V(L)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - V(0)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, at $x = 0$ we have that $w = w' = 0$, so we prescribe $a_1 = a_2 = 0$; furthermore, the test function has to satisfy these essential boundary conditions (or: we do not know the values of $M(0)$ and $V(0)$ in the right hand side) — thus, we omit the first two equations, and since the first two columns are to be multiplied by zeros, we get

$$\frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{-6}{L} \\ \frac{-6}{L} & 4 \end{bmatrix}\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = -M(L)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + V(L)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The boundary conditions at $x = L$ end are obtained from equilibrium: $M(L) = 0$ and $V(L) = -P$, so finally we have

$$\frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{-6}{L} \\ \frac{-6}{L} & 4 \end{bmatrix}\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} -P \\ 0 \end{bmatrix}$$

Solution 4d: Since $q = 0$ and EI is a constant, the governing equation reduces to

$\frac{d^4 w}{dx^4} = 0$, so the exact solution is a cubic polynomial; we also know that the exact solution minimizes the potential energy. The basis functions used to establish the FE–approximation w_h are 4 linear independent cubic polynomials; thus, any cubic polynomial may be expressed as a linear combination of the selected basis functions. Also, the FE–approximation with test functions according to Galerkin, is known to minimize the potential energy. It follows that $w = w_h$, i.e. we get the exact solution in this case. Hence, $\Pi(w) = \Pi(w_h)$.

Department of Applied Mechanics, Chalmers University of Technology

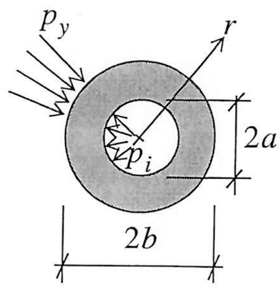
FINITE ELEMENT METHOD (MHA 021) — EXAMINATION
JANUARY 11 2006

Time and location:	$8^{\underline{30}}$ – $12^{\underline{30}}$ in the V building
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that January 11. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses*.
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments.**
Results:	will be posted at the Department of Applied Mechanics no later than January 16. Results are sent for registration January 23 — **course participants that have not completed the computer assignments by this time will be registered as *not approved*.** You may scrutinize the correction (mark up) of your written examination Tuesday January 17 $10^{\underline{00}}$–$11^{\underline{30}}$ and/or Friday January 20 $15^{\underline{30}}$–$16^{\underline{30}}$ (at the office space of Department of Applied Mechanics).

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

1 (svensk uppgiftstext nedan)



Consider a thick–walled pipe with inner radius a and outer radius b, that is loaded by an external pressure p_y and an internal pressure p_i. Assuming plane stress conditions, we may calculate the radial displacement u by solving the boundary value problem

$$\begin{cases} -\dfrac{d}{dr}\left[\dfrac{1}{r}\dfrac{d}{dr}[ru]\right] = 0 \\ \sigma_r(a) = -p_i \\ \sigma_r(b) = -p_y \end{cases}$$

Once solved, the radial and tangential stresses are obtained through

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \qquad \sigma_\varphi = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

a: Let $v = v(r)$ be a test function and show that the weak problem of the boundary value problem may be written

$$\frac{E}{1-\nu^2}\int_a^b \frac{1}{r}\frac{d}{dr}[vr]\frac{d}{dr}[ur]dr - \frac{E}{1+\nu}[v(b)u(b) - v(a)u(a)] = \qquad \text{(2p)}$$
$$= av(a)p_i - bv(b)p_y$$

b: Assume that the problem have been discretized with linear finite elements, so that we have $u \approx u_h = N_1^e a_1^e + N_2^e a_2^e$ on a single element, and that the test functions have been chosen according to Galerkin. Show that the element stiffness matrix of an element with node coordinates r_j and r_{j+1} becomes

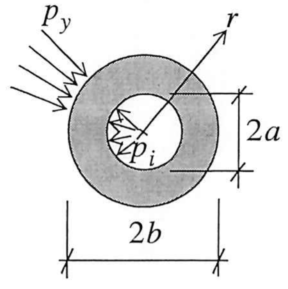
$$\boldsymbol{K}^e =$$

$$= \frac{E}{1-\nu^2}\int_{r_j}^{r_{j+1}} \begin{bmatrix} \dfrac{(N_1^e)^2}{r} + 2N_1^e\dfrac{dN_1^e}{dr} + r\left(\dfrac{dN_1^e}{dr}\right)^2 & \dfrac{N_1^e N_2^e}{r} + N_1^e\dfrac{dN_2^e}{dr} + N_2^e\dfrac{dN_1^e}{dr} + r\dfrac{dN_1^e}{dr}\dfrac{dN_2^e}{dr} \\ \dfrac{N_1^e N_2^e}{r} + N_1^e\dfrac{dN_2^e}{dr} + N_2^e\dfrac{dN_1^e}{dr} + r\dfrac{dN_1^e}{dr}\dfrac{dN_2^e}{dr} & \dfrac{(N_2^e)^2}{r} + 2N_2^e\dfrac{dN_2^e}{dr} + r\left(\dfrac{dN_2^e}{dr}\right)^2 \end{bmatrix} dr$$

provided that $a < r_j < r_{j+1} < b$. (2p)

c: Explain, physically, why the stiffness matrix $\boldsymbol{K}$ for this problem, becomes positive definite although there is no essential boundary condition. (1p)

1 (english text above)



Betrakta ett tjockväggigt rör med radien b och hålets radie a. Om röret är belastat med ett inre tryck p_i och ett yttre tryck p_y, och om plant spänningstillstånd antas, så kan den radiella förskjutningen u beräknas genom att lösa randvärdesproblemet

$$\begin{cases} -\dfrac{d}{dr}\left[\dfrac{1}{r}\dfrac{d}{dr}[ru]\right] = 0 \\ \sigma_r(a) = -p_i \\ \sigma_r(b) = -p_y \end{cases}$$

När u är känd, kan radiella och tangentiella spänningar beräknas som

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \qquad \sigma_\varphi = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

a: Låt $v = v(r)$ vara en testfunktion (viktsfunktion) och visa att den svaga formen av randvärdesproblemet kan skrivas

$$\frac{E}{1-\nu^2}\int_a^b \frac{1}{r}\frac{d}{dr}[vr]\frac{d}{dr}[ur]dr - \frac{E}{1+\nu}[v(b)u(b) - v(a)u(a)] = \qquad \text{(2p)}$$
$$= av(a)p_i - bv(b)p_y$$

b: Antag att problemet FE–diskretiseras med lineära element så att $u \approx u_h = N_1^e a_1^e + N_2^e a_2^e$ på ett element, samt att testfunktionerna väljs enligt Galerkin. Visa att elementstyvhetsmatrisen för ett element med nodkoordinater r_j och r_{j+1} blir

$$\boldsymbol{K}^e =$$

$$= \frac{E}{1-\nu^2}\int_{r_j}^{r_{j+1}} \begin{bmatrix} \dfrac{(N_1^e)^2}{r} + 2N_1^e\dfrac{dN_1^e}{dr} + r\left(\dfrac{dN_1^e}{dr}\right)^2 & \dfrac{N_1^eN_2^e}{r} + N_1^e\dfrac{dN_2^e}{dr} + N_2^e\dfrac{dN_1^e}{dr} + r\dfrac{dN_1^e}{dr}\dfrac{dN_2^e}{dr} \\ \dfrac{N_1^eN_2^e}{r} + N_1^e\dfrac{dN_2^e}{dr} + N_2^e\dfrac{dN_1^e}{dr} + r\dfrac{dN_1^e}{dr}\dfrac{dN_2^e}{dr} & \dfrac{(N_2^e)^2}{r} + 2N_2^e\dfrac{dN_2^e}{dr} + r\left(\dfrac{dN_2^e}{dr}\right)^2 \end{bmatrix} dr$$

om $a < r_j < r_{j+1} < b$. (2p)

c: Ge en fysikalisk förklaring till varför styvhetsmatrisen $\boldsymbol{K}$ (för det givna randvärdesproblemet) blir positivt definit, trots att det inte finns några väsentliga randvillkor. (1p)

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2 (svensk uppgiftstext nedan)

a: Formulate the divergence theorem; explain (define) the notation you introduce. Show how the Green–Gauss theorem, i.e. $\int_\Omega \phi \operatorname{div}(\boldsymbol{q})\, d\Omega = \oint_\Gamma \phi\, \boldsymbol{q}^T \boldsymbol{n}\, d\Gamma - \int_\Omega (\nabla\phi)^T \boldsymbol{q}\, d\Omega$, may be derived from the divergence theorem. You may resort to the 2D case in your solution. (2p)

b: Let $\Omega \subset \Re^2$ be a domain in the (x,y)-plane and let Γ be its boundary. Further, let the potential function $\phi = \phi(x, y)$ be the solution of the boundary value problem $-\operatorname{div}(\boldsymbol{D}\nabla\phi) = f$ in Ω, $\phi = 0$ on Γ; here, f is a given function and $\boldsymbol{D}$ is a symmetric and positive definite constitutive matrix. Use the Green–Gauss theorem to do a variational formulation of the problem; explain how the boundary condition is introduced in the weak form. Also describe restrictions on the choice of test functions. Make a finite element formulation of the problem and show the expression of the $\boldsymbol{B}^e$ matrix in the case of an element with N_e basis functions. (3p)

2 (english text above)

a: Formulera divergens–teoremet och redogör för ingående storheter. Visa hur Green–Gauss sats, dvs. $\int_\Omega \phi \operatorname{div}(\boldsymbol{q})\, d\Omega = \oint_\Gamma \phi\, \boldsymbol{q}^T \boldsymbol{n}\, d\Gamma - \int_\Omega (\nabla\phi)^T \boldsymbol{q}\, d\Omega$, kan härledas ur divergens–teoremet. Problemet får lösas för 2D–fallet. (2p)

b: Låt $\Omega \subset \Re^2$ vara ett område i (x,y)-planet och Γ dess rand. Potentialfunktionen $\phi = \phi(x, y)$ är lösningen till randvärdesproblemet $-\operatorname{div}(\boldsymbol{D}\nabla\phi) = f$ i Ω, $\phi = 0$ på Γ; här är f en given funktion och $\boldsymbol{D}$ en symmetrisk och positivt definit konstitutiv matris. Använd Green–Gauss sats för att variationsformulera problemet; det måste framgå hur randvillkoret påverkar formuleringen. Redogör också för vilka krav som ställs på testfunktionerna (viktsfunktionerna). Finit–elementformulera variationsproblemet och visa hur $\boldsymbol{B}^e$-matrisen för ett element med N_e basfunktioner ser ut. (3p)

3 (svensk uppgiftstext nedan)

Consider the problem described in task 2b above; the abstract form of the weak problem may be written: Find $\phi \in V$ such that $a(\phi,v) = (v,f) \qquad \forall v \in V$

where $(v,f) = \int_{\Omega} vf\mathrm{d}\Omega$, $a(\phi,v) = \int_{\Omega} (\nabla v)^{\mathrm{T}}\boldsymbol{D}(\nabla\phi)\mathrm{d}\Omega$, and V is the space of all admissible functions.

a: Show that $a(\phi, v)$ is symmetric and linear in both arguments. (1p)

b: Given that $a(\phi,v)$ is symmetric and bi–linear: Prove that the solution ϕ of the variational problem is a stationary point of the functional $\Pi(v) = \frac{1}{2}a(v,v) - (v,f)$. Also show that the stationary point is a minimum. (4p)

3 (english text above)

Betrakta problemet i uppgift 2b. På abstrakt form kan den svaga formen skrivas: Bestäm $\phi \in V$ så att $a(\phi,v) = (v,f) \qquad \forall v \in V$

där $(v,f) = \int_{\Omega} vf\mathrm{d}\Omega$ och $a(\phi,v) = \int_{\Omega} (\nabla v)^{\mathrm{T}}\boldsymbol{D}(\nabla\phi)\mathrm{d}\Omega$. Här V är rummet av alla tillåtna funktioner.

a: Visa att $a(\phi, v)$ är symmetrisk och lineär i båda argumenten. (1p)

b: Under antagande om att $a(\phi,v)$ är symmetrisk och bi–lineär: Visa att lösningen ϕ till variationsproblemet gör funktionalen $\Pi(v) = \frac{1}{2}a(v,v) - (v,f)$ stationär; visa att den stationära punkten är ett minimum. (4p)

4 (svensk uppgiftstext nedan)

Equilibrium in a 2D elasticity problem requires that $-\tilde{\nabla}^T \sigma = \boldsymbol{b}$, where $\sigma = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^T$ is the stress vector, $\boldsymbol{b} = \begin{bmatrix} b_x & b_y \end{bmatrix}^T$ a given volume loading, and $\tilde{\nabla}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$. The deformations (strains) $\varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^T$ are (according to Hooke) proportional to the stresses: $\sigma = \boldsymbol{D}\varepsilon$. Kinematics relates deformations and displacements $\boldsymbol{u} = \begin{bmatrix} u_x & u_y \end{bmatrix}^T$ as $\varepsilon = \tilde{\nabla}\boldsymbol{u}$. By combining these three equations, we get two second order partial differential equations from which the unknown displacements may be solved. In the most common case where $\boldsymbol{b} = \boldsymbol{0}$, the weak form of the problem may be written

$$\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) = \oint_{\Gamma} \boldsymbol{v}^T \boldsymbol{t} t \, d\Gamma \tag{1}$$

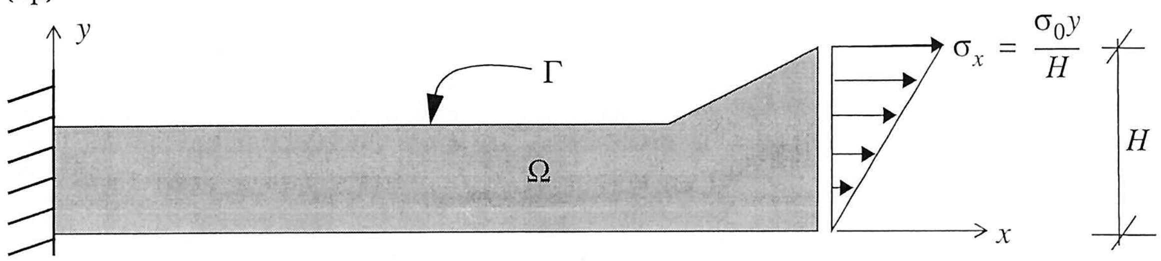
where t is the thickness, $\boldsymbol{t} = \begin{bmatrix} t_x & t_y \end{bmatrix}^T$ is the traction (boundary loading), $\boldsymbol{v} = \begin{bmatrix} v_x & v_y \end{bmatrix}^T$ is a vector with test functions, while the bi-linear symmetric form is given by

$$\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} (\tilde{\nabla}\boldsymbol{v})^T \boldsymbol{D} (\tilde{\nabla}\boldsymbol{u}) t \, d\Omega .$$

a: Specify the boundary conditions that are required to solve the elasticity problem depicted in the illustration below. (2p)

b: Give some example of a modelling error that might have arisen when the physical problem was described by a mathematical model. (1p)

c: Starting from Eq (1), make a finite element formulation of the given problem; select test functions according to Galerkin. From your solution, it should be clear how the boundary conditions affect the formulation. Show the integral that defines the element stiffness matrix of an element that has N_e basis functions — the $\boldsymbol{B}^e$ matrix should also be shown. (2p)



2006-01-11/PWM

4 (english text above)

Jämvikt i ett 2D elasticitetsproblem kräver att $-\tilde{\nabla}^{\mathrm{T}}\sigma = \boldsymbol{b}$ där $\sigma = \begin{bmatrix}\sigma_x & \sigma_y & \tau_{xy}\end{bmatrix}^{\mathrm{T}}$ är spänningsvektorn, $\boldsymbol{b} = \begin{bmatrix}b_x & b_y\end{bmatrix}^{\mathrm{T}}$ en given volymslast, och $\tilde{\nabla}^{\mathrm{T}} = \begin{bmatrix}\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}$. Hookes lag relaterar spänningarna med töjningarna $\varepsilon = \begin{bmatrix}\varepsilon_x & \varepsilon_y & \gamma_{xy}\end{bmatrix}^{\mathrm{T}}$ enligt $\sigma = \boldsymbol{D}\varepsilon$, och kinematiken ger att $\varepsilon = \tilde{\nabla}\boldsymbol{u}$, där $\boldsymbol{u} = \begin{bmatrix}u_x & u_y\end{bmatrix}^{\mathrm{T}}$ är förskjutningsvektorn. Kombineras de tre ekvationerna fås två andra ordningens partiella differentialekvationer, vars lösning ger den primärt obekanta förskjutningsvektorn. Om $\boldsymbol{b} = \boldsymbol{0}$ så kan den svaga formen av problemet skrivas

$$\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) = \oint_{\Gamma} \boldsymbol{v}^{\mathrm{T}} \boldsymbol{t} t d\Gamma \tag{2}$$

där vi infört tjockleken t, randlasten $\boldsymbol{t} = \begin{bmatrix}t_x & t_y\end{bmatrix}^{\mathrm{T}}$, viktsfunktionerna $\boldsymbol{v} = \begin{bmatrix}v_x & v_y\end{bmatrix}^{\mathrm{T}}$, samt den bi-lineära symmetriska formen $\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} (\tilde{\nabla}\boldsymbol{v})^{\mathrm{T}} \boldsymbol{D} (\tilde{\nabla}\boldsymbol{u}) t d\Omega$.

a: Ange randvillkoren som krävs för att lösa elasticitetsproblemet som illustreras av figuren (se föregående sida) (2p)

b: Ge exempel på något modelleringsfel som kan ha uppkommit då det bakomliggande fysiska problemet beskrivits med en matematisk modell. (1p)

c: Utgå från ekv (2) för att finit-elementformulera det givna problemet med viktsfunktioner enligt Galerkin. Det ska framgå hur randvillkoren påverkar formuleringen. Ange också den integral som ger styvhetsmatrisen för ett element med N_{e} basfunktioner — $\boldsymbol{B}^{\mathrm{e}}$-matrisens utseende ska visas. (2p)

Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION

JANUARY 11 2006

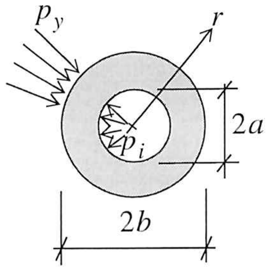
Solutions

Time and location:	$8^{\underline{30}}$ – $12^{\underline{30}}$ in the V building
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that January 11. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses*.
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments**.
Results:	will be posted at the Department of Applied Mechanics no later than January 16. Results are sent for registration January 23 — **course participants that have not completed the computer assignments by this time will be registered as *not approved***. You may scrutinize the correction (mark up) of your written examination Tuesday January 17 $10^{\underline{00}}$–$11^{\underline{30}}$ and/or Friday January 20 $15^{\underline{30}}$–$16^{\underline{30}}$ (at the office space of Department of Applied Mechanics).

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

1



Consider a thick–walled pipe with inner radius a and outer radius b, that is loaded by an external pressure p_y and an internal pressure p_i. Assuming plane stress conditions, we may calculate the radial displacement u by solving the boundary value problem

$$\begin{cases} -\dfrac{d}{dr}\left[\dfrac{1}{r}\dfrac{d}{dr}[ru]\right] = 0 \\ \sigma_r(a) = -p_i \\ \sigma_r(b) = -p_y \end{cases}$$

Once solved, the radial and tangential stresses are obtained through

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \qquad \sigma_\varphi = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

a: Let $v = v(r)$ be a test function and show that the weak problem of the boundary value problem may be written

$$\frac{E}{1-\nu^2}\int_a^b \frac{1}{r}\frac{d}{dr}[vr]\frac{d}{dr}[ur]dr - \frac{E}{1+\nu}[v(b)u(b) - v(a)u(a)] = \qquad \text{(2p)}$$

$$= av(a)p_i - bv(b)p_y$$

b: Assume that the problem have been discretized with linear finite elements, so that we have $u \approx u_h = N_1^e a_1^e + N_2^e a_2^e$ on a single element, and that the test functions have been chosen according to Galerkin. Show that the element stiffness matrix of an element with node coordinates r_j and r_{j+1} becomes

$$K^e =$$

$$= \frac{E}{1-\nu^2}\int_{r_j}^{r_{j+1}} \begin{bmatrix} \dfrac{(N_1^e)^2}{r} + 2N_1^e\dfrac{dN_1^e}{dr} + r\left(\dfrac{dN_1^e}{dr}\right)^2 & \dfrac{N_1^eN_2^e}{r} + N_1^e\dfrac{dN_2^e}{dr} + N_2^e\dfrac{dN_1^e}{dr} + r\dfrac{dN_1^e}{dr}\dfrac{dN_2^e}{dr} \\ \dfrac{N_1^eN_2^e}{r} + N_1^e\dfrac{dN_2^e}{dr} + N_2^e\dfrac{dN_1^e}{dr} + r\dfrac{dN_1^e}{dr}\dfrac{dN_2^e}{dr} & \dfrac{(N_2^e)^2}{r} + 2N_2^e\dfrac{dN_2^e}{dr} + r\left(\dfrac{dN_2^e}{dr}\right)^2 \end{bmatrix} dr$$

provided that $a < r_j < r_{j+1} < b$. (2p)

c: Explain, physically, why the stiffness matrix K for this problem, becomes positive definite although there is no essential boundary condition. (1p)

Solution 1a: Multiply the differential equation by a test function $v(r)$ and integrate over the surface: $-\int_a^b\left(\int_0^{2\pi} v\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}[ru]\right]rd\theta\right)dr = 0 \Rightarrow -2\pi\int_a^b v\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}[ru]\right]rdr = 0$. Integration by parts yields: $\int_a^b (vr)\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}[ru]\right]dr = \left[v\frac{d}{dr}[ru]\right]_a^b - \int_a^b \frac{1}{r}\frac{d}{dr}[rv]\frac{d}{dr}[ru]dr$. In order to account for the boundary conditions, we need evolve the boundary terms:

$$\left[v\frac{d}{dr}[ru]\right]_a^b = \left[v\left(u + r\frac{du}{dr}\right)\right]_a^b = \left[vr\left(\frac{u}{r} + \frac{du}{dr}\right)\right]_a^b = \left[vr\left(\nu\frac{u}{r} + \frac{du}{dr} + (1-\nu)\frac{u}{r}\right)\right]_a^b =$$

$$= \frac{1-\nu^2}{E}\left[vr\frac{E}{1-\nu^2}\left(\nu\frac{u}{r} + \frac{du}{dr} + (1-\nu)\frac{u}{r}\right)\right]_a^b = \frac{1-\nu^2}{E}\left[vr\left(\sigma_r + \frac{E}{(1+\nu)}\frac{u}{r}\right)\right]_a^b =$$

$$= \frac{1-\nu^2}{E}\left[bv(b)\sigma_r(b) + \frac{E}{(1+\nu)}u(b)v(b) - av(a)\sigma_r(a) - \frac{E}{(1+\nu)}u(a)v(a)\right] =$$

$$= \left(\frac{1-\nu^2}{E}\left[av(a)p_i - bv(b)p_y + \frac{E}{(1+\nu)}(u(b)v(b) - u(a)v(a))\right]\right)$$

Hence, multiplication by $\frac{E}{1-\nu^2}$ now gives

$$\frac{E}{1-\nu^2}\int_a^b \frac{1}{r}\frac{d}{dr}[rv]\frac{d}{dr}[ru]dr - \frac{E}{1+\nu}(u(b)v(b) - u(a)v(a)) = av(a)p_i - bv(b)p_y$$

Solution 1b: Let $\boldsymbol{N}^{\mathrm{e}} = \left[N_1^{\mathrm{e}}\ N_2^{\mathrm{e}}\right]$ and $\boldsymbol{a}^{\mathrm{e}} = \left[a_1^{\mathrm{e}}\ a_2^{\mathrm{e}}\right]^{\mathrm{T}}$, so that $u_{\mathrm{h}} = \boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$ on one element.

Substitution into the weak problem and test functions according to Galerkin, i.e. $v = N_1^{\mathrm{e}}$ and $v = N_2^{\mathrm{e}}$, yield:

$\frac{E}{1-\nu^2}\int_a^b \frac{1}{r}\frac{d}{dr}[r(\boldsymbol{N}^{\mathrm{e}})^{\mathrm{T}}]\frac{d}{dr}[r(\boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}})]dr = \frac{E}{1-\nu^2}\int_{r_j}^{r_{j+1}} \frac{1}{r}\frac{d}{dr}[r(\boldsymbol{N}^{\mathrm{e}})^{\mathrm{T}}]\frac{d}{dr}[r(\boldsymbol{N}^{\mathrm{e}})]dr\boldsymbol{a}^{\mathrm{e}} = \boldsymbol{K}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$. We have $\frac{d}{dr}[r(\boldsymbol{N}^{\mathrm{e}})] = \boldsymbol{N}^{\mathrm{e}} + r\frac{d\boldsymbol{N}^{\mathrm{e}}}{dr} = \left[N_1^{\mathrm{e}} + r\frac{dN_1^{\mathrm{e}}}{dr}\ \ N_2^{\mathrm{e}} + r\frac{dN_2^{\mathrm{e}}}{dr}\right]$ and

$$\frac{d}{dr}[r(\boldsymbol{N}^e)^T] = (\boldsymbol{N}^e)^T + r\frac{d(\boldsymbol{N}^e)^T}{dr} = \begin{bmatrix} N_1^e + r\frac{dN_1^e}{dr} & N_2^e + r\frac{dN_2^e}{dr} \end{bmatrix}^T$$. The integrand thus becomes:

$$\frac{1}{r}\frac{d}{dr}[r(\boldsymbol{N}^e)^T]\frac{d}{dr}[r(\boldsymbol{N}^e)] = \frac{1}{r}\begin{bmatrix} N_1^e + r\frac{dN_1^e}{dr} & N_2^e + r\frac{dN_2^e}{dr} \end{bmatrix}^T \begin{bmatrix} N_1^e + r\frac{dN_1^e}{dr} & N_2^e + r\frac{dN_2^e}{dr} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{(N_1^e)^2}{r} + 2N_1^e\frac{dN_1^e}{dr} + r\left(\frac{dN_1^e}{dr}\right)^2 & \frac{N_1^eN_2^e}{r} + N_1^e\frac{dN_2^e}{dr} + N_2^e\frac{dN_1^e}{dr} + r\frac{dN_1^e}{dr}\frac{dN_2^e}{dr} \\ \frac{N_1^eN_2^e}{r} + N_1^e\frac{dN_2^e}{dr} + N_2^e\frac{dN_1^e}{dr} + r\frac{dN_1^e}{dr}\frac{dN_2^e}{dr} & \frac{(N_2^e)^2}{r} + 2N_2^e\frac{dN_2^e}{dr} + r\left(\frac{dN_2^e}{dr}\right)^2 \end{bmatrix}$$

from which the result follows.

Solution 1c: Due to symmetry of revolution, there is no rigid body translation involved in the problem. For instance, in the expression for the stresses we have the term u/r, so even if we would have $u = c$ (constant) there will be stresses and strains.

2

a: Formulate the divergence theorem; explain (define) the notation you introduce. Show how the Green–Gauss theorem, i.e. $\int_\Omega \phi \operatorname{div}(\boldsymbol{q})\, d\Omega = \oint_\Gamma \phi\, \boldsymbol{q}^T\boldsymbol{n}\, d\Gamma - \int_\Omega (\nabla\phi)^T\boldsymbol{q}\, d\Omega$, may be derived from the divergence theorem. You may resort to the 2D case in your solution. (2p)

b: Let $\Omega \subset \Re^2$ be a domain in the (x,y)–plane and let Γ be its boundary. Further, let the potential function $\phi = \phi(x, y)$ be the solution of the boundary value problem $-\operatorname{div}(\boldsymbol{D}\nabla\phi) = f$ in Ω, $\phi = 0$ on Γ; here, f is a given function and $\boldsymbol{D}$ is a symmetric and positive definite constitutive matrix. Use the Green–Gauss theorem to do a variational formulation of the problem; explain how the boundary condition is introduced in the weak form. Also describe restrictions on the choice of test functions. Make a finite element formulation of the problem and show the expression of the $\boldsymbol{B}^e$ matrix in the case of an element with N_e basis functions. (3p)

Solution 2a: Let $\Omega \subset \Re^2$ be a domain in the (x,y)–plane and Γ its boundary, $n = \begin{bmatrix} n_x & n_y \end{bmatrix}^T$ an out-ward unit normal, and let $q = \begin{bmatrix} q_x & q_y \end{bmatrix}^T$ be a vector-valued function. Provided that the functions are sufficiently regular, we have (the divergence theorem): $\int_\Omega \operatorname{div}(\boldsymbol{q})d\Omega = \oint_\Gamma \boldsymbol{q}^T\boldsymbol{n}\,d\Gamma$

To obtain the Green–Gauss theorem, we construct the vector $\phi\boldsymbol{q}$, where ϕ is a sufficiently regular, but otherwise arbitrary, function. Next we calculate the divergence of $\phi\boldsymbol{q}$:

$$\operatorname{div}(\phi\boldsymbol{q}) = \frac{\partial}{\partial x}[\phi q_x] + \frac{\partial}{\partial y}[\phi q_y] = \phi\frac{\partial q_x}{\partial x} + \phi\frac{\partial q_y}{\partial y} + \frac{\partial\phi}{\partial x}q_x + \frac{\partial\phi}{\partial y}q_y = \phi\operatorname{div}(\boldsymbol{q}) + (\nabla\phi)^{\mathrm{T}}\boldsymbol{q}$$

The divergence theorem applied to $\phi\boldsymbol{q}$ and the identity $\operatorname{div}(\phi\boldsymbol{q}) = \phi\operatorname{div}(\boldsymbol{q}) + (\nabla\phi)^{\mathrm{T}}\boldsymbol{q}$, produce the desired result.

Solution 2b: Multiply the differential equation by a test function v and integrate over the domain: $-\int_{\Omega} v\operatorname{div}(\boldsymbol{D}\nabla\phi)d\Omega = \int_{\Omega} vfd\Omega$. Now apply the Green–Gauss theorem (identify v and the vector $\boldsymbol{D}\nabla\phi$, by ϕ and $\boldsymbol{q}$, respectively); we obtain: $\int_{\Omega}(\nabla v)^{\mathrm{T}}\boldsymbol{D}(\nabla\phi)d\Omega = \oint_{\Gamma} v(\boldsymbol{D}\nabla\phi)^{\mathrm{T}}\boldsymbol{n}d\Gamma + \int_{\Omega} vfd\Omega$

In order for this equation to have any meaning, the involved functions have to be regular enough — it must be possible to integrate the square of the functions as well as the squares of their first derivatives; furthermore, we have to chose test functions such that $v = 0$ on Γ, since it would not be possible to calculate the boundary integral otherwise ($\nabla\phi$ is unknown) — thus, the test function has to satisfy *homogenous essential* boundary conditions. Let V be the space of all functions that satisfy these conditions; the variational formulation then becomes: Find $\phi \in V$ such that $\int_{\Omega}(\nabla v)^{\mathrm{T}}\boldsymbol{D}(\nabla\phi)d\Omega = \int_{\Omega} vfd\Omega \qquad \forall v \in V$

Finite element formulation: Chose N basis functions N_i out of V (conform method) and let $V_{\mathrm{h}} \subset V$ be the space of all functions that may be expressed as a linear combination of the basis functions. In particular, we approximate $\phi \approx \phi_{\mathrm{h}} = \sum_{i=1}^{N} N_i a_i$ and calculate the node variables a_i such that the variational problem is satisfied in V_h (Galerkin method). Hence, we have: Find $\phi_{\mathrm{h}} \in V_{\mathrm{h}}$ such that $\int_{\Omega}(\nabla v)^{\mathrm{T}}\boldsymbol{D}(\nabla\phi_{\mathrm{h}})d\Omega = \int_{\Omega} vfd\Omega \qquad \forall v \in V_{\mathrm{h}}$

For an element with N_{e} basis functions, i.e one on which only N_{e} out of N basis functions take on non–zero values, the finite element approximation may be written

$$\phi_{\mathrm{h}} = \left[N_1^{\mathrm{e}}\ N_2^{\mathrm{e}}\ \dots\ N_{N_{\mathrm{e}}}^{\mathrm{e}}\right]\left[a_1^{\mathrm{e}}\ a_2^{\mathrm{e}}\ \dots\ a_{N_{\mathrm{e}}}^{\mathrm{e}}\right]^{\mathrm{T}} = \boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}, \text{ so that}$$

$$\nabla\phi_{\mathrm{h}} = \nabla(\boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}) = (\nabla\boldsymbol{N}^{\mathrm{e}})\boldsymbol{a}^{\mathrm{e}} = \boldsymbol{B}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}, \text{ where, hence,}$$

$$\boldsymbol{B}^{\mathrm{e}} = \nabla \boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} \dfrac{\partial N_1^{\mathrm{e}}}{\partial x} & \dfrac{\partial N_2^{\mathrm{e}}}{\partial x} & \cdots & \dfrac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} \\ \dfrac{\partial N_1^{\mathrm{e}}}{\partial y} & \dfrac{\partial N_2^{\mathrm{e}}}{\partial y} & \cdots & \dfrac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} \end{bmatrix}$$

3

Consider the problem described in task 2b above; the abstract form of the weak problem may be written: Find $\phi \in V$ such that $a(\phi,v) = (v,f) \qquad \forall v \in V$

where $(v,f) = \int_{\Omega} vf\,\mathrm{d}\Omega$, $a(\phi,v) = \int_{\Omega} (\nabla v)^{\mathrm{T}} \boldsymbol{D}(\nabla \phi)\,\mathrm{d}\Omega$, and V is the space of all admissible functions.

a: Show that $a(\phi,\, v)$ is symmetric and linear in both arguments. (1p)

b: Given that $a(\phi,v)$ is symmetric and bi–linear: Prove that the solution ϕ of the variational problem is a stationary point of the functional $\Pi(v) = \frac{1}{2}a(v,v) - (v,f)$. Also show that the stationary point is a minimum. (4p)

Solution 3a: In order to prove symmetry, we utilize the fact that the integrand is scalar and thus equal to its own transpose and that $\boldsymbol{D}^{\mathrm{T}} = \boldsymbol{D}$:

$$a(\phi,v) = \int_{\Omega} (\nabla v)^{\mathrm{T}} \boldsymbol{D}(\nabla\phi)\,\mathrm{d}\Omega = \int_{\Omega} [(\nabla v)^{\mathrm{T}} \boldsymbol{D}(\nabla\phi)]^{\mathrm{T}}\,\mathrm{d}\Omega = \int_{\Omega} (\nabla\phi)^{\mathrm{T}}[(\nabla v)^{\mathrm{T}} \boldsymbol{D}]^{\mathrm{T}}\,\mathrm{d}\Omega =$$
$$= \int_{\Omega} (\nabla\phi)^{\mathrm{T}} \boldsymbol{D}^{\mathrm{T}}(\nabla v)\,\mathrm{d}\Omega = \int_{\Omega} (\nabla\phi)^{\mathrm{T}} \boldsymbol{D}(\nabla v)\,\mathrm{d}\Omega = a(v,\phi)$$

$\therefore a(u,v) = a(v,u)$.

Linearity in the first argument: let $\alpha, \beta \in \Re$. Then we have

$$a(\alpha\phi_1 + \beta\phi_2, v) = \int_{\Omega} (\nabla v)^{\mathrm{T}} \boldsymbol{D}(\nabla\alpha\phi_1 + \nabla\beta\phi_2)\,\mathrm{d}\Omega =$$
$$= \alpha\int_{\Omega} (\nabla v)^{\mathrm{T}} \boldsymbol{D}(\nabla\phi_1)\,\mathrm{d}\Omega + \beta\int_{\Omega} (\nabla v)^{\mathrm{T}} \boldsymbol{D}(\nabla\phi_2)\,\mathrm{d}\Omega = \alpha a(\phi_1,v) + \beta a(\phi_2,v)$$

Linearity in the second argument follows from above:

$$a(\phi,\, \alpha v_1 + \beta v_2) = a(\alpha v_1 + \beta v_2,\, \phi) = \alpha a(v_1,\phi) + \beta a(v_2,\phi) = \alpha a(\phi,\, v_1) + \beta a(\phi,\, v_2)$$

Solution 3b: Let ϕ be the solution of the variational problem and $\Pi(\phi) = \frac{1}{2}a(\phi, \phi) - (\phi, f)$. Further, let $\delta\phi \in V$ be a small variation of ϕ; then $\phi + \delta\phi \in V$. We may then write

$$\Pi(\phi + \delta\phi) = \frac{1}{2}a(\phi + \delta\phi, \phi + \delta\phi) - (\phi + \delta\phi, f) =$$
$$= \frac{1}{2}a(\phi, \phi) + a(\phi, \delta\phi) + \frac{1}{2}a(\delta\phi, \delta\phi) - (\phi, f) - (\delta\phi, f)$$

The variation of Π hence becomes

$$\partial\Pi = \Pi(\phi + \delta\phi) - \Pi(\phi) = a(\phi, \delta\phi) + \frac{1}{2}a(\delta\phi, \delta\phi) - (\delta\phi, f) \approx a(\phi, \delta\phi) - (\delta\phi, f)$$

where we used that $a(\delta\phi, \delta\phi) \approx 0$ (since $\delta\phi$ is small). We have that $a(\phi, \delta\phi) = (\delta\phi, f)$, since ϕ solves the variational problem and $\delta\phi \in V$. Thus, $\partial\Pi = 0$, i.e. Π is stationary when $v = \phi$.

To show that the stationary point is a minimum, we need to show that the second variation is positive: $\partial^2\Pi > 0$:

$$\partial^2\Pi = \partial\Pi(\phi + \delta\phi) - \partial\Pi(\phi) = a(\phi + \delta\phi, \delta\phi) - (\delta\phi, f) - a(\phi, \delta\phi) + (\delta\phi, f) = a(\delta\phi, \delta\phi)$$

Since the constitutive matrix $\boldsymbol{D}$ is positive definite, we have that $a(\delta\phi, \delta\phi) > 0$ (if $\delta\phi \neq 0$ somewhere). Thus, $\partial^2\Pi > 0$.

4

Equilibrium in a 2D elasticity problem requires that $-\tilde{\nabla}^{\mathrm{T}}\sigma = \boldsymbol{b}$, where $\sigma = \begin{bmatrix}\sigma_x & \sigma_y & \tau_{xy}\end{bmatrix}^{\mathrm{T}}$ is the stress vector, $\boldsymbol{b} = \begin{bmatrix}b_x & b_y\end{bmatrix}^{\mathrm{T}}$ a given volume loading, and $\tilde{\nabla}^{\mathrm{T}} = \begin{bmatrix}\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}$. The deformations (strains) $\varepsilon = \begin{bmatrix}\varepsilon_x & \varepsilon_y & \gamma_{xy}\end{bmatrix}^{\mathrm{T}}$ are (according to Hooke) proportional to the stresses: $\sigma = \boldsymbol{D}\varepsilon$. Kinematics relates deformations and displacements $\boldsymbol{u} = \begin{bmatrix}u_x & u_y\end{bmatrix}^{\mathrm{T}}$ as $\varepsilon = \tilde{\nabla}\boldsymbol{u}$. By combining these three equations, we get two second order partial differential equations from which the unknown displacements may be solved. In the most common case where $\boldsymbol{b} = \boldsymbol{0}$, the weak form of the problem may be written

$$a(\boldsymbol{u}, \boldsymbol{v}) = \oint_{\Gamma} \boldsymbol{v}^{T} \boldsymbol{t} t d\Gamma \qquad (1)$$

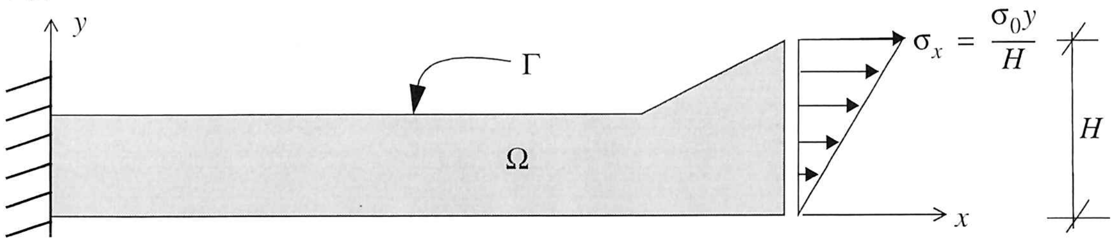
where t is the thickness, $\boldsymbol{t} = \begin{bmatrix} t_x & t_y \end{bmatrix}^{T}$ is the traction (boundary loading), $\boldsymbol{v} = \begin{bmatrix} v_x & v_y \end{bmatrix}^{T}$ is a vector with test functions, while the bi-linear symmetric form is given by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} (\tilde{\nabla}\boldsymbol{v})^{T} \boldsymbol{D} (\tilde{\nabla}\boldsymbol{u}) t d\Omega .$$

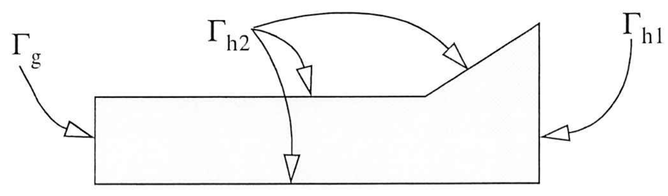
a: Specify the boundary conditions that are required to solve the elasticity problem depicted in the illustration below. (2p)

b: Give some example of a modelling error that might have arisen when the physical problem was described by a mathematical model. (1p)

c: Starting from Eq (1), make a finite element formulation of the given problem; select test functions according to Galerkin. From your solution, it should be clear how the boundary conditions affect the formulation. Show the integral that defines the element stiffness matrix of an element that has N_e basis functions — the $\boldsymbol{B}^{e}$ matrix should also be shown. (2p)



Solution 4a: The problem is described by two second order partial differential equations, so we need two conditions on each and every part of the boundary. Using the notation in the illustration, we have: $\left.\begin{matrix} u_x = 0 \\ u_y = 0 \end{matrix}\right\}$ on Γ_g $\left.\begin{matrix} t_x = \sigma_x = \sigma_0 y/H \\ t_y = 0 \end{matrix}\right\}$ on Γ_{h1} $\left.\begin{matrix} t_x = 0 \\ t_y = 0 \end{matrix}\right\}$ on Γ_{h2}.



The right–hand–side of the variational problem then becomes $\oint_{\Gamma} \boldsymbol{v}^{T} \boldsymbol{t} t d\Gamma = \int_{\Gamma_{h1}} \boldsymbol{v}^{T} \begin{bmatrix} \sigma_x \\ 0 \end{bmatrix} t d\Gamma$, since the test functions have to satisfy the essential boundary conditions (i.e. we have $v_x = v_y = 0$ on Γ_g).

Solution 4b: Uncertainties in material properties (D) and/or the exact loading (σ_x) and/or the actual geometry of the structure (e.g. the thickness t); the boundary conditions may have been approximated — e.g. the clamping has some flexibility that is not accounted for; etc.

Solution 4c: Approximate $u \approx u_h = \begin{bmatrix} u_{hx} & u_{hy} \end{bmatrix}^{\mathrm{T}} = Na$, where $N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}$ and

$a = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & \dots & u_{xn} & u_{yn} \end{bmatrix}^{\mathrm{T}}$. Let V_h be the space of functions that can be written as a linear combination of the selected basis functions N_i; furthermore, the basis functions are selected so that essential boundary conditions are satisfied, i.e. we have $N_i = 0$ on Γ_g. Next select test functions according to Galerkin: $v = \begin{bmatrix} N_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ N_1 \end{bmatrix}, \begin{bmatrix} N_2 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ N_n \end{bmatrix}$; we then obtain: Find $u_h = \begin{bmatrix} u_{hx} & u_{hy} \end{bmatrix}^{\mathrm{T}}$,

$u_{hx}, u_{hy} \in V_h$, such that $\mathrm{a}(u_h, v) = \int_{\Gamma_{h1}} v^{\mathrm{T}} \begin{bmatrix} \sigma_x \\ 0 \end{bmatrix} t d\Gamma \qquad \forall v_x, v_y \in V_h$

On a single element we have the approximation $u_h = N^{\mathrm{e}} a^{\mathrm{e}}$, where a^{e} contains the node variables on the element, and $N^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & 0 & N_2^{\mathrm{e}} & 0 & \dots & N_{ne}^{\mathrm{e}} & 0 \\ 0 & N_1^{\mathrm{e}} & 0 & N_2^{\mathrm{e}} & \dots & 0 & N_{ne}^{\mathrm{e}} \end{bmatrix}$ contains the basis functions that have support on the element. The contribution from the element to left–hand–side in the weak formulation becomes $\int_{\Omega_e} (\tilde{\nabla} N)^{\mathrm{T}} D (\tilde{\nabla} N a) t d\Omega = \int_{\Omega_e} (\tilde{\nabla} N^e)^{\mathrm{T}} D (\tilde{\nabla} N^e) t d\Omega a^e = \int_{\Omega_e} B^{e\mathrm{T}} D B^{e\mathrm{T}} t d\Omega a^e = K^e a^e$

where $B^e = \tilde{\nabla} N^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \dots & \frac{\partial N_{N_e}^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \dots & 0 & \frac{\partial N_{N_e}^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \dots & \frac{\partial N_{N_e}^e}{\partial y} & \frac{\partial N_{N_e}^e}{\partial x} \end{bmatrix}$