Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION
AUGUST 24 2005

Solutions

Time and location:	14^{00} - 18^{00} in the V building
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that August 25. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses*.
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments**.
Results:	will be posted at the Department of Applied Mechanics no later than September 1. Results are sent for registration September 9 — **course participants that have not completed the computer assignments by this time will be registered as *not approved***. You may scrutinize the correction (mark up) of your written examination Friday September 2 10^{00}–11^{30} and Monday September 5 13^{00}–15^{00} (at the office space of Department of Applied Mechanics).

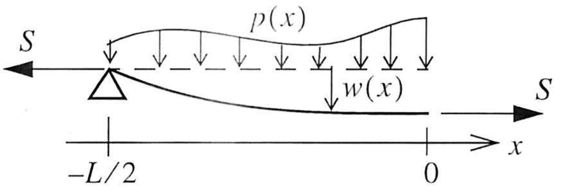
Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

1

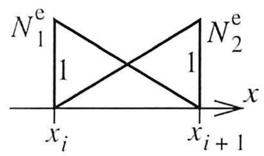
Consider a string of length L that has been pre-tensioned by a force S and that is loaded by a transverse load with intensity $p(x)$. Provided that the problem is symmetric, the deflection $w(x)$ is given by the solution of the boundary value problem

$$(1)\quad \begin{cases} -\dfrac{d^2 w}{dx^2} = \dfrac{p}{S} & -\dfrac{L}{2} < x < 0 \\ w\left(-\dfrac{L}{2}\right) = 0 & \\ \left.\dfrac{dw}{dx}\right|_{x=0} = 0 & \end{cases}$$



a: Derive the weak form (variational formulation) of the problem (1) and make a finite element formulation with test (weight) functions according to Galerkin. It shall be clearly shown how the boundary conditions affect the formulation. Also state regularity requirements of involved functions. (3p)

b: Consider a linear element with length $h = x_{i+1} - x_i$. Show that



the element stiffness matrix becomes $\boldsymbol{K}^e = \dfrac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. (2p)

c: Assume that p is constant and solve the problem with a single linear element. (1p)

d: The energy norm of the finite element approximation in the previous sub-task is

$\|w_h\|_a = \sqrt{a(w_h, w_h)} = \dfrac{pL}{S}\sqrt{\dfrac{L}{32}}$, while for the exact solution it can be shown that

$\|w\|_a = \sqrt{a(w, w)} = \dfrac{pL}{S}\sqrt{\dfrac{L}{24}}$. Calculate the energy norm of the error $e = w - w_h$:

$$\|e\|_a = \sqrt{a(e, e)} = \left[\int_{-L/2}^{0} \left(\frac{de}{dx}\right)^2 dx\right]^{1/2}. \text{ (2p)}$$

Solution 1a: Multiply the differential equation by a test function $v(x)$ and integrate over the interval: $-\displaystyle\int_{-L/2}^{0} v(x)\frac{d^2 w}{dx^2}dx = \int_{-L/2}^{0} v(x)\frac{p}{S}dx$. Partial integration of the left hand side yields

$\displaystyle\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw}{dx}dx - \left(v(0)\frac{dw}{dx}(0) - v\left(\frac{-L}{2}\right)\frac{dw}{dx}\left(\frac{-L}{2}\right)\right) = \int_{-L/2}^{0} v(x)\frac{p}{S}dx$. The first boundary term vanishes, since $w'(0) = 0$; to get rid of the second boundary term, we restrict the formulation to test functions such that $v(-L/2) = 0$. Thus, defining

$V = \left\{ v\colon\ v(-L/2) = 0 \quad \int_{-L/2}^{0} \left(\frac{dv}{dx}\right)^2 dx < \infty,\ \int_{-L/2}^{0} v^2 dx < \infty \right\}$, we are lead to the varia-

tional problem: Find $w \in V$ such that $\displaystyle\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw}{dx}\,dx = \int_{-L/2}^{0} v\frac{p}{S}dx \qquad \forall v \in V$.

Finite element formulation: select a set of basis functions N_i and let V_h be the space of functions that can be expressed as a linear combination of the basis functions. Approximate $w \approx w_h = \sum_i N_i a_i = \boldsymbol{Na}$ (where, hence, $\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & \dots \end{bmatrix}$ and $a = \begin{bmatrix} a_1 & a_2 & \dots \end{bmatrix}^{\mathrm{T}}$). Now, substitute the approximation into the variational problem and use the basis functions as test functions (i.e. the Galerkin method); we obtain:

Find $w_h \in V_h$ such that $\displaystyle\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw_h}{dx}\,dx = \int_{-L/2}^{0} v\,\frac{p}{S}\,dx \qquad \forall v \in V_h$ (or

$\displaystyle\int_{-L/2}^{0} \left(\frac{d\boldsymbol{N}}{dx}\right)^{\mathrm{T}} \frac{d\boldsymbol{N}}{dx}\,dx\,\boldsymbol{a} = \int_{-L/2}^{0} \boldsymbol{N}^{\mathrm{T}}\,\frac{p}{S}dx$).

Solution 1b: The element stiffness matrix is obtained from the left-hand-side of the FE-formulation: $\boldsymbol{K}^{\mathrm{e}} = \displaystyle\int_{x_i}^{x_{i+1}} \left(\frac{d\boldsymbol{N}^{\mathrm{e}}}{dx}\right)^{\mathrm{T}} \frac{d\boldsymbol{N}^{\mathrm{e}}}{dx}\,dx$. Here, $\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & N_2^{\mathrm{e}} \end{bmatrix}$ contains the basis functions that are non-zero on the element. We have $\dfrac{dN_1^{\mathrm{e}}}{dx} = \dfrac{-1}{h}$ and $\dfrac{dN_2^{\mathrm{e}}}{dx} = \dfrac{1}{h}$, so

$$\boldsymbol{K}^{\mathrm{e}} = \int_{x_i}^{x_{i+1}} \begin{bmatrix} \frac{-1}{h} \\ \frac{1}{h} \end{bmatrix} \begin{bmatrix} \frac{-1}{h} & \frac{1}{h} \end{bmatrix} dx = \begin{bmatrix} \frac{1}{h^2} & \frac{-1}{h^2} \\ \frac{-1}{h^2} & \frac{1}{h^2} \end{bmatrix} \int_{x_i}^{x_{i+1}} dx = \begin{bmatrix} \frac{1}{h^2} & \frac{-1}{h^2} \\ \frac{-1}{h^2} & \frac{1}{h^2} \end{bmatrix} h = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Solution 1c: With a single element of length $h = \dfrac{L}{2}$ we get $\dfrac{2}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \dfrac{p}{S}\displaystyle\int_{-L/2}^{0} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$,

where we used the fact that p and S are constants; a_1 and a_2 represents the transverse displacements at $x = -\frac{L}{2}$ and $x = 0$, respectively. Hence, by the condition $v\left(-\frac{L}{2}\right) = 0$ we have that $a_1 = 0$ and the first equation is not valid (since the test function $v = N_1 \neq 0$ at $x = -\frac{L}{2}$. We are thus lead to $2\frac{a_2}{L} = \frac{p}{S}\int_{-L/2}^{0} N_2 dx$. With $\int_{-L/2}^{0} N_2 dx = \frac{L}{4}$ we finally obtain

$$a_2 = \frac{pL^2}{8S}.$$

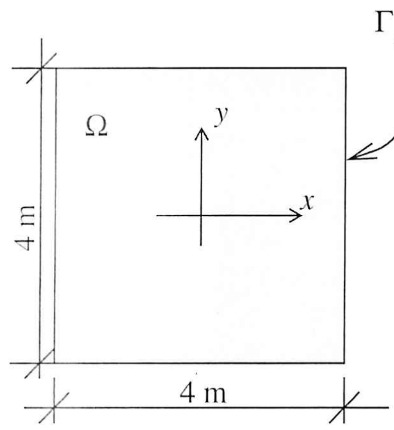
Solution 1d: Using the fact that the energy in the error equals the error in energy, we get

$$a(e, e) = a(w, w) - a(w_h, w_h) = \frac{p^2L^3}{24S^2} - \frac{p^2L^3}{32S^2} = \frac{p^2L^3}{96S^2}.$$ Hence,

$$\|e\|_a = \sqrt{a(e, e)} = \frac{pL}{S}\sqrt{\frac{L}{96}}.$$

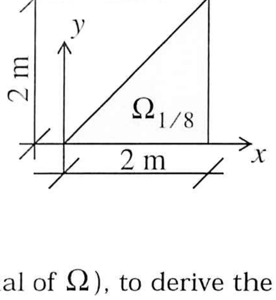
2

Consider a square membrane with side length 4 m and loaded with a distributed load $q = 150\text{ N/m}^2$ in the z-direction; the membrane is pre-tensioned with a force $S = 1000\text{ N/m}$. The deflection $w(x)$ in z-direction of the membrane, satisfies the boundary value problem



$$\begin{cases} -\mathrm{div}(\nabla w) = \frac{q}{S} & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_g \end{cases}$$

a: Using the symmetry of the problem, we can settle with considering only $1/8th$ of the domain. State how to formulate the boundary value problem so as to account for the symmetry in this manner. (1p)

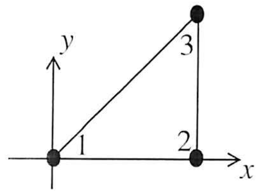


b: Use the Green-Gauss theorem, i.e.

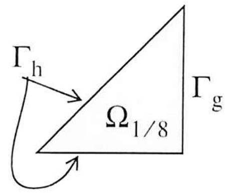
$\int_{\Omega} \psi\ \mathrm{div}(\boldsymbol{q})d\Omega = \oint_{\Gamma} \psi\boldsymbol{q}^T\boldsymbol{n}d\Gamma - \int_{\Omega} (\nabla\psi)^T\boldsymbol{q}d\Omega$ (where ψ is a scalar function, $\boldsymbol{q}$ is a vector valued function, and $\boldsymbol{n}$ is an out-ward normal of Ω), to derive the variational formulation of the problem. Regularity conditions of involved functions should be explicitly given, and it should be clear how the boundary conditions affect the variational problem. (2p)

c: Make a finite element formulation of the problem with test (weight) functions according to the Galerkin method. Show what the $\boldsymbol{B}^{e}$ – matrix looks like, for an element with N_e, basis functions. (2p)

d: Assume that the domain is discretized with a single 3-noded triangular element. With node numbering according to the figure, the basis functions becomes $N_1 = \frac{1}{2}(2-x)$, $N_2 = \frac{1}{2}(x-y)$, and $N_3 = \frac{y}{2}$. Show that this gives the equation system $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$, with $\boldsymbol{K} = [1/2]$, $\boldsymbol{a} = [a_1]$, and $\boldsymbol{f} = [0.1]$. (2p)



Solution 2a: The normal derivative of the deflection has to be zero along the lines of symmetry. Thus, the b.v.p reads



$$\begin{cases} -\mathrm{div}(\nabla w) = \frac{q}{S} & \text{in } \Omega_{1/8} \\ w = 0 & \text{on } \Gamma_g \\ (\nabla w)^T \boldsymbol{n} = 0 & \text{on } \Gamma_h \end{cases}$$

where $\boldsymbol{n}$ is an out-ward unit normal vector of the boundary.

Solution 2b: Multiply both sides of the differential equation by a test function v and integrate over the domain: $-\int_{\Omega_{1/8}} v\,\mathrm{div}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega$. Apply the Green-Green theorem to the left hand side: $\int_{\Omega_{1/8}} (\nabla v)^T(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega + \oint_{\Gamma} v(\nabla w)^T\boldsymbol{n}\,d\Gamma$. Now study the boundary integral; from the boundary condition on Γ_h, we know that the integrand is zero, but we do not know the value of $(\nabla w)^T\boldsymbol{n}$ on Γ_g. Hence, since it is necessary to be able to evaluate the boundary term, we must enforce $v = 0$ on Γ_g. Thus,

$$\oint_{\Gamma} v(\nabla w)^T\boldsymbol{n}\,d\Gamma = \int_{\Gamma_g} v(\nabla w)^T\boldsymbol{n}\,d\Gamma + \int_{\Gamma_h} v(\nabla w)^T\boldsymbol{n}\,d\Gamma = \int_{\Gamma_g} 0\cdot(\nabla w)^T\boldsymbol{n}\,d\Gamma + \int_{\Gamma_h} v\cdot 0\,d\Gamma = 0$$

To be able to evaluate the integrals, all involved functions have to be regular enough: $\int_{\Omega_{1/8}} v^2 d\Omega < \infty$ $\int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla v) d\Omega < \infty$. Let V denote the space of functions that are regular enough and that fulfil $v = 0$ on Γ_{g}. The variational problem may now be expressed as

$$\text{Find } w \in V \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla w) d\Omega = \int_{\Omega_{1/8}} v \frac{q}{S} d\Omega \qquad \forall v \in V$$

Solution 2c: Approximate the unknown function by a linear combination of selected basis functions $N_i(x, y)$: $w \approx w_{\mathrm{h}} = \sum_i N_i a_i = \boldsymbol{N}\boldsymbol{a}$ ($\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & \dots \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots \end{bmatrix}^{\mathrm{T}}$). We substitute this into the variational problem and choose test functions v according to Galerkin, viz. any linear combination of basis functions. If we define the finite element space V_{h} as the space of functions that can be expressed as a linear combination of the basis functions, the FE-formulation according to Galerkin may be expressed as:

$$\text{Find } w_{\mathrm{h}} \in V_{\mathrm{h}} \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla w_{\mathrm{h}}) d\Omega = \int_{\Omega_{1/8}} v \frac{q}{S} d\Omega \qquad \forall v \in V_{\mathrm{h}}$$

If the N_{e} basis functions that are non-zero on an element are collected into a row vector $\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & \dots & N_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix}$, and the corresponding node variables into a column vector $\boldsymbol{a}^{\mathrm{e}} = \begin{bmatrix} a_1^{\mathrm{e}} & \dots & a_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix}^{\mathrm{T}}$, then the FE-approximation on the element becomes $w_{\mathrm{h}} = \boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$. On the element we thus have $\nabla w_{\mathrm{h}} = \nabla(\boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}) = (\nabla \boldsymbol{N}^{\mathrm{e}})\boldsymbol{a}^{\mathrm{e}} = \boldsymbol{B}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$, where, hence,

$$\boldsymbol{B}^{\mathrm{e}} = \nabla \boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} N_1^{\mathrm{e}} & \dots & N_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^{\mathrm{e}}}{\partial x} & \dots & \frac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} \\ \frac{\partial N_1^{\mathrm{e}}}{\partial y} & \dots & \frac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} \end{bmatrix}$$

Solution 2d: From the finite element formulation we have $\int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla \boldsymbol{N}) d\Omega \boldsymbol{a} = \int_{\Omega_{1/8}} v \frac{q}{S} d\Omega$ with $v = N_1$ (the choices $v = N_2$ and $v = N_3$ are not valid, since N_2 and N_3 do not satisfy the essential boundary condition). Furthermore, since the essential boundary condition requires that $a_2 = a_3 = 0$, we obtain $\int\limits_{\Omega_{1/8}} (\nabla N_1)^T(\nabla N_1)d\Omega a_1 = \int\limits_{\Omega_{1/8}} N_1 \frac{q}{S} d\Omega$. Now,

$\nabla N_1 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$, so $\int\limits_{\Omega_{1/8}} (\nabla N_1)^T(\nabla N_1)d\Omega = \frac{1}{4}\int\limits_{\Omega_{1/8}} d\Omega = \frac{1}{4} \cdot \frac{2 \cdot 2}{2} = \frac{1}{2}$. Also, the right

hand side becomes $\int\limits_{\Omega_{1/8}} N_1 \frac{q}{S} d\Omega = \frac{q}{S} \int\limits_{\Omega_{1/8}} N_1 d\Omega = \frac{150}{1000} \cdot \frac{1 \cdot \frac{2 \cdot 2}{2}}{3} = 0.1$ (where we calcu-

lated the integral by calculating the volume enclosed by $\Omega_{1/8}$ and the graph of N_1). Hence,

we have $\frac{1}{2} a_1 = 0.1$.

3

Consider the weak form of the boundary value problem in the previous task; it may be expressed as

$$\text{Find } w \in V \text{ such that } \mathrm{a}(w, v) = \left(v, \frac{q}{S}\right) \qquad \forall v \in V$$

where V is the space of all admissible functions, $\mathrm{a}(.,.)$ is symmetric and linear in both arguments and $(.,.)$ a scalar product of functions. Let $w_h \in V_h$ be a conform FE-approximation of w. Show that the discretization error $e = w - w_h$ is a-orthogonal to the FE-space V_h. (Two functions, say v_1 and v_2, are said to be a-orthogonal if $\mathrm{a}(v_1, v_2) = 0$). (2p)

Solution 3: The FE-formulation reads:

$$\text{Find } w_h \in V_h \text{ such that } \mathrm{a}(w_h, v) = \left(v, \frac{q}{S}\right) \qquad \forall v \in V_h$$

We subtract this from the variational problem go get: $\mathrm{a}(w, v) - \mathrm{a}(w_h, v) = 0 \qquad \forall v \in V_h$.

Since $\mathrm{a}(.,.)$ is linear in its first argument, we can write this as

$\mathrm{a}(w - w_h, v) = 0 \qquad \forall v \in V_h$; but $w - w_h = e$, so $\mathrm{a}(e, v) = 0$ if $v \in V_h$.

4

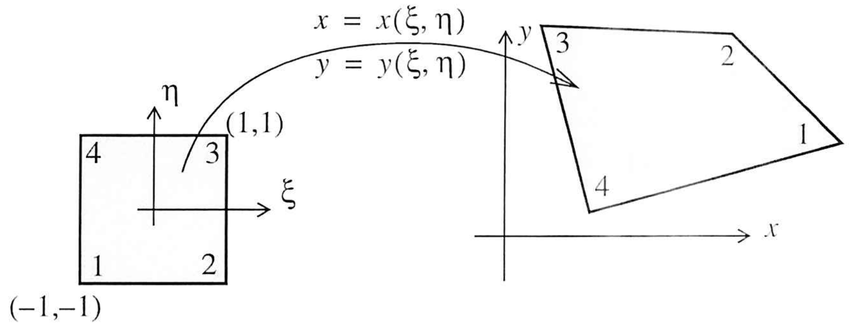
a: The basis functions of a bi-linear element are given by

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

in a local coordinate system (ξ,η). Derive an expression for the derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ for the case of an isoparametric mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. (2p)

b: Give a couple of examples of cases where the mapping is not unique, i.e. cases where $\det(\boldsymbol{J}) = 0$ somewhere inside an element. (1p)

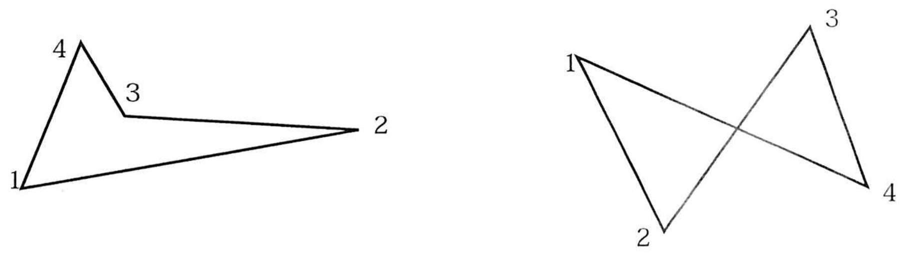


Solution 4a: Isoparametric mapping means that the basis functions are used as shape functions: $x(\xi, \eta) = \sum_{i=1}^{4} x_i N_i(\xi, \eta)$, $y(\xi, \eta) = \sum_{i=1}^{4} y_i N_i(\xi, \eta)$, where (x_i, y_i) is the coordinate of the i:th node on the element. Hence, the derivatives $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$, and $\frac{\partial y}{\partial \eta}$ are straightforward to calculate; $\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} x_i \frac{\partial N_i}{\partial \xi}$, etc. By the chain rule we have

$$\begin{aligned} \frac{\partial N_i}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i}{\partial y} \end{aligned} \quad \text{or} \quad \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}, \text{ where } \boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \text{ Thus,}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \boldsymbol{J}^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

Solution 4b: For the mapping to be unique, the inverse of the Jacobian must exist, i.e. we must have $\det \boldsymbol{J} \neq 0$. This condition is not met if, for instance, the element has an re-entrant vertex or if the nodes in the element definition are numbered in the wrong order:



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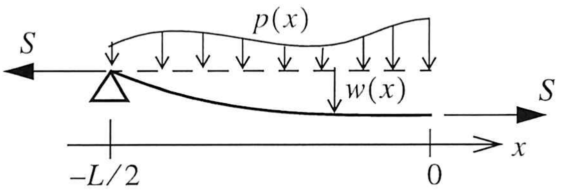

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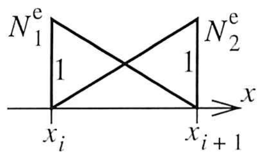

c: Assume that p is constant and solve the problem with a single linear element. (1p)

d: The energy norm of the finite element approximation in the previous sub-task is

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$\|w\|_a = \sqrt{a(w, w)} = \dfrac{pL}{S}\sqrt{\dfrac{L}{24}}$. Calculate the energy norm of the error $e = w - w_h$:

$$\|e\|_a = \sqrt{a(e, e)} = \left[\int_{-L/2}^{0} \left(\frac{de}{dx}\right)^2 dx\right]^{1/2}. \text{ (2p)}$$

Solution 1a: Multiply the differential equation by a test function $v(x)$ and integrate over the interval: $-\displaystyle\int_{-L/2}^{0} v(x)\frac{d^2w}{dx^2}dx = \int_{-L/2}^{0} v(x)\frac{p}{S}dx$. Partial integration of the left hand side yields

$\displaystyle\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw}{dx}dx - \left(v(0)\frac{dw}{dx}(0) - v\left(\frac{-L}{2}\right)\frac{dw}{dx}\left(\frac{-L}{2}\right)\right) = \int_{-L/2}^{0} v(x)\frac{p}{S}dx$. The first boundary term vanishes, since $w'(0) = 0$; to get rid of the second boundary term, we restrict the formulation to test functions such that $v(-L/2) = 0$. Thus, defining

$V = \left\{ v:\ v(-L/2) = 0 \quad \int_{-L/2}^{0} \left(\frac{dv}{dx}\right)^2 dx < \infty,\ \int_{-L/2}^{0} v^2 dx < \infty \right\}$, we are lead to the varia-

tional problem: Find $w \in V$ such that $\displaystyle\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw}{dx} dx = \int_{-L/2}^{0} v\frac{p}{S} dx \qquad \forall v \in V$.

Finite element formulation: select a set of basis functions N_i and let V_h be the space of functions that can be expressed as a linear combination of the basis functions. Approximate $w \approx w_h = \sum_i N_i a_i = \boldsymbol{Na}$ (where, hence, $\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & \ldots \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \ldots \end{bmatrix}^{\mathrm{T}}$). Now, substitute the approximation into the variational problem and use the basis functions as test functions (i.e. the Galerkin method); we obtain:

Find $w_h \in V_h$ such that $\displaystyle\int_{-L/2}^{0} \frac{dv}{dx}\frac{dw_h}{dx} dx = \int_{-L/2}^{0} v\frac{p}{S} dx \qquad \forall v \in V_h$ (or

$\displaystyle\int_{-L/2}^{0} \left(\frac{d\boldsymbol{N}}{dx}\right)^{\mathrm{T}} \frac{d\boldsymbol{N}}{dx} dx\, \boldsymbol{a} = \int_{-L/2}^{0} \boldsymbol{N}^{\mathrm{T}} \frac{p}{S} dx$).

Solution 1b: The element stiffness matrix is obtained from the left-hand-side of the FE-formulation: $\boldsymbol{K}^{\mathrm{e}} = \displaystyle\int_{x_i}^{x_{i+1}} \left(\frac{d\boldsymbol{N}^{\mathrm{e}}}{dx}\right)^{\mathrm{T}} \frac{d\boldsymbol{N}^{\mathrm{e}}}{dx} dx$. Here, $\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & N_2^{\mathrm{e}} \end{bmatrix}$ contains the basis functions that are non-zero on the element. We have $\dfrac{dN_1^{\mathrm{e}}}{dx} = \dfrac{-1}{h}$ and $\dfrac{dN_2^{\mathrm{e}}}{dx} = \dfrac{1}{h}$, so

$$\boldsymbol{K}^{\mathrm{e}} = \int_{x_i}^{x_{i+1}} \begin{bmatrix} \frac{-1}{h} \\ \frac{1}{h} \end{bmatrix} \begin{bmatrix} \frac{-1}{h} & \frac{1}{h} \end{bmatrix} dx = \begin{bmatrix} \frac{1}{h^2} & \frac{-1}{h^2} \\ \frac{-1}{h^2} & \frac{1}{h^2} \end{bmatrix} \int_{x_i}^{x_{i+1}} dx = \begin{bmatrix} \frac{1}{h^2} & \frac{-1}{h^2} \\ \frac{-1}{h^2} & \frac{1}{h^2} \end{bmatrix} h = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Solution 1c: With a single element of length $h = \dfrac{L}{2}$ we get $\dfrac{2}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \dfrac{p}{S}\displaystyle\int_{-L/2}^{0} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$,

where we used the fact that p and S are constants; a_1 and a_2 represents the transverse displacements at $x = -\frac{L}{2}$ and $x = 0$, respectively. Hence, by the condition $v\left(-\frac{L}{2}\right) = 0$ we have that $a_1 = 0$ and the first equation is not valid (since the test function $v = N_1 \neq 0$ at $x = -\frac{L}{2}$. We are thus lead to $2\frac{a_2}{L} = \frac{p}{S}\int_{-L/2}^{0} N_2 dx$. With $\int_{-L/2}^{0} N_2 dx = \frac{L}{4}$ we finally obtain

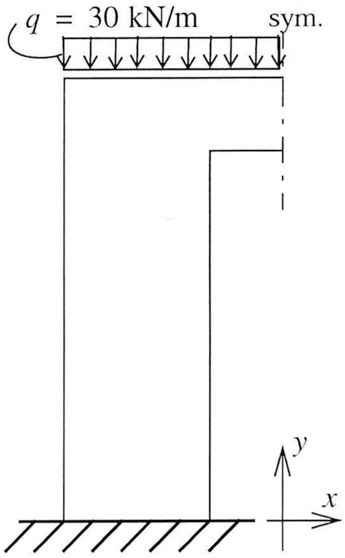
$$a_2 = \frac{pL^2}{8S}.$$

Solution 1d: Using the fact that the energy in the error equals the error in energy, we get

$a(e, e) = a(w, w) - a(w_h, w_h) = \frac{p^2L^3}{24S^2} - \frac{p^2L^3}{32S^2} = \frac{p^2L^3}{96S^2}$. Hence,

$$\|e\|_a = \sqrt{a(e, e)} = \frac{pL}{S}\sqrt{\frac{L}{96}}.$$

2



In the absence of body loading, the weak form of a 2D elasticity problem may be written

$$(2) \quad \int_{\Omega} (\tilde{\nabla} \boldsymbol{v})^{\mathrm{T}} \boldsymbol{D} \tilde{\nabla} \boldsymbol{u} d\Omega = \int_{\Gamma} \boldsymbol{v}^{\mathrm{T}} \boldsymbol{t} d\Gamma$$

where $\boldsymbol{v} = \begin{bmatrix} v_x & v_y \end{bmatrix}^{\mathrm{T}}$ is a vector with test functions,

$\boldsymbol{u} = \begin{bmatrix} u_x & u_y \end{bmatrix}^{\mathrm{T}}$ is the displacement vector,

$$\boldsymbol{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} n_x + \sigma_{xy} n_y \\ \sigma_{xy} n_x + \sigma_{yy} n_y \end{bmatrix}$$

is the traction vector, and $\tilde{\nabla}^{\mathrm{T}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$.

Now consider the symmetric problem depicted in the illustration; assume that the thickness is t.

a: Specify the boundary conditions necessary to solve the illustrated problem. Take care to give conditions on each and every part of the boundary. (2p)

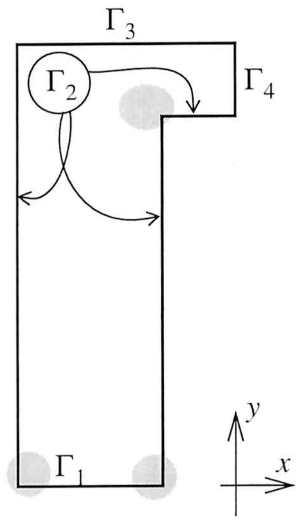
b: Assume that the problem is solved by an adaptive FE–program; in which areas do you expect the program to generate the smallest elements? Why? (2p)

c: Make a finite element formulation of the problem (2) — take the boundary conditions into account. Your solution should show how the unknown vector $\boldsymbol{u}$ is approximated. Also show what the $\boldsymbol{B}^{e}$–matrix looks like for a 3–noded triangular element. (3p)

d: In abstract notation the problem (2) may be written: Find $\boldsymbol{u} \in [V]^2$ such that

$\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma} \boldsymbol{v}^{\mathrm{T}} \boldsymbol{t} d\Gamma \qquad \forall \boldsymbol{v} \in [V]^2$, for some appropriate space V of functions. Consider a conform FE–formulation of the problem, i.e. a FE–space V_h that is a sub–space of V. Show that the error $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$ is energy–orthogonal to V_h (i.e. prove Galerkin orthogonality), and show that the energy in the error is equal to the error in energy: $\mathrm{a}(\boldsymbol{e}, \boldsymbol{e}) = \mathrm{a}(\boldsymbol{u}, \boldsymbol{u}) - \mathrm{a}(\boldsymbol{u}_h, \boldsymbol{u}_h)$. (2p)

Solution 2a: With reference to the notation in the figure, we have that the displacements are zero on Γ_1: $u_x = u_y = 0$. The boundary Γ_2 is free and unloaded: $t_x = t_y = 0$; on Γ_3 there is no load in x-direction, while there is a distributed load (force/length) in negative y-direction: $t_x = 0$, $t_y = -q/t$. On the line of symmetry, Γ_4, the x-displacement have to be zero, while the shear in y-direction must be zero: $u_x = 0$, $t_y = 0$.



(Inserting into the right-hand-side of the variational problem and enforcing the test functions to satisfy the Dirichlet conditions, one obtains $\int_{\Gamma} \boldsymbol{v}^{\mathrm{T}} \boldsymbol{t} d\Gamma = -\int_{\Gamma_3} v_y \frac{q}{t} d\Gamma$).

Solution 2b: One expects smallest elements in regions where second derivatives are large; this typically occurs at singular points. In the present example we have 3 such points: the re-entrant corner and the 2 points with abrupt change in boundary conditions (see encircled areas in the figure).

Solution 2c: Approximate the displacement vector $\boldsymbol{u} \approx \boldsymbol{u}_h = \begin{bmatrix} u_{xh} \\ u_{yh} \end{bmatrix} = \boldsymbol{Na}$, where

$\boldsymbol{a} = \begin{bmatrix} a_{1x} & a_{1y} & a_{2x} & a_{2y} & a_{3x} & \ldots \end{bmatrix}^{\mathrm{T}}$ are the node displacements and $\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \ldots \\ 0 & N_1 & 0 & N_2 & \ldots \end{bmatrix}$

is a matrix with the selected basis functions. Substituting into the variational problem we

obtain $\int\limits_{\Omega} (\tilde{\nabla}\boldsymbol{v})^{\mathrm{T}} \boldsymbol{D}\tilde{\nabla}(\boldsymbol{Na})d\Omega = \int\limits_{\Gamma_3} \boldsymbol{v}^{\mathrm{T}} \begin{bmatrix} 0 \\ -q/t \end{bmatrix} d\Gamma$, or with the notation $\boldsymbol{B} = \tilde{\nabla}\boldsymbol{N}$

$\int\limits_{\Omega} (\tilde{\nabla}\boldsymbol{v})^{\mathrm{T}} \boldsymbol{DB}d\Omega\boldsymbol{a} = \int\limits_{\Gamma_3} \boldsymbol{v}^{\mathrm{T}} \begin{bmatrix} 0 \\ -q/t \end{bmatrix} d\Gamma$. Now select test functions according to Galerkin:

$\boldsymbol{v} = \begin{bmatrix} N_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ N_1 \end{bmatrix}, \begin{bmatrix} N_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ N_2 \end{bmatrix}, \ldots$; we obtain the equation system

$\int\limits_{\Omega} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{DB}d\Omega\boldsymbol{a} = \int\limits_{\Gamma_3} \boldsymbol{N}^{\mathrm{T}} \begin{bmatrix} 0 \\ -q/t \end{bmatrix} d\Gamma$.

For a 3-noded triangle we have $\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & 0 & N_2^{\mathrm{e}} & 0 & N_3^{\mathrm{e}} & 0 \\ 0 & N_1^{\mathrm{e}} & 0 & N_2^{\mathrm{e}} & 0 & N_3^{\mathrm{e}} \end{bmatrix}$ (the basis functions that are

non-zero on the element), so $\boldsymbol{B}^{\mathrm{e}} = \tilde{\nabla}\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} \frac{\partial N_1^{\mathrm{e}}}{\partial x} & 0 & \ldots & 0 \\ 0 & \frac{\partial N_2^{\mathrm{e}}}{\partial y} & \ldots & \frac{\partial N_3^{\mathrm{e}}}{\partial y} \\ \frac{\partial N_1^{\mathrm{e}}}{\partial y} & \frac{\partial N_2^{\mathrm{e}}}{\partial x} & \ldots & \frac{\partial N_3^{\mathrm{e}}}{\partial x} \end{bmatrix}$

Solution 2d: The FE-problem is: Find $\boldsymbol{u}_h \in [V_h]^2$ such that

$\mathrm{a}(\boldsymbol{u}_h, \boldsymbol{v}) = \int\limits_{\Gamma} \boldsymbol{v}^{\mathrm{T}} \boldsymbol{t} d\Gamma \qquad \forall \boldsymbol{v} \in [V_h]^2$. Subtracting from the variational problem, we obtain

$\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) - \mathrm{a}(\boldsymbol{u}_h, \boldsymbol{v}) = 0$, which holds in the common domain of definition, i.e. for test functions $\boldsymbol{v} \in [V_h]^2$. Since a(., .) is linear in its first argument, this amounts to

$\mathrm{a}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = \mathrm{a}(\boldsymbol{e}, \boldsymbol{e}) = 0 \qquad \forall \boldsymbol{v} \in [V_h]^2$ which is referred to as Galerkin orthogonality.

To show that the energy in the error equals the error in energy, we use the fact that $a(.,.)$ is symmetric and linear in both arguments:

$$a(\boldsymbol{e}, \boldsymbol{e}) = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) = a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) =$$

$$a(\boldsymbol{u}, \boldsymbol{u}) - a(\boldsymbol{u}, \boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{u}) + a(\boldsymbol{u}_h, \boldsymbol{u}_h) = a(\boldsymbol{u}, \boldsymbol{u}) + a(\boldsymbol{u}_h, \boldsymbol{u}_h) - 2a(\boldsymbol{u}, \boldsymbol{u}_h) =$$

$$\{\text{use } \boldsymbol{u} = \boldsymbol{u}_h + \boldsymbol{e} \text{ in the last term}\} = a(\boldsymbol{u}, \boldsymbol{u}) + a(\boldsymbol{u}_h, \boldsymbol{u}_h) - 2a(\boldsymbol{u}_h + \boldsymbol{e}, \boldsymbol{u}_h) =$$

$$a(\boldsymbol{u}, \boldsymbol{u}) + a(\boldsymbol{u}_h, \boldsymbol{u}_h) - 2a(\boldsymbol{u}_h, \boldsymbol{u}_h) - 2a(\boldsymbol{e}, \boldsymbol{u}_h) =$$

$$\{\boldsymbol{u}_h \in [V_h]^2 \text{ so according to Galerkin orthogonality the last term vanish}\} =$$

$$a(\boldsymbol{u}, \boldsymbol{u}) - a(\boldsymbol{u}_h, \boldsymbol{u}_h)$$

Hence, $a(\boldsymbol{e}, \boldsymbol{e}) = a(\boldsymbol{u}, \boldsymbol{u}) - a(\boldsymbol{u}_h, \boldsymbol{u}_h)$.

3

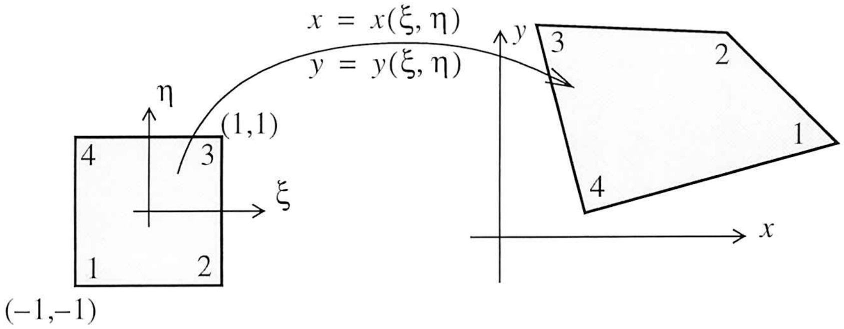
a: The basis functions of a bi-linear element are given by

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

in a local coordinate system (ξ,η). Derive an expression for the derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ for the case of an isoparametric mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. (2p)

b: Give a couple of examples of cases where the mapping is not unique, i.e. cases where $\det(\boldsymbol{J}) = 0$ somewhere inside an element. (1p)

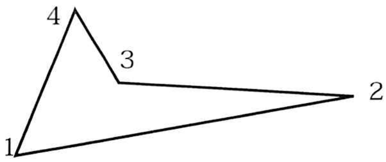


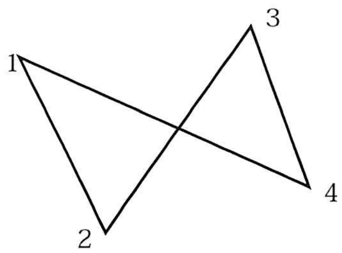
Solution 3a: Isoparametric mapping means that the basis functions are used as shape functions: $x(\xi, \eta) = \sum_{i=1}^{4} x_i N_i(\xi, \eta)$, $y(\xi, \eta) = \sum_{i=1}^{4} y_i N_i(\xi, \eta)$, where (x_i, y_i) is the coordinate of the i:th node on the element. Hence, the derivatives $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$, and $\frac{\partial y}{\partial \eta}$ are straightforward to calculate; $\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} x_i \frac{\partial N_i}{\partial \xi}$, etc. By the chain rule we have

$$\begin{aligned}\frac{\partial N_i}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i}{\partial y}\end{aligned} \quad \text{or} \quad \begin{bmatrix}\frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta}\end{bmatrix} = \boldsymbol{J}\begin{bmatrix}\frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y}\end{bmatrix}, \text{ where } \boldsymbol{J} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}. \text{ Thus,}$$

$$\begin{bmatrix}\frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y}\end{bmatrix} = \boldsymbol{J}^{-1}\begin{bmatrix}\frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta}\end{bmatrix}$$

<u>Solution 3b:</u> For the mapping to be unique, the inverse of the Jacobian must exist, i.e. we must have $\det \boldsymbol{J} \neq 0$. This condition is not met if, for instance, the element has an re-entrant vertex or if the nodes in the element definition are numbered in the wrong order:





Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION
JANUARY 12 2005

Solutions

Time and location:	8^{30} - 12^{30} in the V building
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that January 13. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses*.
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments.**
Results:	will be posted at the Department of Applied Mechanics no later than January 19. Results are sent for registration Friday January 21 — **course participants that have not completed the computer assignments by this time will be registered as *not approved*.** You may scrutinize the correction (mark up) of your written examination Thursday January 20 13^{30}–15^{00} (at the office space of Department of Applied Mechanics).

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

1

a: Formulate the divergence theorem (explain the entities you use) and show how the Green–Gauss theorem, i.e. $\int_\Omega \phi \mathrm{div}(\boldsymbol{q})d\Omega = \oint_\Gamma \phi \boldsymbol{q}^{\mathrm{T}} \boldsymbol{n} d\Gamma - \int_\Omega (\nabla\phi)^{\mathrm{T}} \boldsymbol{q} d\Omega$, may be derived from it. (2p)

b: Consider the boundary value problem $-\mathrm{div}(\boldsymbol{D}\nabla\phi) = f$ in Ω, $\phi = 0$ on Γ, where $\boldsymbol{D}$ is symmetric and positive definite, f is a given function, and Γ is the boundary of $\Omega \subset \Re^2$. Derive the weak formulation of the problem. It should be clear how the boundary condition affects the formulation; also specify requirements on the test (weight) functions. (2p)

c: Make a finite element formulation and show what the $\boldsymbol{B}^{\mathrm{e}}$-matrix looks like for an element with N_{e} basis functions. (2p)

Solution 1a: Let $\boldsymbol{n} = \begin{bmatrix} n_x & n_y \end{bmatrix}^{\mathrm{T}}$ ($|\boldsymbol{n}| = 1$) be an out-ward normal on the boundary Γ and $\boldsymbol{q} = \begin{bmatrix} q_x(x, y) & q_y(x, y) \end{bmatrix}^{\mathrm{T}}$ be a vector field. Provided that the functions are sufficiently smooth, we have (the divergence theorem):

$$\int_\Omega \mathrm{div}(\boldsymbol{q})d\Omega = \oint_\Gamma \boldsymbol{q}^{\mathrm{T}} \boldsymbol{n} d\Gamma$$

To obtain the Green–Gauss theorem, we construct the vector field $\phi\boldsymbol{q}$, where $\phi = \phi(x, y)$ is smooth enough, but otherwise arbitrary. Calculate the divergence of $\phi\boldsymbol{q}$:

$$\mathrm{div}(\phi\boldsymbol{q}) = \frac{\partial}{\partial x}[\phi q_x] + \frac{\partial}{\partial y}[\phi q_y] = \phi\frac{\partial q_x}{\partial x} + \phi\frac{\partial q_y}{\partial y} + \frac{\partial\phi}{\partial x}q_x + \frac{\partial\phi}{\partial y}q_y = \phi\mathrm{div}(\boldsymbol{q}) + (\nabla\phi)^{\mathrm{T}}\boldsymbol{q}$$

Applying the divergence theorem on $\phi\boldsymbol{q}$ and utilizing the identity $\mathrm{div}(\phi\boldsymbol{q}) = \phi\mathrm{div}(\boldsymbol{q}) + (\nabla\phi)^{\mathrm{T}}\boldsymbol{q}$, give the desired result.

Solution 1b: Multiply the differential equation by a test function v and integrate over the domain: $-\int_\Omega v\mathrm{div}(\boldsymbol{D}\nabla\phi)d\Omega = \int_\Omega vf d\Omega$. Identify v and the vector $\boldsymbol{D}\nabla\phi$, with ϕ and $\boldsymbol{q}$, respectively, in the Green–Gauss theorem; The left hand side then becomes:

$\int_\Omega (\nabla v)^{\mathrm{T}}\boldsymbol{D}(\nabla\phi)d\Omega = \oint_\Gamma v(\boldsymbol{D}\nabla\phi)^{\mathrm{T}}\boldsymbol{n}d\Gamma + \int_\Omega vf d\Omega$. For this identity to be meaningful, all functions have to be sufficiently smooth — they have to be square integrable and have square integrable first derivatives; furthermore, we have to choose the test function so that $v = 0$ on Γ since we cannot calculate the boundary integral otherwise ($\nabla\phi$ is unknown) — thus the test function have to satisfy *homogeneous essential* boundary conditions. Let V denote the space of all functions that satisfies these conditions; the variational formulation may the be written: find $\phi \in V$ such that

$$\int_{\Omega} (\nabla v)^{\mathrm{T}} \boldsymbol{D} (\nabla \phi) d\Omega = \int_{\Omega} v f \, d\Omega \qquad \forall v \in V$$

Solution 1c: Choose N basis functions N_i in V (conform method) and let $V_{\mathrm{h}} \subset V$ be the space of all functions that may be expressed as a linear combination of the basis functions. In particular, we approximate $\phi \approx \phi_{\mathrm{h}} = \sum_{i=1}^{N} N_i a_i$; the node variables a_i are determined so that the variational problem is satisfied in V_{h} (Galerkin method). Hence, find $\phi_{\mathrm{h}} \in V_{\mathrm{h}}$ such that

$$\int_{\Omega} (\nabla v)^{\mathrm{T}} \boldsymbol{D} (\nabla \phi_{\mathrm{h}}) d\Omega = \int_{\Omega} v f \, d\Omega \qquad \forall v \in V_{\mathrm{h}}$$

On an element with N_{e} basis functions, i.e. one in which only N_{e} out of the N functions are non-zero, we may write the FE approximation

$\phi_{\mathrm{h}} = \begin{bmatrix} N_1^{\mathrm{e}} & N_2^{\mathrm{e}} & \dots & N_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix} \begin{bmatrix} a_1^{\mathrm{e}} & a_2^{\mathrm{e}} & \dots & a_{N_{\mathrm{e}}}^{\mathrm{e}} \end{bmatrix}^{\mathrm{T}} = \boldsymbol{N}^{\mathrm{e}} \boldsymbol{a}^{\mathrm{e}}$, so that

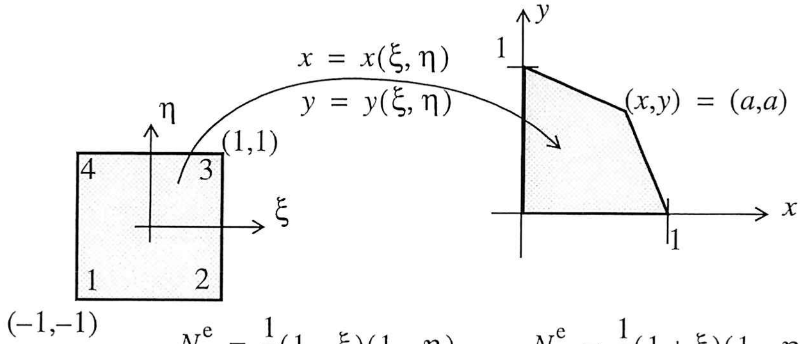
$\nabla \phi_{\mathrm{h}} = \nabla(\boldsymbol{N}^{\mathrm{e}} \boldsymbol{a}^{\mathrm{e}}) = (\nabla \boldsymbol{N}^{\mathrm{e}}) \boldsymbol{a}^{\mathrm{e}} = \boldsymbol{B}^{\mathrm{e}} \boldsymbol{a}^{\mathrm{e}}$, where hence

$$\boldsymbol{B}^{\mathrm{e}} = \nabla \boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} \dfrac{\partial N_1^{\mathrm{e}}}{\partial x} & \dfrac{\partial N_2^{\mathrm{e}}}{\partial x} & \cdots & \dfrac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} \\ \dfrac{\partial N_1^{\mathrm{e}}}{\partial y} & \dfrac{\partial N_2^{\mathrm{e}}}{\partial y} & \cdots & \dfrac{\partial N_{N_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} \end{bmatrix}$$

2

a: When using mapped elements, we have the basis functions defined in terms of some local coordinates (ξ, η), i.e. $N_i^e = N_i^e(\xi, \eta)$. Derive expressions for the derivatives $\dfrac{\partial N_i^e}{\partial x}$ and $\dfrac{\partial N_i^e}{\partial y}$ in the case when the mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ to the global coordinates is isoparametric. (2p)

b: Calculate the determinant of the Jacobian of the mapping, as a function of the parameter a, of the element shown in the illustration. Which values of a yield a unique mapping? (2p)



$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

<u>Solution 2a:</u> Use the chain rule to calculate the derivatives with respect to ξ and η:

$$\left.\begin{aligned} \frac{\partial N_i^e}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i^e}{\partial y} \\ \frac{\partial N_i^e}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial y} \end{aligned}\right\} = \boldsymbol{J}\begin{bmatrix} \dfrac{\partial N_i^e}{\partial x} \\ \dfrac{\partial N_i^e}{\partial y} \end{bmatrix}$$

where, hence, $\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$. We then get the desired derivatives by multiplying both sides by $\boldsymbol{J}^{-1}$: $\begin{bmatrix} \dfrac{\partial N_i^e}{\partial x} \\ \dfrac{\partial N_i^e}{\partial y} \end{bmatrix} = \boldsymbol{J}^{-1}\begin{bmatrix} \dfrac{\partial N_i^e}{\partial \xi} \\ \dfrac{\partial N_i^e}{\partial \eta} \end{bmatrix}$. The components of $\boldsymbol{J}$ are easily calculate as $\dfrac{\partial x}{\partial \xi} = \dfrac{\partial}{\partial \xi}\sum_i x_i N_i^e = \sum_i x_i \dfrac{\partial N_i^e}{\partial \xi}$, etc.; (x_i, y_i) is the coordinate of node i on the element.

Solution 2b: Number the nodes in anti-clock-wise order; starting with the node in the origin, we get the node coordinates $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (1, 0)$, $(x_3, y_3) = (a, a)$, and $(x_4, y_4) = (0, 1)$. Hence;

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_2^e}{\partial \xi} + a\frac{\partial N_3^e}{\partial \xi} = \frac{1}{4}(1 - \eta + a + a\eta) \qquad \frac{\partial x}{\partial \eta} = \frac{1}{4}(-1 - \xi + a + a\xi)$$

$$\frac{\partial y}{\partial \xi} = \frac{1}{4}(-1 - \eta + a + a\eta) \qquad \frac{\partial y}{\partial \eta} = \frac{1}{4}(1 - \xi + a + a\xi)$$

so that $\det \boldsymbol{J} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = \frac{1}{8}(2a - \xi - \eta + a\xi + a\eta)$.

The mapping will be unique provided that $\boldsymbol{J}$ is non-singular, that is $\det \boldsymbol{J} \neq 0$ inside the element. This is obviously the case if a is sufficiently large; the local coordinates ξ and η may only vary between -1 and 1. When a decreases, $\det \boldsymbol{J}$ becomes zero at $(x, y) = (a, a)$ first — i.e. at $(\xi, \eta) = (1, 1)$.

$$\det \boldsymbol{J}(1, 1) = 0 \Rightarrow 2a - 1 - 1 + a + a = 0 \Rightarrow a = \frac{1}{2}$$

Thus, the mapping is unique whenever $a > \frac{1}{2}$.

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a: According to the Euler-Bernoulli theory, beam bending is described by the differential equation $\frac{d^2}{dx^2}\left[EI\frac{d^2 w}{dx^2}\right] = q$, where $w = w(x)$ is the transverse displacement, $q = q(x)$ is the intensity of a distributed load along the beam, and $EI = EI(x)$ is the bending stiffness. Derive the weak form and make a finite element formulation of the problem. Assume that the problem is defined on the interval $0 < x < L$, but leave the boundary conditions unspecified. (2p)

b: A simple beam element has four basis functions, viz. $N_1^e = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$,

$N_2^e = x\left[1 - 2\frac{x}{L} + \left(\frac{x}{L}\right)^2\right]$, $N_3^e = \left(\frac{x}{L}\right)^2\left(3 - 2\frac{x}{L}\right)$, and $N_4^e = x\left(\frac{x}{L}\right)\left(\frac{x}{L} - 1\right)$; $(0 < x < L)$.

Show that the element is complete. (2p)

Hint: The basis functions have been constructed so that $N_1^e(0) = 1$, $N_2^e(0) = N_3^e(0) = N_4^e(0) = 0$, $N_3^e(L) = 1$ and $N_1^e(L) = N_2^e(L) = N_4^e(L) = 0$; the derivatives are such that $\frac{dN_2^e}{dx} = 1$ and $\frac{dN_1^e}{dx} = \frac{dN_3^e}{dx} = \frac{dN_4^e}{dx} = 0$ at $x = 0$, while $\frac{dN_4^e}{dx} = 1$ and $\frac{dN_1^e}{dx} = \frac{dN_2^e}{dx} = \frac{dN_3^e}{dx} = 0$ at $x = L$.

Solution 3a: Multiply the differential equation by a test function v and integrate over the domain (interval): $\int_0^L v\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right]dx = \int_0^L vqdx$. Partial integration (twice) of the left hand side and moving the boundary terms to the right hand side yield:

$\int_0^L \frac{d^2v}{dx^2}EI\frac{d^2w}{dx^2}dx = \left[\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_0^L - \left[v\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]\right]_0^L + \int_0^L vqdx$. The variational problem is thus to find the function w that satisfies this equation for all admissible choices of the test function v. The functions have to be square integrable, have square integrable first and second derivatives. Furthermore, the test functions have to satisfy homogeneous essential boundary conditions, i.e. conditions given on w and its first derivative.vata.

FE formulation: Approximate w by a linear combination of N chosen basis functions $w \approx w_h = \sum_{i=1}^{N} N_i a_i$ and select the basis functions as test functions (Galerkin) to obtain as many equations as unknown node variables a_i:

$$\int_0^L \frac{d^2v}{dx^2}EI\frac{d^2w_h}{dx^2}dx = \left[\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_0^L - \left[v\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]\right]_0^L + \int_0^L vqdx \qquad v = N_1, N_2, \ldots, N$$

Solution 3a: The approximation on the element is $w \approx w_h = \sum_{i=1}^{4} N_i^e a_i^e$. For the element to be complete, we need to be able to: select node variables a_i^e so that w_h becomes constant (on the element); select node variables a_i^e so that w_h' becomes constant; select node variables a_i^e so that w_h'' becomes constant.

Choose for instance $a_1^e = a_3^e = c$ (where c is an arbitrary constant) and $a_2^e = a_4^e = 0$; then

$$w_h = \sum_{i=1}^{4} N_i^e a_i^e = c(N_1^e + N_3^e) = c$$

so the first condition is meet. Now, select $a_2^e = a_4^e = c$, $a_3^e = cL$ and $a_1^e = 0$; then

$$w_h = \sum_{i=1}^{4} N_i^e a_i^e = c(N_2^e + LN_3^e + N_4^e) = cx$$

i.e. the first derivative is a constant.

To find node variables that give a constant second derivative, we first calculate

$$\frac{d^2 N_1^e}{dx^2} = -\frac{6}{L^2} + \frac{12x}{L^3} \qquad \frac{d^2 N_2^e}{dx^2} = -\frac{4}{L} + \frac{6x}{L^2}$$

$$\frac{d^2 N_3^e}{dx^2} = \frac{6}{L^2} - \frac{12x}{L^3} \qquad \frac{d^2 N_4^e}{dx^2} = -\frac{2}{L} + \frac{6x}{L^2}$$

From this we see that the selection $a_2^e = -a_4^e = c$ and $a_1^e = a_3^e = 0$ leads to

$$w_h = \sum_{i=1}^{4} N_i^e a_i^e = c(N_2^e - N_4^e) = c\left(x - \frac{x^2}{L}\right)$$

which has a constant second derivative $(-2c/L)$. $\therefore$ the element is complete.

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Consider the weak form of the boundary value problem in the previous task; provided that we have homogeneous essential boundary conditions, it may be expressed as

$$\text{Find } w \in V \text{ such that } \mathrm{a}(w, v) = (v, f) \qquad \forall v \in V$$

where V is the space of all admissible functions, $\mathrm{a}(.,.)$ is symmetric and linear in both arguments and $(.,.)$ a scalar product of functions.

Let $w_\mathrm{h} \in V_\mathrm{h}$ be a conform FE-approximation of w.

a: Prove Galerkin orthogonality, i.e. show that the discretization error $e = w - w_\mathrm{h}$ is a-orthogonal to the FE-space V_h. (Two functions, say v_1 and v_2, are said to be a-orthogonal if $\mathrm{a}(v_1, v_2) = 0$). (2p)

b: Use Galerkin orthogonality to show that the energy of the error equals the error in energy: $\mathrm{a}(e, e) = \mathrm{a}(w, w) - \mathrm{a}(w_\mathrm{h}, w_\mathrm{h})$. (2p)

c: Let $\boldsymbol{K}$ be the stiffness matrix and $\boldsymbol{a}$ the solution vector (the node variables), so that $w_\mathrm{h} = \boldsymbol{Na}$, where $\boldsymbol{N}$ is a vector containing the basis functions. Express $\mathrm{a}(w_\mathrm{h}, w_\mathrm{h})$ in terms of the node variables and the stiffness matrix. (2p)

Solution 4a: Subtracting the finite element approximation from the weak problem, we get $a(w, v) - a(w_h, v) = 0 \ \forall v \in V_h$, since $V_h \subset V$. Since $a(.,.)$ is linear in its first argument and $w - w_h = e$, one has that $a(w, v) - a(w_h, v) = a(w - w_h, v) = a(e, v)$. Hence, $a(e, v) = 0 \ \forall v \in V_h$.

Solution 4b:

$$a(e, e) = a(w - w_\mathrm{h}, w - w_\mathrm{h}) = a(w, w - w_\mathrm{h}) - a(w_\mathrm{h}, w - w_\mathrm{h}) =$$
$$a(w, w) - a(w, w_\mathrm{h}) - a(w_\mathrm{h}, w) + a(w_\mathrm{h}, w_\mathrm{h}) = a(w, w) + a(w_\mathrm{h}, w_\mathrm{h}) - 2a(w, w_\mathrm{h}) =$$
$$\{\text{substitute } w = w_\mathrm{h} + e \text{ in the last term}\} = a(w, w) + a(w_\mathrm{h}, w_\mathrm{h}) - 2a(w_\mathrm{h} + e, w_\mathrm{h}) =$$
$$a(w, w) + a(w_\mathrm{h}, w_\mathrm{h}) - 2a(w_\mathrm{h}, w_\mathrm{h}) - 2a(e, w_\mathrm{h}) =$$
$$\{w_\mathrm{h} \in V_\mathrm{h} \text{ so the last term is zero by Galerkin orthogonality}\} = a(w, w) - a(w_\mathrm{h}, w_\mathrm{h})$$

Thus, $a(e, e) = a(w, w) - a(w_\mathrm{h}, w_\mathrm{h})$.

Solution 4b: $a(w_\mathrm{h}, w_\mathrm{h}) = a(\boldsymbol{a}^\mathrm{T}\boldsymbol{N}^\mathrm{T}, \boldsymbol{Na}) = \boldsymbol{a}^\mathrm{T}a(\boldsymbol{N}^\mathrm{T}, \boldsymbol{N})\boldsymbol{a} = \boldsymbol{a}^\mathrm{T}\boldsymbol{Ka}$