Solutions

Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION

AUGUST 25 2004

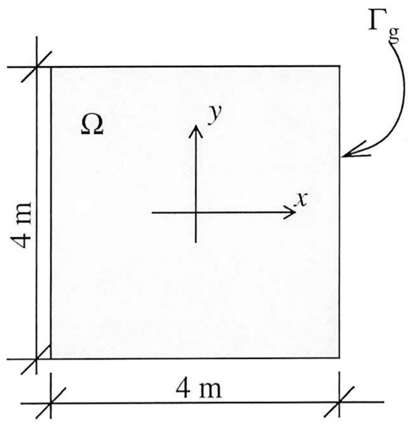
Time and location:	$14^{\underline{15}}$ - $18^{\underline{15}}$ in the V building
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that August 26. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses*.
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments**.
Results:	will be posted at the Department of Applied Mechanics no later than September 3. Results are sent for registration September 8 — **course participants that have not completed the computer assignments by this time will be registered as *not approved***.
	You may scrutinize the correction (mark up) of your written examination Tuesday September 7 $13^{\underline{30}}$–$15^{\underline{00}}$ (at the office space of Department of Applied Mechanics).

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

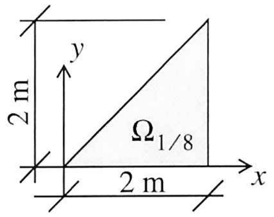
1

Consider a square membrane with side length 4 m and loaded with a distributed load $q = 150\ \mathrm{N/m^2}$ in the z-direction; the membrane is pre-tensioned with a force $S = 1000\ \mathrm{N/m}$. The deflection $w(x)$ in z-direction of the membrane, satisfies the boundary value problem



$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_g \end{cases}$$

a: Using the symmetry of the problem, we can settle with considering only $1/8th$ of the domain. State how to formulate the boundary value problem so as to account for the symmetry in this manner. (1p)

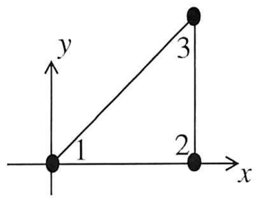


b: Use the Green-Gauss theorem, i.e.

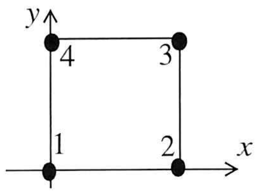
$$\int_\Omega \psi\ \mathrm{div}(\boldsymbol{q})d\Omega = \oint_\Gamma \psi \boldsymbol{q}^T \boldsymbol{n} d\Gamma - \int_\Omega (\nabla\psi)^T \boldsymbol{q} d\Omega$$ (where ψ is a scalar function, $\boldsymbol{q}$ is a vector valued function, and $\boldsymbol{n}$ is an out-ward normal of Ω), to derive the variational formulation of the problem. Regularity conditions of involved functions should be explicitly given, and it should be clear how the boundary conditions affect the variational problem. (2p)

c: Make a finite element formulation of the problem with test (weight) functions according to the Galerkin method. Show what the $\boldsymbol{B}^e$- matrix looks like, for an element with N_e, basis functions. (2p)

d: Assume that the domain is discretized with a single 3-noded triangular element. With node numbering according to the figure, the basis functions becomes $N_1 = \frac{1}{2}(2-x)$, $N_2 = \frac{1}{2}(x-y)$, and $N_3 = \frac{y}{2}$. Show that this gives the equation system $\boldsymbol{Ka} = \boldsymbol{f}$, with $\boldsymbol{K} = [1/2]$, $\boldsymbol{a} = [a_1]$, and $\boldsymbol{f} = [0.1]$. (2p)

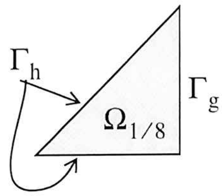


e: If instead we use a 4-noded element to solve the problem on $1/4\,th$ of the domain Ω, we get $\boldsymbol{K} = \begin{bmatrix} 2/3 \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 \end{bmatrix}$, and $\boldsymbol{f} = \begin{bmatrix} 0.15 \end{bmatrix}$. Will this FE-approximation be better or worse that the one obtained with the triangular element in task d above? Motivate your answer. (2p)



Solution 1a: The normal derivative of the deflection has to be zero along the lines of symmetry. Thus, the b.v.p reads

$$\begin{cases} -\mathrm{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega_{1/8} \\ w = 0 & \text{on } \Gamma_g \\ (\nabla w)^{\mathrm{T}}\boldsymbol{n} = 0 & \text{on } \Gamma_h \end{cases}$$



where $\boldsymbol{n}$ is an out-ward unit normal vector of the boundary.

Solution 1b: Multiply both sides of the differential equation by a test function v and integrate over the domain: $-\int_{\Omega_{1/8}} v\,\mathrm{div}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega$. Apply the Green-Green theorem to the left hand side: $\int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega + \oint_{\Gamma} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma$. Now study the boundary integral; from the boundary condition on Γ_h, we know that the integrand is zero, but we do not know the value of $(\nabla w)^{\mathrm{T}}\boldsymbol{n}$ on Γ_g. Hence, since it is necessary to be able to evaluate the boundary term, we must enforce $v = 0$ on Γ_g. Thus,

$$\oint_{\Gamma} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma = \int_{\Gamma_g} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma + \int_{\Gamma_h} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma = \int_{\Gamma_g} 0\cdot(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma + \int_{\Gamma_h} v\cdot 0 d\Gamma = 0$$

To be able to evaluate the integrals, all involved functions have to be regular enough:

$\int_{\Omega_{1/8}} v^2 d\Omega < \infty$ $\quad$ $\int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla v)d\Omega < \infty$. Let V denote the space of functions that are regular enough and that fulfil $v = 0$ on Γ_g. The variational problem may now be expressed as

$$\text{Find } w \in V \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega \qquad \forall v \in V$$

Solution 1c: Approximate the unknown function by a linear combination of selected basis functions $N_i(x, y)$: $w \approx w_h = \sum_i N_i a_i = \boldsymbol{N}\boldsymbol{a}$ ($\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & \dots \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots \end{bmatrix}^T$). We substitute this into the variational problem and choose test functions v according to Galerkin, viz. any linear combination of basis functions. If we define the finite element space V_h as the space of functions that can be expressed as a linear combination of the basis functions, the FE-formulation according to Galerkin may be expressed as:

$$\text{Find } w_h \in V_h \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^T (\nabla w_h) d\Omega = \int_{\Omega_{1/8}} v \frac{q}{S} d\Omega \qquad \forall v \in V_h$$

If the N_e basis functions that are non-zero on an element are collected into a row vector $\boldsymbol{N}^e = \begin{bmatrix} N_1^e & \dots & N_{N_e}^e \end{bmatrix}$, and the corresponding node variables into a column vector $\boldsymbol{a}^e = \begin{bmatrix} a_1^e & \dots & a_{N_e}^e \end{bmatrix}^T$, then the FE-approximation on the element becomes $w_h = \boldsymbol{N}^e \boldsymbol{a}^e$. On the element we thus have $\nabla w_h = \nabla(\boldsymbol{N}^e \boldsymbol{a}^e) = (\nabla \boldsymbol{N}^e)\boldsymbol{a}^e = \boldsymbol{B}^e \boldsymbol{a}^e$, where, hence,

$$\boldsymbol{B}^e = \nabla \boldsymbol{N}^e = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} N_1^e & \dots & N_{N_e}^e \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \dots & \frac{\partial N_{N_e}^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \dots & \frac{\partial N_{N_e}^e}{\partial y} \end{bmatrix}$$

Solution 1d: From the finite element formulation we have $\int_{\Omega_{1/8}} (\nabla v)^T (\nabla \boldsymbol{N}) d\Omega \boldsymbol{a} = \int_{\Omega_{1/8}} v \frac{q}{S} d\Omega$ with $v = N_1$ (the choices $v = N_2$ and $v = N_3$ are not valid, since N_2 and N_3 do not satisfy the essential boundary condition). Furthermore, since the essential boundary condition requires that $a_2 = a_3 = 0$, we obtain $\int_{\Omega_{1/8}} (\nabla N_1)^T (\nabla N_1) d\Omega a_1 = \int_{\Omega_{1/8}} N_1 \frac{q}{S} d\Omega$. Now,

$\nabla N_1 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$, so $\int_{\Omega_{1/8}} (\nabla N_1)^T (\nabla N_1) d\Omega = \frac{1}{4} \int_{\Omega_{1/8}} d\Omega = \frac{1}{4} \cdot \frac{2 \cdot 2}{2} = \frac{1}{2}$. Also, the right hand side becomes $\int_{\Omega_{1/8}} N_1 \frac{q}{S} d\Omega = \frac{q}{S} \int_{\Omega_{1/8}} N_1 d\Omega = \frac{150}{1000} \cdot \frac{1 \cdot \frac{2 \cdot 2}{2}}{3} = 0.1$ (where we calculated the integral by calculating the volume enclosed by $\Omega_{1/8}$ and the graph of N_1). Hence, we have $\frac{1}{2}a_1 = 0.1$.

Solution 1e: With the triangular element we get $a_1 = 0.2$, so the potential energy in $1/8th$ of the domain becomes $-\frac{1}{2}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{f} = -0.01$, and thus in the whole domain $\Pi = -0.08$. With the bilinear element we similarly obtain $\Pi = -0.0675$. Considering that the exact solution minimizes the potential energy in V and that a FE–approximation minimizes Π in V_h which (for a conform method) is a subspace of V, we judge the solution with the triangular element to be the best one.

2

Equilibrium in a 2D elasticity problem requires that $-\tilde{\nabla}^{\mathrm{T}}\sigma = \boldsymbol{b}$, where $\sigma = \begin{bmatrix}\sigma_x & \sigma_y & \tau_{xy}\end{bmatrix}^{\mathrm{T}}$ is the stress vector, $\boldsymbol{b} = \begin{bmatrix}b_x & b_y\end{bmatrix}^{\mathrm{T}}$ a given volume loading, and $\tilde{\nabla}^{\mathrm{T}} = \begin{bmatrix}\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}$. The deformations (strains) $\varepsilon = \begin{bmatrix}\varepsilon_x & \varepsilon_y & \gamma_{xy}\end{bmatrix}^{\mathrm{T}}$ are (according to Hooke) proportional to the stresses: $\sigma = \boldsymbol{D}\varepsilon$. Kinematics relates deformations and displacements $\boldsymbol{u} = \begin{bmatrix}u_x & u_y\end{bmatrix}^{\mathrm{T}}$ as $\varepsilon = \tilde{\nabla}\boldsymbol{u}$. By combining these three equations, we get two second order partial differential equations from which the unknown displacements may be solved. In the most common case where $\boldsymbol{b} = \boldsymbol{0}$, the weak form of the problem may be written

$$\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) = \oint_{\Gamma} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{t}t\,d\Gamma \tag{1}$$

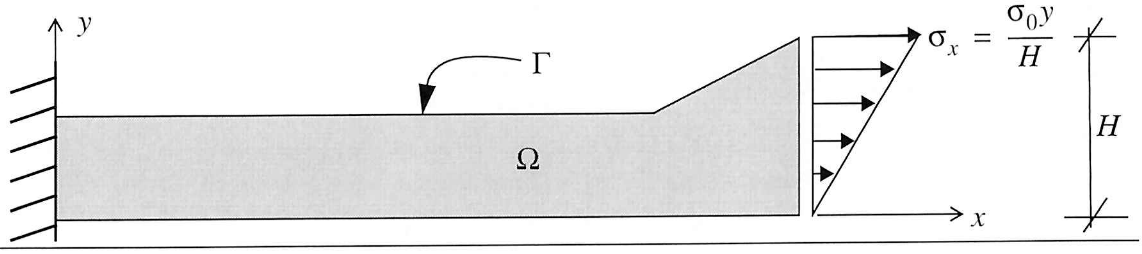
where t is the thickness, $\boldsymbol{t} = \begin{bmatrix}t_x & t_y\end{bmatrix}^{\mathrm{T}}$ is the traction (boundary loading), $\boldsymbol{v} = \begin{bmatrix}v_x & v_y\end{bmatrix}^{\mathrm{T}}$ is a vector with test functions, while the bi–linear symmetric form is given by

$$\mathrm{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} (\tilde{\nabla}\boldsymbol{v})^{\mathrm{T}}\boldsymbol{D}(\tilde{\nabla}\boldsymbol{u})t\,d\Omega .$$

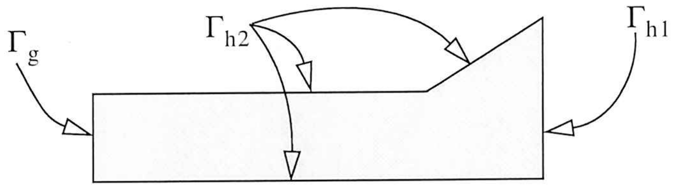
a: Specify the boundary conditions that are required to solve the elasticity problem depicted in the illustration below. (2p)

b: Give some example of a modelling error that might have arisen when the physical problem was described by a mathematical model. (1p)

c: Starting from Eq (1), make a finite element formulation of the given problem; select test functions according to Galerkin. From your solution, it should be clear how the boundary conditions affect the formulation. Show the integral that defines the element stiffness matrix of an element that has N_e basis functions — the $\boldsymbol{B}^e$ matrix should also be shown. (2p)



Solution 2a: The problem is described by two second order partial differential equations, so we two conditions on each and every part of the boundary. Using the notation in the illustration, we have: $\left.\begin{array}{l} u_x = 0 \\ u_y = 0 \end{array}\right\}$ on Γ_g $\left.\begin{array}{l} t_x = \sigma_x = \sigma_0 y/H \\ t_y = 0 \end{array}\right\}$ on Γ_{h1} $\left.\begin{array}{l} t_x = 0 \\ t_y = 0 \end{array}\right\}$ on Γ_{h2}.



The right-hand-side of the variational problem then becomes $\oint_{\Gamma} \boldsymbol{v}^T \boldsymbol{t} t d\Gamma = \int_{\Gamma_{h1}} \boldsymbol{v}^T \begin{bmatrix} \sigma_x \\ 0 \end{bmatrix} t d\Gamma$,

since the test functions have to satisfy the essential boundary conditions (i.e. we have $v_x = v_y = 0$ on Γ_g).

Solution 2b: Uncertainties in material properties ($\boldsymbol{D}$) and/or the exact loading (σ_x) and/or the actual geometry of the structure (e.g. the thickness t); the boundary conditions may have been approximated — e.g. the clamping has some flexibility that is not accounted for; etc.

Solution 2c: Approximate $\boldsymbol{u} \approx \boldsymbol{u}_h = \begin{bmatrix} u_{hx} & u_{hy} \end{bmatrix}^{\mathrm{T}} = \boldsymbol{N}\boldsymbol{a}$, where

$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & \cdots & u_{xn} & u_{yn} \end{bmatrix}^{\mathrm{T}}$. Let V_h be the space of functions that can be written as a linear combination of the selected basis functions N_i; furthermore, the basis functions are selected so that essential boundary conditions are satisfied, i.e. $N_i = 0$ on Γ_g. Next select test functions according to Galerkin:

$v = \begin{bmatrix} N_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ N_1 \end{bmatrix}, \begin{bmatrix} N_2 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ N_n \end{bmatrix}$; we then obtain: Find $\boldsymbol{u}_h = \begin{bmatrix} u_{hx} & u_{hy} \end{bmatrix}^{\mathrm{T}}$, $u_{hx}, u_{hy} \in V_h$, such that $\mathrm{a}(\boldsymbol{u}_h, \boldsymbol{v}) = \int_{\Gamma_{h1}} \boldsymbol{v}^{\mathrm{T}} \begin{bmatrix} \sigma_x \\ 0 \end{bmatrix} t d\Gamma \qquad \forall v_x, v_y \in V_h$.

On a single element we have the approximation $\boldsymbol{u}_h = \boldsymbol{N}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{e}}$, where $\boldsymbol{a}^{\mathrm{e}}$ contains the node variables on the element, and $\boldsymbol{N}^{\mathrm{e}} = \begin{bmatrix} N_1^{\mathrm{e}} & 0 & N_2^{\mathrm{e}} & 0 & \dots & N_{ne}^{\mathrm{e}} & 0 \\ 0 & N_1^{\mathrm{e}} & 0 & N_2^{\mathrm{e}} & \dots & 0 & N_{ne}^{\mathrm{e}} \end{bmatrix}$ contains the basis functions that have support on the element. The contribution from the element to left-hand-side in the weak formulation becomes

$$\int_{\Omega_e} (\tilde{\nabla}\boldsymbol{N})^{\mathrm{T}} \boldsymbol{D}(\tilde{\nabla}\boldsymbol{N}\boldsymbol{a}) t d\Omega = \int_{\Omega_e} (\tilde{\nabla}\boldsymbol{N}^e)^{\mathrm{T}} \boldsymbol{D}(\tilde{\nabla}\boldsymbol{N}^e) t d\Omega \boldsymbol{a}^e = \int_{\Omega_e} B^{e\mathrm{T}} \boldsymbol{D} B^{e\mathrm{T}} t d\Omega \boldsymbol{a}^e = \boldsymbol{K}^e \boldsymbol{a}^e$$

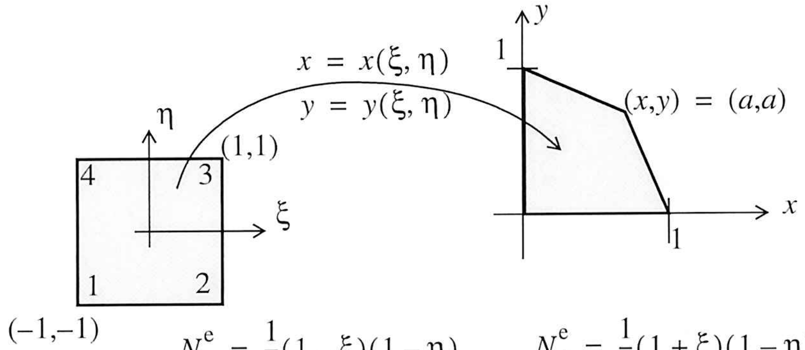
where

$$\boldsymbol{B}^e = \tilde{\nabla}\boldsymbol{N}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \dots & \frac{\partial N_{N_e}^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \dots & 0 & \frac{\partial N_{N_e}^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \dots & \frac{\partial N_{N_e}^e}{\partial y} & \frac{\partial N_{N_e}^e}{\partial x} \end{bmatrix}$$

3

a: When using mapped elements, we have the basis functions defined in terms of some local coordinates (ξ, η), i.e.

$N_i^e = N_i^e(\xi, \eta)$.



$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

Derive expressions for the derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ in the case when the mapping $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ to the global coordinates is isoparametric. (2p)

b: Calculate the determinant of the Jacobian of the mapping, as a function of the parameter a, of the element shown in the illustration. Which values of a yield a unique mapping? (2p)

Solution 3a: Use the chain rule to calculate the derivatives with respect to ξ and η:

$$\left.\begin{aligned} \frac{\partial N_i^e}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N_i^e}{\partial y} \\ \frac{\partial N_i^e}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial y} \end{aligned}\right\} = \boldsymbol{J}\begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix}$$

where, hence, $\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$. We then get the desired derivatives by multiplying both sides by $\boldsymbol{J}^{-1}$: $\begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix} = \boldsymbol{J}^{-1}\begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix}$. The components of $\boldsymbol{J}$ are easily calculate as $\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\sum_i x_i N_i^e = \sum_i x_i \frac{\partial N_i^e}{\partial \xi}$, etc.; (x_i, y_i) is the coordinate of node i on the element.

Solution 3b: Number the nodes in anti-clock-wise order; starting with the node in the origin, we get the node coordinates $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (1, 0)$, $(x_3, y_3) = (a, a)$, and $(x_4, y_4) = (0, 1)$. Hence;

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_2^e}{\partial \xi} + a\frac{\partial N_3^e}{\partial \xi} = \frac{1}{4}(1-\eta+a+a\eta) \qquad \frac{\partial x}{\partial \eta} = \frac{1}{4}(-1-\xi+a+a\xi)$$

$$\frac{\partial y}{\partial \xi} = \frac{1}{4}(-1-\eta+a+a\eta) \qquad \frac{\partial y}{\partial \eta} = \frac{1}{4}(1-\xi+a+a\xi)$$

so that $\det\boldsymbol{J} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = \frac{1}{8}(2a-\xi-\eta+a\xi+a\eta)$.

The mapping will be unique provided that $\boldsymbol{J}$ is non-singular, that is $\det\boldsymbol{J} \neq 0$ inside the element. This is obviously the case if a is sufficiently large; the local coordinates ξ and η may only vary between -1 and 1. When a decreases, $\det\boldsymbol{J}$ becomes zero at $(x, y) = (a, a)$ first — i.e. at $(\xi, \eta) = (1, 1)$.

$$\det\boldsymbol{J}(1,1) = 0 \Rightarrow 2a-1-1+a+a = 0 \Rightarrow a = \frac{1}{2}$$

Thus, the mapping is unique whenever $a > \frac{1}{2}$.

4

Consider the integral $I = \int_{-1}^{1} f(\xi)d\xi$, where $f(\xi)$ is enough smooth to be expressed by a Taylor series. We may then write $f(\xi) = \sum_{i=1}^{n} f_i l_i^{n-1}(\xi) + P(\xi)(\beta_0 + \beta_1\xi + \ldots)$, with $f_i = f(\xi_i)$, ξ_i is an integration point, and

$$l_i^{n-1}(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)\ldots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\ldots(\xi-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)\ldots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\ldots(\xi_i-\xi_n)}$$

is the Lagrange polynomial of degree $n-1$ such that $l_i^{n-1}(\xi_j) = 0$ if $\xi_j \neq \xi_i$ and $l_i^{n-1}(\xi_i) = 1$. Further, $P(\xi) = (\xi-\xi_1)(\xi-\xi_2)\ldots(\xi-\xi_n)$ is a n:th degree polynomial that is zero in the n points ξ_i, while the coefficients β_i depends of the Taylor series expansion of $f(\xi)$.

Derive equations that can be used to solve for the integration points ξ_i in Gauss quadrate with n points. Your solution shall clearly show that a polynomial of degree less or equal to $2n-1$ will be integrated exact. You need not utilize the description given above, if you do not want to. (2p)

Solution 4: We have

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n}\left[f_i\int_{-1}^{1} l_i^{n-1}(\xi)d\xi\right] + \int_{-1}^{1} P(\xi)(\beta_0+\beta_1\xi+\ldots)d\xi =$$

$$= \sum_{i=1}^{n} f_i H_i + \int_{-1}^{1} P(\xi)(\beta_0+\beta_1\xi+\ldots)d\xi \approx \sum_{i=1}^{n} f_i H_i$$

where, hence, the integration weights are $H_i = \int_{-1}^{1} l_i^{n-1}(\xi)d\xi$. With n-point Gauss integration, the coordinates ξ_i are chosen so that the integral of the first n terms in the polynomial $P(\xi)(\beta_0+\beta_1\xi+\ldots)$ become zero. Since these polynomial terms, together with the n Lagrange polynomials, embrace all polynomial terms up to degree $2n-1$, it is obvious that $I = \sum_{i=1}^{n} f_i H_i$ becomes exact if $f(\xi)$ is a polynomial of degree $2n-1$ (or lower).

The condition that the integral of the first n terms of $P(\xi)(\beta_0+\beta_1\xi+\ldots)$ should vanish, yields the n equations

$$\int_{-1}^{1} \xi^i(\xi-\xi_1)(\xi-\xi_2)\ldots(\xi-\xi_n)d\xi = 0 \qquad i = 0, 1, \ldots, n-1$$

from which the n coordinates $\xi_1, \xi_2, \ldots, \xi_n$ may be solved.

Department of Applied Mechanics, Chalmers University of Technology

FINITE ELEMENT METHOD (MHA 021) — EXAMINATION
MARCH 11 2004

Solutions

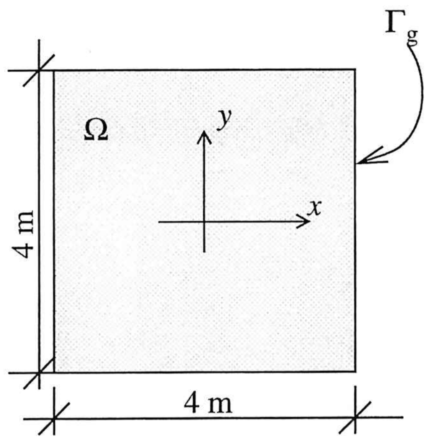
Time and location:	$8^{\underline{45}}$ – $12^{\underline{45}}$ in the M building
Aids:	'Closed books' examination; only dictionaries and a 'standard' calculator allowed.
Teacher:	Peter Möller; phone (772) 1505
Solutions:	will be posted at the entrance of the Department of Applied Mechanics no later that March 12. See also the web pages of the course at *http://www.am.chalmers.se/eng/welcome.html* — follow the link *Education/Undergraduate Courses.*
Grading:	A complete and correct solution on any task grants points as stated in the thesis. Minor errors result in a reduced score. Gross error(s) and/or incomplete solution of a task will not grant any points on that particular task. Maximum score is 20. You need 8, 12 and 16 points to obtain grades 3, 4 and 5 respectively. **NB: the above is for the written examination only — to pass the course you also have to complete 4 computer assignments.**
Results:	will be posted at the Department of Applied Mechanics no later than March 22. Results are sent for registration Monday March 24 — **course participants that have not completed the computer assignments by this time will be registered as *not approved*.** You may scrutinize the correction (mark up) of your written examination Tuesday March 23 $13^{\underline{30}}$–$16^{\underline{00}}$ (at the office space of Department of Applied Mechanics).

Kindly consider:

- The person that corrects your solutions will not try to guess your thoughts, but the grading will be based exclusively on what you have actually written down. Hence, write legible and explain what you are doing.
- Explain/define any notation that you introduce.
- Draw clear illustrations. Use coordinate systems; carefully indicate positive/negative directions on vector entities such as e.g. displacements and forces.
- If you make any assumption apart from what is stated in the respective tasks, you have to state and motivate this explicitly.

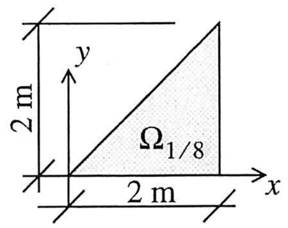
1

Consider a square membrane with side length 4 m and loaded with a distributed load $q = 150\ \mathrm{N/m^2}$ in the z-direction; the membrane is pre-tensioned with a force $S = 1000\ \mathrm{N/m}$. The deflection $w(x)$ in z-direction of the membrane, satisfies the boundary value problem



$$\begin{cases} -\mathrm{div}(\nabla w) = \frac{q}{S} & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_g \end{cases}$$

a: Using the symmetry of the problem, we can settle with considering only $1/8th$ of the domain. State how to formulate the boundary value problem so as to account for the symmetry in this manner. (1p)

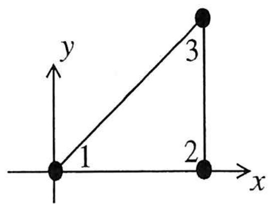


b: Use the Green–Gauss theorem, i.e.

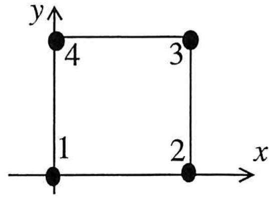
$\int_{\Omega} \psi\ \mathrm{div}(\boldsymbol{q}) d\Omega = \oint_{\Gamma} \psi \boldsymbol{q}^T \boldsymbol{n} d\Gamma - \int_{\Omega} (\nabla\psi)^T \boldsymbol{q} d\Omega$ (where ψ is a scalar function, $\boldsymbol{q}$ is a vector valued function, and $\boldsymbol{n}$ is an out-ward normal of Ω), to derive the variational formulation of the problem. Regularity conditions of involved functions should be explicitly given, and it should be clear how the boundary conditions affect the variational problem. (2p)

c: Make a finite element formulation of the problem with test (weight) functions according to the Galerkin method. Show what the $\boldsymbol{B}^e$ – matrix looks like, for an element with N_e, basis functions. (2p)

d: Assume that the domain is discretized with a single 3–noded triangular element. With node numbering according to the figure, the basis functions becomes $N_1 = \frac{1}{2}(2-x)$, $N_2 = \frac{1}{2}(x-y)$, and $N_3 = \frac{y}{2}$. Show that this gives the equation system $\boldsymbol{Ka} = \boldsymbol{f}$, with $\boldsymbol{K} = \begin{bmatrix}1/2\end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix}a_1\end{bmatrix}$, and $\boldsymbol{f} = \begin{bmatrix}0.1\end{bmatrix}$. (2p)

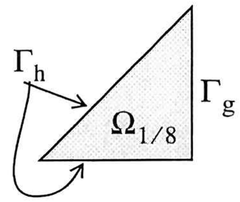


e: If instead we use a 4–noded element to solve the problem on $1/4\ th$ of the domain Ω, we get $\boldsymbol{K} = [2/3]$, $\boldsymbol{a} = [a_1]$, and $f = [0.15]$. Will this FE–approximation be better or worse that the one obtained with the triangular element in task d above? Motivate your answer. (2p)



Solution 1a: The normal derivative of the deflection has to be zero along the lines of symmetry. Thus, the b.v.p reads

$$\begin{cases} -\operatorname{div}(\nabla w) = \dfrac{q}{S} & \text{in } \Omega_{1/8} \\ w = 0 & \text{on } \Gamma_g \\ (\nabla w)^{\mathrm{T}}\boldsymbol{n} = 0 & \text{on } \Gamma_h \end{cases}$$



where $\boldsymbol{n}$ is an out–ward unit normal vector of the boundary.

Solution 1b: Multiply both sides of the differential equation by a test function v and integrate over the domain: $-\int_{\Omega_{1/8}} v\,\operatorname{div}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega$. Apply the Green–Green theorem to the left hand side: $\int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega + \oint_{\Gamma} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma$. Now study the boundary integral; from the boundary condition on Γ_h, we know that the integrand is zero, but we do not know the value of $(\nabla w)^{\mathrm{T}}\boldsymbol{n}$ on Γ_g. Hence, since it is necessary to be able to evaluate the boundary term, we must enforce $v = 0$ on Γ_g. Thus,

$$\oint_{\Gamma} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma = \int_{\Gamma_g} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma + \int_{\Gamma_h} v(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma = \int_{\Gamma_g} 0\cdot(\nabla w)^{\mathrm{T}}\boldsymbol{n}d\Gamma + \int_{\Gamma_h} v\cdot 0 d\Gamma = 0$$

To be able to evaluate the integrals, all involved functions have to be regular enough: $\int_{\Omega_{1/8}} v^2 d\Omega < \infty$ $\quad\int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla v)d\Omega < \infty$. Let V denote the space of functions that are regular enough and that fulfil $v = 0$ on Γ_g. The variational problem may now be expressed as

$$\text{Find } w \in V \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^{\mathrm{T}}(\nabla w)d\Omega = \int_{\Omega_{1/8}} v\frac{q}{S}d\Omega \qquad \forall v \in V$$

Solution 1c: Approximate the unknown function by a linear combination of selected basis functions $N_i(x, y)$: $w \approx w_h = \sum_i N_i a_i = \boldsymbol{N}\boldsymbol{a}$ ($\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & \dots \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots \end{bmatrix}^T$). We substitute this into the variational problem and choose test functions v according to Galerkin, viz. any linear combination of basis functions. If we define the finite element space V_h as the space of functions that can be expressed as a linear combination of the basis functions, the FE–formulation according to Galerkin may be expressed as:

$$\text{Find } w_h \in V_h \text{ such that } \int_{\Omega_{1/8}} (\nabla v)^T (\nabla w_h) d\Omega = \int_{\Omega_{1/8}} v \frac{q}{S} d\Omega \qquad \forall v \in V_h$$

If the N_e basis functions that are non–zero on an element are collected into a row vector $\boldsymbol{N}^e = \begin{bmatrix} N_1^e & \dots & N_{N_e}^e \end{bmatrix}$, and the corresponding node variables into a column vector $\boldsymbol{a}^e = \begin{bmatrix} a_1^e & \dots & a_{N_e}^e \end{bmatrix}^T$, then the FE–approximation on the element becomes $w_h = \boldsymbol{N}^e \boldsymbol{a}^e$. On the element we thus have $\nabla w_h = \nabla(\boldsymbol{N}^e \boldsymbol{a}^e) = (\nabla \boldsymbol{N}^e)\boldsymbol{a}^e = \boldsymbol{B}^e \boldsymbol{a}^e$, where, hence,

$$\boldsymbol{B}^e = \nabla \boldsymbol{N}^e = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} N_1^e & \dots & N_{N_e}^e \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \dots & \frac{\partial N_{N_e}^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \dots & \frac{\partial N_{N_e}^e}{\partial y} \end{bmatrix}$$

Solution 1d: From the finite element formulation we have $\int_{\Omega_{1/8}} (\nabla v)^T (\nabla \boldsymbol{N}) d\Omega \boldsymbol{a} = \int_{\Omega_{1/8}} v \frac{q}{S} d\Omega$ with $v = N_1$ (the choices $v = N_2$ and $v = N_3$ are not valid, since N_2 and N_3 do not satisfy the essential boundary condition). Furthermore, since the essential boundary condition requires that $a_2 = a_3 = 0$, we obtain $\int_{\Omega_{1/8}} (\nabla N_1)^T (\nabla N_1) d\Omega a_1 = \int_{\Omega_{1/8}} N_1 \frac{q}{S} d\Omega$. Now, $\nabla N_1 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$, so $\int_{\Omega_{1/8}} (\nabla N_1)^T (\nabla N_1) d\Omega = \frac{1}{4} \int_{\Omega_{1/8}} d\Omega = \frac{1}{4} \cdot \frac{2 \cdot 2}{2} = \frac{1}{2}$. Also, the right hand side becomes $\int_{\Omega_{1/8}} N_1 \frac{q}{S} d\Omega = \frac{q}{S} \int_{\Omega_{1/8}} N_1 d\Omega = \frac{150}{1000} \cdot \frac{1 \cdot \frac{2 \cdot 2}{2}}{3} = 0.1$ (where we calculated the integral by calculating the volume enclosed by $\Omega_{1/8}$ and the graph of N_1). Hence, we have $\frac{1}{2}a_1 = 0.1$.

Solution 1e: With the triangular element we get $a_1 = 0.2$, so the potential energy in $1/8th$ of the domain becomes $-\frac{1}{2}\boldsymbol{a}^T\boldsymbol{f} = -0.01$, and thus in the whole domain $\Pi = -0.08$. With the bilinear element we similarly obtain $\Pi = -0.0675$. Considering that the exact solution minimizes the potential energy in V and that a FE–approximation minimizes Π in V_h which (for a conform method) is a subspace of V, we judge the solution with the triangular element to be the best one.

2

The weak form of the boundary value problem in the previous task may be expressed as

$$\text{Find } w \in V \text{ such that } \mathrm{a}(w, v) = (v, f) \qquad \forall v \in V$$

where V is the space of all admissible functions, $\mathrm{a}(.,.)$ is symmetric and linear in both arguments and $(.,.)$ a scalar product of functions.

Let $w_h \in V_h$ be a conform FE–approximation of w.

a: Prove Galerkin orthogonality, i.e. show that the discretization error $e = w - w_h$ is a-orthogonal to the FE–space V_h. (Two functions, say v_1 and v_2, are said to be a-orthogonal if $\mathrm{a}(v_1, v_2) = 0$). (2p)

b: Use Galerkin orthogonality to show that the energy of the error equals the error in energy: $\mathrm{a}(e, e) = \mathrm{a}(w, w) - \mathrm{a}(w_h, w_h)$. (2p)

Solution 2a: The FE–formulation reads:

$$\text{Find } w_h \in V_h \text{ such that } \mathrm{a}(w_h, v) = (v, f) \qquad \forall v \in V_h$$

We subtract this from the variational problem go get: $\mathrm{a}(w, v) - \mathrm{a}(w_h, v) = 0 \qquad \forall v \in V_h$.

Since $\mathrm{a}(.,.)$ is linear in its first argument, we can write this as

$\mathrm{a}(w - w_h, v) = 0 \qquad \forall v \in V_h$; but $w - w_h = e$, so $\mathrm{a}(e, v) = 0$ if $v \in V_h$.

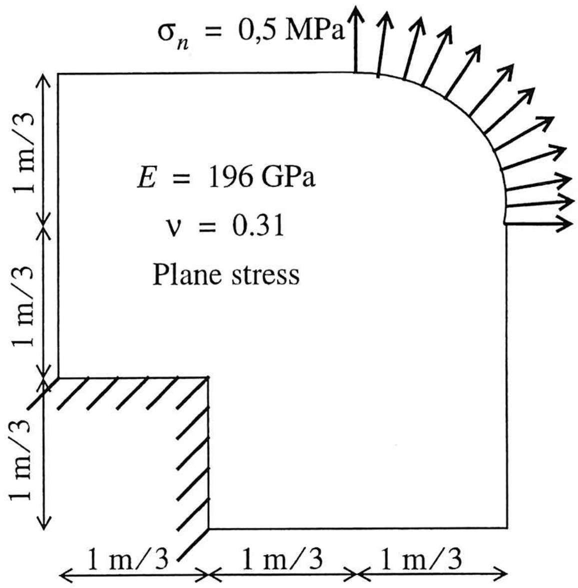
Solution 2b: We use the fact that $a(.,.)$ is symmetric and linear in both arguments:

$$a(e,e) = a(w-w_h, w-w_h) = a(w, w-w_h) - a(w_h, w-w_h) =$$
$$a(w,w) - a(w,w_h) - a(w_h,u) + a(w_h,w_h) = a(w,w) + a(w_h,w_h) - 2a(w,w_h) =$$
$$\{\text{use } w = w_h + e \text{ in the last term}\} = a(w,w) + a(w_h,w_h) - 2a(w_h + e, w_h) =$$
$$a(w,w) + a(w_h,w_h) - 2a(w_h,w_h) - 2a(e,w_h) =$$
$$\{w_h \in V_h \text{ so according to Galerkin orthogonality the last term vanish}\} =$$
$$a(w,w) - a(w_h,w_h)$$

Hence, $a(e,e) = a(w,w) - a(w_h,w_h)$.

3

Assume that the problem depicted in the illustration, is to be solved by a conform FE–method. We start with fairly few linear triangular elements (CST elements) and use an adaptive *h*–method to approximate the solution of the governing system of partial differential equations.



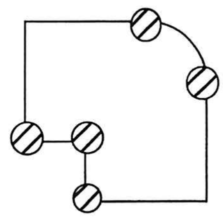
a: Explain what adaptivity means in this context and describe what *h*–method means. (2p)

b: In which area or areas do you expect the FE–program to generate the smallest element sizes? Motivate your answer. (1p)

c: A conform FE–discretization of the problem leads to a system of equations $\boldsymbol{Ka} = \boldsymbol{f}$, where the stiffness matrix $\boldsymbol{K}$ is symmetric and positive definite. Show that the solution $\boldsymbol{a} = \boldsymbol{K}^{-1}\boldsymbol{f}$ minimizes the potential energy $\Pi(\boldsymbol{v}) = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{Kv} - \boldsymbol{v}^T\boldsymbol{f}$, i.e. show that $\Pi(\boldsymbol{a}) \le \Pi(\boldsymbol{v})$, where $\Pi(\boldsymbol{a}) = \Pi(\boldsymbol{v})$ only if $\boldsymbol{a} \equiv \boldsymbol{v}$. (2p)

Solution 3a: An adaptive FE–program estimates the discretization error and, if found to be larger that some given threshold, changes the FE–discretization in accordance with an estimation of how the error is distributed; by resolving the problem one hence gets a better approximate solution. By repeating this process, the discretization error could be reduced to below the given threshold.

h–method means that we try to reduce the discretization error by using smaller elements, i.e. a larger number of elements in the FE–analysis.



Solution 3b: With an adaptive h–method, we expect to get the smallest element sizes in regions where first derivatives of the solution varies fast (i.e. where second derivatives are large). In an elasticity problem we get singular points — where stresses (first derivatives) changes much over small distances — at for instance in-ward corners and at abrupt changes of boundary conditions. In the illustrated problem there are 5 such points, as indicated in the figure to the right.

Solution 3c: Let $\partial v = \varepsilon w$ be an arbitrary, but small ($\varepsilon \ll 1$), variation of the vector v and calculate the variation of Π at v:

$$\partial\Pi(v) = \Pi(v+\partial v) - \Pi(v) = \frac{1}{2}(v+\partial v)^{\mathrm{T}} K(v+\partial v) - (v+\partial v)^{\mathrm{T}} f - \left(\frac{1}{2}v^{\mathrm{T}}Kv - v^{\mathrm{T}}f\right) =$$

$$= \frac{1}{2}\partial v^{\mathrm{T}}Kv + \frac{1}{2}v^{\mathrm{T}}K\partial v + \frac{1}{2}\partial v^{\mathrm{T}}K\partial v - \partial v^{\mathrm{T}}f =$$

$$= \frac{\varepsilon}{2}w^{\mathrm{T}}Kv + \frac{\varepsilon}{2}v^{\mathrm{T}}Kw + \frac{\varepsilon^2}{2}w^{\mathrm{T}}Kw - \varepsilon w^{\mathrm{T}}f$$

Because $\varepsilon \ll 1$, the ε^2–term may be neglected since it is small compared to the terms that are linear in ε. Furthermore, $w^{\mathrm{T}}Kv = v^{\mathrm{T}}Kw$ since K is symmetric. Thus, we arrive at $\partial\Pi(v) = \varepsilon w^{\mathrm{T}}Kv - \varepsilon w^{\mathrm{T}}f = \partial v^{\mathrm{T}}(Kv - f)$. Here we note that $v = a = K^{-1}f$ gives $\partial\Pi = 0$ for arbitrary (but small) variations ∂v; hence, the FE–solution (approximation) makes Π stationary.

To show that the stationary point is a minimum, we have to show that the second variation is positive. We have:

$$\partial(\partial\Pi) = \partial\Pi(v+\partial v) - \partial\Pi(v) = \partial v^{\mathrm{T}}[K(v+\partial v) - f] - \partial v^{\mathrm{T}}(Kv - f) = \partial v^{\mathrm{T}}K\partial v$$

Since K is positive definite one has that $\partial v^{\mathrm{T}}K\partial v > 0$ (for all $\partial v \neq \mathbf{0}$), so the second variation is positive and, thus, any stationary point is a minimum of Π.

Uniqueness ($\Pi(\boldsymbol{a}) = \Pi(\boldsymbol{v})$ only if $\boldsymbol{a} \equiv \boldsymbol{v}$) follows from the fact that $\boldsymbol{K}$ is positive definite: if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ make Π stationary, then, according to above, $\boldsymbol{K}\boldsymbol{a}_1 = \boldsymbol{f}$ and $\boldsymbol{K}\boldsymbol{a}_2 = \boldsymbol{f}$. Subtracting these equations, we get $\boldsymbol{K}(\boldsymbol{a}_1 - \boldsymbol{a}_2) = \boldsymbol{0}$ so $\boldsymbol{a}_1 - \boldsymbol{a}_2 = \boldsymbol{K}^{-1}\boldsymbol{0} = \boldsymbol{0}$; thus, $\boldsymbol{a}_1 = \boldsymbol{a}_2$.

4

Consider the integral $I = \int_{-1}^{1} f(\xi)d\xi$, where $f(\xi)$ is enough smooth to be expressed by a Taylor series. We may then write $f(\xi) = \sum_{i=1}^{n} f_i l_i^{n-1}(\xi) + P(\xi)(\beta_0 + \beta_1\xi + \ldots)$, with $f_i = f(\xi_i)$, ξ_i is an integration point, and

$$l_i^{n-1}(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)\ldots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\ldots(\xi-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)\ldots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\ldots(\xi_i-\xi_n)}$$

is the Lagrange polynomial of degree $n-1$ such that $l_i^{n-1}(\xi_j) = 0$ if $\xi_j \neq \xi_i$ and $l_i^{n-1}(\xi_i) = 1$. Further, $P(\xi) = (\xi-\xi_1)(\xi-\xi_2)\ldots(\xi-\xi_n)$ is a n:th degree polynomial that is zero in the n points ξ_i, while the coefficients β_i depends of the Taylor series expansion of $f(\xi)$.

Derive equations that can be used to solve for the integration points ξ_i in Gauss quadrate with n points. Your solution shall clearly show that a polynomial of degree less or equal to $2n-1$ will be integrated exact. You need not utilize the description given above, if you do not want to. (2p)

Solution 4: We have

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n}\left[f_i \int_{-1}^{1} l_i^{n-1}(\xi)d\xi\right] + \int_{-1}^{1} P(\xi)(\beta_0 + \beta_1\xi + \ldots)d\xi =$$

$$= \sum_{i=1}^{n} f_i H_i + \int_{-1}^{1} P(\xi)(\beta_0 + \beta_1\xi + \ldots)d\xi \approx \sum_{i=1}^{n} f_i H_i$$

where, hence, the integration weights are $H_i = \int_{-1}^{1} l_i^{n-1}(\xi) d\xi$. With n-point Gauss integration, the coordinates ξ_i are chosen so that the integral of the first n terms in the polynomial $P(\xi)(\beta_0 + \beta_1\xi + \ldots)$ become zero. Since these polynomial terms, together with the n Lagrange polynomials, embrace all polynomial terms up to degree $2n-1$, it is obvious that $I = \sum_{i=1}^{n} f_i H_i$ becomes exact if $f(\xi)$ is a polynomial of degree $2n-1$ (or lower).

The condition that the integral of the first n terms of $P(\xi)(\beta_0 + \beta_1\xi + \ldots)$ should vanish, yields the n equations

$$\int_{-1}^{1} \xi^i (\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_n) d\xi = 0 \qquad i = 0, 1, \ldots, n-1$$

from which the n coordinates $\xi_1, \xi_2, \ldots, \xi_n$ may be solved.