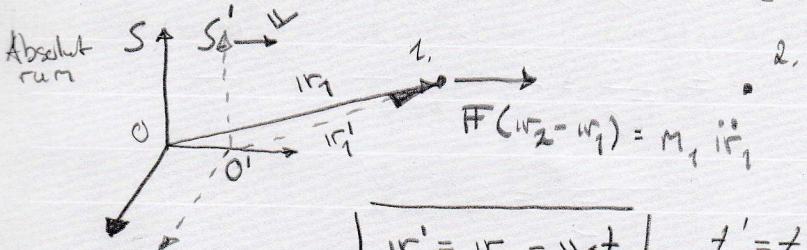
Speciell relativitetsteori

1. 1/11-16

Före Einstein

Tid och rum absoluta. (like för alla)

Newton om rum: referenssystem i vila gentemot stjärnorna. Inerdiala referenssystem nästan lika bra som det absoluta rummet. Rör sig med konstant hastighet $\mathbb{v}$ m.a.p. det absoluta rummet. I ett sådant system är Newtons ekvationer oförändrade.



Nytt system som rör sig med hastighet $\mathbb{v}$ (konstant).

$$\boxed{\mathbb{r}_1' = \mathbb{r}_1 - \mathbb{v}t} \qquad t' = t \quad \text{(tiden lika för alla)}$$

$$\mathbb{F}(\mathbb{r}_2' - \mathbb{r}_1') = m_1 \ddot{\mathbb{r}}_1'$$

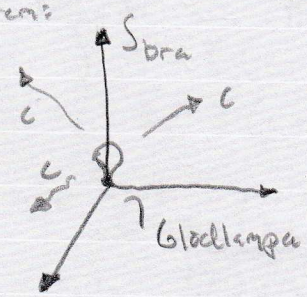
∴ Newtons ekvation oförändrad, oberoende av referenssystem.

~1800 utgjordes fysik av mekanik & elektrodynamik.

Mekanik: alla inertsialla system är lika.

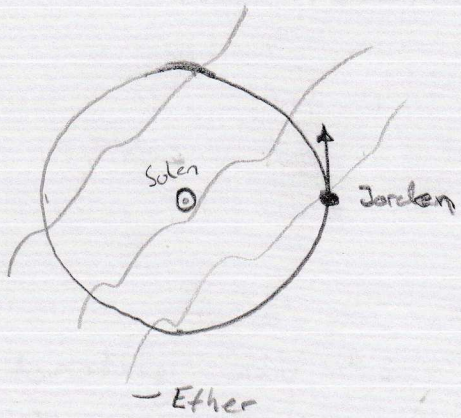
Elektrodynamik: ljusets hastighet, c. Vågrörelse.

Bra system:



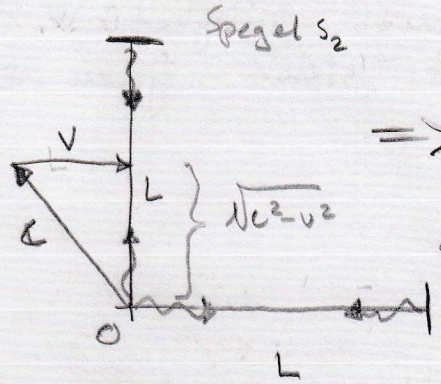
$$c' = c - \mathbb{v} \qquad S_{bra} = S_{absolut}?$$

Ether: mystisk materia med konstiga egenskaper, för ljus vad luft är för ljud. Man tänkte sig att ether var i vila gentemot stjärnorna.



Jorden skulle uppleva en Ethervind.

Interferometerexperiment, antag ethervind.



$\Rightarrow$ v, ethervind

$$t_1 = \frac{L}{v+c} + \frac{L}{v-c} = \frac{L \cdot 2c}{c^2 - v^2} = \frac{2L/c}{1 - v^2/c^2}$$

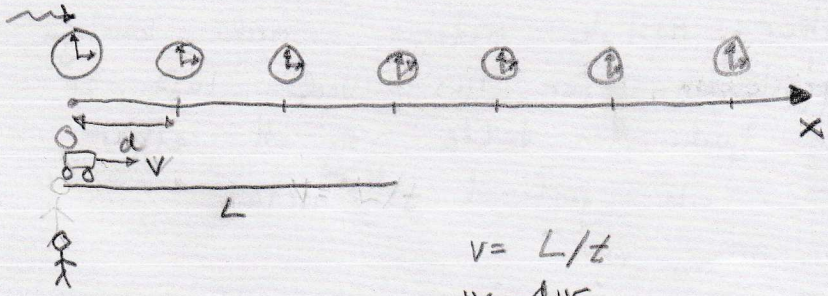
Tid för ljuset att ta sig fram & tillbaka enligt Newton.

$$t_2 = \frac{2L/c}{\sqrt{1 - v^2/c^2}}$$

Pythagoras i figur.

Experimentellt: $t_1 = t_2$! Lorentz: L inte desamma för de olika fallen. Rörelse i ether medför olika längder L_1 och L_2. Relaterade enligt $L_1 = \sqrt{1 - v^2/c^2}\, L_2$ (rätt, men av fel anledning).

S

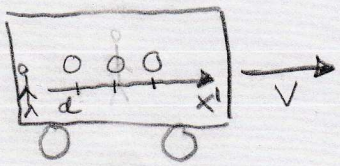


Klockor, fördröjda $\frac{d}{c}, \frac{2d}{c}, \frac{3d}{c}$ osv... Alla klockor startar när en ljusstråle passerar.

$$v = L/t$$

$$a = \frac{dv}{dt}$$

S'



v konstant i S. Samma längdenheter. Klockor som ovan. Tid t'?

Sambandet mellan (x, t) och (x', t')?
(Newton: $x' = x - vt$, $t = t'$. Kan inte stämma!) $\Rightarrow$

$\Rightarrow$ <u>Lorentz-transformation!</u>

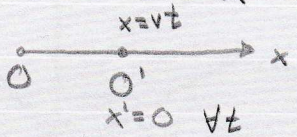
Antag att sambandet är linjärt:

$$x' = Ax + Bt + C.$$

Vi kan sätta $C = 0$ genom att säga att första klockan i S startas $(t = 0)$ då första klockan i S' passerar den. Detta innebär $x = x' = 0$, $O = O'$. Detta kallas standard configuration.

$$x' = Ax + Bt.$$

S:



$$\Rightarrow \overset{=x'}{0} = A\overset{=x}{vt} + Bt \Rightarrow B = -Av$$

Newton: $A = 1$, $B = -v$

$$x' = A(x - vt) = \gamma(x - vt)$$

(namnbyte p.g.a. konvention)

$$\begin{cases} x' = \gamma(x - vt) \\ x = \gamma(x' + vt') \end{cases}$$

γ måste vara samma i båda ekvationerna.
Einsteins relativitetsprincip: $\begin{cases} x = ct \\ x' = ct' \end{cases}$ $\Rightarrow$

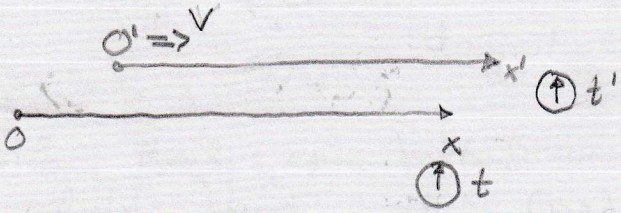
$$\Rightarrow \begin{cases} ct' = \gamma(c - v)t \\ ct = \gamma(c + v)t' \end{cases}$$

med produkt $c^2 tt' = \gamma^2 (c^2 - v^2) tt' \Rightarrow$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$ Lorentzfaktor.

2. 2/11

Lorentztransformation



Standard configuration: $x \parallel x'$, $y \parallel y'$, $z \parallel z'$, $v \parallel x, x'$

$O \equiv O' \Rightarrow x = x' = 0$ när $t = 0$.

Första gången: $x' = Ax + Bt$

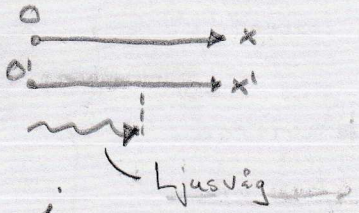
$x' = 0 \Leftarrow x = vt$

$\Rightarrow B = -vA = -v\gamma$

$$\because \begin{cases} x' = \gamma(x - vt) & (1) \\ x = \gamma(x' + vt') & (2) \end{cases}$$

~~Newton: $t' = t \Rightarrow \gamma = 1$~~

$t = t' = 0$



$$\left.\begin{matrix} x = ct \\ x' = ct' \end{matrix}\right\} \Rightarrow \begin{cases} ct' = \gamma(ct - vt) \\ ct = \gamma(ct' + vt') \end{cases} \Rightarrow$$

Samma c

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad v < c.$$

$t' = \alpha t + \beta x$ $\quad (1) \to (2)$ ger $x = \gamma(\underbrace{\gamma(x - vt)}_{x'} + vt') = \ldots \Rightarrow$ $\quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

$$\Rightarrow \begin{cases} t' = \gamma\left(t - \frac{v}{c^2}x\right) \\ t = \gamma\left(t' + \frac{v}{c^2}x'\right) \end{cases}$$

Från och med nu byter vi enheter och sätter $c = 1$. $[t] = s$, $[x] =$ ljussekund. 1 nanosekund $\sim$ 30 cm.

$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma(t - vx) \end{cases} \text{ eller } \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\gamma^2 - (v\gamma)^2 = 1$$
$$\Leftrightarrow$$
$$\frac{1}{1 - v^2} - \left(\frac{v^2}{1 - v^2}\right) = 1$$

Inför ϕ, rapidity. $\gamma = \cosh\phi$, $v\gamma = \sinh\phi$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

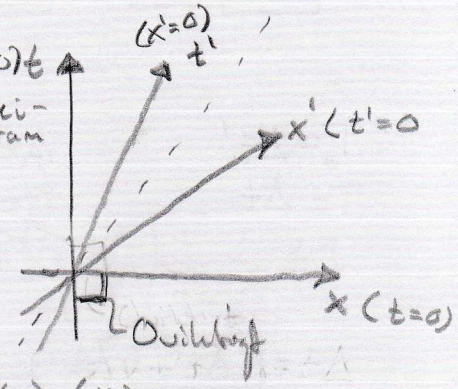
$t' = \gamma t - v\gamma x$ ($t' = 0$)

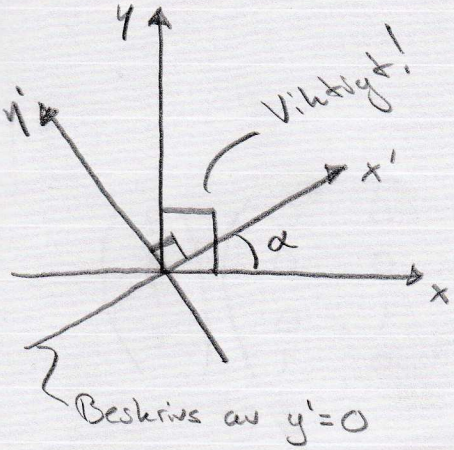
$t = vx$

hathet $\Rightarrow$ $\cosh\phi = \mathrm{ch}\,\phi$, $\sinh\phi = \mathrm{sh}\,\phi$.

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\,\phi & -\mathrm{sh}\,\phi \\ -\mathrm{sh}\,\phi & \mathrm{ch}\,\phi \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

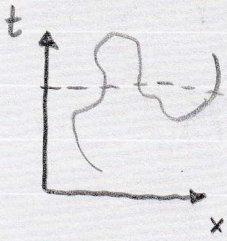
Minkowski-diagram



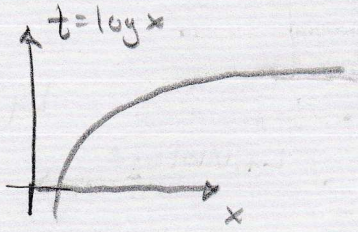


$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Partikel



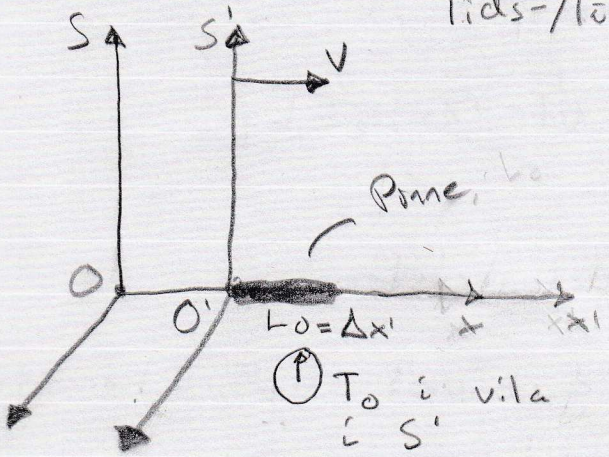
Dåligt! Samma bild, olika platser.



lutning i diagrammen

$\frac{dt}{dx} > 1$, $\frac{dx}{dt} = v \leq 1$

Tids-/längddilatation



(!) T_0 i vila i S'

$L = L_0/\gamma < L_0$ $\quad \gamma > 1$

$T = T_0\gamma > T_0$

$\Delta x' = \gamma(\Delta x - v\Delta t)$

$\Delta t' = \gamma(\Delta t - v\Delta x)$

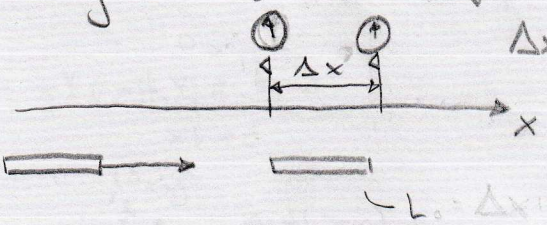
$\Delta x = x_2 - x_1$ $\quad \Delta x = dx$

$\vdots$

$\lim_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}$

längden i S definieras som Δx när $\Delta t = 0$.

Δx mäts alltså då $\Delta t = 0$.



$$L_0 = \Delta x' = \gamma(\Delta x - v\Delta t) = \gamma L$$

$T_0 = \Delta t'$ $\qquad$ $\Delta x' = 0$ (!)

$T = \Delta t$

~~$\Delta x = 0$~~

~~Något galet?~~

$$\Delta t = \gamma(\Delta t' + v\Delta x')$$
$$\overset{\shortparallel}{T} = \gamma(T_0 + 0)$$

$$\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt \\ y' = y \\ z' = z \end{cases}, \quad \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Samma längd i $S = S'$.

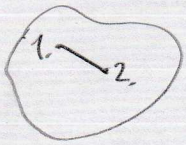
Pseudoparadox:

Får pinnen plats om den rör sig fort?

Fel lösning: $L_1 \longrightarrow L_1/\gamma < L_2$

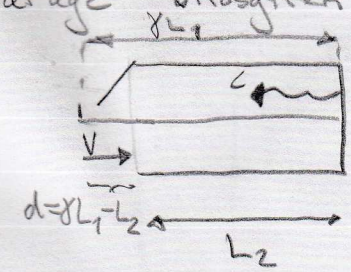
$$\cancel{\sqrt{1 - v^2/c^2}\, L_1 = L_2}, \quad \cancel{\sqrt{1 - v^2/c^2}\, L_2 = L_1} < L_2$$

Inga stela kroppar i relativitetsteori.



$\|\mathbf{r}_1 - \mathbf{r}_2\| =$ konst. ej bevarat under Lorentztransf.

Garage vilosyften



Information om stavens läge kan inte färdas snabbare än ljuset.

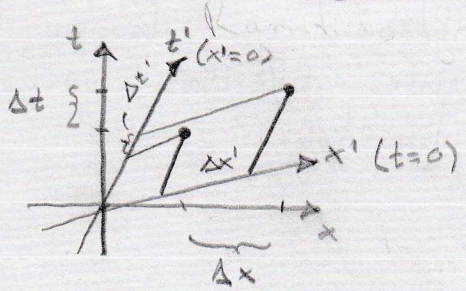
$t = \frac{L_2}{c}$, tiden det tar att nå dörren.

Även: $t = \frac{d}{v}$ $\quad \because \quad \frac{\gamma L_1 - L_2}{v} = \frac{L_2}{c} \Rightarrow \frac{L_2}{L_1} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} < 1$

$$\frac{v}{c} = \frac{L_1^2 - L_2^2}{L_1^2 + L_2^2}$$

3. 8/11

Rumsticldiagram



Två händelser.

$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma(t - vx) \end{cases}$$

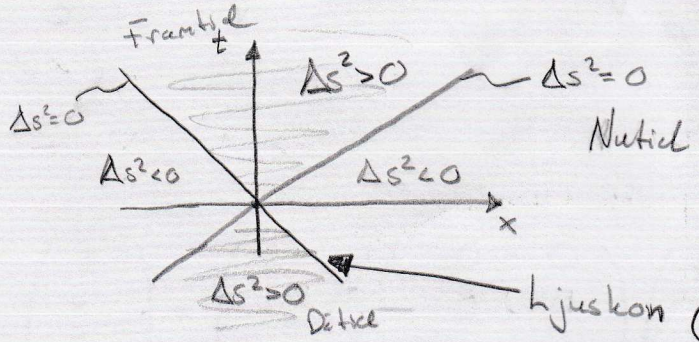
Intervall:

$$\Delta t^2 - \Delta x^2 = \Delta t'^2 - \Delta x'^2 \equiv \Delta s^2$$

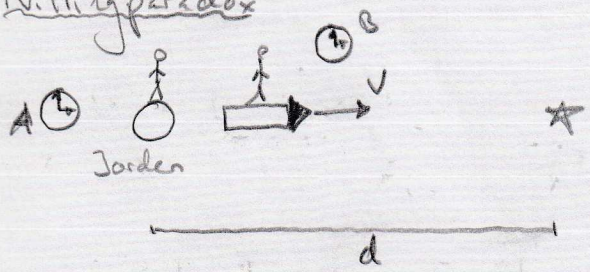
∵ Samma oavsett ref.system.

(jmf. med $\Delta x^2 + \Delta y^2 = \Delta x'^2 + \Delta y'^2 = \Delta r^2$, klassisk rotation)

$$\Delta s^2 \begin{cases} >0 & \text{tidslik} \\ =0 & \text{ljuslik} \\ <0 & \text{rumslik} \end{cases}$$



Tvillingparadox"

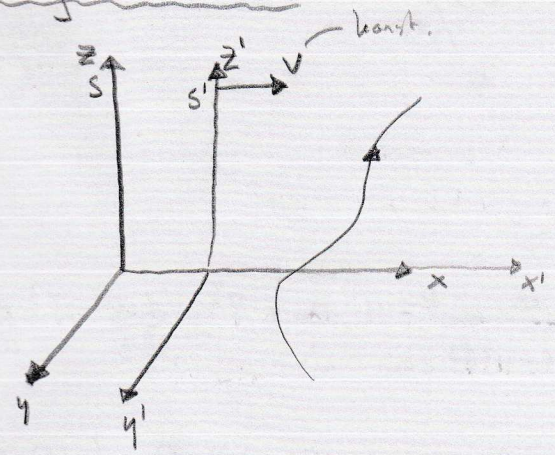


Enligt klocka på Jorden: $t_A = \frac{2d}{v}$

(Stämmer!)

-"- raketen: $t_B = \sqrt{1 - v^2/c^2} \cdot t_A < t_A$

Hastighetsaddition



$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - vx) \end{cases}$$

$$\mathbf{u} = \frac{d\mathbf{r}}{dt} \qquad \mathbf{u}' = \frac{d\mathbf{r}'}{dt}$$

Samband?

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - vdx)} = \frac{dy/dt}{\gamma(dt - vdx)/dt} = \frac{u_y}{\gamma(1 - vu_x)}$$

$$u_x' = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)/dt}{\gamma(dt - vdx)/dt} = \frac{u_x - v}{1 - vu_x}$$

$$a_x' = \frac{du_x'}{dt'} \ ?$$

$$\frac{du_x'}{dt} = \frac{d}{dt}\left(\frac{u_x - v}{1 - vu_x}\right) = \frac{\frac{du_x}{dt}\left((1 - vu_x) - (u_x - v)\left(-v\frac{du}{dt'}x\right)\right)}{(1 - vu_x)^2} = \frac{\frac{du_x}{dt'} - v^2\frac{du_x}{dt'}}{(1 - vu_x)^2}$$

$$= \frac{(1 - v^2)\frac{du_x}{dt'}}{(1 - vu_x)^2}$$

Mult. med dt' $\Rightarrow$ $du_x' = \frac{(1 - v^2)\,du_x}{(1 - vu_x)^2}$

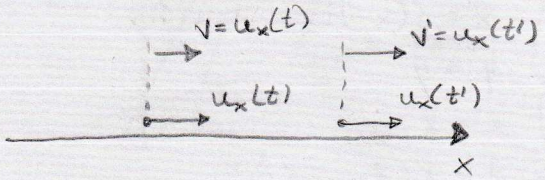
$$\therefore \quad a_x' = \frac{du_x'}{dt} = \frac{(1 - v^2)\,du_x/dt}{\gamma(dt - vdx)(1 - vu_x)^2/dt} = \frac{(1 - v^2)\,a_x}{\gamma(1 - vu_x)^3} = \frac{a_x}{\gamma^3(1 - vu_x)^3}$$

Dim. ok! ($c = 1$)

Newton: Konst. acc. $\Rightarrow$ $u_x = at$, $x = \frac{1}{2}a_x t^2$

$\frac{du_x}{dt} = a_x$. Konstant accelerationer i spec. rel.? (Nej, men..)

Egenacceleration (kan vara konstant)



$$\alpha = a' = \frac{a}{\gamma^3(1-vu)^3} = \left\{ \begin{matrix} v(t) = \\ = u(t) \end{matrix} \right\} = \frac{a}{\gamma^3(1-v^2)^3} = \gamma^3 a = \gamma^3 \cdot \frac{du}{dt} = \gamma^3 \frac{dv}{dt} =$$

egenacc. = konst.

Möjligt?

$\gamma = \frac{1}{\sqrt{1-v^2}}$

$$= \frac{d}{dt}(\gamma v) \Rightarrow \gamma v = \alpha t + \cancel{\text{konst.}} \Rightarrow$$

välj till 0. $(v(0)=0)$

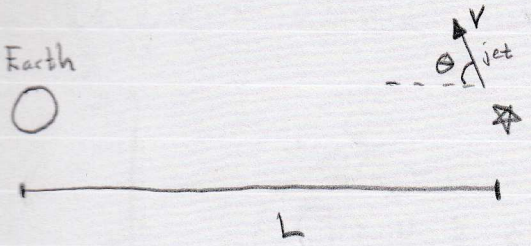
$$\Rightarrow \frac{dx}{dt} = v = \frac{\alpha t}{\sqrt{1+\alpha^2 t^2}}, \quad \lim_{t\to\infty} v = 1 \quad \text{bra!} \Rightarrow$$

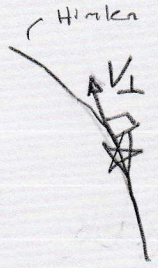
$$\Rightarrow x\alpha = \sqrt{1+\alpha^2 t^2} + \text{konst.}, \quad \text{konst.} = 0 \Rightarrow x^2 - t^2 = \frac{1}{\alpha^2}, \quad x = \frac{1}{2}\alpha t^2 + \ldots$$

Vi vill välja konstanten så att $x(t=0) = 0$. $\Rightarrow$ konst. $= -1$.

$$\therefore \quad x = \frac{c^2}{\alpha}\left(\sqrt{1+\frac{\alpha^2 t^2}{c^2}} - 1\right) = \frac{c^2}{\alpha}\left(\cancel{1} + \frac{1}{2}\frac{\alpha^2 t^2}{c^2} + \ldots - \cancel{1}\right) = \frac{1}{2}\alpha t^2 + \ldots \quad ☺$$

2.21



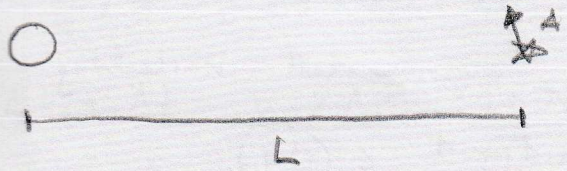


$V_\perp > c!$ => Fel.

$$= \frac{v \sin\theta}{1 - \frac{v}{c}\cos\theta}$$

Visa: $V_\perp = \dfrac{c \sin\theta}{c/v - \cos\theta}$. $\theta = 45 \Rightarrow V_\perp = \dfrac{v/\sqrt{2}}{1 - \frac{v}{c}\cdot\frac{1}{\sqrt{2}}} = c \Rightarrow$

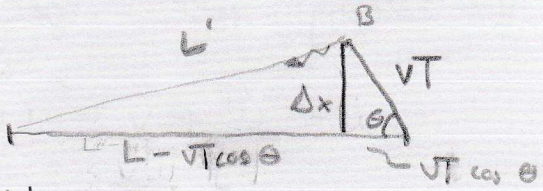
$\Rightarrow v = \dfrac{c}{\sqrt{2}} < c$ $\quad \because \dfrac{c}{\sqrt{2}} < v < c \Rightarrow V_\perp \geq c$ (Oroande om man inte tänker efter).



$t = 0$: Objekt (atom) A skickas ut från stjärnan.

$t_1 = L/c$: A syns från jorden.

T: A befinner sig i B.



$L' \approx L - vT\cos\theta$

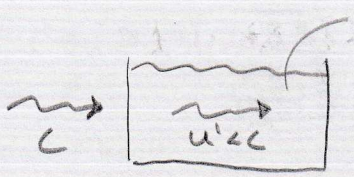
$$t_2 = T + \frac{L'}{c} = T + \frac{L - vT\cos\theta}{c}$$

$$V_\perp = \frac{\Delta x}{\Delta t} = \frac{vT\sin\theta}{t_2 - t_1} = \frac{vT\sin\theta}{T - \frac{vT\cos\theta}{c}} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}$$

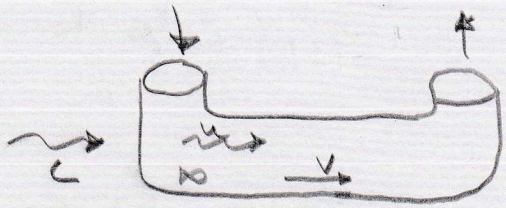
4. 11/11

Optik

1) Drag effect
2) Doppler effect
3) Aberration of light (4.13 & 4.14)

① 

Refraktionsindex: $n = \frac{c}{u'} > 1$



$v \ll c$

experimentell härledning

$$u = u' + kv \quad , \quad k = 1 - \frac{1}{n^2} \in (0,1)$$

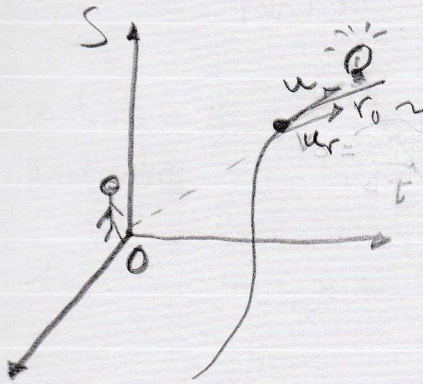
Härledning med spec. rel.:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \overset{\text{Taylor}}{=} (u'+v)\left(1 - \frac{u'v}{c^2}\right) = u' + v - \frac{u'^2 v}{c^2} - \cancel{\frac{u'v^2}{c^2}} =$$

försumbar

$$= u' + v\left(1 - u'^2/c^2\right)$$

② Doppler effect



~ Frekvens i lampans momentana viloram.

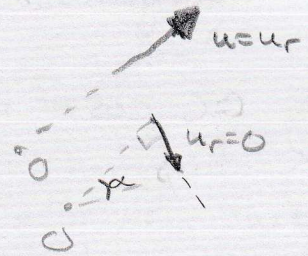
Tidsdilitation i S: $\Delta t = \gamma \Delta \tau$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}, \quad \nu = \frac{1}{\Delta t}?$$

Extra avstånd för informationen (från lampa till origo) $= u_r \cdot \Delta t = u_r \cdot \gamma \cdot \Delta \tau$

$$\Delta t_s = \Delta t + \frac{u_r \gamma \Delta \tau}{c} = \frac{1 + u_r/c}{\sqrt{1-u^2/c^2}} \Delta \tau$$

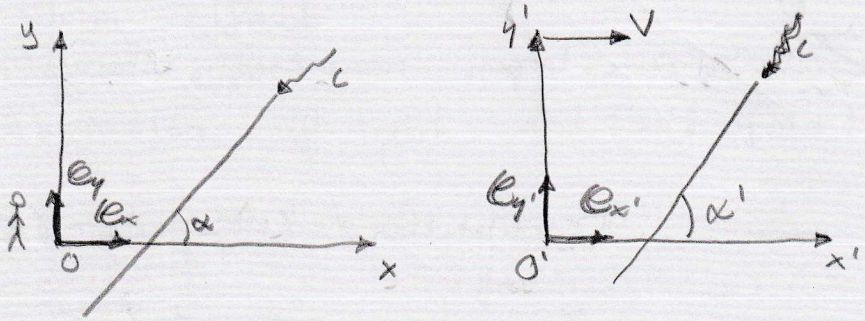
$$\nu_s = \frac{1}{\Delta t_s} = \frac{\sqrt{1-u^2/c^2}}{1+u_r/c} \nu_0 \quad . \quad \text{Om } u_r = u$$



$\Rightarrow$

$$\Rightarrow \nu_0 = \sqrt{\frac{1+u/c}{1-u/c}} \nu_s$$

③ Aberation av ljus, partiklar och vågor
(4.13) (4.14)



Standard config.

Samband mellan α & α'?

$$-\underbrace{c\,(\cos\alpha\, e_x}_{u_x} + \overbrace{\sin\alpha\, e_y}^{u_y}) - \underbrace{c\,(\cos\alpha'\, e_{x'}}_{u_{x'}} + \overbrace{\sin\alpha'\, e_{y'}}^{u_{y'}})$$

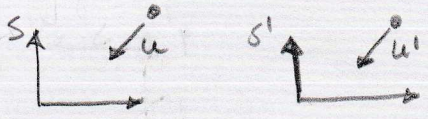
Hastighetsaddition:

$$\begin{cases} (x) & -c\cos\alpha' = \dfrac{-c\cos\alpha - v}{1-(-c\cos\alpha)v/c^2} \quad (1) \\ (y) & -c\sin\alpha' = \dfrac{-c\sin\alpha}{\gamma(1-(-c\cos\alpha)v/c^2} \quad (2) \end{cases}$$

$$\frac{(2)}{(1)} = \tan\alpha' = \frac{\sin\alpha}{\gamma(\cos\alpha + v/c)} \Rightarrow \tan(\alpha'/2) = \sqrt{\frac{1-v/c}{1+v/c}}\tan(\alpha/2)$$

(Fyra olika samband)

Partikel istället:



$$-u(\cos\alpha\, \mathbb{e}_x + \sin\alpha\, \mathbb{e}_y) \quad - u'(\cos\alpha'\, \mathbb{e}_{x'} + \sin\alpha'\, \mathbb{e}_{y'})$$

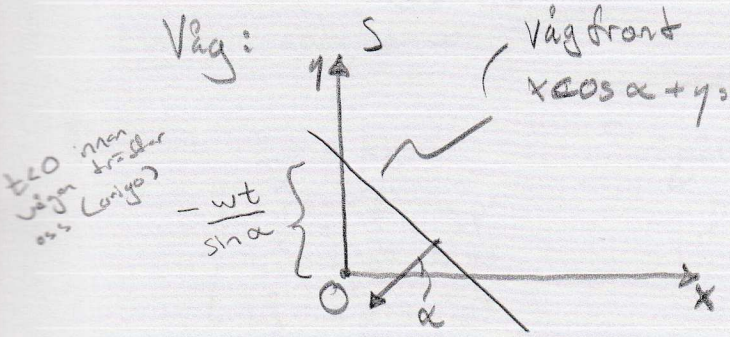
$$\left\{\begin{array}{ll} (x) & -u'\cos\alpha' = \dfrac{-u\cos\alpha - v}{1-(-u\cos\alpha)\,v/c^2} \qquad (1) \\[2ex] (y) & -u'\sin\alpha' = \dfrac{-u\sin\alpha}{\gamma(1-(-u\sin\alpha)\,v/c^2} \qquad (2) \end{array}\right.$$

Ej lösning (än!).

$$\frac{(2)}{(1)} = \tan\alpha' = \frac{\sin\alpha}{\gamma(\cos\alpha + v/u)}$$

$$\cancel{\tan(\alpha'/2) = \sqrt{\frac{1-v/u}{1+v/u}}\,\tan(\alpha/2)}$$ gäller ej här!

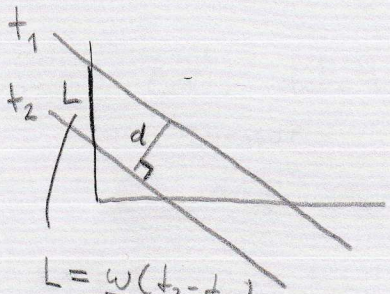
Våg:



vågfront

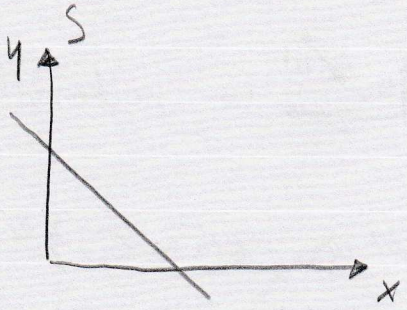
vågfrontens hastighet

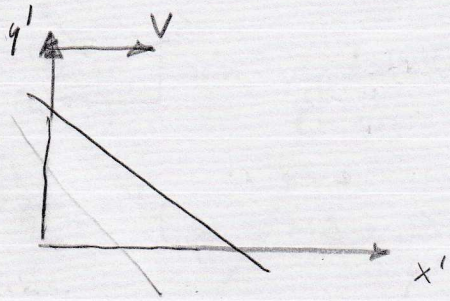
$x\cos\alpha + y\sin\alpha = \cancel{\text{konst.}}\ -wt \Rightarrow y = -\cot\alpha \cdot x + \cancel{\text{konst.}}\ wt$



$$\frac{d}{t_2 - t_1} = \frac{L\sin\alpha}{t_2 - t_1} = w$$

$$L = \frac{w(t_2 - t_1)}{\sin\alpha}$$





$x\cos\alpha + y\sin\alpha = -\omega t$ $\qquad$ $x'\cos\alpha' + y'\sin\alpha' = -\omega' t'$

Samband α & α'?

$$\gamma(x'+vt')\cos\alpha + y'\sin\alpha = -\omega\gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$x'\underbrace{\gamma\left(\cos\alpha + \frac{\omega v}{c^2}\right)}_{A\cos\alpha'} + y'\underbrace{\sin\alpha}_{A\sin\alpha'} = \underbrace{-\gamma(\omega + v\cos\alpha)t'}_{-A\omega' t}$$

$$\frac{\cancel{A}\sin\alpha'}{\cancel{A}\cos\alpha'} = \underbrace{\boxed{\tan\alpha' = \frac{\sin\alpha}{\cos\alpha + \omega v/c^2}}}_{\text{Våg } \omega} \quad \underbrace{\boxed{\frac{\sin\alpha}{\gamma(\cos\alpha + v/u)}}}_{\text{Partikel } u}$$

Ljus är både partikel & våg..?

$\omega = u \neq c \Rightarrow$ ingen likhet

$\omega = u = c \Rightarrow$ likhet ☺

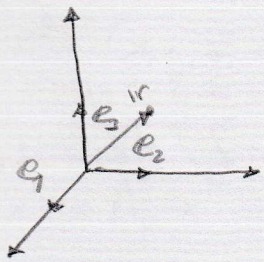
5. 15/11

4-vektorer

$\mathbb{V}$, $\mathbb{r}$: 3-vektorer

I boken: $\underbrace{\mathbb{V}, \mathbb{R}}$: 4-vektorer

Dålig notation. Vi kommer att använda en annan.



$$\mathbb{r} = \sum x^i e_i$$

$$\mathbb{F} = m\mathbb{a} \longrightarrow F^i = ma^i \ , \ \forall i = 1,2,3$$

Komposant i 4-vektor: $\mu = \underbrace{1,2,3}_{\text{rum}}, 4$ (tid)

Även vanligt med $\mu = \underset{\text{tid}}{0}, \underbrace{1,2,3}_{\text{rum}}$.

Einstein-notation

Ex. $\mathbb{L} = \mathbb{r} \times \mathbb{p}$

$$\begin{cases} L_1^1 = x^2 p^3 - x^3 p^2 \\ L_2^2 = x^3 p^1 - x^1 p^3 \\ L_3^3 = x^1 p^2 - x^2 p^1 \end{cases} \qquad L_i^i = (RHS)_i^i = ?$$

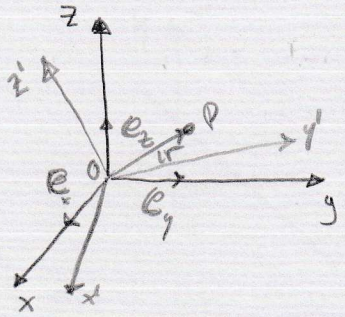
Inför ϵ^{ijk}, totalt antisymmetrisk: $\epsilon^{ijk} = -\epsilon^{jik} = \ldots$, $\epsilon^{123} = 1$.

$\llcorner$ tensor

$$\Rightarrow L^i = \epsilon_{ijk} x^j p^k$$

$i = 1$: $\epsilon_{1jk} x^j p^k = \epsilon_{123} x^2 p^3 + \epsilon_{132} x^3 p^2 = x^2 p^3 - x^3 p^2$

Skillnad mellan index uppe & nere? (Ja!)



Aktiv transformation: flytta enbart punkt.
Passiv transformation: basbyte, punkten "stannar", men koord.systemet ändras.

Passiv: $\mathbb{r} = \underbrace{x^i e_i}_{\text{ändras}} = x'^j e'_j$

Samma

Basbyte: $e_i = L_i^j e'_j$ Kräver L_i^j inverterbar

$e_1 = L_1^1 e'_1 + L_1^2 e'_2 + L_1^3 e'_3$

$x^i e_i = x'^j e'_j$

$\parallel$

$x^i L_i^j e'_j = x'^j e'_j \Rightarrow x^i L_i^j = x'^j$, $e_i = L_i^j e'_j$

$\Updownarrow$ $\Updownarrow$

$x^i = (L^{-1})_j^i x'^j$ $e'_j = (L^{-1})_j^i e_i$

Index uppe: Kontravariant
Index nere: Kovariant

Observera skillnaden ovan.

$\mathbb{L} = \mathbb{r} \times \mathbb{p} \rightarrow L_i = \epsilon_{ijk} x^j p^k$

$\mathbb{L} = L^i e_i$

$W = \mathbb{F} \cdot \mathbb{r} = \underbrace{F^1 x^1 + F^2 x^2 + F^3 x^3}_{= \delta_{ij} F^i x^j} \neq F^i x^i$??

$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

W transformeras under ett basbyte.

Villkor på L för att $\mathbb{F}' \cdot \mathbb{r}' = \mathbb{F} \cdot \mathbb{r}$

$e_i = L_i^j e'_j$, $x'^i = L_j^i x^j$
$F'^k = L_m^k F^m$

$\mathbb{F}' \cdot \mathbb{r}' = \mathbb{F} \cdot \mathbb{r}$

$\mathbb{F}' \cdot \mathbb{r}' = \delta_{ki} F'^k x'^i = \delta_{ki} L_m^k F^m L_j^i x^j = \mathbb{F} \cdot \mathbb{r} = \delta_{ij} F^i x^j$

$\forall \mathbb{F}, \mathbb{r}$ $\delta_{ki} L_m^k L_j^i F^m x^j = \delta_{mj} F^m x^j$

$\therefore \forall m, j \; \delta_{ki} L_m^k L_j^i = \delta_{mj}$

L ortogonal.

Sammanfattat

EH godtycklig basbyte kräver bara $LL^{-1} = \mathbb{1}$. (invertera bar)

Ska även t.ex. $\mathbb{F} \cdot \mathbb{F}$ vara bevarade krävs (skalärprodukter)

även $L^T = L^{-1}$.

ortogonal

Nu ska vi generalisera till 4-vektorer.

$$x^\mu = (x, y, z, ct = t)$$

$$\mu = 1, 2, 3, 4$$

$$x^1 \; x^2 \; x^3 \; x^4$$

$$x'^\nu = \Lambda^\nu_\mu x^\mu$$

Lorentztransformation = $\begin{cases} x' = \gamma(x - vt) = x'^1 = \gamma(x^1 - v x^4) \\ y' = y \quad = x'^2 = x^2 \\ z' = z \quad = x'^3 = x^3 \\ t' = \gamma(t - vx) = x'^4 = \gamma(x^4 - v x^1) \end{cases}$ Standard config.

$$\Rightarrow \Lambda^{\text{standard}}{}^{1}_{1} = \gamma \begin{pmatrix} \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 & \Lambda^1_4 \\ \vdots & \ddots & & 1 \\ \Lambda^4_1 & \cdots & & \Lambda^4_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

$$(\Delta x^4)^2 - \Delta(x^1)^2 - \Delta(x^2)^2 - \Delta(x^3)^2 =$$

$$= \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \quad , \quad \eta_{\mu\nu} = \begin{cases} 0 & \text{om } \mu \neq \nu \\ -1 & \text{om } \mu = \nu = 1,2,3 \\ 1 & \text{om } \mu = 4 \end{cases}$$

Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}$$

$$\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = \eta_{\mu\nu} \Delta x'^\mu \Delta x'^\nu = \eta_{\mu\nu} \underbrace{\Lambda^\mu_\rho \Delta x^\rho}_{\Delta x'^\mu} \underbrace{\Lambda^\nu_\sigma \Delta x^\sigma}_{\Delta x'^\nu} =$$

Mkste gälla

$$\Delta x'^\mu = \Lambda^\mu_\rho \Delta x^\rho$$

$$= \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \Delta x^\rho \Delta x^\sigma$$

Byt index i V.L. $\Rightarrow$ $\eta_{\rho\sigma} \Delta x^\rho \Delta x^\sigma = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \Delta x^\rho \Delta x^\sigma$

$$\Rightarrow \eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma$$ Lorentztransformation

FORAY WHITELINES

So far: 4-vektor: $x^\mu = (x, y, z, t)$

$\mathbf{v} = \frac{d\mathbf{r}}{dt}$ 3-hastighet

4-hastighetsvektor?

Gissning: $\cancel{\frac{dx^\mu}{dt}} = (v^x, v^y, v^z, 1)$

$\cancel{dx^\mu}$

$d\tau$: proper time. $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

$$\frac{dx^\mu}{d\tau} = U^\mu$$

18/11

4-vektor: $\mathbb{R} = (x^1, x^2, x^3, x^4 = ct) = X^{\mu=1,2,3,4}$

def: $X'^{\nu} = \underset{\text{4x4-matris}}{\Lambda^{\nu}_{\mu}} X^{\mu}$ (1 $S = S'$, standard config.)

$$ds^2 = \underbrace{c^2dt^2 - dx^2 - dy^2 - dz^2}_{\eta_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\mu\nu} dx'^{\mu} dx'^{\nu}} = c^2 d\tau^2$$

$\eta_{\mu\nu}$ = metric, $g_{\mu\nu}$ i boken

$c = 1 \Rightarrow ds^2 = d\tau^2$

$$\eta = g = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}$$

η_{11}, η_{44}

$g_{\mu\nu} = \Lambda^{\rho}_{\mu} \Lambda^{\sigma}_{\nu} g_{\rho\sigma}$ (Kraw for Lorentztransformation)

4-hastighet $\frac{dX'^{\nu}}{d\tau'} = \Lambda^{\nu}_{\mu} \frac{dX^{\mu}}{d\tau}$ $\quad U'^{\nu} = \Lambda^{\nu}_{\mu} U^{\mu}$

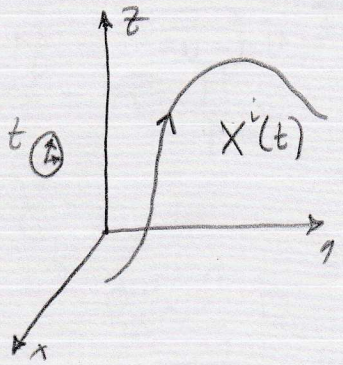
$d\tau^2 = d\tau'^2$

$S \quad S'$

$$\frac{dX^{\mu}}{dt} = U^{\mu}$$

$$U'^{\mu} = \frac{dx'^{\mu}}{d\tau'}$$

$U = (U^1, U^2, U^3, U^4)$



$$\frac{d\tau^2}{dt^2} = \frac{dt^2}{dt^2} - \frac{dr^2}{dt^2} \Rightarrow$$

$dx^2 + dy^2 + dz^2$

$$\Rightarrow \left(\frac{d\tau}{dt}\right)^2 = 1 - v^2$$

$$\Rightarrow \frac{dt}{d\tau} = \gamma$$

18/11

4-vektor: $\mathbb{R} = (x^1, x^2, x^3, x^4 = ct) = X^{\mu=1,2,3,4}$

def: $X'^{\nu} = \Lambda^{\nu}_{\mu} X^{\mu}$ (I $S = S'$, standard config.)

4x4-matris

$$ds^2 = \underbrace{c^2dt^2 - dx^2 - dy^2 - dz^2}_{\eta_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\mu\nu} dx'^{\mu} dx'^{\nu}} = c^2 d\tau^2$$

$\eta_{\mu\nu}$ = metric — $g_{\mu\nu}$ i boken

$$\eta = g = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}$$

η_{11}, η_{44}

$c = 1 \Rightarrow ds^2 = d\tau^2$

$g_{\mu\nu} = \Lambda^{\rho}_{\mu} \Lambda^{\sigma}_{\nu} g_{\rho\sigma}$ (Krav für Lorentztransformation)

4-hastighet $\frac{dX'^{\nu}}{d\tau'} = \Lambda^{\nu}_{\mu} \frac{dx^{\mu}}{d\tau}$ $\quad U'^{\nu} = \Lambda^{\nu}_{\mu} U^{\mu}$

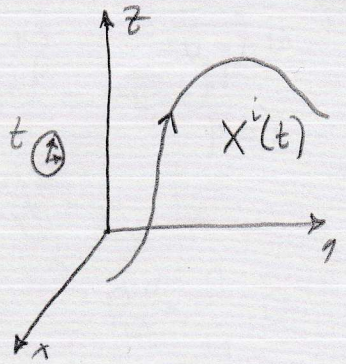
$d\tau^2 = d\tau'^2$

$S \qquad S'$

$$\frac{dX^{\mu}}{dt} = U^{\mu}$$

$$U'^{\mu} = \frac{dx'^{\mu}}{d\tau'}$$

$U = (U^1, U^2, U^3, U^4)$



$$\frac{d\tau^2}{dt^2} = \frac{dt^2}{dt^2} - \frac{d\mathbf{r}^2}{dt^2} \Rightarrow$$

$dx^2 + dy^2 + dz^2$

$$\Rightarrow \left(\frac{d\tau}{dt}\right)^2 = 1 - v^2$$

$$\Rightarrow \frac{dt}{d\tau} = \gamma$$

$$U^1 = \frac{dx^1}{d\tau} = \frac{dt}{d\tau} \cdot \frac{dx^1}{dt} = \gamma v^1$$

$$\therefore \quad \mathbb{U} = (\gamma v^1, \gamma v^2, \gamma v^3, \gamma) = \gamma(\mathbb{v}, 1)$$

4-acceleration, $\quad \mathbb{A} = (A^1, A^2, A^3, A^4) = A^\mu = \frac{dU^\mu}{d\tau}$

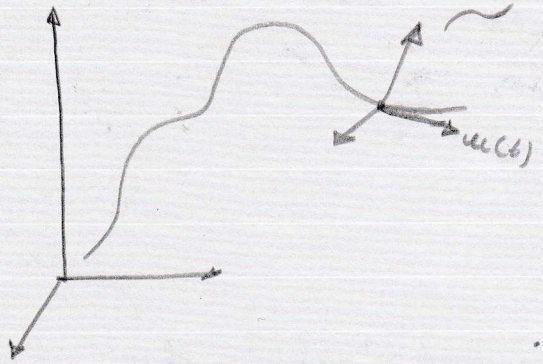
$$A^1 = \frac{dU^1}{d\tau} = \frac{d\gamma v^1}{d\tau} = \frac{dt}{d\tau} \cdot \frac{d\gamma v^1}{dt} = \gamma \cdot \frac{d\gamma v^1}{dt} =$$

$$= \gamma(\dot{\gamma} v^1 + \gamma a^1)$$

$$\mathbb{R} = x^\mu, \quad \mathbb{U} = U^\mu = \gamma(\mathbb{v}, 1), \quad \mathbb{A} = A^\mu = \gamma(\dot{\gamma}\mathbb{v} + \gamma \mathbb{a}, \dot{\gamma})$$

$$\mathbb{U}^2 = \mathbb{U} \cdot \mathbb{U} = \underbrace{(U^4)^2 - \dots (U^1)^2}_{\substack{\gamma^2 - \gamma^2 u^{1\,2} - \gamma^2 u^{2\,2} - \gamma^2 u^{3\,2} = \\ = \gamma^2(1-u^2) = 1}} = g_{\mu\nu} U^\mu U^\nu = 1$$

$\underline{\mathbb{r}(t)}$



momentant referensystem, $\mathbb{u}(t) = 0$. ~~$\mathbb{a}(t) = 0$~~?

$$\dot{\gamma} = \frac{d}{dt} \frac{1}{\sqrt{1-u^2}} = -\frac{\mathbb{u} \cdot \mathbb{a}}{(1-u^2)} = 0$$

3-skalärprodukt

i det momentana systemet $\quad \gamma = 1$

$$\therefore \quad U^\mu = (0, 0, 0, 1)$$

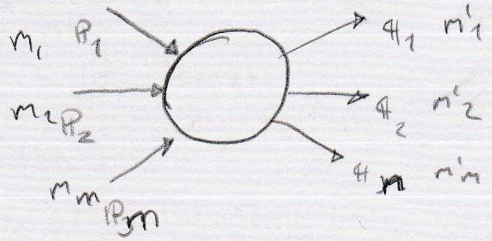
$$A^\mu = (\mathbb{a}, 0)$$

$$A^2 = -a^2 = -\alpha^2$$

Generalisera $\mathbb{F} = m\mathbb{a}$?

$$\left.\begin{array}{l}\mathbb{F} = m\mathbb{a} \\ \mathbb{F}_{12} = -\mathbb{F}_{21}\end{array}\right\} \Rightarrow$$

$$\mathbb{F} = \dot{\mathbb{P}}$$



$$\sum_m \mathbb{P}_i = \sum_m \mathbb{q}_i \xrightarrow[\text{Generalisera}]{} ?$$

Krav:
- Lorentzinvariant
- Resultatet ska reducera till Newtons ekvationer om $v \ll c$.

$$\mathbb{P} = m\mathbb{v} \xrightarrow[\text{till 4-vekt.}]{\text{Gen.}} P^\mu = m\,U^\mu \quad \text{(4-rörelsemängd)}$$

Lorentzinvariant storhet.

$U^1 = \gamma u^1$ | boken: m_0, "vilomassa".

$$P^1 = m_0 \gamma u^1 = m u^1$$

$m \Rightarrow$ "relativistiska massa", dåligt begrepp, egentligen en energi.

∴ Använd inte m_0, det finns bara en massa, m.

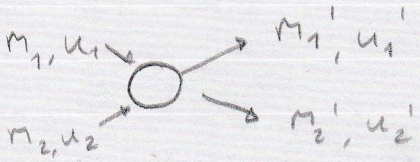
Gissning $\sum \mathbb{P}_i = \sum \mathbb{Q}_i \Leftrightarrow \sum_{i=1}^{m} P_i^\mu = \sum_{i=1}^{n} Q_i^\mu \quad \begin{cases} \mu = 1,2,3 \\ \mu = 4 \end{cases}$

- $m(\neq 0)$

$$P_i^\mu = m_i U_i^\mu$$

$$Q_j^\mu = m_j U_j^\mu$$

$$\sum_{i=1}^{m} P_{(i)}^4 = \sum_{j=1}^{n} Q_{(j)}^4$$



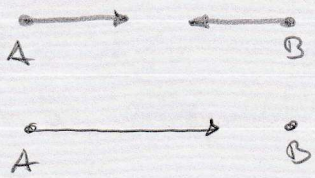
$$\therefore \underbrace{m_1 \gamma(u_{(1)})}_{P_{(1)}^4} + \underbrace{m_2 \gamma(u_{(2)})}_{P_{(2)}^4} = m_1' \gamma(u_{(1)}') + m_2' \gamma(u_{(2)}')$$

$$= Q_{(1)}^4 + Q_{(2)}^4$$

$$\underbrace{m\gamma(u)}_{} = \frac{m}{\sqrt{1-u^2}} \approx m\left(1+\frac{1}{2}u^2+O(u^4)\right) = m + \frac{1}{2}mu^2 + O(mu^4)$$

(energi!
$(mc^2\gamma(u))$

Ex.



zero momentum frame

lab frame

$p^\mu = m U^\mu = (\mathbf{p}\ E)$, $\mathbf{p} = m\gamma \mathbf{u}$ rel. 3-rörelsemängd

$E = \underset{c=1}{m\gamma}$ -"- kinetisk energi

$$p^{\mu 2} = m^2 \underbrace{U^{\mu 2}}_{1} = m^2$$

$$\parallel$$

$$\boxed{E^2 - \mathbf{p}\cdot\mathbf{p} = m^2}$$

$$E = \sqrt{c^2\mathbf{p}^2 + m^2c^4}$$

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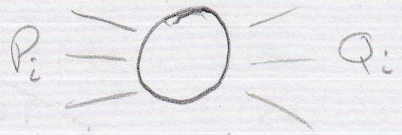
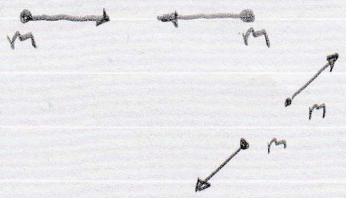
$$P^\mu = (\mathbb{P}, E) = m\, U^\mu = (m\gamma u, m\gamma) \qquad u = \mathbb{P}/E$$
$$m \neq 0 \qquad \gamma = E/m$$

$m \to 0$: $\quad m\gamma u = \mathbb{P}$ konst. ($m \to 0$, $\gamma \to \infty$ $(u \to c)$, $u \to c$)

$$\sum P_i^\mu = \sum Q_j^\mu$$



"Relativistisk biljard"



Elastisk stöt (bevarar identitet, proton + proton $\longrightarrow$ proton + proton)

Zero momentum frame $\equiv$ masscentrum $= P^\mu_{tot\,cm} = P^\mu_{1cm} + P^\mu_{2cm} =$

cm: centre of mass

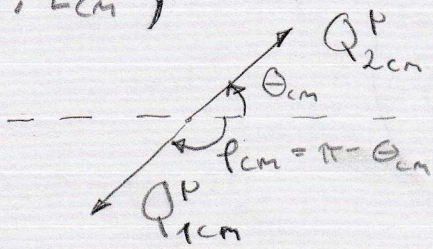
$= (0, \neq 0)$

$P^{\mu 2} = m^2 \Rightarrow E = \sqrt{P^2 + m^2}$

Före studs: $P^\mu_{1cm} = (P_{cm}, 0, 0, \underbrace{\sqrt{P_{cm}^2 + m^2}}_{E_{cm}})$

$$P^\mu_{2cm} = (-P_{cm}, 0, 0, E_{cm})$$

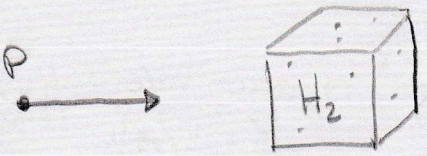
Efter studs:

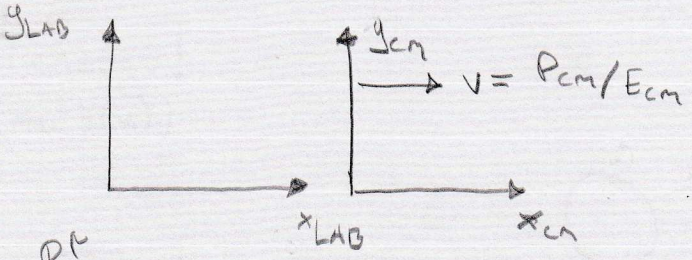


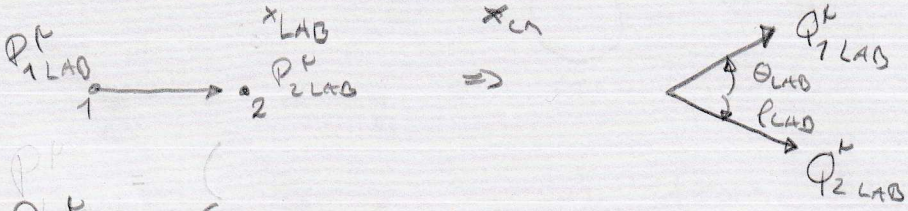
$$Q^\mu_{1cm} = (P_{cm} \cos\varphi_{cm}, -\varphi_{cm} \sin\varphi_{cm}, 0, E_{cm})$$

$$Q^\mu_{2cm} = (P_{cm} \cos\theta_{cm}, -P_{cm} \sin\varphi_{cm}, 0, E_{cm})$$

Lab reference system







$$P^\mu_{2\,LAB} = (0,0,0,m)$$

$$P_{2\,cm} \longrightarrow P_{2\,LAB}\ ?$$

$$\underbrace{\begin{pmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{pmatrix}\begin{pmatrix} -P_{cm} \\ E_{cm} \end{pmatrix}}_{\begin{pmatrix} \gamma(-P_{cm}+vE_{cm}) \\ \gamma(-vP_{cm}+E_{cm}) \end{pmatrix}} = \begin{pmatrix} 0 \\ m \end{pmatrix}$$

$\because\ v = \dfrac{P_{cm}}{E_{cm}}$ confirmed.

$$Q_{1\,cm} \longrightarrow Q_{2\,cm}\ ?$$

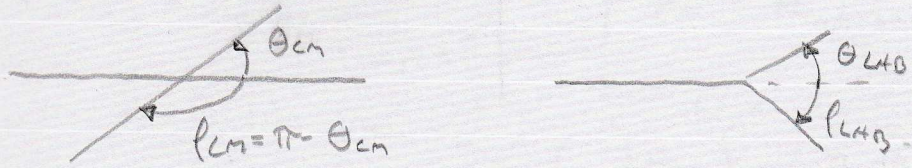
$$\begin{pmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{pmatrix}\begin{pmatrix} P_{cm}\cos\varphi_{cm} \\ E_{cm} \end{pmatrix} = \begin{pmatrix} \overbrace{\gamma(P_{cm}\cos\varphi_{cm} + vE_{cm})}^{\gamma P_{cm}(1-\cos\theta_{cm})\ +\cos\varphi_{cm}} \\ \dots \end{pmatrix} =$$

$$= \begin{pmatrix} P_{LAB}\cos\varphi_{LAB} \\ \dots \end{pmatrix}$$

y-Ind:

$$Q^y_{1\,LAB} = Q^y_{1\,cm} \Rightarrow P_{LAB}\sin\varphi_{LAB} = P_{cm}\sin\varphi_{cm}$$

Sammanstall $\Rightarrow$ $\boxed{\tan\varphi_{LAB} = \dfrac{\sin\theta_{cm}}{\gamma(1-\cos\theta_{cm})}}$



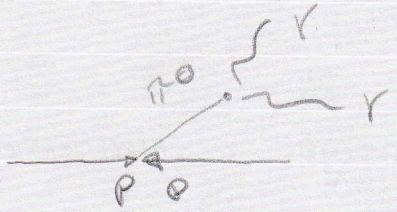
$$\theta_{cm}, \varphi_{cm} \longrightarrow \theta_{LAB}, \varphi_{LAB} \begin{cases} \tan \varphi_{LAB} = \dfrac{\sin \theta_{cm}}{\gamma(1-\cos\theta_{cm})} \\ \tan \theta_{LAB} = \dfrac{\sin \theta_{cm}}{\gamma(1+\cos\theta_{cm})} \end{cases} \Rightarrow$$

$$\Rightarrow \tan(\varphi_{LAB}) \cdot \tan(\theta_{LAB}) = \frac{1}{\gamma^2} \longrightarrow 0$$

$$p + p \longrightarrow p + p + \underbrace{\overset{\text{pion}}{\pi^0}}_{\text{sonderfall}} \longrightarrow \gamma\gamma$$

$$P^\mu_{\gamma_1} = (\mathbb{P}_{\gamma_1}, E_{\gamma_1} = |\mathbb{P}_{\gamma_1}| c)$$

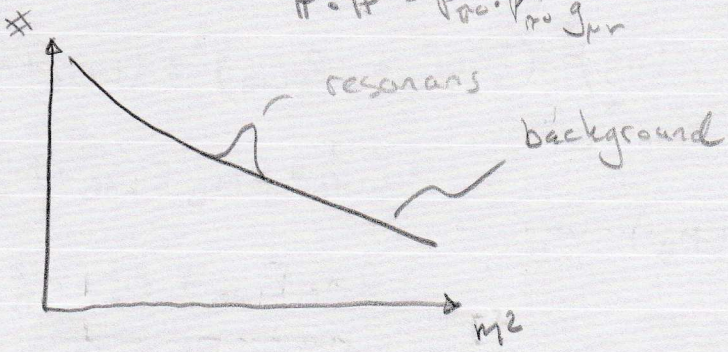
$$P^\mu_{\gamma_2} = (\mathbb{P}_{\gamma_2}, E_{\gamma_2} = |\mathbb{P}_{\gamma_2}| c)$$



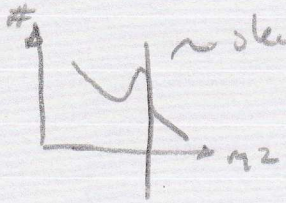
$$P^\mu_{\pi^0} = P^\mu_{\gamma_1} + P^\mu_{\gamma_2}$$

$$\underbrace{P^{\mu\,2}_{\pi^0}} = m^2_{\pi^0} = (P^\mu_{\gamma_1} + P^\mu_{\gamma_2})^2$$

$$\mathbb{P} \cdot \mathbb{P} = P^\mu_{\pi^0} \cdot P^\nu_{\pi^0} g_{\mu\nu}$$



$1 \to 2$ – sönder fall $\Rightarrow$

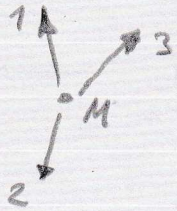


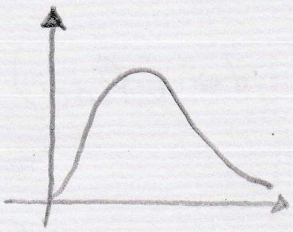
~ skarp linje i puckeln

$$(0\ 0\ 0\ M) = (\mathbb{P}, \underbrace{E_1}_{\sqrt{|\mathbb{P}|^2 + m_1^2}}) + (-\mathbb{P}, E_2)$$

$$|\mathbb{P}| = f(M, m_1, m_2)$$

$1 \to 3$ – sönder fall





Tröskelenergi

$$\underset{m_A}{A} + \underset{m_B}{B} \longrightarrow \underset{m_{c_1} + \dots + m_{c_n}}{C_1 + \dots + C_n}$$

$$P_A^\mu + P_B^\mu = \sum P_{c_i}^\mu$$

I cm: $P_A^\mu = (\mathbb{P}, E_A)$
$P_B^\mu = (-\mathbb{P}, E_B)$

$$S = (P_A + P_B)^2 = (\mathbb{0}, E_A + E_B)^2 =$$

$$= (E_A + E_B)^2 \Rightarrow \sqrt{S} = \text{total energi i masscentrum.}$$

LAB

$P_A^\mu = (P_{LAB}, 0, 0, E_{LAB})$

$P_B^\mu = (0, 0, 0, m_B)$

$(P_{LAB}, 0, 0, E_{LAB} + m_B)$

$$\underset{E_{CM}^2}{(\sqrt{S})^2} = (P_{A\,LAB} + P_{B\,LAB})^2 = (E_{LAB} + m_B)^2 - P_{LAB}^2 =$$

$$= E_{LAB}^2 + 2m_B E_{LAB} + m_B^2 - P_{LAB}^2 =$$

$$= 2m_B E_{LAB} + m_A^2 + m_B^2$$

$$\therefore E_{CM} = \sqrt{2m_B E_{LAB} + m_A^2 + m_B^2}$$

$$P_A^\mu + P_B^\mu = \sum P_{c_i}^\mu$$

$(0,0,0,m_{c_i})$ $\quad$ $(\ldots, E_{c_j} = m_{c_j}\gamma_{c_j})$

$$S = (P_A + P_B)^2 = \left(\sum P_{c_i}^\mu\right)^2 = \underbrace{\sum_i P_{c_i}^2}_{} + 2\sum_{i<j} P_{c_i}\cdot P_{c_j}$$

γ_{c_i} for product j in ref. system i (vila)

$$\sum_i m_{c_i}^2 + 2\sum_{i<j} m_{c_i}\cdot m_{c_j}\cdot\gamma_{c_j} \geq$$

$$\geq \sum_i m_{c_i}^2 + 2\sum_{i<j} m_{c_i} m_{c_j} =$$

$$= \sum_i m_{c_i}^2$$

$E_{cm}^2 > \left(\sum m_{c_i}\right)^2$ i cm-system.

$$2 m_B E_{LAB} + m_A^2 + m_B^2 > \left(\sum m_{c_i}\right)^2$$

$$E_{Tröskel}^{LAB} = \frac{\left(\sum m_{c_i}\right)^2 - m_A^2 - m_B^2}{2m_B}$$

$$\underset{M}{p} + p \longrightarrow p + p + \underset{m}{\pi^0}$$

$$E_{Tröskel} = \frac{(M+M+m)^2 - M^2 - M^2}{2M} = M + 2m + \frac{m^2}{2M}$$

effektivitet

$$T_{min} = E_{Tröskel} - M, \quad k = \frac{m}{T_{min}}$$

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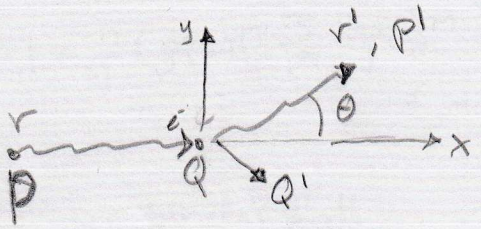
Comptoneffekt:

$\gamma + e^- \longrightarrow \gamma + e^-$

ljus med frekv. ν, består av odelbara partiklar med energi $= h\nu$.



Kan vi beräkna θ? Nej, endast sannolikheten för olika θ.

$\Rightarrow$ Vi antar att θ är känd och tar fram ett samband mellan ν e ν'.

$$P^\mu + Q^\mu = P'^\mu + Q'^\mu$$

$m_{foton} = 0$

$$\begin{cases} P^\mu = (h\nu, 0, 0, h\nu) \\ Q^\mu = (0, 0, 0, mc^2 = m) \\ P'^\mu = (h\nu'\cos\theta, h\nu'\sin\theta, 0, h\nu') \\ Q'^\mu = \ldots, \text{ behövs ej här.} \end{cases}$$

$P^{\mu 2} = m^2$

$\underbrace{P^{0 2}} - P^{3 2} - P^{2 2} - P^{1 2} = m^2$

$E^2 - |\mathbb{P}^2| = m^2$

$E = \pm\sqrt{|\mathbb{P}^2| + m^2}$

antipartiklar

Foton: $P^2 = P'^2 = 0$ "On shell conditions"

$Q^2 = Q'^2 = m^2$

$$(Q')^2 = (P + Q - P)^2$$
$$m^2 = P^2 + Q^2 + P'^2 + 2P\cdot Q - 2P\cdot P' - 2Q\cdot P'$$
$$m^2 = 0 + m^2 + 0 + 2h\nu m - 2h^2\nu\nu'(1 - \cos\theta)$$
$$(\nu - \nu')m = h\nu\nu'(1 - \cos\theta)$$

$2\sin^2(\theta/2)$

$\vdots$

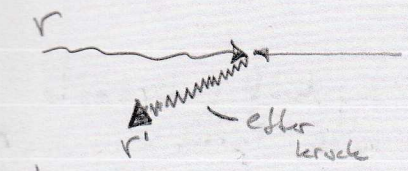
$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{2h}{mc}\sin^2(\theta/2)$$

$\therefore \quad \lambda' - \lambda = 2\cdot \lambda_c \sin^2(\theta/2)$, λ_c Comptons våglängd

$$\lambda_c = \frac{1}{m}\left(\frac{h}{c}\right)$$

Invers Comptoneffekt

$k_B T = h\nu$



$\nu' > \nu$

Proton

$$P = (\underbrace{-\gamma m u}_{P_x}, 0, 0, \gamma m)$$

$$-P_x^2 + \underbrace{E^2}_{(\gamma m)^2} = m^2 \Rightarrow P_x = -\gamma m u$$

Proton: γ upp till 10^{11}

CMBR-proton: $E = h\nu = 10^{-3}$ eV
Cosmic Background

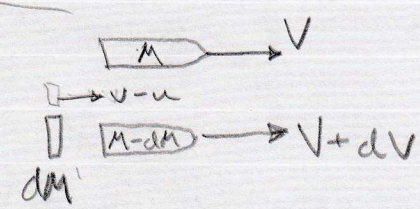
$E' = h\nu' = 4 \cdot 10^{19}$ eV (!)

6.6

Tsiolkovskyekvationen

Hur mycket bränsle för att lämna jorden?

Newton:



$$M_{in} = M_{raket} + M_{bränsle}$$

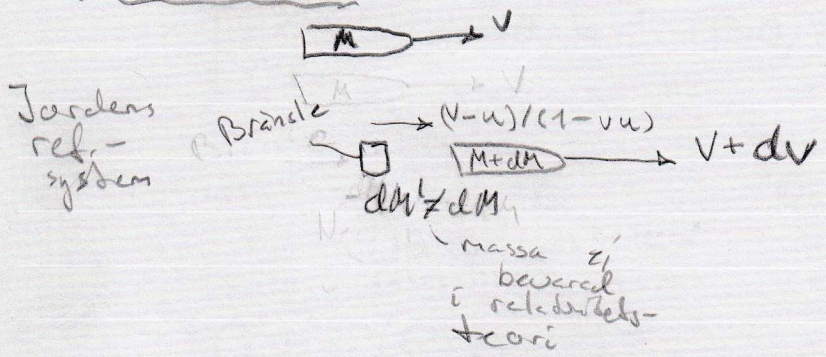
$$Mv = dM(v-u) + (M-dM)(v+dV)$$

$$\cancel{Mv} = v\cancel{dM} - u\,dM + \cancel{MV} - V\cancel{dM} + M\,dV - \cancel{dM\,dv}$$ försummas

$$\int_{v_{in}}^{v_{fin}} dV = u\int_{M_{in}}^{M_{fin}} \frac{dM}{M}$$

$$V_{fin} - V_{in} = u\log\left(\frac{M_{fin}}{M_{in}}\right)$$

Relativistiskt:



$(dM<0)$

P: $$M_v\gamma(v) = (M+dM)(v+dv)\gamma(v+dv) + dM'\left(\frac{v-u}{1-vu}\right)\gamma\left(\frac{v-u}{1-vu}\right) \quad (1)$$

E: $$M\gamma(v) = (M+dM)\gamma(v+dv) + dM'\gamma\left(\frac{v-u}{1-vu}\right) \quad (2)$$

$$\gamma(v+dV) = \frac{1}{\sqrt{1-(v+dv)^2}} \overset{\text{Taylor}}{\approx} \gamma(v) + \gamma'(v)\,dv = \gamma(v) + v\gamma^3(v)\,dV$$

$$\gamma\left(\frac{v-u}{1-uv}\right) = \frac{1}{\sqrt{1-\left(\frac{v-u}{1-vu}\right)^2}} = (1-uv)\gamma(u)\cdot\gamma(v)$$

$$(1) \Rightarrow \quad v\,dM + M\gamma^2(v)\,dv + (v-u)\,\gamma(u)\,dM' = 0$$

$$(2) \Rightarrow \quad dM + Mv\,\gamma^2(v)\,dv + (1-uv)\,\gamma(u)\,dM' = 0$$

$$(1-uv)\,(1) - (v-u)\,(2) \Rightarrow u\,dM + \gamma^2(v)\,M\,dV = 0$$

$$-u\int_{M_{in}}^{M_f} \frac{dM}{M} = \int_{v_{in}}^{v_{fin}} \gamma^2(v)\,dV$$

$$v_{in} = 0 \Rightarrow \quad \frac{M_{in}}{M_{fin}} = \left(\frac{1+v_{fin}/c}{1-v_{fin}/c}\right)^{\frac{c}{2u}}$$

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4-kraft

Generalisera Newton II: $\mathbf{f} = m\mathbf{a}$

Krav:

1) Lorentzinvarians
2) Att uttrycket reduceras tillbaka till vid v<<c.

Två alternativ:

$$F_1^\mu = m A^\mu = m \frac{dU^\mu}{d\tau} = m \frac{d^2 x^\mu}{d\tau^2} \quad \text{eller}$$

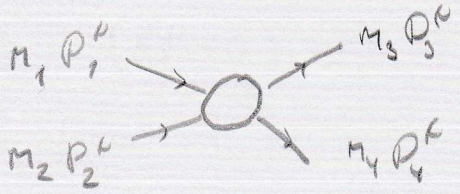
$$F_2^\mu = \frac{d}{d\tau}(mU^\mu) = \frac{dm}{d\tau} U^\mu + F_1^\mu$$

Både $\frac{d}{d\tau}(mU^\mu)$ & mA^μ är 4-vektorer.

$$A^\mu \xrightarrow[c\to\infty]{} (a_i, \cdot), \quad \frac{d}{d\tau} = \underbrace{\frac{dt}{d\tau}}_{\gamma} \cdot \frac{d}{dt} \xrightarrow[c\to\infty]{} \frac{d}{dt}$$

Finns några bra exempel på krafter?
Vi kommer att använda definitionen av F_2^μ för 4-kraft.

$$F^\mu = \frac{d}{d\tau}(mU^\mu) = \frac{d}{d\tau} P^\mu$$



$$P_1^\mu + P_2^\mu = P_3^\mu + P_4^\mu$$

$$\frac{d}{dt}(P_1^\mu + P_2^\mu) = F_1^\mu + F_2^\mu$$

$\therefore$ Om $F_1^\mu + F_2^\mu = 0 \Rightarrow P_1^\mu + P_2^\mu$ bevarad.

1) Mer explicit uttryck för F.
2) Exempel (EM-Kraft / Lorentzkraft), endast ett exempel som är fysikaliskt relevant.

$$F^\mu = \frac{d}{d\tau} p^\mu = \frac{dt}{d\tau} \cdot \frac{d}{dt}(\mathbb{P}, E) = \gamma\left(\frac{d\mathbb{P}}{dt}, \frac{dE}{dt}\right) = \gamma\left(\mathbb{f}_{rel}, \frac{dE}{dt}\right)$$

En definition till:

Relativistisk 3-kraft:

$$\mathbb{f}_{rel} = \frac{d}{dt}\mathbb{P} = \frac{d}{dt}(m\gamma \mathbb{v})$$

Specialfall: massbevarande kraft, ytterligare villkor på F^μ.

$$F^\mu = \frac{d}{d\tau}(mU^\mu) = \frac{dm}{d\tau}U^\mu + mA^\mu$$

$$U_\mu F^\mu = \frac{dm}{d\tau}\underbrace{U_\mu U^\mu}_{1} + m\underbrace{U_\mu A^\mu}_{0} = \frac{dm}{d\tau}$$

Massbevarande kraft omm $\boxed{\frac{dm}{d\tau} = 0}$.

Lorentzkraft (EM-kraft) är massbevarande, $U^\mu F_\mu = 0$.

Måste kollas för Lorentz.

Mer explicit:

$$F_\mu U^\mu = \underbrace{\gamma\left(\mathbb{f}_{rel}, \frac{dE}{dt}\right)}_{F^\mu} \cdot \underbrace{\gamma(\mathbb{u}, 1)}_{U^\mu} = \gamma^2\left(\frac{dE}{dt} - \mathbb{f}_{rel} \cdot \mathbb{u}\right) = 0$$

3D-skalärprodukt

$$\therefore \quad \frac{dE}{dt} = \mathbb{f}_{rel} \cdot \mathbb{u} \iff F_\mu U^\mu = 0$$

Elektromagnetism

Notation

$$A'^i = \underbrace{P^i_j}_{n\times n\text{-matris}} A^j$$

$(A^1, \dots, A^n)$

$$\underbrace{U^{\mu'}}_{S'} = \underbrace{\Lambda^\mu_\nu}_{S} U^\nu$$

4x4 Lorentztransform
$\mu, \nu = 1,2,3,4$

$$x^{\mu'} = \Lambda^\mu_\nu U^\nu \Rightarrow$$

$$\Rightarrow \Lambda^\mu_\nu = \frac{\partial x^{\mu'}}{\partial x^\nu}$$

Maxwells ekv. på relativistisk form (+ Lorentzkraft)

Ej SI-enheter, $\cancel{\epsilon_0}$ $\cancel{\mu_0}$. $\sqrt{\epsilon_0 \mu_0} = c$ (naturliga enheter)

Maxwell i 3D

$$\begin{cases} \nabla \cdot \mathbb{E} = 4\pi\rho \\ \nabla \cdot \mathbb{B} = 0 \\ \nabla \times \mathbb{E} = -\frac{1}{c}\frac{d\mathbb{B}}{dt} \\ \nabla \cdot \mathbb{B} = \frac{1}{c}\frac{d\mathbb{E}}{dt} + \frac{4\pi}{c}\mathbb{J} \end{cases}$$

Lorentzkraft

Ändras i relativitetsteori

$$\mathbb{F} = q\left(\mathbb{E} + \frac{1}{c}\mathbb{u} \times \mathbb{B}\right) = m\mathbb{a}$$

Partikel med massa m och el. laddning q som rör sig i ett EM-fält.

Vi vill skriva allt i 4D-notation. Vi börjar med Lorentzkraften.

F_μ?

1) Lorentzvektor
2) 3D-komposanterna i $F_\mu \xrightarrow[c\to\infty]{} \mathbb{F}_{Lorentz}$
3) $F_\mu U^\mu = 0$ (massbevarande)
4) Linjär i både EM-fältet och hastigheten

Ansats:

$$F_\mu = \frac{q}{\cancel{c}} F_{\mu\nu} \overset{\text{4-hastighet}}{U^\nu}$$

Ny tensor, blanda inte ihop med F_μ (Lorentzkraft).
I boken: $F_{\mu\nu} = E$.

Uppfyller F_μ att $F_\mu U^\mu = 0$?

$$F_\mu U^\mu = q \underbrace{F_{\mu\nu} U^\nu U^\mu}_{\text{2 index}} = 0 \qquad (*)$$

Vi letar efter en 2-tensor som uppfyller *.
Ett tillräckligt villkor för att $F_{\mu\nu} U^\mu U^\nu = 0$ är att $F_{\mu\nu} = -F_{\nu\mu}$.
Visa att $F_{\mu\nu} = -F_{\nu\mu} \Rightarrow F_{\mu\nu} U^\nu U^\mu = 0$.

$$F_{\mu\nu} U^\nu U^\mu = -F_{\nu\mu} U^\mu U^\nu = -F_{\nu\mu} U^\nu U^\mu \xrightarrow[\text{Byta index}]{} -F_{\mu\nu} U^\mu U^\nu$$

$\therefore$ $\underline{F_{\mu\nu} U^\nu U^\mu = 0}$

$$F_{\mu\nu} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} = \begin{pmatrix} 0 & -B_3 & B_2 & -E_1 \\ B_3 & 0 & -B_1 & -E_2 \\ -B_2 & B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix}$$

$$F_{\mu=1} = q \cdot F_{1\nu} U^\nu = q\left(\cancel{F_{11} U^1} + F_{12} U^2 + F_{13} U^3 + F_{14} U^4\right) =$$

$$= q\left(-B_z \cdot \gamma u_y + B_y \gamma u_z - E_x \gamma\right) = -q\gamma\left(E_x + (\mathbf{u} \times \mathbf{B})_x\right)$$

$$(F_1, F_2, F_3) = -\gamma \mathbb{F}_{Lorentz}$$

$$(F^1, F^2, F^3) = \gamma \mathbb{F}_{Lorentz}$$

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Förra gången: Maxwells & Lorentzkraft.

4D-kraft: $F^\mu = \frac{d}{d\tau} p^\mu$

$F^{1,2,3} = \underbrace{\frac{dt}{d\tau}}_{\gamma} \cdot \underbrace{\frac{d}{dt} p^{1,2,3}}_{f_{rel}^{1,2,3}}$, i 3D-notation: $\mathbb{f}_{rel} = \frac{d}{dt}\mathbb{P} = \frac{d}{dt}(m\gamma \mathbb{u})$!

Lorentz: $\mathbb{F} = q(\mathbb{E} + \mathbb{u} \times \mathbb{B})$

Vi visade att $F_\mu = q F_{\mu\nu} U^\nu$, där $F_{\mu\nu} = \begin{pmatrix} 0 & -B^z & B^y & E^x \\ & 0 & -B^x & -E^y \\ & & 0 & -E^z \\ & & & 0 \end{pmatrix}$

EM-tensor

$U^\nu = \gamma(\mathbb{u}, 1)$

$F^\mu = \gamma F^{1,2,3}_{Lorentz}$

Hur skriver man Maxwells ekvationer på 4D-form?

Använd $F_{\mu\nu}$ och $\frac{\partial}{\partial x^\mu} = \partial_\mu$ index nere!

index uppe!

Förklaring index uppe/nere: $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow \Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$

$\partial'_\nu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \cdot \frac{\partial}{\partial x^\nu} = \Lambda^{-1}{}^\nu{}_\mu \partial_\nu$

$\nabla, \frac{\partial}{\partial t} \rightsquigarrow \partial_\mu$

$\mathbb{E}, \mathbb{B} \rightsquigarrow F_{\mu\nu}$

$\mathbb{J}, \rho \rightsquigarrow J_\mu$ $\quad -\nabla \cdot \mathbb{J} = \frac{\partial \rho}{\partial t}$, kontinuitetsekvationen

Om vi sätter $J^\mu = (\mathbb{J}, \overset{\mu=4}{\rho})$ blir kontinuitetsekvationen att $\partial_\mu J^\mu = 0$.

För att visa att J^μ transformeras som en 4-vektor:

1) ∂_μ transformeras som en (kovariant) 4-vektor.

2) Kontinuitetsekvationen: $\partial_\mu J^\mu = 0$.

3) Kont.ekv. gäller i alla ref.system (d.v.s. laddningen är bevarad i alla ref.system).

$\Rightarrow$ $\partial_\mu J^\mu$ måste vara invariant (skalär)

kovariant vektor — kontravariant vektor

3-vekt. → 4-vekt.

$\mathbb{P} \rightarrow P^\mu = (\mathbb{P}, P^4 = E)$

$\mathbb{J} \rightarrow J^\mu = (\mathbb{J}, J^4 = \rho)$

$\cancel{\mathbb{E} \rightarrow E^\mu = (\mathbb{E}, ?)}$ Nej! Man kan inte allmänt föra över alla 3-vektorer till 4-vektorer.

$\left.\begin{matrix}\mathbb{E}\\ \mathbb{B}\end{matrix}\right\} \longrightarrow F_{\mu\nu}$

Med ∂_μ, $F_{\mu\nu} \doteq J_\mu$ fås Maxwells ekvationer på 4D-form som:

$$\begin{cases} \partial_\mu F^{\mu\nu} = 4\pi J^\nu & (1) \\ \underbrace{\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}}_{\text{cyklisk perm.}} = 0 & (2) \end{cases}$$

$\nu\rho$

Ta (1) med $\nu = 4$ och visa att detta blir $\nabla \cdot \mathbb{E} = 4\pi\rho$

$H.L = 4\pi J^4 = 4\pi\rho$ ok!

$\partial^\mu = (\nabla, \frac{\partial}{\partial t})$

$J_\mu = (\mathbb{J}, \rho)$

$V.L = \partial_\mu F^{\mu 4} = \partial_1 F^{14} + \partial_2 F^{24} + \partial_3 F^{34} + \partial_4 F^{44}$

kontravariant

$\partial^\mu = g^{\mu\nu} \partial_\nu$

$J_\mu = g_{\mu\nu} J^\nu$

kovariant

Index kan alltid höjas/sänkas med en metrik.

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$$\left.\begin{aligned} \partial_\mu &= \left(-\nabla, \frac{\partial}{\partial t}\right) \\ J^\mu &= (\mathbb{J}, \rho) \end{aligned}\right\} \Rightarrow \quad \partial_\mu J^\mu = \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 + \partial_4 J^4 =$$

$$= -\frac{\partial}{\partial x} J^x - \frac{\partial}{\partial y} J^y - \frac{\partial}{\partial z} J^z + \frac{\partial}{\partial t}\rho = -\nabla \cdot \mathbb{J} + \frac{\partial \rho}{\partial t} = 0$$

Vi vill testa att $\partial_\mu F^{\mu 4} = 4\pi J^4 \Rightarrow \nabla \cdot \mathbb{E} = 4\pi\rho$.
Vi måste höja index på $F_{\mu\nu} \to F^{\mu\nu}$.

$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}$ (jämför med t.ex. $\partial^\mu = g^{\mu\nu}\partial_\nu$)

$$F^{\mu\nu} = \begin{pmatrix} 0 & -B_z & B_y & E_x \\ +B_z & 0 & -B_x & E_y \\ -B_y & B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix}, \quad F_{12} = -B_x, \quad F^{12} = ?$$

$$F^{12} = g^{1\rho} g^{2\sigma} F_{\rho\sigma} = (-1)\cdot(-1)\cdot F_{12} = F_{12} = -B_x$$

$$F_{14} = -E_x, \quad F^{14} = g^{1\rho} \cdot g^{4\sigma} F_{\rho\sigma} = g^{11} \cdot g^{44} \cdot F_{14} = 1 \cdot (-1) \cdot F_{14} = -F_{14} = E_x$$

Varje gång man höjer/sänker ett rumsindex får man $\cdot(-1)$.
T.ex. $T^{\mu\nu}{}_\rho$: $T^{14}{}_3 = -T_1{}^4{}_3 = T_1{}^{43} = T_{14}{}^3$

$r=1$: $\partial_\mu F^{\mu 1} = \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} + \partial_4 F^{41} = 0 + \frac{\partial}{\partial y} B_z + \frac{\partial}{\partial z}(-B_y) +$

$+ \frac{\partial}{\partial t}(-E_x) = 4\pi J_x$ $\quad \left((\nabla \times \mathbb{B})_x - \frac{\partial}{\partial t} E_x = 4\pi J_x, \text{ yay!} \right)$

$\mu = 1, \quad \nu = 2, \quad \rho = 3$

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \frac{\partial}{\partial x}(-B_x) + \frac{\partial}{\partial y}(-B_y) + \frac{\partial}{\partial z}(-B_z) = 0$$

$$-\nabla \cdot \mathbb{B} = 0$$

$$\underline{\nabla \cdot \mathbb{B} = 0}$$

$\mu = 1, \quad \nu = 1, \quad \rho = 2$

$$\underbrace{\partial_1 F_{12} + \partial_1 F_{21}}_{F_{12} = -F_{21} \Rightarrow 0} + \partial_2 \underbrace{\cancel{F_{11}}}_{0} = 0$$

$\Rightarrow$ För att få en icke-trivial ekvation måste alla tre index vara annorlunda.

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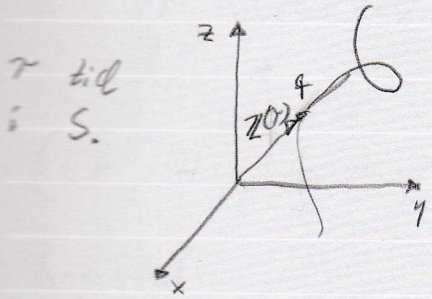
7.9 (LW-potentialer)

Fältet från en partikel i rörelse.



$A^\mu(x,t)$ Först måste vi införa begreppet "EM-potential":

$$A_\mu(x)$$

Man kan "lösa" $\begin{cases} \nabla \cdot \mathbb{B} = 0 \\ \nabla \times \mathbb{E} = -\frac{d\mathbb{B}}{dt} \end{cases}$

med hjälp av V (skalär), $\mathbb{A}$ (3-vektor)

$$\mathbb{B} = \nabla \times \mathbb{A}, \quad \mathbb{E} = -\nabla V - \frac{d}{dt}\mathbb{A}.$$

4D? Ansats: $A^\mu = (\mathbb{A}, V)$

$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$ motsvarar $\nabla \cdot \mathbb{B} = 0$ & $\nabla \times \mathbb{E} = -\frac{d\mathbb{B}}{dt}$

Kan lösas m.h.a. A^μ:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Kolla att $F_{\mu\nu} = -F_{\nu\mu}$

Vi vill visa att $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ löser $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$

$$Y = \partial_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \partial_\nu(\partial_\rho A_\mu - \partial_\mu A_\rho) + \partial_\rho(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

Observera att $\partial_\mu \partial_\nu X \equiv \partial_\nu \partial_\mu X \Rightarrow Y = 0$

Kolla också att $F_{\mu\nu} = \begin{pmatrix} 0 & -B_z & B_y & -E_z \\ \vdots & & & \vdots \end{pmatrix}$

$A_\mu = (-\mathbb{A}, V)$

$$-B_z = F_{12} \overset{?}{=} \partial_1 A_2 - \partial_2 A_1 = \frac{\partial}{\partial x}(-A_y) - \frac{\partial}{\partial y}(-A_x) = -(\nabla \times \mathbb{A})_z$$

Vi såg att $A_\mu \longrightarrow F_{\mu\nu}$

$$\overset{?}{\longleftarrow}$$

$$A_\mu \rightleftarrows F_{\mu\nu}$$

Låt A_μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Gaugetransformation

$A'_\mu = A_\mu + \underbrace{\partial_\mu \alpha}_{\text{godt. derivata av skalär funktion}}$ ger samma $F_{\mu\nu}$.

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu \alpha) - \partial_\nu(A_\mu + \partial_\mu \alpha) =$$

$$= \partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu \alpha} - \partial_\nu A_\mu - \cancel{\partial_\nu \partial_\mu \alpha} = F_{\mu\nu}$$

Det finns fler lösningar till Maxwells ekvationer i termer av $A_\mu \Rightarrow$ vi kan välja en av dem (Gaugeval).

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \partial_\mu F^{\mu\nu} = J^\nu$$

$$\Downarrow$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\nu \Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = J^\nu$$

Välj det A_μ som uppfyller $\partial_\mu A^\mu = 0$ (Lorenz-gauge)

$$\partial_\mu \partial^\mu A^\nu = J^\nu$$

$$\Box A^\nu = J^\nu$$

A^μ: fält från partikel i rörelse. (Se figur)

$z(\tau)$ talar om för oss vart partikeln befinner sig vid tiden τ (i S).

τ, $z(\tau)$ (partikel)
t, x (fält?)

(Ett sätt: lös $\Box A^\mu = 4\pi J^\mu$)

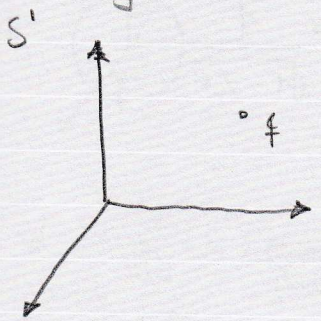
Annat sätt:

1) Vi byter till ett lämpligt referenssystem.

2) Vi skriver svaret i ett 4D Lorentz-invariant sätt.

$\Rightarrow$ Då gäller det i alla ref.system (även S) ☺

① Välj momentan vilo-ref.-ram.



Vid τ (i S) befinner sig q i vila.

Antagande: fältet i (x,t) bestäms av partikeln när $\tau = t - |z(\tau) - x|/c$.

Inför $R^\mu = (x - z(\tau), t - \tau) = x^\mu - z^\mu$. uppfyller $R_\mu R^\mu = 0$

$R_\mu R^\mu = -|x - z(\tau)|^2 + (t-\tau)^2 = 0$ ljuslik

Om q är i vila:

$$A = 0, \quad V = \frac{q}{|z(\tau) - x|} \quad \text{(Coulombpot.)}$$

Om q är i rörelse har vi två 4-vektorer att "leka" med.

$$\begin{cases} R^\mu = (x - z, t - \tau) \\ U^\mu = \gamma(\dot{z}, 1), \quad \text{där } \dot{z} = \frac{dz}{d\tau} \quad (\text{i vila: } U^\mu = (0,0,0,1)) \end{cases}$$

När $U^\mu = (0,0,0,1)$ vet vi att $A^\mu = (0,0,0,v)$

$$A^\mu = k U^\mu \Rightarrow v = k \cdot 1 = k$$
$$= \frac{q}{|z - x|} U^\mu$$

Hur skriver man $|x - z|$ på ett Lorentzinvariant sätt? $(t > \tau)$

Betrakta storheten (i partiklens vilosystem) $U^\mu R_\mu = R_4 = R^4 = t - \tau = |x - z|$

$$\Rightarrow A^\mu = k U^\mu = \frac{q}{|z - x|} U^\mu = \frac{q U^\mu}{U^\mu R_\mu}$$

$$A(x,t) = \left. \frac{q \gamma \dot{z}(\tau)}{|x - z(\tau)|} \right|_{\tau = t - |x - z(\tau)|}, \quad \gamma = \frac{1}{\sqrt{1 - \dot{z}^2}}$$

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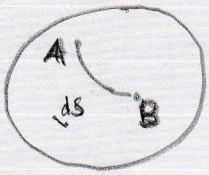
Kort om allmän relativitetsteori

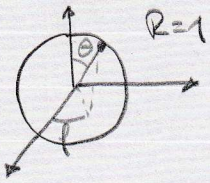
$$m \frac{d^2 x^\mu}{d\tau^2} = q F^{\mu\nu} \frac{dx_\nu}{d\tau}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix} \longrightarrow \text{allmän } 4\times 4: \begin{pmatrix} g_{11}(x) & \cdots \\ \vdots & \ddots \end{pmatrix}$$

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

Geodesiska ekvation Kortaste vägen mellan två punkter A & B.



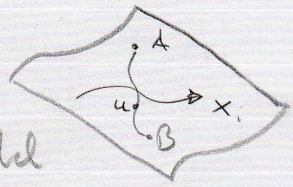


$$ds^2 = d\theta^2 + \sin^2\theta\, d\theta$$

ds – avståndet mellan (θ, φ) & $(\theta + d\theta, \varphi + d\varphi)$

$$ds^2 = g_{ij}(x) \frac{dx^i}{du}\frac{dx^j}{du}\, du^2$$

Mångfald



$$g_{\theta\theta} = 1$$

$$g_{\varphi\varphi} = \sin^2\theta$$

Vi vill hitta x^i: $u \in [u_A, u_B] \longrightarrow M$.

Minimera avståndet mellan A & B,

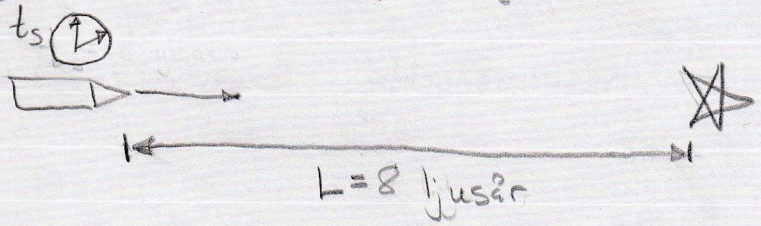
Minimera $$\int_A^B ds = \int_{u_A=0}^{u_B=1} \sqrt{g_{ij}(x(u)) \frac{dx^i}{du}\frac{dx^j}{du}}\, du$$

ty oberoende av val av u

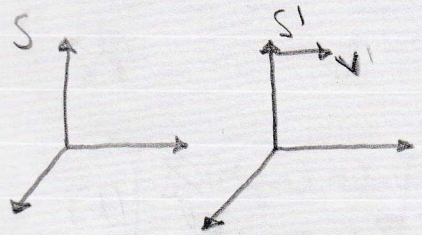
$$x^i(u) \longrightarrow x^i(u) + \delta x^i(u) \quad , \quad \delta x^i(u_A, u_B) = 0$$

Tentauppgifter

2.9 A spaceship travels to a star 8 light-years away, in a time its crew conciders to be 8 years. What is the ship's speed?



Vid ankomst: $t_s = 8$ år.



Skeppet upplever att sträckan är $L' = \frac{L}{\gamma}$ på grund av längdkontraktion.

$s = vt$

$\Rightarrow L' = v \cdot t_s$

$$\because \quad v = \frac{L'}{t_s} = \frac{L/\gamma}{t_s} = \frac{L}{\gamma t_s} = \underbrace{\frac{L}{t_s}}_{A} \cdot \sqrt{1-v^2} \qquad (c=1)$$

$$\frac{v}{\sqrt{1-v^2}} = A$$

$$\frac{v^2}{1-v^2} = A^2$$

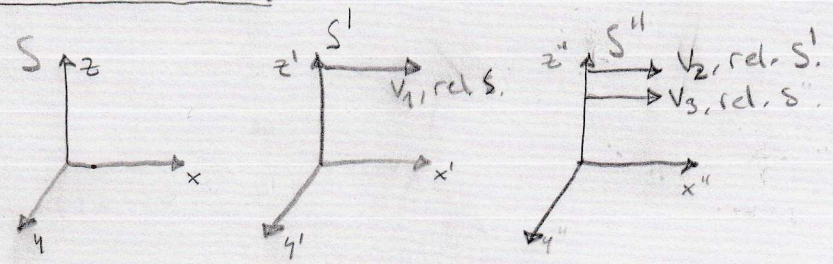
$$v^2 = A^2 - A^2 v^2$$

$$v^2(1+A^2) = A^2 \quad \Rightarrow \quad v = \frac{A}{\sqrt{1+A^2}} = \frac{L/t_s}{\sqrt{1+A^2}} = \left\{ L/t_s = 1 \text{ ljusår} \right\} =$$

återinför c

$$= \frac{1}{\sqrt{2}} c$$

2.16 Three initial frames S, S', S'' in standard config. S' has velocity v_1 relative to S, S'' has velocity v_2 relative to S'. Prove that S'' has velocity $(v_1+v_2)/(1+v_1v_2/c^2)$ relative to S.



$$\boxed{\text{Rapiditet: } \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}} \quad \boxed{\begin{aligned} \gamma &= \cosh\phi \\ \gamma v &= \sinh\phi \end{aligned}} \ (*)$$

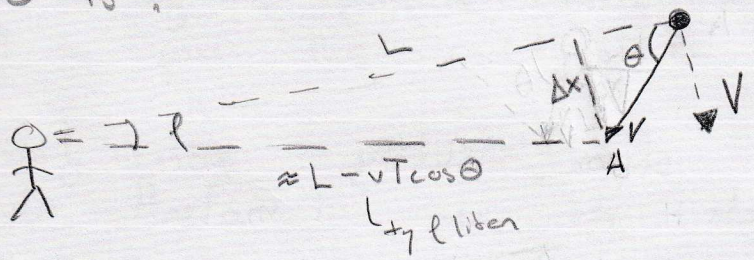
$$\left(\begin{aligned} &\text{Här: } \begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh\phi'' & \sinh\phi'' \\ \sinh\phi'' & \cosh\phi'' \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}, \quad \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi' & \sinh\phi' \\ \sinh\phi' & \cosh\phi' \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\ &\text{, söker } \phi''' \text{ s.a.: } \begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh\phi''' & \sinh\phi''' \\ \sinh\phi''' & \cosh\phi''' \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \end{aligned} \right)$$

Rapiditeten är additiv $\Rightarrow$ $\phi''' = \phi' + \phi''$. (1)

$$(*) \Rightarrow v_3 = \frac{\sinh\phi'''}{\gamma} = \frac{\sinh\phi'''}{\cosh\phi'''} \overset{(1)}{=} \frac{\sinh(\phi'+\phi'')}{\cosh(\phi'+\phi'')} = \tanh(\phi'+\phi'') \overset{\text{tips}}{=}$$

$$= \frac{\tanh\phi' + \tanh\phi''}{1 + \tanh\phi' \cdot \tanh\phi''} \overset{(*)}{=} \frac{v_1 + v_2}{1 + v_1 v_2} = \{c=1\} = \underline{\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}}$$

2.21 A stellar object at some known large distance ejects a "jet" at speed v towards an observer obliquely, making an angle θ with the line of sight. To the observer the jet appears to be ejected sideways at speed V. Prove that $V = \dfrac{c \cdot \sin\theta}{c/v - \cos\theta}$ and show that this can exceed c, for example, when $\theta = 45°$.



$L = ct_1$ (t_1 är tiden innan observatören ser avfyrningen).

Efter tiden T har projektilen nått punkt A. Observatören ser detta efter tiden $t_2 = T + \underbrace{\dfrac{L - vT\cos\theta}{c}}_{\text{typ liten}}$.

$$V = \frac{\Delta x}{\Delta t} = \frac{vT\sin\theta}{T - \frac{vT\cos\theta}{c}} = \frac{v\sin\theta}{1 - \frac{v\cos\theta}{c}} = \frac{c\cdot\sin\theta}{c/v - \cos\theta}$$

$$\theta = 45° \Rightarrow V = \frac{(1/\sqrt{2})\cdot c}{c/v - 1/\sqrt{2}} = \frac{(1/\sqrt{2})\cdot v}{1 - \frac{v}{\sqrt{2}}\cdot\frac{1}{c}}$$

$$\frac{(1/\sqrt{2})\cdot v}{1 - \frac{v}{\sqrt{2}}/c} \geq c$$

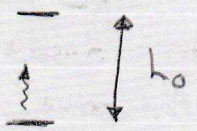
$$\frac{v}{\sqrt{2}} \geq c - \frac{v}{\sqrt{2}}$$

$$v\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \geq c$$

$$v \geq \frac{c}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} \overset{c=1}{=} \frac{c}{2/\sqrt{2}} = \frac{1}{\sqrt{2}}c$$

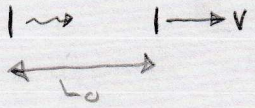
A "light-clock" moving, considering only length contraction and invariance of c, prove that such a clock is slowed by the γ-factor in the case of: i. longitudinal motion
ii. transverse motion

3.9



Ingen rörelse av speglarna $\Rightarrow T_0 = 2 \cdot \frac{L_0}{c}$

i.

① 

Längdkontraktion $\Rightarrow$ Upplevd längd blir $L_i = \frac{L_0}{\gamma}$.

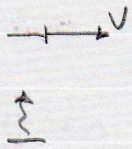
②

Spegeln rör sig i samma riktning som c-fotonen $\Rightarrow$
$\Rightarrow v_i = c - v$. Spegeln rör sig i motsatt riktning $\Rightarrow$
$\Rightarrow v_i = c + v$. (v_i är hastigheten för fotonen relativt spegeln)

$$\Rightarrow T_i = \frac{L_i}{v_i} = \frac{L_0/\gamma}{c+v} + \frac{L_0/\gamma}{c-v} = \frac{L_0}{\gamma}\left(\frac{c-v}{c^2-v^2} + \frac{c+v}{c^2-v^2}\right) = 2 \cdot \frac{L_0}{\gamma} \cdot \frac{c}{c^2-v^2} =$$

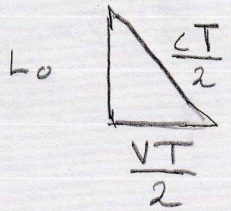
$$= \frac{2 \cdot L_0}{c \cdot \gamma} \cdot \underbrace{\left(\frac{1}{1 - v^2/c^2}\right)}_{\gamma^2} = 2 \cdot \frac{L_0}{c} \cdot \gamma = \underline{T_0 \gamma}$$

ii.

① 

②

Här bildar fotonens rörelse en triangel:



(En "studs" $\Rightarrow$ faktor $\frac{1}{2}$)

Pythagoras sats $\Rightarrow L_0^2 + \left(\frac{vT}{2}\right)^2 = \left(\frac{cT}{2}\right)^2$

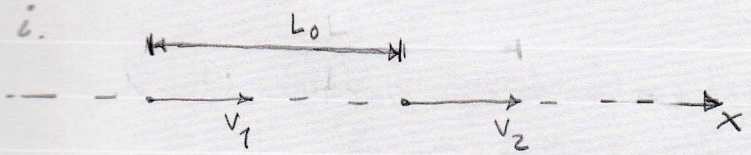
$$L_0^2 = \frac{T^2}{4}(c^2 - v^2) \Rightarrow T = \frac{2L_0}{\sqrt{c^2-v^2}} = \frac{2L_0}{c} \cdot \frac{1}{\sqrt{1-v^2/c^2}} = \frac{2L_0}{c} \cdot \gamma =$$

$$= \underline{T_0 \cdot \gamma}$$

3.12 i. Two particles move along the x-axis of S at velocities 0,8 c and 0,9 c, respectivily, the faster one momentarily 1 m behind the slower one. How many seconds elapse before collision?

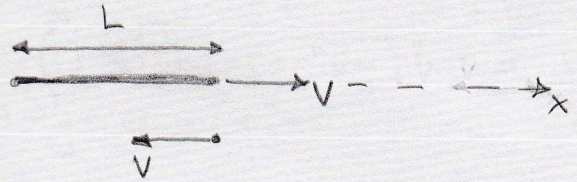
ii. A rod of proper length 10 cm moves longitudinally along the x-axis of S at speed $\frac{1}{2}$ c. How long (in S) does it take a particle, moving oppositely at the same speed, to pass the rod?

i.



$$v_{tot} = v_1 - v_2 \quad \Rightarrow \quad t = \frac{L_0}{v_1 - v_2}$$

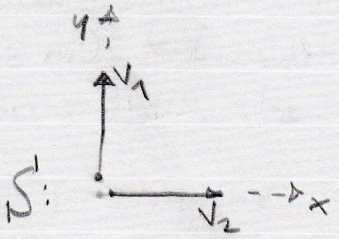
ii.



$v_{tot} = 2v$. Längdkontraktion ger att L i S upplevs ha längden $L_S = \frac{L}{\gamma}$.

$$\Rightarrow \quad T = \frac{L_S}{v_{tot}} = \frac{L/\gamma}{2v}$$

3.13 In a given inertial frame, two particles are shot out simultaneously from a given point, with equal speeds v, in orthogonal directions. What is the <u>speed</u> of each particle relative to the other?



$$\mathbf{v}_1 = (\underbrace{0}_{v_{1x}}, \underbrace{v}_{v_{1y}}) \; ; \; \mathbf{v}_2 = (v, 0)$$

Inför ett nytt referenssystem där partikel 2 alltid är i vila. Kalla detta system S'.
Vi transformerar $\mathbf{v}_1$ till detta system:

$$v_{1x}' = \frac{v_{1x} - v_{2x}}{1 - \frac{v_{1x}v_{2x}}{c^2}} = \frac{0 - v}{1 - 0} = -v$$

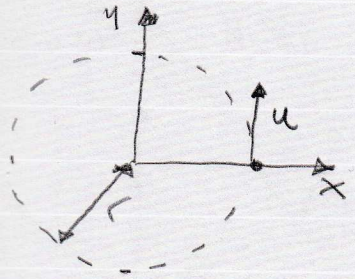
$$v_{1y}' = \frac{v_{1y} - v_{2y}}{\gamma\left(1 - \frac{v_{1y}v_{2y}}{c^2}\right)} = \frac{v}{\gamma - 0} = \frac{v}{\gamma} = v\sqrt{1 - v^2} \qquad (c = 1)$$

För att få fram partikel 1s hastighet relativt partikel 2 beräknar vi beloppet av $\mathbf{v}_1'$ ($\mathbf{v}_1$ i S').

$$|\mathbf{v}_1'| = \sqrt{v_{1x}'^2 + v_{1y}'^2} = \sqrt{v^2 + v^2(1 - v^2)} = \sqrt{2v^2 - v^4} \quad \left(= \sqrt{2v^2 - \frac{v^4}{c^2}} \right)$$

3.20 A particle moves at uniform speed around a circular path of radius r. Find its proper acceleration.

S



$$\begin{cases} x = r\cdot\cos\omega t, & u_x = -r\omega\sin\omega t, \quad a_x = -r\omega^2\cos\omega t \\ y = r\cdot\sin\omega t, & u_y = r\omega\cos\omega t, \quad a_y = -r\omega^2\sin\omega t \end{cases}$$

För att hitta partikelns egenacceleration studerar vi en "snäll" punkt: $x=r$, $y=0$, $t=0$. I den punkten har vi:

$$\mathbf{u} = (0,\ r\omega) = (0, u)$$
$$\mathbf{a} = (-r\omega^2, 0) = (-u^2/r, 0)$$

$$\alpha_x = a'_x = \frac{a_x}{\gamma^2(u)\cdot(1-u^2)^2} = -\frac{u^2}{r}\cdot\frac{(1-u^2)}{(1-u^2)^2} = \frac{-u^2}{r(1-u^2)}$$

$$\alpha_y = a'_y = 0 \quad (a_y = 0)$$

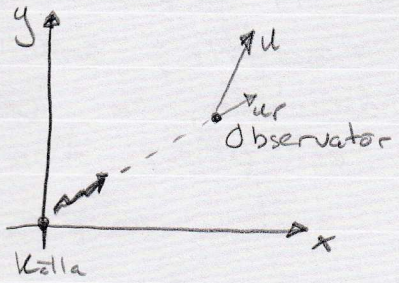
$$\therefore\ \alpha = \alpha_x = \frac{-u^2}{r(1-u^2)} = -\frac{\gamma^2 u^2}{r}$$

(riktad inåt mot cirkelns mitt)

(Varför bara γ^2 = inte γ^3 ?)

3.25 Given that g, the acceleration of gravity at the earth's surface, is $\sim 9{,}8\ ms^{-2}$, and that a year has $\sim 3{,}2 \cdot 10^7 s$, verify that, in units of

4.2 A source of monochromatic light of proper frequency ν_0 is fixed in a frame S. An observer travels through S with instantaneous velocity u and radial velocity u_r relative to the source. What frequency does the observer see?



Tiden mellan två pulser: $\Delta t_0 = 1/\nu_0$.

Tiden det tar för observatören att se en puls blir $\Delta t_S = \Delta t_0 + u_r \cdot \Delta t_S$. $(c=1)$

$$\therefore \Delta t_S = \frac{\Delta t_0}{1-u_r}$$

För att omvandla detta till observatörens referenssystem behöver vi ta hänsyn till tidsdilation. Tiden som observatören upplever mellan pulserna är

$$\Delta t = \frac{\Delta t_S}{\gamma(u)} = \frac{\Delta t_0}{(1-u_r)\gamma(u)} \Rightarrow$$

$$\Rightarrow \nu = \frac{1}{\Delta t} = \frac{(1-u_r)\gamma(u)}{\Delta t_0} = \nu_0 (1-u_r) \cdot \frac{1}{\sqrt{1-u^2}} \qquad (c=1)$$

4.10 Writing $\alpha' = \alpha + d\alpha$, and assuming $v \ll c$, derive the approximate aberration formula $d\alpha = -(v/c)\cdot \sin\alpha$. From it, and given that the earth's orbital speed is ~30 km/s, prove that each fixed star's apparent position relative to the ecliptic (that is, the earth's orbit) changes in the course of one year by ~41" (seconds of arc) in the direction parallel to the ecliptic, and by ~(41")$\sin\alpha$ in the orthogonal direction, α being the star's elevation relative to the ecliptic, that is, its "ecliptic latitude".

Vi utgår från den exakta formeln:

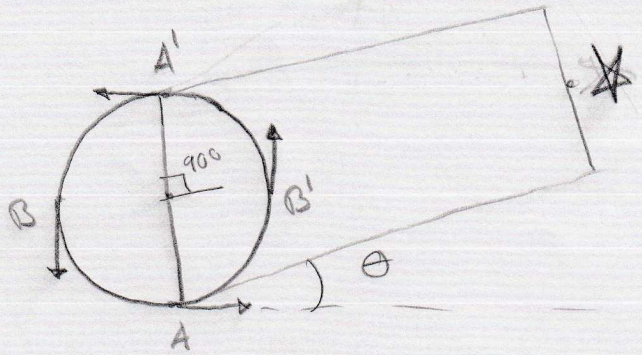
$$\cos\alpha' = \frac{\cos\alpha + v/c}{1 + \frac{v}{c}\cdot\cos\alpha} \quad \text{för} \quad v \ll c$$

$$\cos\alpha' \approx \cos\alpha + \frac{v}{c}(1-\cos^2\alpha) = \cos\alpha + \frac{v}{c}\sin\alpha.$$

Vi har $\alpha' = \alpha + d\alpha$,

$$\cos\alpha' = \cos(\alpha + d\alpha) = \cos\alpha\cdot\cos(d\alpha) - \sin\alpha\cdot\sin(d\alpha) = \{\text{Taylor}\} =$$
$$= \cos\alpha\cdot 1 - \sin\alpha\cdot d\alpha$$

$$\therefore \quad \cos\alpha - \sin\alpha\cdot d\alpha = \cos\alpha + \frac{v}{c}\sin\alpha \Rightarrow \underline{d\alpha = -\frac{v}{c}\cdot\sin\alpha}$$



$$\frac{v}{c} = \frac{30\,000 \text{ m/s}}{300\,000\,000 \text{ m/s}} \approx 20{,}6''$$

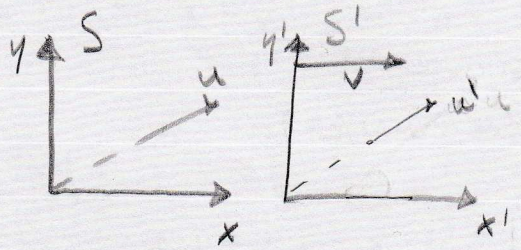
$A \leftrightarrow A'$: $\Delta\alpha = \pm\frac{v}{c}\cdot\sin\theta = \pm 20{,}6''\cdot\sin\theta$

$B \leftrightarrow B'$: $\Delta\alpha = \pm\frac{v}{c}\cdot\sin 90° = \pm\frac{v}{c}$

4.13 A particle moves uniformly in a frame S with velocity $\mathbf{u}$ making an angle α with the positive x-axis. If α' is the corresponding angle in the usual second frame S', prove the 'particle aberration formula'

$$\tan\alpha' = \frac{\sin\alpha}{\gamma(v)(\cos\alpha - v/u)}$$

and compare this with (4.7) and (4.8).



$$\begin{cases} u = u_x + u_y \\ u' = u'_x + u'_y \end{cases}$$

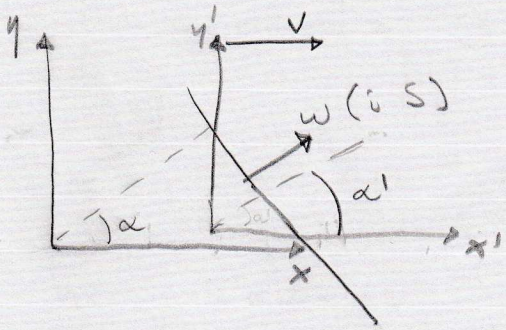
$$\left.\begin{array}{l} \begin{cases} u_x = u\cdot\cos\alpha \,, & u_y = u\cdot\sin\alpha \\ u'_x = u'\cdot\cos\alpha' \,, & u'_y = u'\cdot\sin\alpha' \end{cases} \\ u'_x = \dfrac{u_x - v}{1 - u_x v} \,, \quad u'_y = \dfrac{u_y}{\gamma(v)(1 - u_x v)} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{cases} u'\cdot\cos\alpha' = \dfrac{u\cdot\cos\alpha - v}{1 - uv\cdot\cos\alpha} & (1) \\ u'\cdot\sin\alpha' = \dfrac{u\cdot\sin\alpha}{\gamma(v)\cdot(1 - uv\cdot\cos\alpha)} & (2) \end{cases}$$

$$(2)/(1) \Rightarrow \tan\alpha' = \frac{\sin\alpha}{\gamma(v)\cdot(\cos\alpha - v/u)}$$

4.14 In a frame S, consider the equation $x \cdot \cos\alpha + y \cdot \sin\alpha = wt$. For fixed α it represents a plane propagating in the direction of its normal with speed w, that direction being parallel to the xy-plane and making an angle α with the positive x-axis. We can evidently regard this plane as a wave front. Now transform x, y and t directly to the usual frame S'. From the resulting equation, deduce the following aberration formula for the wave normal:

$$\tan\alpha' = \frac{\sin\alpha}{\gamma(v)\,(\cos\alpha - wv/c^2)}$$



| S: $x \cdot \cos\alpha + y \cdot \sin\alpha = wt$ (1)

Lorentztransform av (1) $\Rightarrow$ $\gamma(v) \cdot (x' + vt') \cdot \cos\alpha + y' \cdot \sin\alpha =$
$= w \cdot \gamma(v) \cdot (t' + vx')$

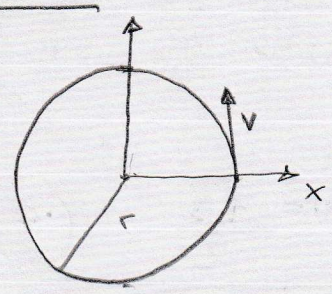
$$\gamma(v) \cdot \cos\alpha\,(x' + vt') + y' \cdot \sin\alpha = \gamma(v) \cdot w\,(t' + vx')$$

$$x' \cdot \gamma(v)\,(\cos\alpha - vw) + y' \cdot \sin\alpha = t' \cdot \gamma(v)\,(w - v \cdot \cos\alpha)$$

$\because$ $\cos\alpha' = \gamma(v)\left(\cos\alpha - \frac{vw}{c^2}\right)$, $\sin\alpha' = \sin\alpha$ $\Rightarrow$

$$\Rightarrow \tan\alpha' = \frac{\sin\alpha'}{\cos\alpha'} = \frac{\sin\alpha}{\gamma(v)\left(\cos\alpha - \frac{vw}{c^2}\right)}$$

5.9 Muons circle a storage ring of radius 10 m at a speed that makes $\gamma = 30$. Verify that their proper accelerations are $\sim 0{,}8 \cdot 10^{18}\, g$.



v konstant $\Rightarrow \mathbb{A} = \gamma^2(v) \cdot (a_c, 0)$

$$\alpha^2 = \mathbb{A}^2 = \gamma^4 a^2 = \gamma^4 \cdot \frac{v^4}{r^2} \Rightarrow \alpha = \gamma^2 \cdot \frac{v^2}{r} = \gamma^2 \cdot \frac{\gamma^2 - 1}{\gamma^2} \cdot \frac{1}{r} = \frac{\gamma^2 - 1}{r}$$

$$\therefore \alpha = \frac{\gamma^2 - 1}{r} \cdot c^2$$

5.10 A particle moves in an inertial frame (for a while after $t=0$) according to the equations

$$\begin{cases} x = \omega t \\ y = \frac{1}{2} g t^2 \\ z = 0 \end{cases} ; \quad \omega, g \text{ constant.}$$

Find its proper acceleration as a function of time.

$x = \omega t \Rightarrow v_x = \omega \Rightarrow a_x = 0$. $\quad y = \frac{1}{2} g t^2 \Rightarrow v_y = gt \Rightarrow a_y = g$.
$z = 0 \Rightarrow v_z = 0 \Rightarrow a_z = 0$.

$$\gamma = 1 / \sqrt{1 - \omega^2 - g^2 t^2}, \quad \dot{\gamma} = g^2 t / (1 - \omega^2 - g^2 t^2)^{3/2}$$

$$\mathbb{A} = \left(\frac{g^2 t \omega}{(1-\omega^2-g^2t^2)^2}, \frac{g^3 t^2}{(1-\omega^2-g^2t^2)^2} + \frac{g}{1-\omega^2-g^2t^2}, 0, \frac{g^2 t}{(1-\omega^2-g^2t^2)^2} \right) =$$

$$= \frac{1}{(1-\omega^2-g^2t^2)^2} \left(g^2 t \omega, g(1-\omega^2), 0, g^2 t \right)$$

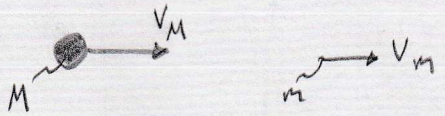
$$-\mathbb{A} \cdot \mathbb{A} = \alpha^2 \Rightarrow \alpha = \frac{g(1-\omega^2)^{1/2}}{(1-\omega^2-g^2t^2)^{3/2}}$$

6.1 How fast must a particle move before its kinetic energy equals its rest energy?

$E_{kin} = \gamma mc^2 - mc^2$, $E_{vila} = mc^2$

$$\left.\begin{array}{l} E_{kin} = E_{vila} \\ \gamma mc^2 - mc^2 = mc^2 \Rightarrow \gamma = 2 \\ \gamma = \dfrac{1}{\sqrt{1-v^2}} \end{array}\right\} \Rightarrow \frac{1}{1-v^2} = 4 \Rightarrow v = \frac{\sqrt{3}}{2} \quad (\cdot c)$$

6.2 How fast must a 1 kg cannon ball move to have the same kinetic energy as a cosmic-ray proton moving with γ-factor 10^{11}?



$E_{k,M} = \underbrace{(\gamma(v_M)-1)M}_{\approx \frac{Mv^2}{2} \text{ om } v \ll c}$, $E_{k,m} = (\gamma(v_m)-1)m$

$$E_{k,M} = E_{k,m} \Rightarrow \frac{Mv^2}{2} = (\gamma(v_m)-1)m \Rightarrow v = \sqrt{\frac{2(\gamma(v_m)-1)m}{M}} \quad (\cdot c)$$

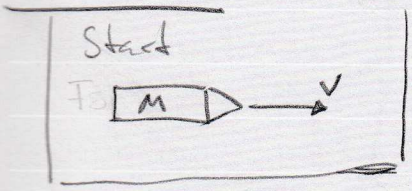
Kolla om $\ll c$!

6.3 The mass of a hydrogen atom is 1,00814 amu, that of a neutron is 1,00898 amu, and that of a helium (two hydrogen atoms and two neutrons) is 4,00388 amu. Find the binding energy as a fraction of the total energy of a helium atom.

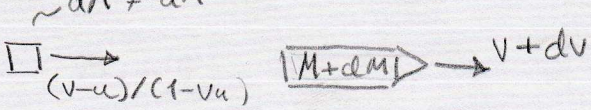
$$\frac{2M_H + 2M_n - M_{He}}{M_{He}} = 0{,}0076 = 0{,}76\ \%$$

6.6 A rocket propels itself rectilinearly by giving portions of its mass a constant (backward) velocity U relative to its instantaneous rest-frame. Prove that the ratio of the initial to the final rest-mass of the rocket is given by

$$\frac{M_i}{M_f} = \left(\frac{c+V}{c-V}\right)^{c/2U}$$



I jordens ref.system:



Rörelsemängd bevarad: $Mv\gamma(v) = (M+dM)(v+dv)\gamma(v+dv) +$

$$+ dM'((v-u)/(1-uv))\gamma((v-u)/(1-uv)) \qquad (1)$$

Dito för energin: $M\gamma(v) = (M+dM)\gamma(v+dv) + dM'\gamma\left((v-u)/(1-vu)\right)$ (2)

$$\gamma(v+dv) = \frac{1}{\sqrt{1-(v+dv)^2}} \approx \gamma(v) + \gamma'(v)dv = \gamma(v) + \gamma^3(v)dv$$

$$\gamma\left(\frac{v-u}{1-vu}\right) = \frac{1}{\sqrt{1-\left(\frac{v-u}{1-uv}\right)^2}} = (1-uv)\gamma(u)\gamma(v)$$

$$\Rightarrow \quad (1): \quad v\,dM + M\gamma^2(v)\,dv + (v-u)\gamma(u)M' = 0$$

$$(2): \quad dM + Mv\gamma^2(v)\,dv + (1-uv)\gamma(u)\,dM' = 0$$

$$\because \quad (1-uv)\cdot(1) - (v-u)\cdot(2) = u\,dM + \gamma^2(v)M\,dv = 0$$

$$u\frac{dM}{M} = -\gamma^2(v)\,dv$$

$$u\int_{M_i}^{M_f}\frac{dM}{M} = -\int_{v_i}^{v_f}\gamma^2(v)\,dv$$

$$v_i = 0 \quad \Rightarrow \quad \underline{\frac{M_i}{M_f} = \left(\frac{c+v}{c-v}\right)^{c/2u}}$$

6.7 Consider a head-on elastic collision of a "bullet" of rest-mass M with a stationary "target" of rest-mass m. Prove that the post-collision γ-factor of the bullet cannot exceed $(m^2+M^2)/2mM$. This is radically different in non-relativistic cases, where the ratio of the pre- and post-collision velocities is

$$\frac{u}{u'} = \frac{M+m}{M-m}, \quad \text{prove this.}$$

Icke-relativistiskt:

Före: $M \xrightarrow{v}$ $\cdot\, m$ Efter: $M \xrightarrow{v'}$ $m \xrightarrow{v'}$

Rörelsemängd bevarad: $Mv = Mv' + mv'$
Energi bevarad: $\frac{Mv^2}{2} = \frac{Mv'^2}{2} + \frac{mv'^2}{2}$ $\Big\} \Rightarrow v' = \frac{M-m}{M+m} v$

Relativistiskt:

I lab-systemet:

Före: $P_L^\mu = (E_L = \sqrt{p_{Lab}^2 + M^2},\ p_L)$; $Q_L^\mu = (m, 0)$

$M \xrightarrow{p_L}$ $\cdot\, m$

Efter: $P_L'^\mu = (\sqrt{p_L'^2 + M^2},\ p_L')$; $Q_L'^\mu = (\sqrt{q_L'^2 + m^2},\ q'_{Lab})$

$M \xrightarrow{p_L'}$ $m \xrightarrow{q_L'}$

I masscentrum-systemet:

Före: $P_{cm}^\mu = (\sqrt{p_{cm}^2 + M^2},\ p_{cm})$; $Q_{cm}^\mu = (\sqrt{p_{cm}^2 + m^2},\ -p_{cm})$

$M \xrightarrow{p_{cm}}$ $\xleftarrow{-p_{cm}}\ m$

Efter: $P_{cm}'^\mu = (\sqrt{p_{cm}^2 + M^2},\ -p_{cm})$; $Q_{cm}'^\mu = (\sqrt{p_{cm}^2 + m^2},\ p_{cm})$

$\xleftarrow{-p_{cm}}\ M$ $m \xrightarrow{p_{cm}}$

$(P'^\mu - Q^\mu)^2$ invariant under Lorentz-transformation.

Lab: $(P'^\mu - Q^\mu)^2 = P'^{\mu 2} + Q^{\mu 2} - 2P'^\mu Q^\mu = M^2 + m^2 - 2\sqrt{p_L'^2 + M^2} \cdot m = M^2 + m^2 - 2mM\gamma(v')$

Cm: $(P'^\mu - Q^\mu)^2 = (\sqrt{p_{cm}^2 + M^2} - \sqrt{p_{cm}^2 + m^2})^2 - (p_{cm} - p_{cm})^2 \geq 0$

$\Rightarrow M^2 + m^2 - 2mM\gamma(v') \geq 0 \Rightarrow \gamma(v') \geq \frac{M^2+m^2}{2mM}$

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6.9 Generalise $k = m\,(2m + m^2/2M)^{-1} = 2/(4 + (m/M)$ for the efficiency of an elastic bombardment, to the case where the target has a different rest-mass N from that of the bullet, M. Then note that for sufficiently large N, k can be arbitrarily close to unity.

Förloppet är oelastiskt.

Lab (före): $M \xrightarrow{p_L} \cdot N$ $\quad P_L^\mu = (E_L, p_L),\ Q_L^\mu = (N, 0)$

Masscentrum (före): $M \xrightarrow{p_{cm}} \xleftarrow{-p_{cm}} N$ $\quad P_{cm}^\mu = (\sqrt{p_{cm}^2 + M^2}, p_{cm}),\ Q_{cm}^\mu = (\sqrt{p_{cm}^2 + N^2}, -p_{cm})$

$$(P_{Lab}^\mu + Q_L^\mu)^2 = (P_{cm}^\mu + Q_{cm}^\mu)^2$$

Masscentrum (efter): m, $M \cdot\cdot N$

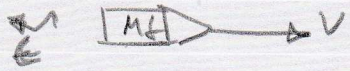
$$\left.\begin{aligned} &(\underbrace{E_L}_{\sqrt{M^2+p_L^2}} + N)^2 - \underbrace{p_L^2}_{= N^2 + M^2 + 2NE_L} = \left(\sqrt{M^2 + p_{cm}^2} + \sqrt{N^2 + p_{cm}^2}\right)^2 \\ &\text{Energibevaring i cm: } \sqrt{M^2 + p_{cm}^2} + \sqrt{N^2 + p_{cm}^2} = M + N + m \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow E_L = \frac{(M+N+m)^2 - N^2 - M^2}{2N} \Rightarrow k^{-1} = \frac{E_L - M}{m} = 1 + \frac{2M + m}{2N}$$

6.13 A rocket propels itself rectilinearly by emitting radiation in the direction opposite to its motion. If v is its final velocity relative to the final rest-frame, prove ab initio that the ratio of the initial to the final rest-mass of the rocket is given by

$$\frac{M_i}{M_f} = \left(\frac{c+v}{c-v}\right)^{1/2}$$

Före:

Efter: 

$\epsilon = h\nu$ för varje foton.

Energikonservering: $M_i = M_f \gamma(v) + n\epsilon$

Rörelsemängdk bevarad: $0 = M_f v \gamma(v) - n\epsilon$

$M_f v \gamma(v) = n\epsilon$

$\Rightarrow$

$$\Rightarrow M_i = M_f (1+v)\gamma(v) = M_f \frac{(1+v)}{\sqrt{1-v^2}} = M_f \sqrt{\frac{1+v}{1-v}}$$

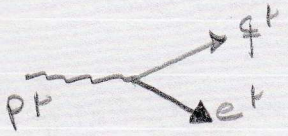
$$\therefore \frac{M_i}{M_f} = \sqrt{\frac{1+v}{1-v}} = \left(\frac{1+v}{1-v}\right)^{1/2}$$

6.14 In an inertial frame S, two photons of frequencies ν_1 and ν_2 travel in the positive and negative x-direction respectively. Find the velocity of the ZM-frame of these photons.

$P_1^\mu = (h\nu_1, h\nu_1)$ $\quad$ $P_2^\mu = (h\nu_2, -h\nu_2)$

$$V_{ZM} = \frac{P_1^1 + P_2^1}{P_1^0 + P_2^0} = \frac{V_1 - V_2}{V_1 + V_2}$$

6.16 Show that a photon cannot spontaneously disintegrate into an electron-positron pair. But in the presence of a stationary nucleus (acting as a kind of catalyst) it can. If the rest-mass of the nucleus is N, and that of the electron and positron is m, what is the threshold frequency of the photon? Verify that for large N the efficiency is ~100 %, so that the nucleus then comes close to being a pure catalyst.



4-rörelsemängd bevarad: $p^\mu = q^\mu + e^\mu$
För foton: $p^2 = 0$ $\Big\} \Rightarrow$

$\Rightarrow \quad p^2 = (q+e)^2 = q^2 + e^2 + 2qe > 0 \quad \therefore$ Motsägelse!

Däremot fungerar fallet:

Före: $\underset{\nu}{\rightsquigarrow}$ $\overset{\bullet}{N}$ | Lab: $P^\mu_{Tot} = P^\mu_\gamma + P^\mu_N = (\nu + m_N, \nu)$

Efter: $\overset{\bullet}{N}$ $\overset{\bullet e^+}{\bullet e^-}$ | CM: $P^\mu_{Tot} = (m_N + 2m, 0)$

P^μ_{tot} bevarad = Lorentzinvariant $\Rightarrow (\nu + m_N)^2 - \nu^2 = (m_N + 2m)^2$

$\nu^2 + 2\nu m_N + m_N^2 - \nu^2 = (m_N + 2m)^2 \Rightarrow \nu = \dfrac{(m_N + 2m)^2 - m_N^2}{2m_N} = \dfrac{4mm_N + 4m}{2m_N} \approx 2m$

Effektivitet: $\dfrac{2m}{\nu} = \dfrac{m_N}{m_N + m} \approx 100\,\%$ (ty $m \ll m_N$)

6.17 If one neutron and one pion are to emerge from the collision of a photon with a stationary proton, find the threshold frequency of the photon in terms of the rest-mass m of a proton or neutron (here assumed equal) and that, m, of a pion.

Fore: $\xrightarrow{\nu}$ $\cdot\, m_N$, | lab: $p^\mu_{tot} = (h\nu + m_N, h\nu)$

Efter: $\cdot\, m_N$ $\cdot\, m_\pi$, | cm: $p'^\mu_{tot} = (m_N + m_\pi, 0)$

$$p^{\mu 2}_{tot} = p'^{\mu 2}_{tot} \Rightarrow (h\nu + m_N)^2 - (h\nu)^2 = (m_N + m_\pi)^2$$

$$\cancel{(h\nu)^2} + 2(h\nu)\cdot m_N + \cancel{m_N^2} - \cancel{(h\nu)^2} = \cancel{m_N^2} + 2 m_N m_\pi + m_\pi^2$$

$$\Rightarrow h\nu = \frac{m_\pi^2 + 2 m_N m_\pi}{2 m_N}$$

6.18 A particle of rest-mass m decays from rest into a particle of rest-mass m' and a photon. Find the separate energies of these end products.

Före: $\overset{\bullet}{m}$ Efter: $\overset{\bullet}{m'}$ $\xrightarrow{\quad}$ ν

Här sammanfaller lab- & cm-systemen, ty endast en partikel rör sig.

Före: $p^\mu = (m, 0)$, Efter: $p'^\mu = (E' + \nu, p' + \nu)$ $E^2 = p^2 + m^2$

$$p^\mu = p'^\mu \Rightarrow \begin{cases} p' = -\nu \\ m = \sqrt{p'^2 + m'^2} + \nu \end{cases}$$

$$\therefore \quad m = \sqrt{\nu^2 + m'^2} + \nu$$

$$(m - \nu)^2 = \nu^2 + m'^2$$

$$m^2 - 2m\nu + \nu^2 = \nu^2 + m'^2 \Rightarrow \nu = \frac{m^2 - m'^2}{2m} \quad \text{(fotonen)}$$

Partikeln: $E' = \sqrt{p'^2 + m'^2} = \sqrt{\left(\frac{m'^2 - m^2}{2m}\right)^2 + m'^2} = \sqrt{\frac{m'^4 - 2m'^2m^2 + m^4}{4m^2} + m'^2} =$

$$= \sqrt{\frac{m'^4 - 2m'^2m^2 + m^4 + 4m'^2m^2}{4m^2}} = \sqrt{\frac{m'^4 + 2m'^2m^2 + m^4}{4m^2}} = \sqrt{\left(\frac{(m'^2 + m^2)}{2m}\right)^2} =$$

$$= \frac{m^2 + m'^2}{2m}$$

6.19 A fast electron having rest-mass m and velocity u decelerates in a collision with a heavy stationary nucleus of rest-mass M and emits a bremsstrahlung photon of frequency ν. Prove that the maximum energy of the photon is given by

$$h\nu = \frac{c^2 m M(\gamma(u)-1)}{M + m\gamma(u)(1-u/c)}$$

and occurs when the photon is emitted in the forward direction and the electron and the nucleus travels 'as a lump' after the collision.

Före: $m \xrightarrow{P^\mu}$ $\quad M, Q^\mu \longrightarrow x$

Efter: $m \to P'^\mu$, $M \to Q'^\mu$, $\rightsquigarrow \nu, N^\mu$ $\qquad Q^\mu + P^\mu = Q'^\mu + N^\mu + P'^\mu$ (1) oavsett system.

I lab-systemet:
$$\begin{cases} P^\mu = m\gamma(u)(1,u,0,0) \\ Q^\mu = (M,0) \\ N^\mu = h\nu(1,\hat{n}) \end{cases}$$

(1) gäller. Betrakta $(Q^\mu + P^\mu - N^\mu)^2 = (Q'^\mu + P'^\mu)^2$

$$Q^{\mu 2} + (P^\mu - N^\mu)^2 + 2Q^\mu(P^\mu - N^\mu) = M^2 + m^2 + 0 - 2Mh\nu + 2Mm\gamma(u) + 2m\gamma(u)h\nu(1-un_x) = M^2 + m^2 + 2Q'_\mu P'_\mu$$

$$-Mh\nu + Mm\gamma(u) - m\gamma(u)h\nu(1-un_x) = \underbrace{Q'_\mu P'_\mu}_{\text{Invariant!}}$$

I kärnans referenssystem är

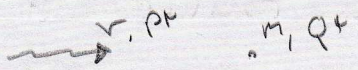
$$Q'_\mu P'_\mu = mM\gamma(\underbrace{v_{rel}}_{\text{mellan } m \,\&\, M})$$

$$\Rightarrow \quad h\nu = \frac{mM(\gamma(u) - \gamma(v_{rel}))}{M + m\gamma(u)(1-un_x)}, \quad \text{störst för } n_x = 1,\ v_{rel} = 0.$$

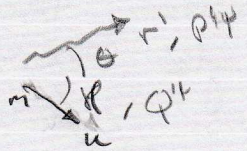
6.22 For the 'Compton collision', prove the relation:

$$\tan\phi = \frac{v' \cdot \sin\theta}{v - v' \cdot \cos\theta}$$

Före:



Efter:



$p^\mu = h\nu\,(1,1,0,0)$, $Q^\mu = (m,0)$

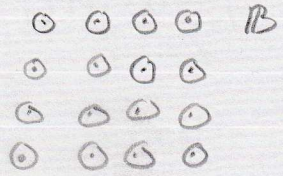
$p'^\mu = h\nu'\,(1,\cos\theta,\sin\theta,0)$, $Q'^\mu = m\gamma(u)\,(1, u\cos\phi, -u\cdot\sin\phi, 0)$

$p^\mu + Q^\mu = p'^\mu + Q'^\mu \quad \Rightarrow$

$$\Rightarrow \begin{cases} h\nu + m = h\nu' + m\gamma(u) \\ h\nu = h\nu'\cos\theta + m\gamma(u)\cdot u\cdot\cos\phi \\ 0 = h\nu'\sin\theta - m\gamma(u)\,u\sin\phi \end{cases} \Rightarrow \tan\phi = \frac{v'\sin\theta}{v + v'\cos\theta}$$

7.6 (i) A particle of rest-mass m_0 and charge q is injected at velocity $\mathbf{u}$ into a constant magnetic field $\mathbf{B}$ at right angles to the field lines. Use the Lorentz force law to establish that the particle will trace out a circle of radius $cm_0 u \cdot \gamma(u)/qb$ with period $2\pi c m_0 \gamma(u)/qb$.

(ii) If the particle is injected into the field with the same velocity but at an angle $\theta < \pi/2$ to the field lines, prove that the path is a helix, of smaller radius, but that the period for one complete cycle is the same as before.



Här: $q\mathbf{u}\times\mathbf{B} = E\mathbf{u}$ $\quad (\mathbf{u}\cdot\mathbf{B}=0)$

Ansätt $\mathbf{u} = \underbrace{u}_{\text{konst.}}(\sin\theta\, \mathbf{e}_x + \cos\theta\, \mathbf{e}_y) \Rightarrow$

$\Rightarrow \dot{\mathbf{u}} = u\dot{\theta}(\cos\theta\, \mathbf{e}_x - \sin\theta\, \mathbf{e}_y) \Rightarrow quB(-\sin\theta\, \mathbf{e}_y + \cos\theta\, \mathbf{e}_x) =$

$= Eu\dot{\theta}(\cos\theta\, \mathbf{e}_x - \sin\theta\, \mathbf{e}_y) \Rightarrow \dot{\theta} = \omega = \frac{qB}{E} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi E}{qB}$

$\int \mathbf{u} = \mathbf{r} = \frac{u}{\omega}(-\cos\omega t\, \mathbf{e}_x + \sin\omega t\, \mathbf{e}_y) \Rightarrow$ Radie: $r = \frac{u}{\omega}$

Skriv $E = mc^2\gamma(u)$ och återinför $c \Rightarrow$

$$\Rightarrow r = \frac{cmu\gamma(u)}{qB}, \quad T = \frac{2\pi c m \gamma(u)}{qB}$$

(ii) Om en z-komponent finns förblir den konstant, ty $(\mathbf{u}\times\mathbf{B})_z = 0 \Rightarrow$ spiral

7.9 Obtain

$$\phi = \left[\frac{q}{r(1+u_r/c)}\right], \quad W = \left[\frac{q u_r}{r(1+u_r/c)}\right]$$

for a moving point-charge q, where the brackets indicate that the enclosed expressions are to be evaluated at the retarded event at the charge; r is the distance of that event from the observer and u_r is the radial velocity of the charge away from the observer.

7.16 There are good reasons for saying that an electromagnetic field is 'radiative' if it satisfies $\mathbb{e} \cdot \mathbb{b} = 0$ and $e = b$, that is, if its is null. Show that the field of a very fast moving point charge is essentially radiative in a plane which contains that charge and is orthogonal to its motion.

För en laddning i rörelse gäller alltid att $\mathbb{B} = \mathbb{v} \times \mathbb{E} \Rightarrow \mathbb{B} \cdot \mathbb{E} = 0$ gäller alltid, och för $|\mathbb{v}| \approx 1$ gäller det att $\mathbb{B} \approx \mathbb{E}$.

$$\mathbb{E} = \frac{q\mathbb{r}}{\gamma^2 r^3 (1 - v^2 \sin^2\theta)^{3/2}} \xrightarrow[\substack{\theta \neq \pi/2 \\ v \to 1}]{} 0 \quad (\text{ty } \gamma \to \infty)$$

For $\theta = \pi/2$: $|E| = \left| \frac{q \gamma \mathbb{r}}{r^3} \right| \to \infty$