Mathematical Physics

Allowed material: No material is allowed.

1. Describe methods to evaluate the following integrals (1p each):

(a) $\int_{-\infty}^{\infty} dx \int_0^{\infty} dy \, \frac{e^{-y}}{x^2+y}$

(b) $\int_0^{\infty} \frac{dx}{(x+1)(x+2)(x+3)}$ using the residue theorem

2. The differential equation

(removed)

has two linearly independent solutions $y_1(x)$ and $y_2(x)$ which have the properties $\lim_{x\to 0^-} y_1(x) = 0$, $\lim_{x\to 0^+} y_1(x) = +\infty$, and $\lim_{x\to 0} y_2(x) = 1$.

(a) Describe a general method that can be used to find the solution $y_1(x)$. (2p)

(b) Describe a different method that can be used to find the solution $y_2(x)$. (2p)

3. Describe Laplace's method (*e.g.*, method of steepest descent) of evaluating integrals approximately. Remark on both real and complex integrals involving real and complex variables. (4p)

4. What is a Green's function? Describe their uses and ways to obtain the Green's function for a particular problem. (4p)

5. Give an example of a mathematical problem involving differential equations that may be solve using singular perturbation theory, and describe a method to solve the problem. (3p)

6. What is an integral equation with a separable kernel? Describe a method to solve them. (3p)

7. (a) Evaluate the functional derivative $\frac{\delta \mathcal{A}}{\delta f(x)}$ where (2p)

$$\mathcal{A}[f] = \int_{-\infty}^{\infty} dx \left[f(x)f''(x) + e^{-x} f(x) + \frac{1}{2}(f(x))^2 + e^{-f'(x)} - 1 \right]$$

(b) [problem no. 70 in the problem collection]

You do not need to solve this problem, but you need to set up a system of equations whose solution would yield the solution. (5p)

Mathematical Physics FTF131

Allowed material: No material is allowed. A pocket calculator with no connection to external memories is OK.

1. *Differential equations*

Describe methods to find approximate solutions to the following differential equations for the variable and parameter ranges given:

a) For $x \to 0^+$, equation (3p)

$$x^3 y''(x) - 3xy'(x) + 2x^{-1}y(x) = 0$$

b) *removed*

2. *Integrals*

a) How would you find an approximate value of the integral $\int_0^{2\pi} dt\,(1+t^2)e^{x\cos(t)}$, $x \gg 1$? (2p)

b) Evaluate $\lim_{\eta\to 0^+} \int_{-\infty}^{\infty} dx\, \frac{e^{-x^2}}{x+i\eta}$. One point for a correct answer, and one point for showing that the answer is correct. (2p)

3. *Sums*

a) How would you determine a reasonable value for the divergent series $\sum_{n=0}^{\infty} n(-x)^n$? (2p)

b) How would you obtain *approximate* expressions for the sum of the convergent series $\sum_{n=-\infty}^{\infty} \frac{\cos(nx)}{n^2+x^2}$ for large and small $|x|$, respectively? (3p)

4. *Integral equations*

a) How would you solve the integral equation

$$f(x) = g(x) + \int_{-1}^{1} dy\,(\lambda x - 2x^2 y) f(y)$$

where λ is a constant and $g(x)$ is a function, both assumed to be known. (3p)

b) Which conditions must λ and $g(x)$ satisfy in order for the equation to be solvable? (3p)

c) How would you go about finding the solution of

$$-\frac{\hbar^2}{2m} y''(x) + V \frac{1}{\cosh(kx)} y(x) = E y(x)$$

for a given $E \gg |V|$ and subject to the condition $y(x)e^{-ikx} \to 1$ as $x \to -\infty$ where $k = \sqrt{2mE}/\hbar$? (3p)

5. *Singular perturbation theory*

Describe the WKB method — in what types of problems is it applicable, which steps are involved in the application of the method, and what are typical applications of the method. (5p)

6. *Functional derivatives and variational calculus*

a) Evaluate the functional derivative $\frac{\delta \mathcal{A}}{\delta f(x)}$ where (2p)

$$\mathcal{A}[f] = 2f(1) + \int_{-\infty}^{\infty} dx \left\{ 2x\,[f(x)]^3 - \left[\frac{d^2 f(x)}{d^2 x}\right]^2 \right\} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dx'\, f(x) K(x,x') f(x')$$

b) Fermat's principle of least times states that a ray of light between two points travels along a path which it can traverse in the shortest time. Consider a region in which the index of refraction varies linearly with altitude, $n = n_0(1 + \alpha z)$. Describe how you would determine the apparent position (angle θ in the Figure) of the object A as seen by the observer P. Note that the speed of light in refractive medium is given by $c' = c/n$ where c is the speed of light in vacuum. (4p)

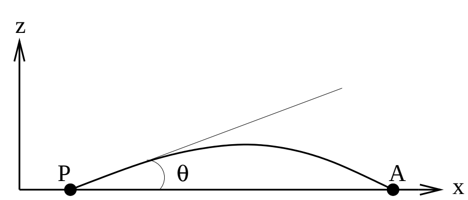

FIG. 1: Illustration for problem no. 6.

Mathematical Physics

Allowed material: No material is allowed.

1. *General problems.* (15p)

(a) What is saddle point approximation and when is it useful?

(b) Describe at least three ways of summing infinite series.

(c) What are advanced and retarded Green's functions and when are they useful?

(d) Consider an oil spill in the Baltic Sea. Assume that you know the density of oil in different parts of the polluted area, and assume that you have a fixed length of barriers that prevent the spreading of the spill. Describe a mathematical model that allows you to place the barriers in the most efficient way.

(e) Give an example of a problem that requires singular perturbation theory and describe a method to solve the problem.

2. Show that the the Schrödinger equation

$$[H_0 + V(\mathbf{r}) - E_\alpha]\,\psi_\alpha(\mathbf{r}) = 0$$

is equivalent with the Lippmann-Schwinger equation

$$\psi_\alpha(\mathbf{r}) = \psi_\alpha^{(0)}(\mathbf{r}) + \int d^d r' G_0(E_\alpha, \mathbf{r}, \mathbf{r}') V(\mathbf{r}')\psi_\alpha(\mathbf{r}')$$

where $\psi_\alpha^{(0)}(\mathbf{r})$ is the solution to the equation with $V(\mathbf{r}) = 0$. (5p)

3. For which values of λ does the equation

$$f(x) = g(x) + \lambda \int_0^1 dy\,(1 - 3xy) f(y)$$

have a solution for general $g(x)$? Find the solution! (5p)

4. Find the function $u(x)$ that minimizes the functional

$$I[u] = \int_0^{2\pi} dx\, \left[(u'(x))^2 - (u(x))^2\right]$$

and satisfies the boundary conditions $u(x) = u(x + 2\pi)$. (5p)