CHALMERS UNIVERSITY OF TECHNOLOGY 2011-03-17
Microtechnology och nanoscience 14.00-18.00

Exam in FMI036, Superconductivity and low temperature physics

Teacher: Per Delsing, phone: 772 3317, 070-3088317
Allowed aids: Tefyma, Physics Handbook, Stand Math Tables and similar handbooks, calculator, and one A4 sheet with handwritten notes.
Max points: 30p
Grading: Grade 3: 5p on problem 1 and 15p in total, (Grade 4: 20p, Grade 5: 25p)
Motivate your answer in a logical way. Illustrate with readable diagrams and/or figures.

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1. Short questions to test the understanding of concepts.
Give short descriptions or definitions of the following, use diagrams or figures if appropriate:

Superconductors

a) Explain why relatively bad conductors like lead and tin become superconducting at low temperature, whereas good conductors like silver and cupper do not.
b) The are three different types of superconductors which have Tc above 30K. Describe briefly each of these types.
c) Explain the concept of plasma oscillations in a Josephson junction, give an expression for the frequency.
d) What are the main differences between a type I and type II superconductor, what determines the type?
e) Explain why there is no thermoelectric effect in superconductors, i.e. why is the Seebeck coefficient zero.
f) Sketch the excitation energy for a superconductor as a function of k-vector, and explain what is meant by a hole-like excitation.

Helium

g) Describe the properties of the A1 phase in superfluid ^{3}He. Under what conditions can it be observed, how do the atoms pair up into Cooper pairs.
h) Explain the concept of the Landau velocity of a superfluid, what does it mean and what sets its value?

Cryogenics

i) Explain the difference between a primary thermometer and a secondary thermometer and give two examples of each type.
j) Sketch the resistivity for copper as a function of temperature, state the approximate temperature dependencies in different parts of the curve and what is the origin of the different parts.

2. Ginzburg-Landau: The coherence length
The first Ginzburg-Landau equation is given by: $\alpha\,\psi + \beta\,|\Psi|^2\Psi + \frac{1}{4\,m}(-i\hbar\bar{\nabla} + 2e\bar{A}) = 0$

a) Explain the London gauge, when can it be applied and how does it simplify the description of superconductors. (1p)
b) Imagine a superconductor normal metal interface and consider small changes in the order-parameter ψ. Derive an expression for how the order-parameter changes in the normal metal. (1.5p)
c) Derive an expression for how the order-parameter changes in the superconductor (1.5p)

3. BCS Theory

The expectation value for the energy of the BCS ground state is given by

$$\langle E\rangle = \sum_k 2v_k^2\varepsilon_k + \sum_{k,k'} v_k u_{k'} u_k v_{k'} V_{k,k'} .$$

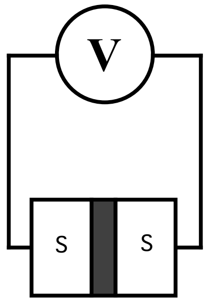
a) Explain what the different terms in this expectation value means, what are v_k, u_k, ε_k and $V_{k,k'}$ (1p)

b) The (k-dependent) energy gap can be defined as $\Delta_k = -\sum_{k'} u_{k'} v_{k'} V_{k,k'}$

Minimize the expectation value with respect to v_k, and u_k and derive an expression for the energy gap Δ_k as a function of v_k, u_k and ε_k. (3p)

4. Josephson effect

Two superconductors of the same material are separated by a thin tunnel barrier and a voltage is applied between them as shown in the figure below. The system can be described by two coupled Schrödinger equations, where μ is the chemical potential of the material, Ψ_1 and Ψ_2 are the orderparameters for the left and the right side respectively.



$$i\hbar\frac{\partial}{\partial t}\Psi_1 = (\mu + eV)\Psi_1 + K\Psi_2$$

$$i\hbar\frac{\partial}{\partial t}\Psi_2 = (\mu - eV)\Psi_2 + K\Psi_1$$

a) Derive the two Josephson relations, explaining each step you take. (3p)

b) Derive an expression for the Josephson inductance from the two Josephson relations. (1p)

5. Bose-Einstein condensation

When bosonic atoms (like ^{4}He-atoms) are cooled to low temperature a substantial fraction of the atoms "condense" into the lowest energy state.

a) Explain how this picture can be connected to the two-fluid model. Sketch how the densities of the different fluids vary with temperature. (1p)

b) The density of states is given by $N(\varepsilon) = \frac{V}{4\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\sqrt{\varepsilon}$, and the Bose-Einstein distribution is $n(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$. Derive an expression for the critical (Bose) temperature at which the condensation occurs assuming that the chemical potential is close to the lowest energy state. (3p)

6. Low temperature properties of materials.

When the lattice of a solid is heated the energy is stored in the phonons. Calculate the lattice specific heat of the phonons in a solid, assuming that the density of states follows the Debey model. (You can ignore the zero point fluctuations since they do not contribute to the specific heat.) (4p)

Good luck !

Solutions for the exam FMI036, March 2011

Part A

1. Superconductors

a) Explain why relatively bad conductors like lead and tin become superconducting at low temperature, whereas good conductors like silver and cupper do not.

This depends on the strength of the electron-phonon interaction in the different metals. A material like copper which has a weak e-ph interaction has low resistance at room temperature since the resistance at that temperature depends on the e-ph scattering. The e-ph scattering is not strong enough to form Cooper-pairs.
Tin on the other hand has relatively strong e-ph interaction which leads to high resistance at high temperature. The strong e-ph interaction leads to Cooper pairing and SC at low temperature

b) Explain the concept of Josephson inductance.

Using the Josephson relations we can write the time derivative of the current as

$$\frac{dI}{dt} = I_C \cos\delta \cdot \dot{\delta} = \frac{\hbar}{2e}\cos\delta \cdot V \equiv \frac{V}{L_J}$$

We see that the voltage across the Josephson junction is proportional to the time derivative of the current. This is precisely what an inductance does. Thus the Josephson junction acts like a inductance, which is parametric, i.e. I depends on the phase across the junction which can be modified by the current through the junction.

$$L = \frac{2e}{\hbar I_C \cos\delta} = \frac{2e}{\hbar I_C \sqrt{1-\left(\frac{I}{I_C}\right)^2}} = \frac{2e}{\hbar\sqrt{I_C^2 - I^2}}$$

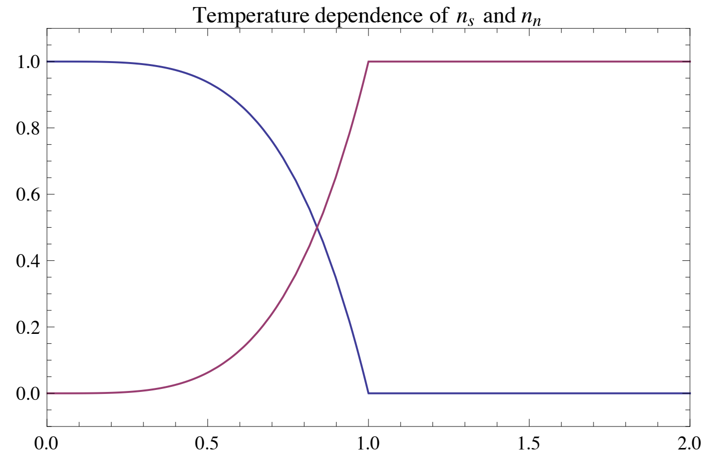
c) What are the main differences between a type I and type II superconductor, what determines the type?

The main difference is the behavior when a magnetic field is applied. A type one superconductor is in the Meisner state all the way up to the thermodynamical crtical field H_C. In a type II superconductor the field starts to penetrate into the SC as individual flux quanta at a lower field H_{C1}. The sign of the interface energy between a normal region and a superconducting region determins the type. If the penetration depth is short compared to the coherence length $\lambda<\xi/\sqrt{2}$, the interface energy is positive and we get a type I SC. In the opposite case $\lambda>\xi/\sqrt{2}$, the interface energy is negative and we get a type II SC. Figure M vs. H.

d) Explain the difference between a quasi-particle and an electron.

A quasi-particle is an excitation of a Cooper-pair. Breaking up a Cooper pair gives two quasi particles. Creating an electron can be done with the creation operator $c^\dagger$. Creating a quasi particle is done by the Bogoliubov creation operator $\gamma^\dagger = u_k c_{k\uparrow}^\dagger - e^{-i\theta} v_k c_{-k\downarrow}$, using the annihilation operator one can remove a quasi particle $\gamma = u_k c_{k\uparrow} - e^{i\theta} v_k c_{-k\downarrow}^\dagger$. A quasi-particle can be partly hole and partly electron. Figure excitations.

e) Sketch how the different densities vary with temperature (from zero to 2xT_C) in the two-fluid model for a superconductor. What do the different densities correspond to?



Helium

f) What is the meaning of the second sound in a superfluid? What is different from ordinary sound?

Second sound is a Temperature Entropy wave whereas ordinary sound is a Density pressure wave. Second sound is generated by a time varying temperature. The variation of entropy and temperature can be described in the two-fluid model. The total density is constant, however the ratio between superfluid and normal fluid varies in space and time.

g) Describe the three different superfluid phases in ^{3}He? How do the different phases differ in terms of symmetries.

The A phase is described by a wave function which is a super position of spin-up spin-up and spin-down spin down. The energy gap is anisotropic with zeros in some directions.
The A1 phase is similar to the A phase, it is described by a wave function which is either of spin-up spin-up or spin-down spin down depending on the external field. The energy gap is anisotropic with zeros in some directions.
The B phase is described by a wave function which is a super position of spin-up spin-down and spin-down spin up. The energy gap is isotropic equal in all directions.

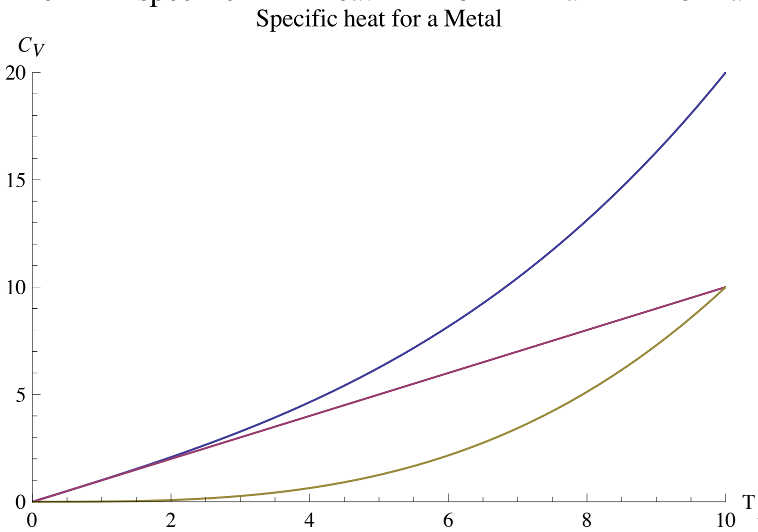
Cryogenics

h) How can a dilution refrigerator be used for cooling, what temperatures and cooling powers can be reached?

In a dilution refrigerator a mixture of He-3 and He-4 is used. At low temperature the mixture is separated in the mixing chamber into two separate phases; a light He-3 rich phase which “floats” on top of the heavier He-4 rich phase which is called the dilute phase. He-3 is circulated in the system by evaporating it from a Still which is a tt approximately 0.7K. At this temperature the vapor pressure of He-4 is extremely small, whereas the vapor pressure for He-3 is substantial. Extracting He-3 from the still reduces the concentration of He-3 in the dilute phase in the mixing chamber. To maintain the equilibrium concentration of He-3 in the dilute phase at 6.4% atoms are forced from the He-3 rich phase into the dilute phase. A latent heat is absorbed in this process which cools the mixing chamber.
The minimum temperature of a dilution refrigerator is in the range 5-30 mK. The cooling power is measured at 100 mK and is typicaly in the range 50-1000 μW.

i) Describe the specific heat of a normal metal at low temperature, draw a figure and indicate temperature dependences and the origin of different parts of the specific heat.

The specific heat of a normal metal has two con



tributions. One part comes from the electron system and it increases linearly with the temperature, this one dominates at low temperature. The contribution from the phonon system increases with T^3 and dominates at higher temperature.

Part B

2. **Ginzburg-Landau: The coherence length**

The first Ginzburg-Landau equation is given by $\alpha\Psi + \beta|\Psi|^2\Psi - \frac{1}{4m}\left(-i\hbar\overline{\nabla} + 2e\overline{A}\right)^2\Psi = 0$

a) Explain the London gauge, when can it be applied and how does it simplify the description of superconductors. (1p)

When describing the electromagnetic response in terms of scalar potential V and vector potential $\overline{A}$, there is a degree of freedom to shift between the scalar and vector potentials, this is called to adopt a certain gauge. The London gauge is defined by $\overline{\nabla} \bullet \overline{A} = 0$. The result of this gauge is that the phase is the same everywhere in the superconductor, and it can conveniently be choosen to zero. This means that the wavefunction Ψ becomes real everywhere in the superconductor.
The London gauge can only be adopted for a simply connected superconductor, not for insatance when there is a whole in the superconductor.

b) Imagine a superconductor normal metal interface and consider small changes in the order-parameter Ψ. Derive an expression for how the order-parameter changes in the normal metal. (1.5p)

c) Derive an expression for how the order-parameter changes in the superconductor (1.5p)

See LectureNotesSC5

3. **BCS Theory**

The expectation value for the energy of the BCS ground state is given by

$$\langle E \rangle = \sum_k 2v_k^2\varepsilon_k + \sum_{k,k'} v_k u_{k'} u_k v_{k'} V_{k,k'} .$$

a) Explain what the different terms in this expectation value means, what are v_k, u_k, ε_k and $V_{k,k'}$ (1p)

The first term describes the kinetic energy of all the electrons, the second term describes the pairwise interaction between all electrons.

v_k is the probability amplitude for finding state k occupied
u_k is the probability amplitude for finding state k empty
ε_k is the kinetic energy for an electron in state k
V_{kk} is the interaction energy for between an electron in state k and an electron in state k

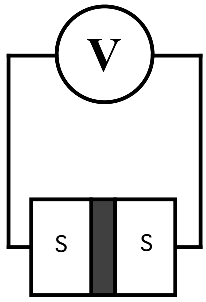
b) The (k-dependent) energy gap can be defined as $\Delta_k = -\sum_{k'} u_{k'} v_{k'} V_{k,k'}$

Minimize the expectation value with respect to v_k, and u_k and derive an expression for the energy gap Δ_k as a function of v_k, u_k and ε_k. (3p)

See LectureNotesSC7

4. Josephson effect

Two superconductors of the same material are separated by a thin tunnel barrier and a voltage is applied between them as shown in the figure below. The system can be described by two coupled Schrödinger equations, where μ is the chemical potential of the material, Ψ_1 and Ψ_2 are the orderparameters for the left and the right side respectively.



$$i\hbar\frac{\partial}{\partial t}\Psi_1 = (\mu + eV)\Psi_1 + K\Psi_2$$

$$i\hbar\frac{\partial}{\partial t}\Psi_2 = (\mu - eV)\Psi_2 + K\Psi_1$$

a) Derive the two Josephson relations, explaining each step you take. (3p)

See LectureNotesSC8

b) Derive an expression for the Josephson inductance from the two Josephson relations. (1p)

Using the Josephson relations $I = I_C \sin\delta,\ V = \frac{\hbar}{2e}\frac{d\delta}{dt}$, we can write the time derivative of the current as $\frac{dI}{dt} = I_C \cos\delta \cdot \frac{d\delta}{dt}$. Next we combine this with the second Josephson relation and get

$$\frac{dI}{dt} = I_C \cos\delta \cdot \frac{2e}{\hbar}V$$

We see that the voltage is proportional to the time derivative of the phase. Thus the Josephson junction acts like a parametric inductance depending on the current through the junction

$$L = \frac{2e}{\hbar I_C \cos\delta} = \frac{2e}{\hbar I_C\sqrt{1-\left(\frac{I}{I_C}\right)^2}} = \frac{2e}{\hbar\sqrt{I_C^2 - I^2}}$$

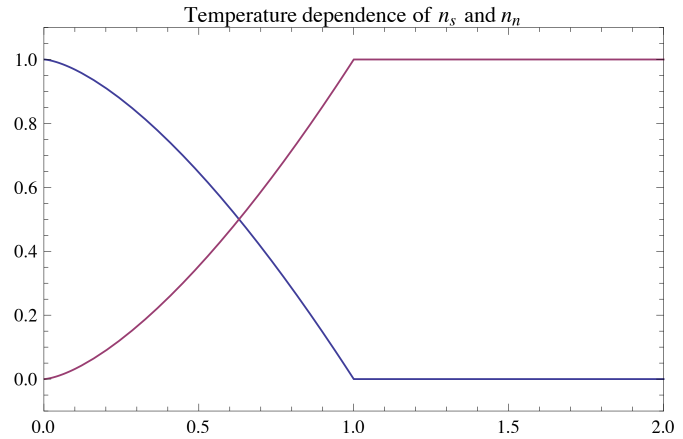
5. Bose-Einstein condensation

When bosonic atoms (like ^{4}He-atoms) are cooled to low temperature a substantial fraction of the atoms "condense" into the lowest energy state.

a) Explain how this picture can be connected to the two-fluid model. Sketch how the densities of the different fluids vary with temperature. (1p)

All bosons in the ground state have the same wave function and therefore the same phase, they can be thought of as the superfluid. The density of the superfluid is then given by $\rho_s = \frac{N_{groundstate}}{N_{total}}$

The bosons which are exited, will then occupy many different states and have different wave functions. They can be thought of as the normal fluid with density $\rho_n = \frac{N_{exited}}{N_{total}}$



b) The density of states is given by $N(\varepsilon) = \frac{V}{4\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\sqrt{\varepsilon}$, and the Bose-Einstein distribution is $n(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}-1}$. Derive an expression for the critical (Bose) temperature at which the condensation occurs assuming that the chemical potential is close to the lowest energy state. (3p)

See LectureNotesHe4b

6. **Low temperature properties of materials.**
When the lattice of a solid is heated the energy is stored in the phonons. Calculate the lattice specific heat of the phonons in a solid, assuming that the density of states follows the Debey model. (You can ignore the zero point fluctuations since they do not contribute to the specific heat.) (4p)

See LectureNotesCr3