Quantum Mechanics FKA081/FIM400

Final Exam 29 October 2014

Next review time for the exam: November 28 between 14-18 in my room.
NB: If you want to come to the review you must collect your exam before at the "Kansli" in Origo 5th floor. (This info is also available on the course homepage.)

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Allowed material during the exam:

- The course textbook J.J. Sakurai and Jim Napolitano, Modern Quantum Mechanics Second Edition.
  NB: The old red cover version: J.J. Sakurai, Modern Quantum Mechanics is *also* allowed.
- A Chalmers approved calculator.

Write the final answers clearly marked by Ans: ...
and underline them.

You may use without proof any formula in the book.

The grades are assigned according to the table in the course homepage.

Problem 1

Consider a finite dimensional Hilbert space $\mathcal{H}$ and a Hermitian operator H on this space. H is called "non-negative" if

$$\langle\psi|H|\psi\rangle \geq 0, \text{ for all states } |\psi\rangle \in \mathcal{H}. \tag{1}$$

Q1 (*1 points*) Given any operator $A : \mathcal{H} \to \mathcal{H}$, show that $H = A^\dagger A$ is non-negative.
Q2 (*1 points*) Given a state $|\phi\rangle \in \mathcal{H}$, show that $H = |\phi\rangle\langle\phi|$ is non-negative.
Q3 (*1 points*) Show that the trace of a non-negative operator obeys $\mathrm{Tr}(H) \geq 0$.

Problem 2

Consider three particles of spin $s_1 = 1$, $s_2 = 2$ and $s_3 = 3$.
Q1 (*1 points*) Add the spins of the first and the second particle, i.e., write all possible values for the total spin of the system: $\mathbf{S}_{12} = \mathbf{S}_1 + \mathbf{S}_2$.
Q2 (*2 points*) To each irreducible representation of $\mathbf{S}_{12}$ in **Q1** add the third spin and summarize all the possible spins (irreducible representations) of $\mathbf{S}_{123} = \mathbf{S}_{12} + \mathbf{S}_3$ *including their degeneracy.*
Q3 (*2 points*) Repeat the same procedure as **Q1** and **Q2** but starting first with the pair $\mathbf{S}_{23} = \mathbf{S}_2 + \mathbf{S}_3$ and then adding $\mathbf{S}_1$ and show that you get the same result.
Q4 (*2 points*) Check that the product of the dimensions of the Hilbert spaces for the three particles is equal to the sum of the dimensions of all resulting irreducible representations of $\mathbf{S}_{123}$.

NB: No Clebsch-Gordan coefficients are needed. You do not have to write down the states explicitly!

Problem 3

Consider a spinless particle of mass m in a one dimensional infinite well:

$$H_0 = \frac{p^2}{2m} + U(x), \tag{2}$$

where

$$U(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise} \end{cases} \tag{3}$$

It is known that the energy eigenvalues of H_0 are $E_n^{(0)} = \frac{n^2\pi^2\hbar^2}{2mL^2}$

The presence of a membrane in the middle of the well can be described by the perturbation

$$V(x) = \lambda\delta(x - L/2) \tag{4}$$

where $\lambda > 0$ is a small parameter and δ the Dirac delta-function.

Q1 (*1 points*) Compute the corrections $E_n^{(1)}$ to the energy eigenvalues of $H_0 + V$ in first order perturbation theory and show that the corrections vanish for n even.

Q2 (*2 points*) Show that the second order corrections $E_n^{(2)}$ also vanishes for n even. (You do not need to compute the value for the case where n is odd.)

Q3 (*3 points*) Discuss the reason for the results obtained in **Q1** and **Q2**. Show that the exact solution is in fact unchanged in going from H_0 to H_0+V for the case n even. What is the qualitative behavior of the exact ground state solution?

Problem 4

Consider a one dimensional harmonic oscillator of angular frequency ω. Let $|n\rangle$ denote its eigenstates. For $t > 0$ it is subjected to a potential

$$V(x) = \lambda e^{-t/\tau} x \tag{5}$$

where x is the coordinate operator and λ a small constant.

Q1 (*2 points*) Discuss what are the transitions $m \to n$ allowed to first order perturbation theory.

Q2 (*3 points*) Assuming that at $t < 0$ the system is in the ground state $|0>$ Find the probability of finding the oscillator in the first excited state at $t \to +\infty$ to first order in perturbation theory.

Problem 5

Consider two *degenerate* states $|\psi_1\rangle$ and $|\psi_2\rangle$ both of energy $E > 0$ mutually orthogonal and properly normalized. At times $t < 0$ the system is in the state $|\psi_1\rangle$. At time $t = 0$ we suddenly turn on an interaction H' mixing the two states, i.e.

$$\langle\psi_1|H'|\psi_2\rangle = \langle\psi_2|H'|\psi_1\rangle = a, \quad \langle\psi_1|H'|\psi_1\rangle = \langle\psi_2|H'|\psi_2\rangle = 0. \tag{6}$$

(a real.)
Q1 (*2 points*) What is the probability of finding the system in the state $|\psi_2\rangle$ at $t > 0$? (Note: Use the sudden approximation.)
Q2 (*2 points*) Suppose I want to turn the system from the state $|\psi_1\rangle$ to the state $|\psi_2\rangle$ by switching on H' at $t = 0$ and then switching it off again at time T. What value of T should I choose?

Problem 6

Consider a spinless particle in three dimensions subjected to the central potential

$$V(r) = -\frac{K}{\sqrt{r}} \tag{7}$$

where K is a positive constant.
Q1 (*3 points*) Estimate the ground state energy by using the following variational ansatz:

$$\psi(r, \theta, \phi) = e^{-r/a} \tag{8}$$

(independent on θ and ϕ.)
Q2 (*2 points*) Do a dimensional analysis check on your answer. (Even if you think you got the wrong answer you can still get credit on this question by checking if the dimensional analysis works or not.)

NOTE: The following integrals are useful:

$$\int_0^\infty r^2 e^{-2r/a}\, \mathrm{d}r = \frac{a^3}{4} \tag{9}$$

$$\int_0^\infty r^{3/2} e^{-2r/a}\, \mathrm{d}r = \frac{3\sqrt{\pi}}{16\sqrt{2}} a^{5/2} \tag{10}$$

PROBLEM 1.

Q1: $\langle\psi|H|\psi\rangle = \langle\psi|A^{\dagger}A|\psi\rangle = \| A|\psi\rangle \|^2 \geq 0$

Q2: $\langle\psi|H|\psi\rangle = \langle\psi|\phi\rangle\langle\phi|\psi\rangle = |\langle\phi|\psi\rangle|^2 \geq 0$

Q3: $Tr(H) = \sum_n \langle n|H|n\rangle \geq 0$

PROBLEM 2.

Q1: Denote by $\mathcal{H}_s$ the Hilbert space of the spin s particle.

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3.$$

Q2: $(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) \otimes \mathcal{H}_3 =$

$$= \mathcal{H}_1 \otimes \mathcal{H}_3 + \mathcal{H}_2 \otimes \mathcal{H}_3 + \mathcal{H}_3 \otimes \mathcal{H}_3 =$$

$$= \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus$$

$$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$$

$$\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5 \oplus \mathcal{H}_6 =$$

$$\mathcal{H}_0 \oplus 2\times\mathcal{H}_1 \oplus 3\times\mathcal{H}_2 \oplus 3\times\mathcal{H}_3 \oplus 3\times\mathcal{H}_4 \oplus 2\times\mathcal{H}_5 \oplus \mathcal{H}_6$$

degeneracy.

Q3:

$$\mathcal{H}_2 \otimes \mathcal{H}_3 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$$

$$\mathcal{H}_1 \otimes (\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5) =$$

$$= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus$$
$$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus$$
$$\mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus$$
$$\mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$$
$$\oplus \mathcal{H}_4 \oplus \mathcal{H}_5 \oplus \mathcal{H}_6$$

$$= \mathcal{H}_0 \oplus 2\times\mathcal{H}_1 \oplus 3\times\mathcal{H}_2 \oplus 3\times\mathcal{H}_3 \oplus 3\times\mathcal{H}_4 \oplus 2\times\mathcal{H}_5 + \mathcal{H}_6$$

Same as before

Q4. Recall: $\dim \mathcal{H}_s = 2s+1$.

$$\dim(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) = 3\times 5\times 7 = 105.$$

$$\dim(\mathcal{H}_0 \oplus 2\times\mathcal{H}_1 \oplus 3\times\mathcal{H}_2 \oplus 3\times\mathcal{H}_3 \oplus 3\times\mathcal{H}_4 \oplus 2\times\mathcal{H}_5 \oplus \mathcal{H}_6)$$

$$= 1 + 2\times 3 + 3\times 5 + 3\times 7 + 3\times 9 + 2\times 11 + 13$$

$$= 105 \checkmark$$

PROBLEM 3

Q1: $\langle \psi_n | V | \psi_n \rangle = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right)^2 \cdot \lambda\left(x - \frac{L}{2}\right) dx$

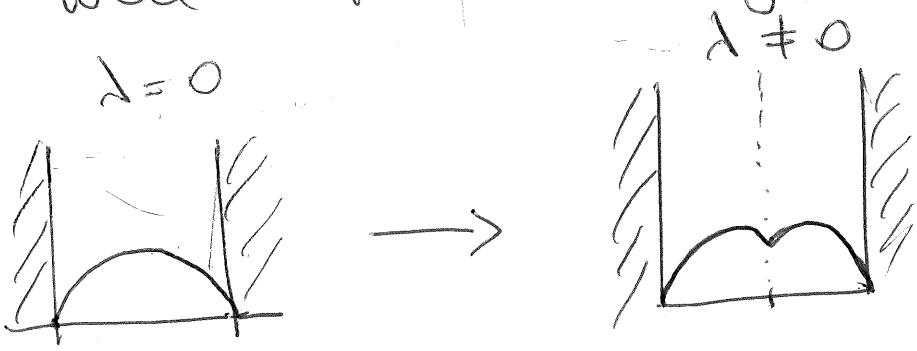
$$= \frac{2\lambda}{L} \sin^2\left(\frac{n\pi}{\cancel{L}} \cdot \frac{\cancel{L}}{2}\right) = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2\lambda}{L} & \text{for } n \text{ odd.} \end{cases}$$

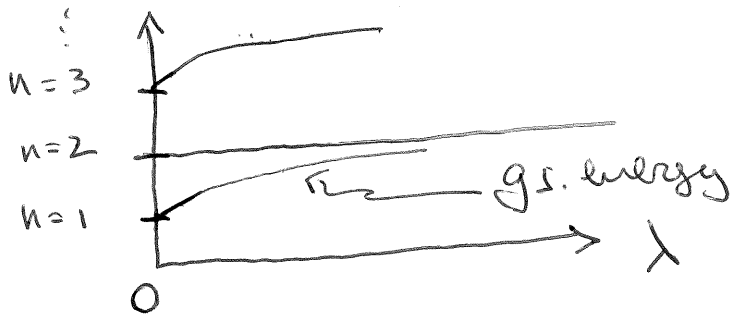
Q2: $\langle \psi_m | V | \psi_n \rangle = 0$ for the same reason if one of the two q. nrs. n or m is zero $\Rightarrow E_n^{(2)} = 0$.

Q3: For n even (set $n = 2k$).

$$V \psi_{2k}(x) = \lambda \delta\left(x - \frac{L}{2}\right) \cdot \sqrt{\frac{2}{L}} \sin \frac{2k\pi x}{L} = 0$$

So ψ_{2k} solves $(H_0 + V)\psi_{2k}(x) = E_{2k}^{(0)} \psi_{2k}(x)$ as well. For the ground state:





PROBLEM 4.

Q1: $\langle m|V|n\rangle \propto \langle m|x|n\rangle$ is non zero iff $m = n \pm 1$.
So the transitions allowed are only between adjacent levels.

Q2: $$P_1(t) = \frac{1}{\hbar^2}\left|\int_0^{+\infty} \langle 1|V(t)|0\rangle e^{i\omega_{10}t}\,dt\right|^2$$

where $\omega_{10} = \dfrac{\hbar\omega(1+\frac{1}{2}) - \hbar\omega(0+\frac{1}{2})}{\hbar} = \omega$.

and $\langle 1|V|0\rangle = \lambda e^{-t/\tau}\langle 1|x|0\rangle = \lambda e^{-t/\tau}\sqrt{\dfrac{\hbar}{2m\omega}}$

$$P_1(t) = \frac{1}{\hbar^2}\cdot\lambda^2\frac{\hbar}{2m\omega}\cdot\left|\int_0^{\infty} e^{-\frac{t}{\tau}+i\omega t}\,dt\right|^2 =$$

$$= \frac{\lambda^2}{2\hbar m\omega}\cdot\left|\frac{1}{\frac{1}{\tau}-i\omega}\right|^2 = \underbrace{\frac{\lambda^2}{2\hbar m\omega}\cdot\frac{1}{\frac{1}{\tau^2}+\omega^2}}_{\text{dimensionless}}.$$

PROBLEM 5

Q1: for $t<0$ $H = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$

$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

for $t>0$: $H = \begin{pmatrix} E & a \\ a & E \end{pmatrix}$. whose eigenv.

are: $|\psi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$

with $E_+ = E + a$

and $|\psi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$

with $E_- = E - a$.

At time $t=0$ $|\psi\rangle \equiv |\psi_1\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle)$

evolving as:

$$|\psi, t\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{(E+a)t}{\hbar}i}|\psi_+\rangle + e^{-\frac{(E-a)t}{\hbar}i}|\psi_-\rangle\right).$$

$$P_2(t) = |\langle\psi_2|\psi, t\rangle|^2 =$$

$$= \left| \frac{1}{\sqrt{2}} \left(\langle \psi_+ | - \langle \psi_- | \right) \cdot \frac{1}{\sqrt{2}} \left(e^{-i\frac{E+a}{\hbar}t} |\psi_+\rangle + e^{-i\frac{E-a}{\hbar}t} |\psi_-\rangle \right) \right|^2$$

$$= \frac{1}{4} \left| e^{-i\frac{E+a}{\hbar}t} - e^{-i\frac{E-a}{\hbar}t} \right|^2 = \sin^2 \frac{at}{\hbar}$$

Q2: I need $\frac{aT}{\hbar} = \frac{\pi}{2}$ (or periodic values).

$$\Rightarrow T = \frac{\hbar\pi}{2a}.$$

$$\Rightarrow E = \frac{\hbar^2}{2m}\left(\frac{3km\sqrt{\pi}}{8\sqrt{2}\,\hbar^2}\right)^{\frac{4}{3}} - \frac{3k\sqrt{\pi}}{4\sqrt{2}}\cdot\left(\frac{3km\sqrt{\pi}}{8\sqrt{2}\,\hbar^2}\right)^{\frac{1}{3}} =$$

$$= -\underbrace{\frac{3^{7/3}\cdot\pi^{2/3}}{2^{17/3}}}_{\simeq -0{,}548}\cdot k^{\frac{4}{3}} m^{\frac{1}{3}} \hbar^{-\frac{2}{3}}.$$

Q2: $[\hbar] = M\cdot L^2\cdot T^{-1}$

$[m] = M$

$[k] = M\,L^{\frac{5}{2}}\cdot T^{-2}$

$$\left[k^{\frac{4}{3}} m^{\frac{1}{3}} \hbar^{-\frac{2}{3}}\right] = M^{\frac{4}{3}}\cdot L^{\frac{10}{3}}\cdot T^{-\frac{8}{3}} \times M^{\frac{1}{3}} \times$$

$$\times M^{-\frac{2}{3}} \times L^{-\frac{4}{3}} \times T^{\frac{2}{3}} =$$

$$= M\cdot L^2\cdot T^{-2} \quad \checkmark$$

PROBLEM 6

Q1: $H = -\frac{\hbar^2}{2m} \cdot \frac{1}{r^2} \partial_r r^2 \partial_r - \frac{k}{\sqrt{r}} = T + V$

$$\langle \psi | \psi \rangle = 4\pi \cdot \int_0^\infty dr\, r^2 e^{-\frac{2r}{a}} = \pi a^3$$

$$\langle \psi | T | \psi \rangle = -\frac{\hbar^2}{2m} \cdot 4\pi \cdot \int_0^\infty dr\, \cancel{r^2} e^{-r/a} \cdot \frac{1}{\cancel{r^2}} \partial_r r^2 \partial_r e^{-r/a} =$$

$$= (\text{by parts}) + \frac{\hbar^2}{2m} \cdot 4\pi \int_0^\infty dr\, r^2 (\partial_r e^{-r/a})^2 =$$

$$= \frac{2\pi\hbar^2}{ma^2} \cdot \int_0^\infty dr\, r^2 e^{-\frac{2r}{a}} = \frac{\pi \hbar^2 a}{2m}$$

$$\langle \psi | V | \psi \rangle = -k 4\pi \int_0^\infty dr\, r^{2-\frac{1}{2}} e^{-\frac{2r}{a}} = -k \cdot \frac{3\pi^{\frac{3}{2}}}{4\sqrt{2}} a^{5/2}$$

$$\bar{H} = \frac{\hbar^2}{2m} a^{-2} - \frac{3k\sqrt{\pi}}{4\sqrt{2}} a^{-1/2}$$

$$\frac{\partial \bar{H}}{\partial a} = -\frac{\hbar^2}{m} a^{-3} + \frac{3k\sqrt{\pi}}{8\sqrt{2}} a^{-\frac{3}{2}} = 0$$

$$\Rightarrow \quad a = \left(\frac{3km\sqrt{\pi}}{8\sqrt{2}\,\hbar^2} \right)^{-2/3}$$