Quantum Mechanics FKA081/FIM400

Final Exam January 15 2014

Next review time for the exam: 17 February 12-13 in my room. NB: If you want to come to the review you must collect your exam before at the "Kansli" in Origo 5th floor. (This info is also available on the course homepage.)

Examinator: Gabriele Ferretti tel. 7723168, 0739687998 ferretti@chalmers.se

Allowed material during the exam:

- The course textbook J.J. Sakurai and Jim Napolitano, Modern Quantum Mechanics Second Edition (2010).
  NB: The old red cover version: J.J. Sakurai, Modern Quantum Mechanics Revised Edition (1994) is *also* allowed.
- A Chalmers approved calculator.

**Write the final answers clearly marked by Ans: ...
and underline them.**

You may use without proof any formula in the book.

The grades are assigned according to the table in the course homepage.

Problem 1

A one-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2 \tag{1}$$

is perturbed by an Hamiltonian

$$H' = \lambda x \tag{2}$$

Q1 (*1 points*) Show that to first order in perturbation theory there is no effect on the energy spectrum.
Q2 (*2 points*) Use second order perturbation theory to calculate the change in the energy levels.
Q3 (*2 points*) Solve the problem exactly and compare the result with the perturbative ones.

Problem 2

The electron in the ground state of a hydrogen atom is subjected at time $t \geq 0$ to an exponentially decreasing time dependent electric field along the z-axis, that is, a perturbation

$$H' = -eE_0 e^{-\Gamma t} z \tag{3}$$

Q1 (*2 points*) What angular momentum states m and l of L_z and $\mathbf{L^2}$ respectively can be excited by this perturbation?
Q2 (*3 points*) Write the expression for the transition probability to the $n = 2$ state at $t = \infty$.

NOTE: You may ignore the electron spin. (It just doubles the degeneracy of the levels in this approximation.) Use the following integrals (a_0 is the Bohr radius).

$$\int_0^\infty dr\, r^2\, \mathcal{R}^*_{2,1}(r) \cdot r \cdot \mathcal{R}_{1,0}(r) = \frac{2^8 a_0}{3^4\sqrt{6}} \tag{4}$$

$$\int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\phi\, Y^*_{\ell,m}(\theta,\phi) \cdot \cos\theta \cdot Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{3}}\delta_{\ell,1}\delta_{m,0} \tag{5}$$

Problem 3

An electron in an atom is in a state of orbital angular momentum $l = 1$. Define $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L}$ is the orbital angular momentum operator and $\mathbf{S}$ the spin operator.

Q1 (*2 points*) What are the allowed values for $\mathbf{J}^2$ and $\mathbf{J_z}$?

Q2 (*2 points*) Write down $|J = 1/2, M = 1/2\rangle$ in terms of spin wave-functions and spherical harmonics.

NOTE: I give you the following CG decomposition:

$$|J = 3/2, M = 1/2\rangle = \sqrt{\frac{1}{3}}\, Y_{1,1}\chi_- + \sqrt{\frac{2}{3}}\, Y_{1,0}\chi_+ \tag{6}$$

where $Y_{l,m}$ are the spherical harmonics and $\chi_\pm$ the spin up/down wave-functions. (J and M refer to $\mathbf{J}$.)

Q3 (*2 points*) Compute $\langle J = 3/2, M = 1/2|H'|J = 1/2, M = 1/2\rangle$, where

$$H' = -\mu B(L_z + 2S_z) \tag{7}$$

is the Hamiltonian representing the interaction with a uniform magnetic field along the z-axis.

Problem 4

Let x and p be the usual one dimensional position and momentum operators, obeying $[x, p] = i\hbar$.

Q1 (*3 points*) Compute the commutator of the following two operators

$$A = xp \tag{8}$$

$$B = p^2 + \alpha x^n \tag{9}$$

(α real and n positive integer)

Q2 (*1 points*) What is the dimension of α in cm, g, s?

Problem 5

Consider a spin *one* atom in a constant magnetic field along the z-axis. The spin wave-function evolves according to the Scrödinger equation

$$\omega S_z|\psi(t)\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle \tag{10}$$

(ω some given real constant.)

At time $t=0$ we measure S_x with eigenvalue zero.

Q1 (*2 points*) Write the exact wave-function $|\psi(t)\rangle$ for $t>0$.

Q2 (*2 points*) What is the probability of measuring again S_x with eigenvalue zero after a time t?

NOTE: You may use the following representation:

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{11}$$

You may set $\hbar = 1$ everywhere if you wish.

Problem 6

True of False?

(To get points, if true give a proof, if false give a counterexample!)

Q1 (*2 points*) If two observables A and B both commute and anti-commute, one of them must be zero.

Q2 (*2 points*) The only operator that is both unitary and hermitian is the identity operator.

Q3 (*2 points*) If an operator A is both hermitian and nilpotent it must be zero. (An operator A is called "nilpotent" if there exist a positive integer n such that $A^n = 0$.)

For simplicity you can consider the above statements for operators in a finite dimensional Hilbert space.

PROBLEM 1.

Q1: Let $|n\rangle$ be the eigenfunctions of H : $H|n\rangle = \omega(n+\frac{1}{2})|n\rangle \equiv E_n^{(0)}|n\rangle$

$$E_n^{(1)} = \langle n|\lambda x|n\rangle = 0.$$

Q2: $$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m|\lambda x|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}} =$$

$$= \lambda^2 \cdot \sum_{m\neq n} \frac{\left|\sqrt{\frac{\hbar}{m\omega}}\left(\sqrt{\frac{n}{2}}\,\delta_{m,n-1} + \sqrt{\frac{n+1}{2}}\,\delta_{m,n+1}\right)\right|^2}{\hbar\omega\left(n+\frac{1}{2}\right) - \hbar\omega\left(m+\frac{1}{2}\right)} =$$

$$= \frac{\lambda^2}{2m\omega^2}\left(\frac{n}{n-(n-1)} + \frac{n+1}{n-(n+1)}\right) = -\frac{\lambda^2}{2m\omega^2}$$

Q3: $$H_{TOT} = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \lambda x =$$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}\Big(\underbrace{x + \frac{\lambda}{m\omega^2}}_{\equiv x'}\Big)^2 - \frac{\lambda^2}{2m\omega^2}$$

$$\left(p = -i\frac{d}{dx} = -i\frac{d}{dx'} \equiv p'\right).$$

$$\Rightarrow E_{TOT} = \omega\left(n+\frac{1}{2}\right) - \frac{\lambda^2}{2m\omega^2}$$

PROBLEM 2

Q1: Let $|n,\ell,m\rangle$ be a generic state (ground state : $|1,0,0\rangle$).

$\langle n,\ell,m|Z|1,0,0\rangle \neq 0$ for $\ell=1, m=0$ only

By W.E. being Z the $q=0$ comp. of a vector operator.

Q2: Let $\psi_{n\ell m} = R_{n\ell} Y_{\ell m}$.

$\langle 2\ell m|H'|100\rangle =$ // $Z = r\cos\vartheta$ //:

$$-eE_0 e^{-\Gamma t} \cdot \int_0^\infty dr\, r^2 R_{2\ell}\, r \cdot R_{10} \,\cdot$$

$$\cdot \int d\theta\, d\varphi \sin\theta\, Y_{\ell m}^* \cos\vartheta\, Y_{00} =$$

$$= -eE_0 e^{-\Gamma t} \int_0^\infty dr\, r^2 R_{2\ell}\, r R_{10} \cdot \frac{1}{\sqrt{3}} \delta_{\ell,1} \delta_{m,0}$$

$$= -eE_0 e^{-\Gamma t} \cdot \frac{2^8 a_0}{3^4\sqrt{6}} \cdot \frac{1}{\sqrt{3}}$$

($\ell=1, m=0$ only) from now on.

$$P = \frac{1}{\hbar^2} \cdot \left| \int_0^\infty dt \, \langle \psi_{210} | H' | \psi_{100} \rangle e^{i\omega_{2,1} t} \right|^2 =$$

$$= \frac{2^{15} e^2 E_0^2 a_0^2}{3^{10} \hbar^2} \cdot \left| \int_0^\infty dt \, e^{-\Gamma t} \cdot e^{i\omega_{2,1} t} \right|^2 =$$

$$= \frac{2^{15} e^2 E_0^2 a_0^2}{3^{10} \hbar^2} \cdot \frac{1}{\Gamma^2 + \omega_{21}^2}$$

where $\omega_{2,1} = \dfrac{E_2 - E_1}{\hbar} = -\dfrac{e^2}{2\hbar a_0} \cdot \left(\dfrac{1}{4} - 1\right)$

$$= \frac{3e^2}{8\hbar a_0}.$$

PROBLEM 3 (Set $\hbar = 1$)

Q1: $\mathbb{J}^2 = J(J+1)$ with $|1-\frac{1}{2}| \leq J \leq 1+\frac{1}{2}$.

for $J = \frac{1}{2}$ $\mathbb{J}^2 = \frac{3}{4}$ and $M = \pm\frac{1}{2}$

for $J = \frac{3}{2}$ $\mathbb{J}^2 = \frac{15}{4}$ and $M = \pm\frac{3}{2}, \pm\frac{1}{2}$.

Q2: $|J=\frac{1}{2}, M=\frac{1}{2}\rangle$ must be orthogonal to $|J=\frac{3}{2}, M=\frac{1}{2}\rangle$. Up to a phase:

$$|J=\tfrac{1}{2}, M=\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{\sqrt{3}} Y_{11}\chi_- - \frac{1}{\sqrt{3}} Y_{10}\chi_+.$$

Q3: $\langle \frac{3}{2}, \frac{1}{2} | H' | \frac{1}{2}, \frac{1}{2} \rangle =$

$$\left\langle \left(\frac{1}{\sqrt{3}} Y_{11}\chi_- + \sqrt{\frac{2}{3}} Y_{10}\chi_+\right) \middle| -\mu B(L_z + 2S_z) \middle| \left(\frac{\sqrt{2}}{\sqrt{3}} Y_{11}\chi_- - \frac{1}{\sqrt{3}} Y_{10}\chi_+\right)\right\rangle$$

$$= -\mu B\left(\frac{\sqrt{2}}{3}\cdot 1 - \frac{\sqrt{2}}{3}\cdot 0 + \frac{\sqrt{2}}{3}\cdot 2\left(-\frac{1}{2}\right) - \frac{\sqrt{2}}{3} 2\cdot\frac{1}{2}\right) =$$

$$= \mu B\cdot\frac{\sqrt{2}}{3}$$

PROBLEM 4

Q1: $[xp, p^2] = [x, p^2]p =$

$= ([x,p]p + p[x,p])p = 2i\hbar p^2.$

$[xp, \alpha x^n] = \alpha x [p, x^n] =$

$\alpha x \left(-i\hbar \frac{d}{dx} x^n\right) = -i\hbar n \alpha \, x \cdot x^{n-1}$

$= -i\hbar n \alpha x^n.$

Q2: $[p^2] = g^2 \cdot cm^2 \cdot s^{-2}$

$[x^n] = cm^n$

$\Rightarrow [\alpha] = g^2 \, cm^{2-n} \cdot s^{-2}.$

PROBLEM 5 $(\hbar = 1)$

Let $|+1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
be the eigenvectors of S_z.

in terms of those, the (properly normalized) initial state is:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle - |-1\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Q1: $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle e^{i\omega t} - |-1\rangle e^{-i\omega t}\right)$

Q2: $\langle \psi_0 | \psi(t)\rangle = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

$= \cos \omega t$

$\Rightarrow P(t) = \cos^2 \omega t.$

PROBLEM 6

Q1: <u>FALSE</u> : ex: $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ $a, b \neq 0$.

$B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$

Q2: <u>FALSE</u> ex. $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Q3: <u>TRUE</u> proof. Let λ be any of the eigenvalues of A (hermitian). Since $A^n = 0$ I must have $\lambda^n = 0$

$\Rightarrow \lambda = 0 \Rightarrow$ all eigenvalues $\equiv 0$

$\Rightarrow A \equiv 0$.