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Exam in FFR105/FIM711 (Stochastic optimization algorithms), 2022-10-26, 14.00-18.00

The examiner will visit the exam room twice, around 15.00 and around 17.00. It will be possible to review your results (exam and home problems) from Nov. 17 and onwards.

In the exam, it is allowed to use an approved ("typgodkänd") calculator. No other calculators are allowed. Furthermore, it is *not* allowed to use the course book, lecture notes, or slides from the course, during the exam. It is allowed to use mathematical tables (e.g. Beta, Standard Math etc.), as long as no notes have been added.

Note! For Problem 1 below, you should *only* give the answers, i.e., a single letter (A), (B), (C), (D), or (E) for each question, all collected on a single page. For Problems 2 – 5 you should provide a clear description of how you arrived at your answer, including all relevant intermediate steps. *Only giving the answer will not give any points for Problems 2 – 5.*

There are five problems in the exam, and the maximum number of points is 25.

1) This problem consists of ten questions, each of which is associated with five alternative answers, of which one is correct. For each of the 10 questions below, you should therefore *only give one answer*, in the form of a letter: (A), (B), (C), (D), or (E). Answer the questions in order (1.1, 1.2, …,1.10) and mark your answers *clearly* (e.g., "1.1: A"). For this problem, the number of points awarded is determined as follows: 10 correct answers: 10p, 9 correct answers: 8p, 8 correct answers: 6p, 7 correct answers: 5p, 6 correct answers: 4p, 5 correct answers: 3p, 4 correct answers: 2p, 3 correct answers: 1p, 2 or fewer correct answers: 0p.

1.1. When using PSO, it is essential to keep the swarm coherent. Which of the following statements is true?

A. Both velocities and positions should be restricted.
B. Only positions should be restricted.
C. Positions should be restricted if particles venture outside the initial range.
D. Only velocities should be restricted.
E. Positions should be restricted if the inertia is smaller than 1.

1.2. Consider the function $f(x_1, x_2) = 4x_1^2 + 2x_2^2 - x_1x_2$ and let H denote the Hessian matrix. Which of the following statements is true?

A. The eigenvalues of H are both negative, so f is convex.
B. The eigenvalues of H are both positive, so f is convex.
C. One eigenvalue of H is positive, and one is negative, so f is not convex.
D. The eigenvalues of H are both negative, so f is not convex.
E. The eigenvalues of H are both positive, so f is not convex.

1.3. Consider a genetic algorithm in which individuals are selected using tournament selection, with a given value of p_{tour} and with a tournament size of three. In a single selection step (selecting one individual from the tournament with three individuals), what is the probability (p) of selecting the second-best individual.

A. $p = p_{\text{tour}}^2$
B. $p = (1 - p_{\text{tour}})p_{\text{tour}}$
C. $p = p_{\text{tour}}^3$
D. $p = (1 - p_{\text{tour}})p_{\text{tour}}^2$
E. $p = (1 - p_{\text{tour}})^2$

1.4. Consider a case in which the Lagrange multiplier method is applied to the problem of finding the minima of a continuously differentiable function $f(x_1, x_2)$ subject to a continuously differentiable equality constraint $h(x_1, x_2) = 0$, both defined in a bounded region (i.e., such that it can be contained in a disc of radius R, where R is finite). Which of the following statements is true? For the problem at hand, the points found by the Lagrange multiplier method …

A. … contain local minima and maxima, but not any global minima or maxima.
B. … contain local and global minima, but no maximum, neither local nor global.
C. … contain local and global maxima, but no minimum, neither local nor global.
D. … contain local and global optima, both minima and maxima.
E. … contain only global minima and maxima.

1.5. What are the stationary points of the function $f(x) = 2x^3 - 3x^2 - 12x + 4$?

A. The stationary points are x = 4 and x = 12.
B. The stationary points are x = -2 and x = 1.
C. The stationary points are x = -1 and x = 2.
D. The stationary points are x = 3 and x = 6.
E. The stationary points are x = -1 and x = 1.

1.6. Consider a genetic algorithm with a population of five individuals, with fitness values $F_1 = 1$, $F_2 = 3$, $F_3 = 4$, $F_4 = 6$, and $F_5 = 10$. Assuming that tournament selection is used, with $p_{\text{tour}} = 0.75$, what is the probability (in a single selection step) of selecting individual 2 (whose fitness is equal to 3)?

A. 0.12
B. 0.10
C. 0.18
D. 0.24
E. 0.16

1.7. Consider a binary chromosome in a GA, which has been generated via selection and crossover, and is then going to be mutated using the standard mutation procedure, with mutation rate $p_{\text{mut}} = 1/m$, where m is the chromosome length. Which of the following statements is true?

A. The number of genes that undergo mutation will range from 0 to m
B. The number of genes that undergo mutations will range from 0 to m/2
C. Either 0 or 1 gene will undergo mutation
D. Either 0, 1, or 2 genes will undergo mutation
E. Exactly 1 gene will undergo mutation

1.8. The Ant system (AS) algorithm can be applied in many different problems. Let τ_{ij} denote the pheromone matrix and η_{ij} the visibility matrix. Which of the following statements is true: For any problem where AS is applied, at the end of a run lasting a given (finite) number of iterations …

A. … the pheromone matrix is symmetric, whereas the visibility matrix is not.
B. … the visibility matrix is symmetric, whereas the pheromone matrix is not.
C. … the symmetry (or lack thereof) of either matrix depends on the problem.
D. … both the visibility matrix and the pheromone matrix are symmetric.
E. … neither the visibility matrix nor the pheromone matrix is symmetric.

1.9. Consider a standard genetic algorithm (GA) where either tournament selection (TS) or (standard) roulette-wheel selection (RWS), without fitness ranking, is used, and where the fitness values (f) are in the range [0,1]. Which of the following statements is true? An individual whose fitness is equal to 0 …

A. … can be selected with TS but not with RWS.
B. … can be selected with RWS but not with TS.
C. … cannot be selected with either method.
D. … can be selected with both methods.
E. … can be selected with both methods, but only in the first generation.

1.10. Consider a case in which ant system (AS) is being used for solving the travelling salesperson problem (TSP), and where three nodes remain to be selected to complete the path of a given ant. From the current node, the visibilities of those nodes are 1, 2, and 5, respectively. Moreover, the parameters α and β are both equal to 1, and the pheromone level (τ) is equal to 1 on all edges. What is the probability of selecting the third node (with visibility 5) as the next node?

A. 0
B. 2/17
C. 3/8
D. 5/8
E. 1

2) Using the Lagrange multiplier method, find the minimum value and the maximum value of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint

$$x^4 + y^4 + z^4 = 1.$$

List also the locations (x, y, z) of *all* points where the function takes either the minimum value or the maximum value. When solving this problem, make sure to include all relevant intermediate steps, and to carefully motivate the steps in the calculations. (4p)

1	2	1	6	3	3	2	2	1	2	1	4	1	3	2	3	1	1	1	5	1	2	1	3

Figure 1: The chromosome for Problem 3.

3) Consider the LGP chromosome shown in Figure 1. As usual in LGP, the chromosome defines a list of instructions, each consisting of four genes. The first gene defines the operator from the set $O = \{+, -, *, /\}$ (i.e., the four standard arithmetic operators), such that $1 \Leftrightarrow +, 2 \Leftrightarrow -$, and so on. The second gene defines the destination register, from the set $R = \{r1, r2, r3\}$. The third and fourth genes define the operands, from the set $A = \{r1, r2, r3, c1, c2, c3\}$, where the constant registers take the values $c1 = 1, c2 = 2$, and $c3 = -1$. When evaluating the chromosome for a general variable x, the variable registers are initialized as follows: $r1 = x, r2 = r3 = 0$, and the output, namely a function $g(x)$, is taken as the content of $r2$ at the end of the evaluation. What function $g(x)$ is represented by the chromosome in Fig. 1? Write your answer in the form of a polynomial, $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ (3p)

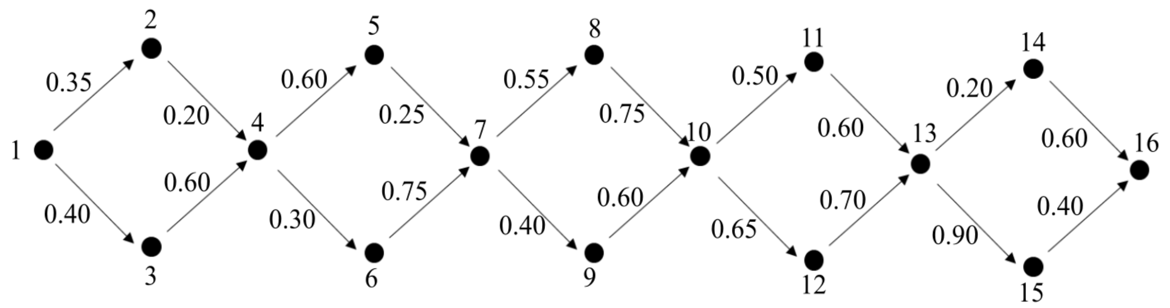

Figure 2: The chain construction graph for Problem 4. The nodes are enumerated from 1 to 16. The decimal numbers on the edges represent the pheromone levels.

4) Chain construction graphs can be used when applying the ant system (AS) algorithm in function optimization tasks. Consider a simple case where the task is to find the minimum of the function

$$f(x) = \left(x - \frac{3}{16}\right)^2$$

The variables x are obtained by artificial ants traversing the chain construction graph in Figure 2, following the standard procedure for such graphs: At the central nodes (1, 4, 7 etc.), the ant uses the standard ACO equation for $p(e_{ij}|S)$ (with $\alpha = \beta = 1$) to take either an up-move or a down-move. An up-move generates a 1 and a down-move generates a 0. Then, the ant moves deterministically to the next central node, and so on, until the last central node (16) is reached. The resulting bit string, which is denoted $a_1 a_2 \ldots$, is then converted to a decimal number in the open range [0,1[, as $x = \sum_j a_j 2^{-j}$ where the sum runs from 1 to 5. Note that the visibility is equal to 1 for all edges, in this case, and the pheromone levels are given in Figure 2.

Consider a population consisting of five ants. The population is evaluated (once) in its entirety, without any pheromone changes. What is the probability that at least one of the five ants generates the value of x that corresponds to the global minimum of $f(x)$? (4p)

5) Consider a simple, one-dimensional application of PSO, in which the goal is to minimize the function $f(x) = (x - (1/4))$^2, using a swarm size of three. Initially, the three particles are located at $x = -1/3$ (particle 1), $x = 0$ (particle 2), and $x = 3/4$ (particle 3), and their speeds are $v = 3$ (particle 1), $v = 1/4$ (particle 2), and $v = -1$ (particle 3). The parameters α and Δt are both equal to 1, w is (here) kept *constant* at the value 1, and $c_1 = c_2 = 2$. Moreover, assume (somewhat unrealistically) that the random numbers q and r are (always) both equal to 1. The initial range $[x_{\min}, x_{\max}]$ is equal to $[-2,2]$, and the particle speeds are restricted to a maximum of 4. Given these parameters, determine, under the PSO algorithm

(a) the velocities and positions of all particles after one iteration (i.e. one updating step for both velocities and positions). (2p)
(b) the velocities and positions of all particles after two iterations. (2p).
Note: For part (b), points are only given if the results from part (a) are correct.

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Solutions to the exam (2022-10-26)

1. 1.1 D. Only velocities should be restricted.

   1.2 B. Both eigenvalues are positive, so the function is convex.

   1.3 B. The probability equals $p = (1 - p_{\text{tour}})p_{\text{tour}}$.

   1.4 D. The list of points found by applying the Lagrange multiplier method contains all optima of f subject to the constraint, both local and global ones.

   1.5 C. The stationary points are $x = -1$ and $x = 2$.

   1.6 E. The probability equals $p_2 = \frac{1}{25}\left(2\frac{3}{4} + 6\frac{1}{4} + 1\right) = 0.16$.

   1.7 A. Any number of genes (in the range $[0, m]$) can mutate, in principle, since the mutation is applied on a gene-by-gene basis.

   1.8 C. No general statement can be made regarding the symmetry of either matrix.

   1.9 A. An individuals with fitness 0 will not be selected with RWS, in which selection is directly proportional to the fitness values. However, it can be selected with TS, which only considers the fitness values (of the participants in the tournament) relative to each other.

   1.10 D. Since the pheromone levels are the same on all edges, the pheromone need not be considered. With $\beta = 1$, the probability of selecting node 3 equals

$$p = \frac{5}{1+2+5} = \frac{5}{8} \tag{1}$$

2. With f and h as specified in the problem formulation, L takes the form

$$L = x^2 + y^2 + z^2 + \lambda(x^4 + y^4 + z^4 - 1). \tag{2}$$

so that, setting the partial derivatives of L to 0, one obtains

$$\frac{\partial L}{\partial x} = 2x + 4\lambda x^3 = 0, \tag{3}$$

$$\frac{\partial L}{\partial y} = 2y + 4\lambda y^3 = 0, \tag{4}$$

$$\frac{\partial L}{\partial z} = 2z + 4\lambda z^3 = 0, \tag{5}$$

$$\frac{\partial L}{\partial \lambda} = x^4 + y^4 + z^4 - 1 = 0. \tag{6}$$

It is clear that λ cannot be equal to 0, since that would give $x = y = z = 0$ which violates the constraint. With $\lambda \neq 0$ one gets the following potential solutions for x

$$x = 0 \tag{7}$$

or

$$x^2 = -\frac{1}{2\lambda}, \tag{8}$$

and similar potential solutions for y and z. First, consider the case where two of the variables, say x and y, take the value 0. In that case, one obtains (using the constraint equation)

$$z = \pm 1. \tag{9}$$

so that the following stationary points are found

$$P_{1,2} = (0, 0, \pm 1). \tag{10}$$

Using the fact that the problem is symmetric in the three variables, one also obtains

$$P_{3,4} = (\pm 1, 0, 0) \tag{11}$$

if $y = z = 0$, and

$$P_{5,6} = (0, \pm 1, 0) \tag{12}$$

if $x = z = 0$. Next, consider the case where one of the variables takes the value 0. Starting, for example, with $x = 0$ and using the equations (see above)

$$y^2 = -\frac{1}{2\lambda} \tag{13}$$

and

$$z^2 = -\frac{1}{2\lambda}, \tag{14}$$

one gets $y^2 = z^2$ so that, $y = \pm z$. Using the constraint equation one then finds

$$y = \pm 2^{-1/4}, \tag{15}$$

thus giving the following stationary points of L

$$P_{7,8,9,10} = (0, \pm 2^{-1/4}, \pm 2^{-1/4}). \tag{16}$$

Using again the symmetry of the problem, one also finds additional stationary points at

$$P_{11,12,13,14} = (\pm 2^{-1/4}, 0, \pm 2^{-1/4}) \tag{17}$$

if $y = 0$ and

$$P_{15,16,17,18} = (\pm 2^{-1/4}, \pm 2^{-1/4}, 0) \tag{18}$$

for $z = 0$. Finally, consider the case where none of the variables takes the value zero. In that case

$$x^2 = y^2 = z^2 = -\frac{1}{2\lambda}, \tag{19}$$

so that the constraint gives $3x^4 = 1$, from which one obtains the following stationary points of L

$$P_{19,20} = (\pm 3^{-1/4}, \pm 3^{-1/4}, \pm 3^{-1/4}). \tag{20}$$

The values of the function f at these points are: $f = 1$ at $P_1, \ldots, P_6$, $f = \sqrt{2}$ at $P_7, \ldots, P_{18}$, and $f = \sqrt{3}$ at $P_{19,20}$. Thus, the minimum value of f is equal to 1, and is attained as $P_1, \ldots, P_6$, whereas the maximum value is $\sqrt{3}$, and is attained at $P_{19,20}$. Note: This problem can also be solved by substituting $x^2 = a$, $y^2 = b$ and $z^2 = c$, and then solving for (a, b, c), while making sure to include both possibilities (e.g. $x = \pm a$) when finding the stationary points.

3. Given the specification in the problem formulation, the instructions (here denoted $I_j, j = 1, \ldots 6$) can be decoded as

I_1	1216:	$r_2 := r_1 + c_3$
I_2	3322:	$r_3 := r_2 \times r_2$
I_3	1214:	$r_2 := r_1 + c_1$
I_4	1323:	$r_3 := r_2 + r_3$
I_5	1115:	$r_1 := r_1 + c_2$
I_6	1213:	$r_2 := r_1 + r_3$

Starting from $r_1 = x$, $r_2 = r_3 = 0$, one then gets

	r_1	r_2	r_3
I_1	x	$x-1$	0
I_2	x	$x-1$	$(x-1)^2$
I_3	x	$x+1$	$(x-1)^2$
I_4	x	$x+1$	$x+1+(x-1)^2$
I_5	$x+2$	$x+1$	$x+1+(x-1)^2$
I_6	$x+2$	$x+2+x+1+(x-1)^2$	$x+1+(x-1)^2$

Hence, the function obtained (from r_2) at the end of the calculation is

$$g(x) = x + 2 + x + 1 + (x-1)^2 = 2x + 3 + x^2 - 2x + 1 = x^2 + 4. \tag{21}$$

4. The minimum of the simple quadratic function $f(x)$ is clearly at $x = 3/16$. Considering the decoding procedure specified in the problem, this corresponds to the chromosome 00110 (which will then be decoded to give $x = 2^{-3} + 2^{-4} = 3/16$). In order to generate this chromosome, an ant must traverse the construction graph such that it first makes two down-moves (at Nodes 1 and 4), and two up-moves (at Nodes 7 and 10), and then one down-move (at Node 13). Note that, at Nodes $2, 3, 5, 6, \ldots$, the move to the next node (i.e. $4, 7, \ldots$) is deterministic and does not produce any output. Consider now the tour of one ant. Use a simplified notation, such that $p(e_{ij}|S)$ is denoted $p_{i,j}$.

   At Node 1, the ant can either go to Node 2 or to Node 3. Noting that the visibility (η_{ij}) is equal to 1 for all edges, the probability of making the required down-move (i.e. going to Node 3, so as to generate $a_1 = 0$) can be computed as

$$p_{3,1} = \frac{\tau_{31}^{\alpha}}{\tau_{31}^{\alpha} + \tau_{21}^{\alpha}} = \frac{0.40}{0.40 + 0.35} = \frac{8}{15}, \tag{22}$$

   where, in the second step, the fact that $\alpha = 1$ has been used. The ant then moves from Node 2 to Node 4, in preparation for the next bit generation step. At Node 4, the probability of making a down-move (to output $a_2 = 0$) equals

$$p_{6,4} = \frac{0.30}{0.30 + 0.60} = \frac{1}{3}. \tag{23}$$

   Next, after moving to Node 7, the ant should then move to Node 8 (to yield $a_3 = 1$). The probability for this move equals

$$p_{8,7} = \frac{0.55}{0.55 + 0.40} = \frac{11}{19}. \tag{24}$$

   Then, after reaching Node 10, the ant should move to Node 11 (to generate $a_4 = 1$). The probability for this move equals

$$p_{11,10} = \frac{0.50}{0.50 + 0.65} = \frac{10}{23}. \tag{25}$$

   After going to Node 13, the ant should then move to Node 15, to yield $a_5 = 0$. This probability for making this move is

$$p_{15,13} = \frac{0.90}{0.20 + 0.90} = \frac{9}{11}. \tag{26}$$

Thus, the probability P of generating 001100 as output equals

$$P = p_{3,1} \times p_{6,4} \times p_{8,7} \times p_{11,10} \times p_{15,13} \approx 0.036613. \tag{27}$$

Now, the population consists of five ants that generate their paths independently of each other, and with the same pheromone levels, as specified in the problem formulation. For any given ant, the probability of *not* finding the required path is equal to $1 - P$. The probability that no ant finds this path thus equals $(1 - P)^5$ and therefore the probability Π that *at least* one ant finds the path is

$$\Pi = 1 - (1 - P)^5 = 0.1701, \tag{28}$$

and this is the answer.

5. (a) Initially, the function values are 49/144 (particle 1), 1/16 (particle 2), and 1/4 (particle 3). Thus, the swarm best position is equal to the position of particle 2 (i.e. $x = 0$). With the simplifications, the velocity update takes the form

$$v_i \leftarrow v_i + 2(x_i^{\text{pb}} - x_i) + 2(x^{\text{sb}} - x_i),\ \ i = 1, 2, 3. \tag{29}$$

One then obtains:

$$v_1 = 3 + 2(0 - (-1/3)) + 2(0 - (-1/3)) = 11/3, \tag{30}$$

$$v_2 = 1/4 + 2(0 - 0) + 2(0 - 0) = 1/4, \tag{31}$$

and

$$v_3 = -1 + 2(3/4 - 3/4) + 2(0 - 3/4) = -5/2. \tag{32}$$

All computed speed values have magnitudes below the limit ($v_{\max} = 4$). Thus, using the equation $x \leftarrow x + v$, the new positions become

$$x_1 = -1/3 + 11/3 = 10/3, \tag{33}$$

$$x_2 = 0 + 1/4 = 1/4, \tag{34}$$

$$x_3 = 3/4 - 5/2 = -7/4. \tag{35}$$

(b) In the second iteration, the swarm best position is $x = 1/4$, i.e. the position of particle 2 (which, of course, also is the particle best position for that particle). The particle best position is unchanged for particle 1 and particle 3, since the function values at their new positions exceed those obtained at their initial positions. Using the same equations as above, one obtains

$$v_1 = 11/3 + 2(-1/3 - 10/3) + 2(1/4 - 10/3) = -59/6. \tag{36}$$

However, this value exceeds (in magnitude) the maximum (negative) speed of -4, meaning that the actual speed of the particle will be $v_3 = -4$ instead. For particle 2 one gets

$$v_2 = 1/4 + 2(1/4 - 1/4) + 2(1/4 - 1/4) = 1/4 \tag{37}$$

and for particle 3

$$v_3 = -5/2 + 2(3/4 - (-7/4)) + 2(1/4 - (-7/4)) = 13/2. \tag{38}$$

This value is larger than the limit of 4, so that the actual speed will be $v_3 = 4$ instead. Thus, finally, one obtains

$$x_1 = 10/3 - 4 = -2/3, \tag{39}$$

$$x_2 = 1/4 + 1/4 = 1/2, \tag{40}$$

and

$$x_3 = -7/4 + 4 = 9/4. \tag{41}$$