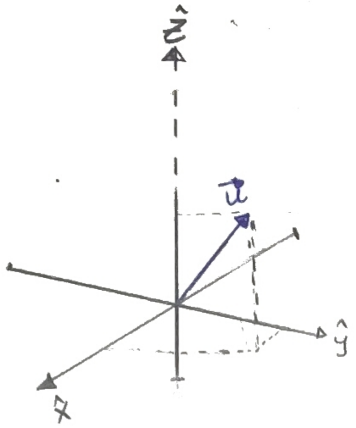
20/03

Repetition vektor Algebra:

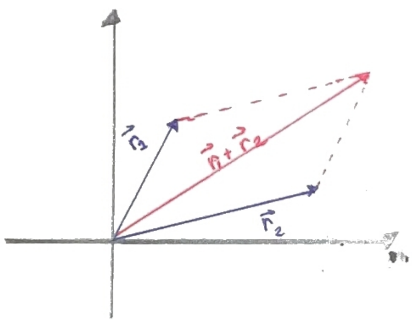
Vektor:



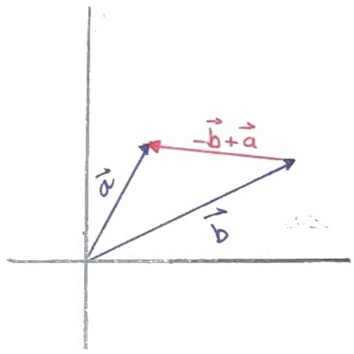
$$\vec{u} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{u} = [x, y, z]$$

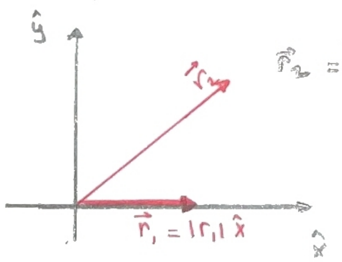
Vektorer kan adderas:



Vektorer kan substraheras



Geometri $\Longleftrightarrow$ Vektor algebra



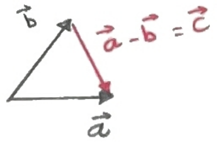
$$\vec{r}_2 = |r_2| \cos\theta \; \hat{x} + |r_2| \sin\theta \; \hat{y}.$$

Skalärprodukt av två vektorer:

$$\vec{r}_1 \cdot \vec{r}_2 = |r_1|\,\hat{x} \cdot (|r_2| \cos\theta\, \hat{x} + |r_2| \sin\theta\, \hat{y})$$

$$= |r_1|\,|r_2| \cos\theta$$

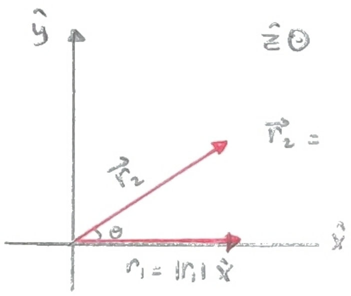
Exempel: (härledning av cosinussatsen)



$$\vec{c}^{\,2} = (\vec{a} - \vec{b})(\vec{a} - \vec{b}) = \vec{a}^{\,2} + \vec{b}^{\,2} - 2\vec{a}\,\vec{b}$$

$$\vec{c}^{\,2} = \vec{a}^{\,2} + \vec{b}^{\,2} - 2ab\cos\theta \implies \cos(\theta) = \frac{a^2 + b^2 - c^2}{2ab} = \text{cosinussatsen}$$

Kryssprodukt:



$\hat{z} \odot$

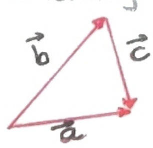
$$\vec{r}_2 = |r_2| (\cos\theta\, \hat{x} + \sin\theta\, \hat{y})$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ |\vec{r}_1| & 0 & 0 \\ |\vec{r}_2| \sin\theta & |\vec{r}_2| \cos\theta & 0 \end{bmatrix} = \hat{z}\, |r_1|\,|r_2| \sin\theta$$

$(\vec{r}_1 \times \vec{r}_2) \perp \vec{r}_1$

$(\vec{r}_1 \times \vec{r}_2) \perp \vec{r}_2$

Exempel: (härledning av sinussatsen)



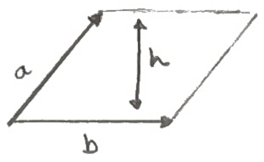
$$|\vec{a} \times \vec{c}| = |\vec{a} \times (\vec{a} - \vec{b})|$$

$$|a||c| \sin\theta_b = |\vec{a} \times \vec{b}| = |a||b| \sin\theta_c$$

$$\implies \frac{\sin\theta_b}{|b|} = \frac{\sin\theta_c}{|c|}$$

Kryssprodukt tolkning:

$|\vec{a} \times \vec{b}| = ab \sin\theta$



arean $= |b| \cdot h = |b||a| \sin(\theta) = |\vec{a} \times \vec{b}|$

normalvektor till $\vec{a}, \vec{b}$ planet

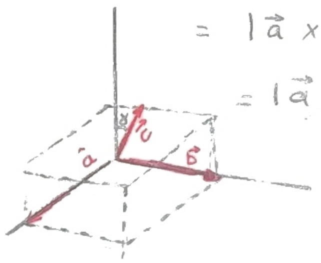
$$\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \hat{n}$$

Volym $=$ arean $\cdot$ h

$= |\vec{a} \times \vec{b}| \; |c| \cos(\gamma)$

$= |\vec{a} \times \vec{b}| \; \vec{c} \cdot \hat{z}$

$= |\vec{a} \times \vec{b}| \; \left| \vec{c} \cdot \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} \right|$



$= |\vec{c}(\vec{a} \times \vec{b})| = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$\cos(\gamma) = \frac{|\vec{c} \cdot (\vec{a} \times \vec{b})|}{|\vec{a} \times \vec{b}||c|}$$

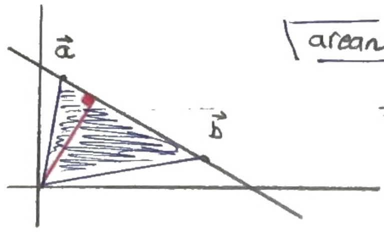
Kryss triple produkt: $\vec{a} \times \vec{b} \times \vec{c}$

Obs! $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ (ej associativ)

Jacobi identitet: $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

lagrangsidentitet: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

Exempel: avstånd från origo till en linje:

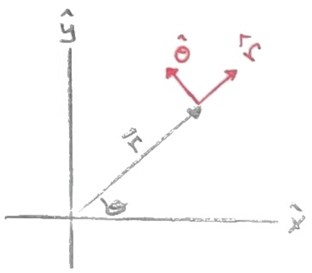


$$|\text{arean}| = |\vec{a} - \vec{b}| \cdot d = \tfrac{1}{2} |\vec{a} \times \vec{b}|$$

$$\Rightarrow d = \frac{|\vec{a} \times \vec{b}|}{|\vec{a} - \vec{b}|}$$

Kap 5.

Polära koordinat:



$$\hat{r} = \cos(\theta_t)\,\hat{x} + \sin(\theta_t)\,\hat{y}$$

$$\hat{\theta} = -\sin(\theta_t)\,\hat{x} + \cos(\theta_t)\,\hat{y}$$

Obs: θ är beroende av tiden t

$$\frac{d\hat{\theta}}{dt} = -\hat{r}\,\dot{\theta}$$

$$\frac{d\hat{r}}{dt} = -\dot{\theta}\sin(\theta_t)\,\hat{x} + \dot{\theta}\cos(\theta_t)\,\hat{y}$$

$$= \dot{\theta}\,(\sin(\theta_t)\,\hat{x} + \dot{\theta}\cos(\theta_t)\,\hat{y}$$

$$= \dot{\theta}\,\hat{\theta}$$

Hela vektorn $\vec{r}$

$$\vec{r}_{(t)} = r_{(t)}\,\hat{r}_{(t)}$$

differentiera map tiden

$$= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \boxed{\dot{r}\hat{r} + r\omega\hat{\theta} = \frac{d\vec{r}}{dt}}$$

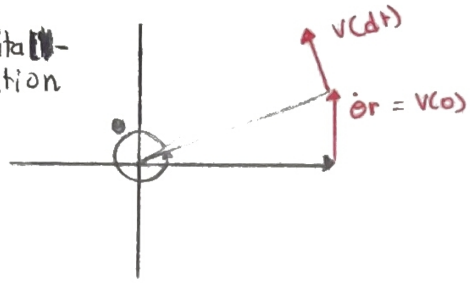
$$\vec{a} = \frac{d}{dt}\frac{d\vec{r}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r}$$

$$\boxed{\vec{a} = (\ddot{r} - \dot{\theta}^2 r)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}}$$

där $\dot{\theta}^2 r$ är centripitalacceleration
$\dot{r}\dot{\theta}$ är corioliskraften

obs! r enkelskrivet betyder beloppet $|r|$

Centripital-
-acceleration

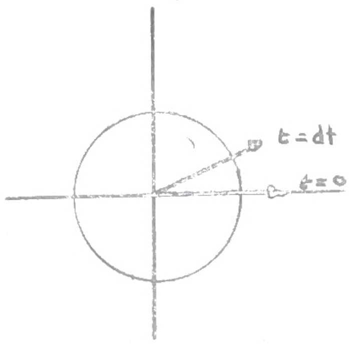


$dv = V(dt) - V(0)$

$= \dot{\theta} r \hat{\theta}(dt) - \dot{\theta} r \hat{\theta}(t)$

$\frac{dv}{dt} = -\dot{\theta} r \frac{d\theta}{dt} = -\dot{\theta}^2 r \hat{r}$

Coriolis kraft:



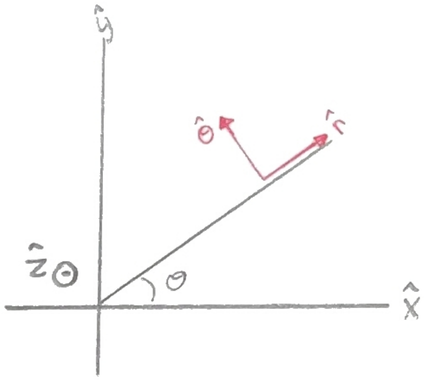
radie ändring → hastighet V_θ ändring

$dV_\theta = \dot{\theta} dr + \dot{r}\, d\theta$

riktning ändring pga rotation

$a_\theta = \frac{dV_\theta}{dt} = \dot{\theta}\frac{dr}{dt} + \frac{\dot{r}\,d\theta}{dt} = 2\dot{r}\dot{\theta}$

Kap 5 i 3dimensioner



$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$

$\hat{z}\times\hat{r} = \cos\theta\,\hat{z}\times\hat{x} + \sin\theta\,\hat{z}\times\hat{y}$

$= \cos\theta\,\hat{y} - \sin\theta\,\hat{x} = \hat{\theta}$

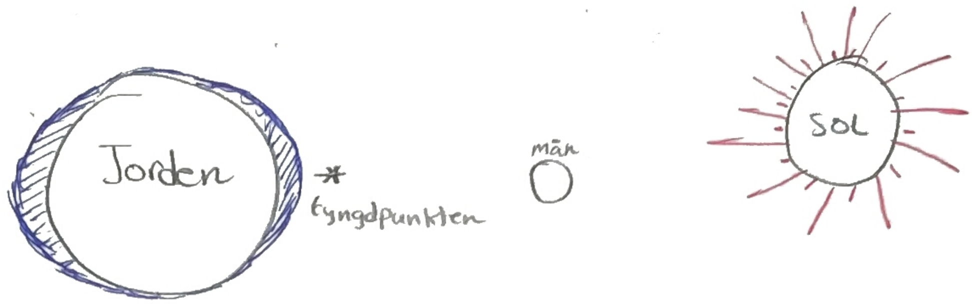
Koordinat oberoende

$\vec{r} = r\hat{r}$ där r = konstant

$$\frac{d\vec{r}}{dt} = r\dot{\theta}\hat{\theta} = r\dot{\theta}(\hat{z}\times\hat{r}) = (\dot{\theta}\hat{z})\times(r\hat{r})$$

$$\boxed{\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}}$$ för en konstant radie

Centripitalkraftens inverkan (tidvatten)



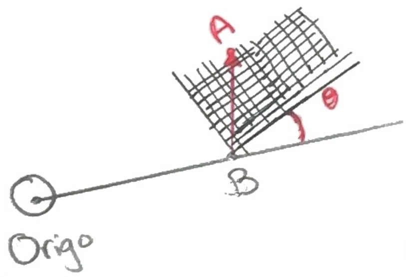
tidvatten sker var 12:e timma
pga solen och månen

Tidvatten tar energi!
tar rörelsenergi och gör om den till värme!
tar energi från jordens rotation!

Obs! (Månen hade en rotation en gång i tiden! men iom att "tidsvattens-
-effekten" är större på månen så upphörde rotationen)

Sammansatt rörelse

Rotation & translation!



$$\vec{V}_a = \vec{V}_B + \vec{V}_{A/B} \iff \vec{V}_A = \vec{V}_B + (\vec{V}_A - \vec{V}_B)$$

$$\vec{a}_A = \vec{a}_B + \vec{a}_{A/B}$$
$$= \vec{a}_B + \left(\vec{a}_{rel} + \omega \times r_{A/B} + 2\omega \times V_{rel} + \omega \times (\omega \times \hat{r})\right)$$

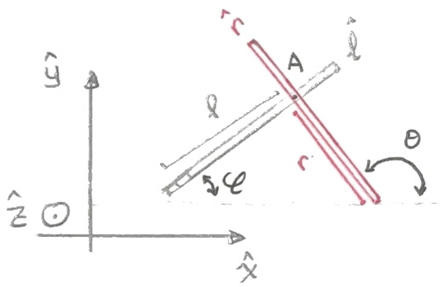
Viktigt specialfall

$$\vec{V}_A - \vec{V}_B = \vec{V}_{A/B}$$

$$= \vec{\omega} \times \vec{r}_{A/B} + \dot{r}_{A/B}\, \hat{r}_{A/B}$$

$$= \vec{\omega} \times \vec{r}_{A/B}$$ (om $\dot{r}_{A/B} = 0$, dvs om radien är konstant)

Räkneexempel:



Fråga va blir $\dot{r}$ $\circeq$ $\dot{\varphi}$ under detta läge? (som funktion av $\dot{\theta}$)

$$\vec{V}_A = \underbrace{\underbrace{\dot{\varphi}\,\hat{z} \times \ell\hat{\ell}}_{\text{sett från } \varphi} = \underbrace{\dot{r}\hat{r} + \dot{\theta}\,\hat{z} \times (r\hat{r})}_{\text{sett från } \theta}}_{\text{Iom A's rotation måste vara samma vid båda refferenspunkterna}}$$

2 okända $\circeq$ 1 ektion, men det är ok ty 2d-ekvation

$$\Longrightarrow \dot{\varphi}\ell\,(\hat{z} \times \hat{\ell}) = \dot{r}\hat{r} + \dot{\theta}r\,(\hat{z} \times \hat{r})$$

(I) $\hat{\ell}\,(\hat{z} \times \hat{\ell}) = \hat{z}\,(\hat{\ell} \times \hat{\ell}) = 0$ vi multiplicerar allt med $\hat{\ell}$ så att vi blir av med $V \cdot \ell$

$$\Rightarrow \hat{l}\,(\dot{\varphi} l \,.\, \hat{z} \times \hat{l}) = \hat{l}\,\dot{r}\hat{r} + \hat{l}\,\dot{\theta} r\,(\hat{z}\times\hat{r})$$

$$0 = \hat{l}\,\dot{r}\hat{r} + \hat{l}\,\dot{\theta} r\,(\hat{z}\times\hat{r}) =$$

$$= \dot{r}\cos(\theta-\varphi) + \dot{\theta} r\,\hat{z}\,(\hat{r}\times\hat{l})$$

$$= \dot{r}\cos(\theta-\varphi) - \dot{\theta} r \sin(\theta-\varphi)$$

$$\Rightarrow \boxed{\dot{r} = \dot{\theta} r \tan(\theta-\varphi)}$$

$\dot{\varphi}$ lösas analogt

$$\dot{\varphi} l\,\hat{r}\,.\,(\hat{z}\times\hat{l}) = \dot{r}$$

$$\dot{\varphi} l\,(\hat{z}(\hat{l}\times\hat{r})) = \dot{r}$$

$$\dot{\varphi} l \sin(\theta-\varphi) = \dot{\theta} r \quad \frac{\sin(\theta-\varphi)}{\cos(\theta-\varphi)}$$

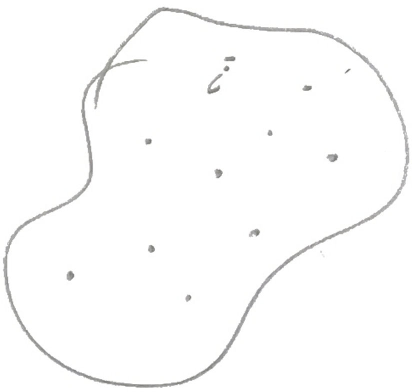
$$\Rightarrow \boxed{\dot{\varphi} = \frac{\dot{\theta} r}{l\cos(\theta-\varphi)}}$$

Kapitel 6 - Intro

Notation:

$\vec{G}$ = rörelsemängd (momentum)
$\vec{F}$ = kraft (force)
$\vec{M}$ = Vridmoment (Torque)
$\vec{H}$ = rörelsemängdsmoment (Angular momentum)

Partikelsystem: (index i) indexerar de enskilda partiklar (kropparna)



Total kraft : $\vec{F}_{tot} = \sum_i \vec{F}_i$

$$= \sum_i m_i \vec{a}_i$$

$$= \sum_i m_i \frac{d}{dt}\vec{V}_i = \frac{d}{dt} \underbrace{\sum m_i \vec{V}_i}_{\vec{G}}$$

$$\Longrightarrow \boxed{\vec{F}_{tot} = \frac{d}{dt}\vec{G}}$$

$$\vec{G}_{tot} = \sum_i m_i \frac{d}{dt}\vec{r}_i$$

massCentrum

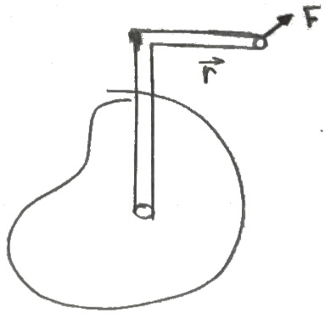
$$= \frac{d}{dt}\sum_i m_i \vec{r}_i = \frac{d}{dt}\left(\frac{1}{m_{tot}} \sum_i m_i \vec{r}_i\right) m_{tot} = m_{tot}\frac{d}{dt}\vec{R}_{cm}$$

$$= m_{tot}\vec{V}_{cm} = \vec{G}_{tot}$$

$$\vec{F}_{tot} = m_{tot}\dot{\vec{V}}_{tot} = \dot{\vec{G}}$$

$$\dot{R}_{cm} \cdot m_{tot} = \vec{G}$$

Arbete och energi



Arbete: $\int \vec{F} \cdot d\vec{r}$

$$\Longrightarrow \quad \frac{d\vec{U}}{dt} = (\vec{r} \times \vec{F}) \cdot (\hat{z}\, \frac{d\theta}{dt}) = \vec{\omega} \cdot (\vec{r} \times \vec{F})$$

$$= \underbrace{\vec{\omega} \cdot \vec{M}_{tot}}_{\text{Effekt}}$$

$$dU = \vec{F}\, d\vec{r} = \vec{F} \cdot (\hat{z} \times \vec{r}\, d\theta)$$
$$= \hat{z} \cdot (\vec{r} \times \vec{F})\, d\theta = \vec{M}(\hat{z}\, d\theta)$$

Kinetisk energi

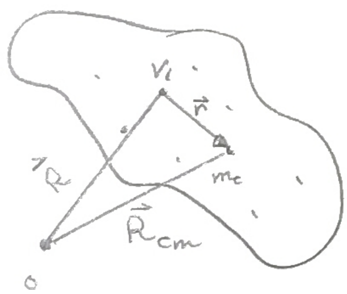
$$T = \sum \frac{1}{2} m_i \vec{V}_i^2$$

$$= \sum_i \frac{1}{2} m_i \left(\vec{V}_{cm} + \vec{V}_i\right)^2$$

$$= \sum_i \frac{1}{2} m_i \left(\vec{V}_{cm}^2 + \underbrace{\cancel{2\vec{V}_{cm} \cdot \vec{V}_i}}_{=0} + \vec{V}_i^2\right)$$

$$= \sum_i \frac{1}{2} m_i V_{cm}^2 + \frac{1}{2} \sum_i m_i \vec{V}_i^2$$

$$= \underbrace{\frac{1}{2} m_{tot} \vec{V}_{cm}}_{\text{Ek för masscentrum}} + \underbrace{\frac{1}{2} \sum m_i \vec{V}_i^2}_{\text{Intern kinetisk energi}} \quad \Longleftrightarrow \quad T_{tot} = T_0 + T_{cm}$$



$$\vec{R} = \vec{R}_{cm} + \vec{r}$$

ty: $\vec{V} = V_{cm} + V_i$

$$T_{cm} = \frac{1}{2} \sum_i m_i \, (\vec{\omega} \times \vec{r}_i)^2$$

om den roterar

Specialfall: $\left.\begin{array}{l} \vec{\omega} = (0, 0, \omega) \\ \vec{r} = (x, y, z) \end{array}\right\}$ $\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & z \end{bmatrix} = \omega (x\hat{y} - y\hat{x})$

$$\Longrightarrow T_{cm} = \frac{1}{2} \sum_i m_i \, \omega^2 \, (x_i^2 \times y_i^2)$$

$$= \frac{1}{2} \omega^2 \sum_i m_i \, (x_i^2 \times y_i^2) = \frac{1}{2} \omega^2 I_{zz}$$

$$\Longrightarrow \boxed{E_k = \frac{1}{2} m_{tot} V^2 + \frac{1}{2} \omega^2 I}$$

Sammanfattning

Linjära koordinator	Rotationer
$\vec{V}$: hastighet	$\vec{\omega}$: vikelhastighet
$\vec{G}$: rörelsemängd	$\vec{H}$: rörelsemängdsmoment
$\vec{F}$: kraft	$\vec{M}$: Vridmoment
m : tröghet massa	I Tröghetmoment

Kinetisk-energi : $\frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$

Effekt : $\vec{F} \cdot \vec{V} + \vec{M} \cdot \omega$

$\vec{F} = \dot{\vec{G}}$ $\qquad$ $\vec{M} = \dot{\vec{H}}$

$\vec{V}_{tot} = \vec{V}_A + \vec{V}_B$ $\qquad$ $\vec{\omega}_{tot} = \vec{\omega}_A + \vec{\omega}_B$

Men !! $\vec{r}_{tot} = \vec{r}_A + \vec{r}_B$ $\qquad$ $\theta_{tot} \neq \theta_A + \theta_B$

Tröghetsmoment: Ex

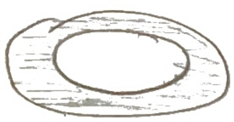
$$I_{zz} = \iiint \rho (x^2 + y^2)\, dx dy dz$$

Ex | Disk!

$$I_{disk} = \frac{1}{2} \cdot m r^2 = \frac{1}{2} (\rho r^2) r^2 = \frac{1}{2} \rho r^4$$

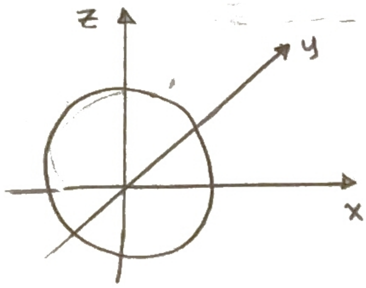
Om ihålig, där inre radie: r, yttre radie: R

$$I = \frac{1}{2} \frac{\cancel{\rho} (R^4 - r^4) m}{\cancel{\rho} (R^2 - r^2)}$$



$$\Longrightarrow I = \frac{1}{2} m (R^2 + r^2)$$

Ex 3:



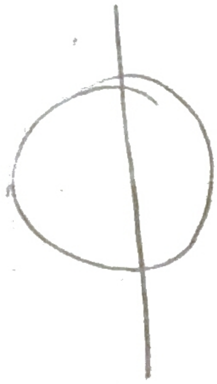
$$I = \oint ds\, r^2 \cos^2 \theta \rho$$

$ds = r d\theta$

$$= \oint x^2 \rho\, ds = \oint \frac{1}{2} \rho (x^2 + y^2)\, ds$$

$$= \frac{1}{2} r^2 \int \rho\, ds = \frac{1}{2} m r^2$$

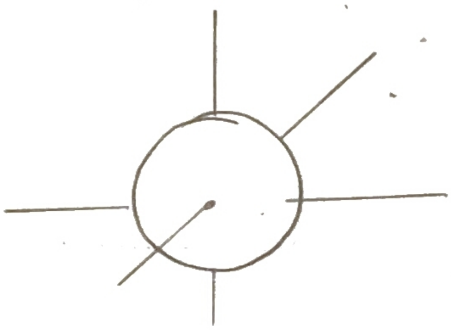
Ex 4



Rotation av disk runt axel

$$I_{disk} = \frac{1}{2} I_{ring} = \frac{1}{4} mr^2$$

Ex 5



$$I_{sfär} = \iint dA\,(x^2+y^2)\,\rho$$

$$= \iint dA\,\frac{2}{3}(x^2+y^2+z^2)$$

$$= \iint dA\left(\frac{2}{3}r^2\right)\rho$$

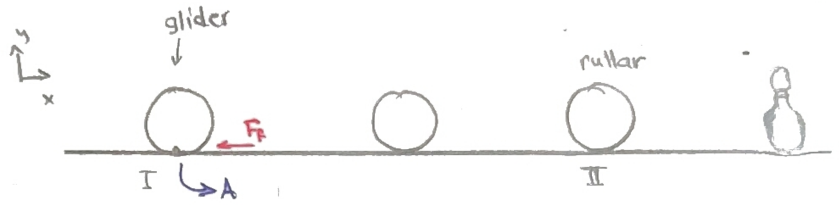
$$= \frac{2}{3}r^2 \iint \rho\, dA = \frac{2}{3}mr^2$$

Tröghetsmoment: $I = \int r^2\, dm$ masselement

"Om massan är ett mått på motstånd för translationell acceleration så är tröghetsmoment motstånd för rotationell accelleration."

Om densiteten är konstant då ges

$$I = \rho \int r^2\, dV$$ Volymelement



klottet glider först sedan rullar

Vad är bevarat?

↳ rörelsemängdmoment ? Ja kring A

genom att räkna ut rörelsemängmoment kring ett punkt

så blir $\vec{r} \times \vec{F} = 0$

$\Longrightarrow \mathbb{H}_0 =$ konstant

1. glider först med fart v

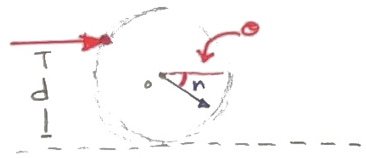
2. Hur snabbt rullar den i slutet

Högerhandsregel ty Vi x-led, r i y-led

$$H_I = -m v r \hat{z}$$

$$H_{II} = (-m\hat{v} r - I\omega)\hat{z} = -m\hat{v} r - \frac{2}{5} m r^2 \omega_{II}$$

Ex: Biljardboll :



Var ska vi stöta kulan så den rullar iväg utan att glida ?

$$F = m \cdot a_{cm}$$

Vridmoment $M_0 = I_0 \ddot{\theta}$ ← tröghetsmoment

$I_0 = \frac{2}{5} m r^2$ (för homogent klot)

införa att den rullar utan att glida

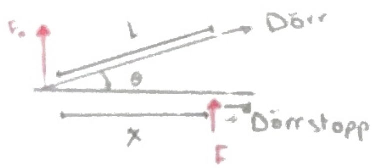
$\Longrightarrow \left.\begin{matrix} V = \dot{\theta} r \\ a = \ddot{\theta} r \end{matrix}\right\}$ rullar ej glider

$$\Longrightarrow F = m \cdot \ddot{\theta} r$$

$$M = F(d-r) = \frac{2}{5} m r^2 \ddot{\theta}$$

$$\Longrightarrow \frac{F(d-r)}{F} = \frac{2}{5} \frac{m r^2}{m r} \qquad \boxed{d = \left(1 + \frac{2}{5}\right) r}$$

Ex:



Var ska dörrstoppet placeras för att inte belasta gångjärnet?

$$F = m\, a_{mc} = m \cdot \frac{L}{2} \ddot{\theta} \qquad \text{Va blir } x \text{ så att } F_0 = 0$$

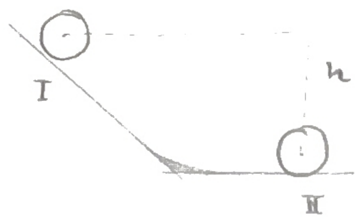
$$M_{mc} = \left(x - \frac{L}{2}\right) F = I_{mc} \cdot \ddot{\theta}$$

$$I = \frac{1}{12} m L^2$$

$$\frac{\left(x - \frac{L}{2}\right) F}{F} = \frac{I_{cm} \ddot{\theta}}{m \frac{L}{2} \ddot{\theta}} \Longrightarrow x - \frac{L}{2} = \frac{I_{cm}}{m \frac{L}{2}} = \frac{\frac{1}{12} m L^2}{\frac{1}{2} m L}$$

$$\Longrightarrow x - \frac{L}{2} = \frac{1}{6} L \Longrightarrow x = \left(\frac{1}{6} + \frac{1}{2}\right) L = \frac{8}{12} L = \frac{2}{3} L$$

Ex:



Vad blir sluthastigheten när kulan släpps i vila

Energi bevaras

$$U_I = KE_{II}$$

$$I = kmr^2$$

$$mgh = \frac{1}{2} m V_{cm}^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m V_{cm}^2 + \frac{1}{2} k m \underbrace{r^2 \omega^2}_{V_{cm}^2}$$

$$= \frac{1}{2} m (1+k) V_{cm}^2$$

$$\Longrightarrow V_{cm}^2 = \frac{2gh}{1+k} \Longrightarrow V_{cm} = \sqrt{\frac{2gh}{1+k}}$$

Kapitel 7

Linjära algebra:

$$\vec{y} = A\vec{x}$$

$$y = A\,x$$

Komponentvis.

$$y_i = \sum_{j=1}^{N} A_{ij}\,x_j \quad , \quad A = \begin{bmatrix} A_{11} & A_{12} & & A_{1n} \\ \vdots & & & \vdots \\ A_{N1} & & & A_{Nn} \end{bmatrix}$$

Einsteins Summeringskonvention

$$y_i = A_{ij}\,x_j = A_{ik}\,x_k$$

Summa över alla index som förekommer exakt 2 gånger

Matrismultiplikation

$$(AB)_{ij} = \sum_{k} A_{ik}\,B_{kj} = A_{il}\,B_{lj}$$

Skalär produkt:

$$a \cdot b = a_i b_i = a_k \delta_{kj} b_j = a_k b_k$$

Kryssprodukt:

$$(a \times b)_i = \sum_{jk} \epsilon_{ijk} \quad , \text{ där } \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \text{ \& } \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$$

och alla repeterade index komponenter = 0

Exampel: $(a \times b)_3 = a_i \, b_j \, \epsilon_{3ij}$

i,j kan vara $(1,2)$ eller $(2,1)$

$1 = \epsilon_{312} = -\epsilon_{321}$

$\Longrightarrow (a \times b)_3 = a_1 b_2 - a_2 b_1$

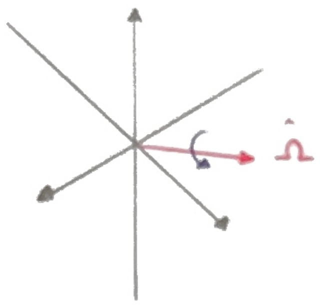
- - - - - - - - - - - - - - - - - -

Vi har $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$

$\epsilon_{ijk} \, \epsilon_{iln} = \delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}$

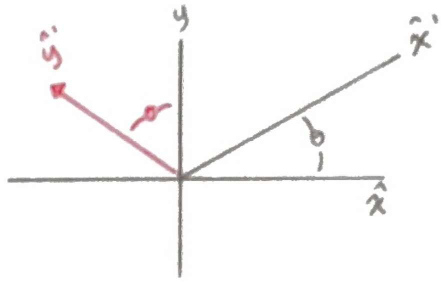
Rotationsmatriser

$R_{\hat{\Omega}}(\theta)$ $\longrightarrow$ Betecknar rotation kring axeln $\hat{\Omega}$ med vinkeln θ



Specialfall: $R_{\hat{z}}(\theta) = R \leftarrow$ Kallar den bara R

$$\hat{x}' = R\hat{x} = x(\cos\theta\ \hat{x} + \sin\theta\ \hat{y})$$
$$\hat{y}' = R\hat{y} = y(-\sin\theta\ \hat{x} + \cos\theta\ \hat{y})$$
$$R\hat{z} = z\hat{z}$$
om $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$



$$\Rightarrow R\hat{z}\,\vec{r} = \cos(\theta)(x\hat{x} + y\hat{y} + z\hat{z}) + \sin(\theta)(x\hat{y} - y\hat{x}) + z\hat{z} - \cos(\theta) z\hat{z}$$

$$\Rightarrow \cos(\theta)\cdot\vec{r} + (1 - \cos(\theta))\,\hat{z}\,(\hat{z}\cdot\vec{r}) + \sin(\theta)(\hat{z} \times \vec{r})$$

På Allmän form:

$$\boxed{R_{\hat{\Omega}}(\theta)\vec{r} = \cos(\theta)\vec{r} + (1 - \cos(\theta))\hat{\Omega}(\hat{\Omega}\cdot\vec{r}) + \sin(\theta)(\hat{\Omega} \times \hat{r})}$$

En allmän rotation av en vektor $\vec{r}$ runt $\hat{\Omega}$ med vinkel θ

Komponenter:

$$(R_{\hat{\Omega}}(\theta)\vec{r})_i = \cos(\theta)\,\vec{r}_i + (1-\cos(\theta))(\hat{\Omega}\cdot\vec{r})\,\hat{\Omega}_i + \epsilon_{ijk}\hat{\Omega}_j \vec{r}_k \sin(\theta)$$

Vill skriva på matrisform:

$(A_{ik})\vec{r}_k \longrightarrow$ en matris vektor

$$(A_{ik})\vec{r}_k = \cos(\theta)\,\delta_{ik}\,\vec{r}_k + (1-\cos(\theta))\,\hat{\Omega}_i(\hat{\Omega}_k\cdot\vec{r}_k) + \sin(\theta)\,\epsilon_{ijk}\,\hat{\Omega}\,\vec{r}$$

$$= \left(\cos(\theta)\,\delta_{ik} + (1-\cos(\theta))\,\hat{\Omega}_i\hat{\Omega}_k + \sin(\theta)\sum_j \epsilon_{ijk}\hat{\Omega}_j\right)\cdot\vec{r}_k$$

varje komponent i rotationsmatrisen!

$$\boxed{(R_{\hat{\Omega}}(\theta))_{ik} = \cos(\theta)\,\delta_{ik} + (1-\cos(\theta))\,\hat{\Omega}_i\hat{\Omega}_k - \sum_j \sin\theta\,\epsilon_{ikj}\,\hat{\Omega}_j}$$

Rotationsmatris runt $\hat{\Omega}$ med vinkeln θ, $\hat{\Omega}$ är en enhetsvektor

Om vi har en godtycklig matris

$R_{\hat{\Omega}}(\theta)$ Hur räknar vi ut $\hat{\Omega}$ och θ?

Först Def: spåret av en matris $A \Longrightarrow sp(A) = \sum_i A_{ii}$

dvs summan av diagonalen

$$Sp(R_{\hat{\Omega}}(\theta)) = 3\cos(\theta) + (1-\cos(\theta)) \sum_i \hat{\Omega}_i \hat{\Omega}_i$$

$$= 3\cos(\theta) + (1-\cos(\theta)) \cdot 1$$

$$= \boxed{1 + 2\cos(\theta)}$$

→ med denna kan man får fram θ som den har roterat

Vad blir Ω ?

titta på $(R_{\hat{\Omega}})_{ik} - (R_{\hat{\Omega}})_{ki} = \sin(\theta)(\epsilon_{ijk}\Omega_j - \epsilon_{kji}\Omega_j)$

$$= \boxed{2\sin(\theta)\epsilon_{ijk} \cdot \Omega_j}$$

→ med denna får man enkelt fram sin vektor Ω

<u>Ex:</u> $(R_{\hat{\Omega}})_{12} - (R_{\hat{\Omega}})_{21} = \sum 2\sin(\theta)\,\epsilon_{1j2}\,\Omega_j$

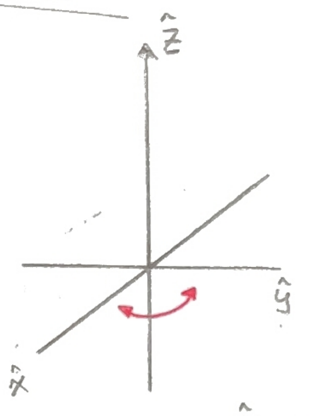
$$= -2\sin(\theta)\,\Omega_3$$

kan endast vara 3

Rotationer:

Ex:

$R_{\hat{z}}(\pi/2)$



$\hat{x} \to \hat{y}$

$\hat{y} \to -\hat{x}$

$\hat{z} \to \hat{z}$

Hur ställer man upp matrisen?

$$R_{\hat{z}}(\pi/2) = \begin{bmatrix} \boxed{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}} \end{bmatrix}$$

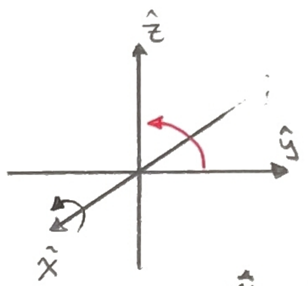
Vad som händer med x-axeln

Vad som händer med y-axeln

Vad som händer med z-axeln

Ex:

$R_{\hat{x}}(\pi/2)$



$$\Rightarrow R_{\hat{x}}(\pi/2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\hat{x} \to \hat{x}$

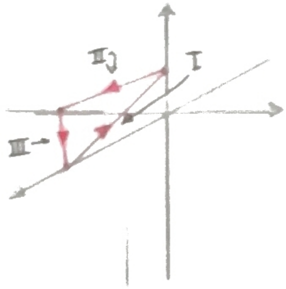
$\hat{y} \to \hat{z}$

$\hat{z} \to -\hat{y}$

Ex:

$R_{\hat{x}}(\pi/2)\ R_{\hat{z}}(\pi/2)$

roterar först kring $\hat{x}$ sedan $\hat{z}$



I $R(\hat{x}) = \hat{z}$

II $R(\hat{z}) = -\hat{y}$

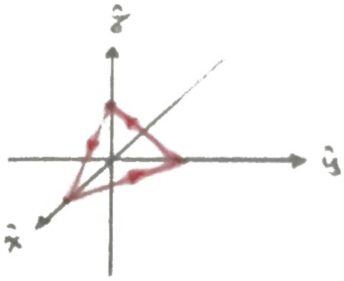
III $R(-\hat{y}) = \hat{z}$

$\hat{x} \to \hat{y},\ \hat{y} \to \hat{z} \Rightarrow \hat{x} \to \hat{z}$

$\hat{z} \to \hat{z},\ \hat{z} \to -\hat{y} \Rightarrow$

$$R_{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}}\left(\frac{2\pi}{3}\right) \qquad R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_{\hat{z}}(\pi/2)\ R_{\hat{x}}(\pi/2)$

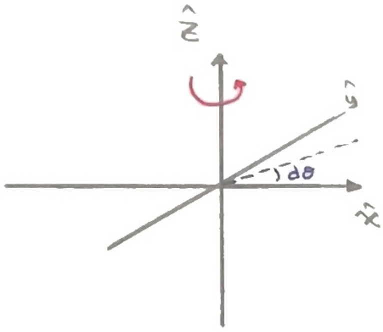


$$\Rightarrow R_{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}}\left(\frac{2\pi}{3}\right)$$

$$\Longrightarrow R_{\Omega_1}(\theta_1)\, R_{\Omega_2}(\theta_2) \neq R_{\Omega_2}(\theta_2)\, R_{\Omega_1}(\theta_1)$$

$\Rightarrow$ Rotationer är ej kommutativt

Betrakta infinitesimal rotation:



$$R_I : \vec{r} \to \vec{r} + d\theta \; z \times \vec{r}$$
$$= \vec{r} + d\vec{\theta}_1 \times \vec{r}$$

$\boxed{\text{där } d\vec{\theta} \equiv d\theta \, \hat{z}} \Rightarrow$ får bara görs vid infinitesimala steg

$$R_{II} : \vec{r} \to \vec{r} + d\vec{\theta}_2 \times \vec{r}$$

Kollar $R_2 R_1 : \vec{r} = R_2(\vec{r} + d\vec{\theta}_1 \times \vec{r}) = (\vec{r} + d\vec{\theta}_1 \times \vec{r}) + d\vec{\theta}_2 \times (\vec{r} + d\theta_1 \times \vec{r})$

$$= \vec{r} + (d\vec{\theta}_1 + d\vec{\theta}_2) \times \vec{r} + $$

till linjär ordning $\longrightarrow$ släpa $(d\vec{\theta}_1 + d\vec{\theta}_2) \times \vec{r}$

$$\frac{d\vec{r}}{dt} = \left(\frac{d\vec{\theta}_1}{dt} + \frac{d\vec{\theta}_2}{dt}\right) \times \vec{r} \Rightarrow \frac{d\vec{r}}{dt} = (\vec{\omega}_1 + \vec{\omega}_2) \times \vec{r}$$

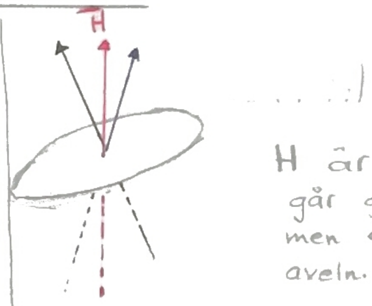
Roterande Kroppar - Godtycklig axel

Tröghetsmoment kring godtycklig axel:

$$\vec{H} = \vec{I}\,\vec{\omega}$$

$$H = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

Antag rotationen kring en fixerad axel



H är noll om axeln går genom mitten men ej för den brutna axeln.

$\Longrightarrow$ $\vec{H}$ är i allmänhet inte parallellt med $\vec{\omega}$

$$\vec{v}_\alpha = \vec{\omega}_\alpha \times \vec{r}_\alpha$$

$$\Longrightarrow \vec{H} = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

$$= \sum_\alpha m_\alpha \left(\vec{\omega} (\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha (\vec{\omega} \cdot \vec{r}_\alpha) \right)$$

$$\vec{H}_i = \sum_\alpha m_\alpha \left(\vec{\omega}_i (\vec{r}_\alpha)_j (\vec{r}_\alpha)_j - (\vec{r}_\alpha)_i \vec{\omega}_j (\vec{r}_\alpha)_j \right)$$

$$\Longrightarrow \boxed{\vec{H}_i = \sum_i I_{ij} \vec{\omega}}$$

$$= \sum_\alpha m_\alpha \left(\delta_{ij} \vec{\omega} \vec{r}_\alpha^{\,2} - \vec{\omega}_j (r_\alpha)_i (\vec{r}_\alpha)_j \right)$$

$$\vec{H}_i = \left[\sum_j \sum_\alpha \left(\delta_{ij} \vec{r}_\alpha^{\,2} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j \right) m_\alpha \right] \vec{\omega}_j = \sum_\kappa \vec{I}_{ij} \vec{\omega}$$

$$\Longrightarrow I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} (\vec{r}_\alpha^{\,2}) - (r_\alpha)_i (r_\alpha)_j \right)$$

$$\boxed{\vec{I}_{ij} = \iiint d^3 r\, \rho(r) \left(\delta_{ij} r^2 - r_i r_j \right)}$$

$$\vec{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Ex[1]: $I_{zz} = \iiint \rho(r)\,(x^2+y^2+z^2-z^2)\,dx\,dy\,dz$

$$= \iiint \rho(x^2+y^2)\,dx\,dy\,dz$$

Ex[2]:

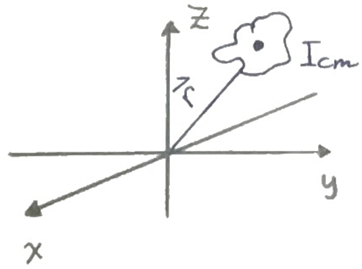
$$\boxed{I_{xy} = -\iiint \rho\, xy\, dx\,dy\,dz} \Rightarrow$$ det som inte är på diagonalen har negativ tecken!

— — — — — — —

Tröghetsmoment

$$\vec{I}_{ij} = \iiint \rho(r)\,(\delta_{ij}\,\vec{r}^{\,2} - \vec{r}_i\,\vec{r}_j)$$

- Parallellförflyttning (Steinerssats)



$$\vec{I}_o = I_{cm} + M_{tot}\,(\delta_{ij}\,\vec{R}^{\,2}_{cm} - \vec{R}_i\,\vec{R}_j)$$

18/04

förra föreläsning:

Tröghetsmoment:

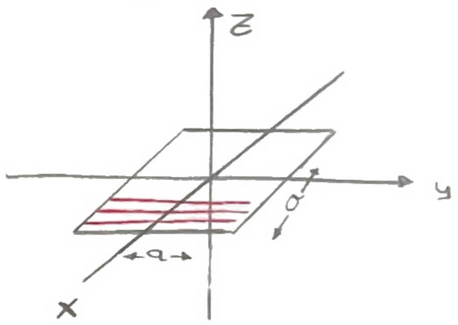
$$\bar{I}_{ij} = \iiint \rho(r) \, (\delta_{ij} \vec{r}^{\,2} - \vec{r}_i \, \vec{r}_j$$

Parallel förflytning; Steinerssats

$$I_{\theta} = m_{\text{totalt}} \, (\delta_{ij} \vec{R}^{2}_{cm} - \vec{R}_i \vec{R}_j) + \bar{I}_{cm}$$

Exempel:

Fyrkant



* $I_{xx} = \iint (\delta_{xx}(x^2 + y^2) - xx) \, \rho \, dx \, dy$

$$= \iint_{-a/2}^{a/2} y^2 \rho \, dx \, dy$$

$$= \rho a \int_{a/2}^{a/2} y^2 \, dy$$

$$= \rho a \frac{1}{3} (y^3) \Big|_{-a/2}^{a/2}$$

$$= \rho \frac{a}{3} \left(\frac{y}{2}\right)^3 2 = \frac{a^4 \rho}{12} = \underbrace{\frac{1}{12}}\, m a^2$$

man känner igen den från stavens tröghetsmoment

* och av semmetri skäl så är $I_{yy} = \frac{1}{12} ma^2$

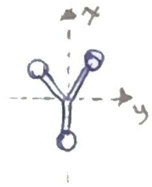
* $I_{zz} = \iint (x^2 + y^2) \, \rho \, dx \, dy = \underbrace{I_{xx} + I_{yy}}$

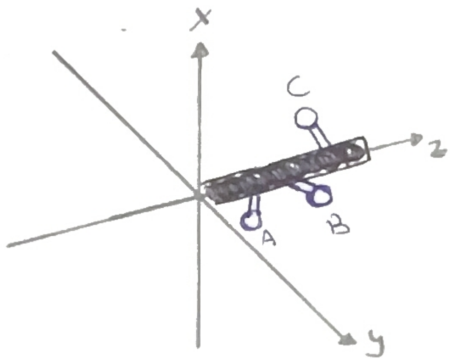
allmänt för en plan yta i $z=0$ planet

$I_{disk} = ?$

$$I_{zz}^{disk} = \frac{1}{2} m r^2 \xrightarrow[\text{skäl}]{\text{av symmetri}} I_{xx} = I_{yy} = \frac{1}{4} m r^2$$

Exempel <u>7.142</u>





<u>Givet:</u> masslös axel, avstånd mellan kulorna (a) och kula A och origo (a)

$$I_{\theta} = \sum_{\alpha} (\delta_{ij} r_{\alpha}^2 - r_{\alpha i} r_{\alpha j}) m$$

$r_A = (-1, 0, 1) a$

$r_B = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 2) a$

$r_C = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 3) a$

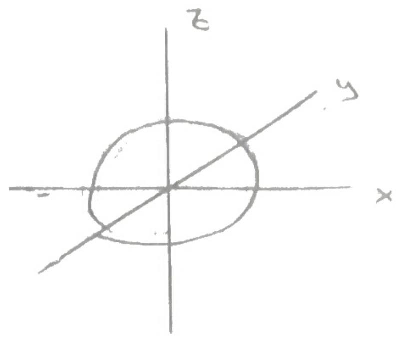
$$17 = (\underset{r_A}{1^2 + 1^2}) + (\underset{r_B}{2^2 + 1^2}) + (\underset{r_C}{3^2 + 1^2})$$

$$= \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix} ma^2 - \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}_A ma^2 - \begin{bmatrix} 3/4 & \sqrt{3}/4 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 4 \end{bmatrix}_B ma^2 - \begin{bmatrix} 3/4 & 0 & 3\sqrt{3}/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_C ma^2$$

$$\bar{I} = \begin{bmatrix} 3\frac{1}{2} & 0 & 2 \\ 0 & 3\frac{1}{2} & \sqrt{3}/2 \\ -3/2 & \sqrt{3}/2 & 3 \end{bmatrix}$$

$$KE = \frac{1}{2} \vec{\omega}\, \bar{I}\, \vec{\omega} = \frac{1}{2} \sum_{ij} \omega_i I_{ij} \omega_j = \frac{1}{2} \omega_z I_{zz} \omega_z = \frac{3}{2} m a^2 \omega^2$$

Exempel : Disk :



$$\underline{I} = \frac{1}{4} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

fall I : rotera runt z-led

$$\vec{H} = \vec{I}\,\vec{\omega} = \frac{1}{4} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \frac{1}{2} m r^2 \omega \hat{z}$$

fall II : rotera runt x-led

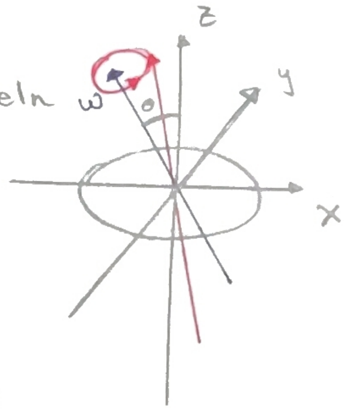
$$\vec{H} = \frac{1}{4} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} m r^2 \omega \hat{x}$$

Obs: $\vec{H} \parallel \vec{\omega}$

Allmänt om $I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$

$\Longrightarrow \vec{H} \parallel \vec{\omega}$ om $\vec{\omega}$ ligger längs en av koordinater

Vi roterar runt den röda axeln

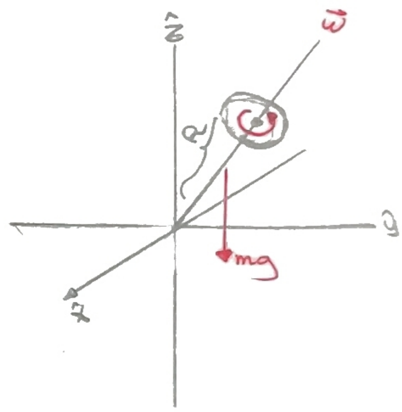


$$\vec{H} = \frac{1}{4} ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix} \omega$$

$$= \frac{1}{4} ma^2 \begin{bmatrix} \sin\theta \\ 0 \\ 2\cos\theta \end{bmatrix} \not\parallel \vec{\omega}$$

$$\Longrightarrow \frac{d\vec{H}}{dt} \neq 0 \Longrightarrow \vec{M} \neq 0$$

Exempel precession:



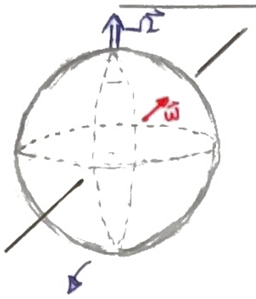
$$\vec{H} = I \, \vec{\omega} = \frac{1}{2} m r^2 \, a \, \vec{\omega}$$

$$\dot{H} = \vec{r} \times F$$

$$\bar{\omega} = w \, (\cos(\theta) \, \hat{z} + \sin(\theta) \, \hat{y})$$

$$\vec{M} = \vec{r} \times \vec{F} = mg \, R \sin\theta \, (-\hat{x})$$

Symetriska snurror!



Jorden roterar med $\vec{\Omega}$ en till rotation $\vec{\omega}$ sker på Jorden

Rotation i ett inertialt system blir sammansatt av $\vec{\Omega} + \vec{\omega}$

Betrakta $\vec{\Omega}$ roterande koordinatsys

Alla vektorer

$$\frac{d\vec{A}}{dt} = \frac{\partial A}{\partial t} + \vec{R} \times \vec{A}$$

och vi vet $\vec{\omega}$ är en vektor

och $\vec{H}$ är en vektor

Vi vet $\frac{d\vec{H}}{dt} = \frac{\partial \vec{H}}{\partial t} + \vec{R} \times \vec{H}$

$\downarrow \vec{M}$ $\downarrow \vec{P}$

=0, om endast roterande

För en stel kropp gäller: $\vec{H} = \vec{I}\,\vec{\omega} = \frac{\partial H}{dt} + \vec{\Omega} \times (I\,\vec{\omega})$

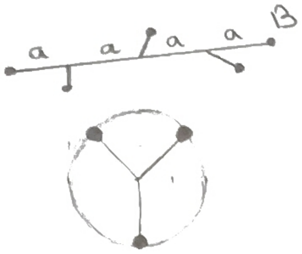
$$\Longrightarrow \vec{M} = \frac{d\vec{H}}{dt} = \vec{\Omega} \times (I\,\vec{\omega})$$

Speciall fall

$\vec{\Omega} = \vec{\omega}$ där får vi $\vec{M} = (\vec{\Omega} \times \vec{I}\,\vec{\Omega})$

Ex

Vad blir $\vec{M}$?



$$\vec{M} = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \times \begin{bmatrix} 31/2 & x & -\frac{3}{2} \\ x & x & \sqrt{3}/2 \\ x & x & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} ma^2$$

$$= \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \times \begin{bmatrix} -\frac{3}{2} \\ \sqrt{3}/2 \\ 3 \end{bmatrix} \Omega ma^2$$

$$\Rightarrow \vec{M} = \Omega^2 \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ -\frac{3}{2} & \frac{\sqrt{3}}{2} & 3 \end{bmatrix} ma^2 = \Omega^2 \left(-\frac{\sqrt{3}}{2}\hat{x} - \frac{3}{2}\hat{y} + 0\hat{z}\right)$$

$$= 4a \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ R_x & R_y & R_z \end{bmatrix} = (4a\hat{z} \times (R_x, R_y, R_z))$$

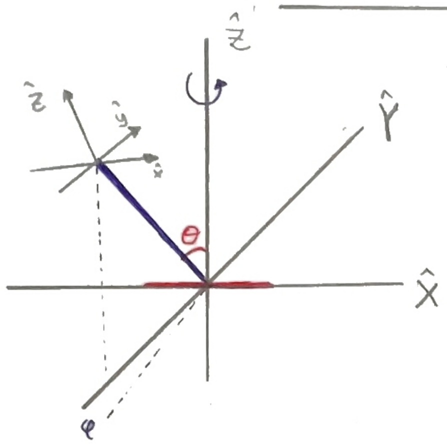
$$R_x = \frac{-\frac{3}{2}\Omega^2 ma^2}{4a} = -\frac{3}{8}\Omega^2 ma$$

$$R_y = \frac{\frac{\sqrt{3}}{2}\Omega^2 ma^2}{4a} = \frac{\sqrt{3}}{8}\Omega^2 ma$$

$$R_z = 0$$

Resulterande krafterna i punkten B

Analysera en Symetrisk snurra



- Stora bokstäver = rumskoord.

blå stav $\in YZ$-planet som roterar kring röd axeln i x-axeln & kring Z-axeln

- snurra längst ut på staven, rotera med ϕ
- hela systemet roterar med φ

Euler Vinklar

θ
φ
ϕ
$\Omega \equiv \dot{\varphi}$

kroppen snurrar runt egen axel med $\dot{\phi}$

Vad blir $\vec{\omega}_{tot}$? $= \dot{\phi}\, z + \dot{\theta}\, \hat{X} + \underbrace{\Omega\, \hat{Z}}$

$\vec{\omega}$: omega summeras som vektor

$\dot{\phi}\, z$: i kroppens-koordinater

$\dot{\theta}\, \hat{X} + \Omega\, \hat{Z}$: i rums-koordinater

Vi vill inte ha ett tröghetsmomet som ändras över tid

$\Longrightarrow$ vi måste uttrycka alla rörelser i kroppens koordinater

Symetrisk snurra $\Longrightarrow I_{xx} = I_{yy}$ så vi slipper rotera tillsammans med kroppen runt $\hat{Z}$

Räkna fram $\vec{H}$ för att ta fram $\vec{M}$

$\Longrightarrow$ behöver $\vec{\omega}_{tot}$! i kroppskoordinater

Vi ta fram $\vec{\omega}_{tot}$ i kroppskoordinater

Vi har enligt uppgiften $\hat{\mathbb{X}} = \hat{x}$

$$\hat{\mathbb{Z}} = \cos(\theta)\,\hat{z} + \sin(\theta)\,\hat{y}$$

$$\Longrightarrow \vec{\omega}_{tot} = \dot{\theta}\,\hat{x} + \Omega \sin\theta\,\hat{y} + (\Omega\cos\theta + \dot{\phi})\,\hat{z}$$

Symmetrisk snurra $\longrightarrow$ $I = \begin{bmatrix} I_\perp & & \\ & I_\perp & \\ & & I \end{bmatrix}$

Specialfall:

Konstant precision (dvs θ = konstant & Ω konstant)

$$\vec{M} = \frac{\partial \vec{H}}{dt} + \vec{\Omega} \times (\vec{I}\,\vec{\omega}_{tot})$$

$$= \frac{\partial}{\partial t}(I\,\vec{\omega}_{tot}) + \vec{\Omega} \times (I\,\vec{\omega}_{tot})$$

$$= I\,\frac{\partial \vec{\omega}_{tot}}{dt} + \vec{\Omega} \times (I\,\vec{\omega}_{tot})$$

$$= I\,\frac{\partial \vec{\omega}_{tot}}{dt} + \vec{\Omega} \times (I\,\vec{\omega}_{tot}$$

$$\Longrightarrow \vec{M} = \begin{bmatrix} \dot{\theta} \\ \Omega\sin\theta \\ \Omega\cos\theta \end{bmatrix} \times \begin{bmatrix} I_\perp\,\dot{\theta} \\ I_\perp\,\Omega\sin\theta \\ I(\Omega\cos\theta + \dot{\phi}) \end{bmatrix}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 \\ \Omega\sin\theta \\ \Omega\cos\theta \end{bmatrix} \times \begin{bmatrix} 0 \\ I_\perp\,\Omega\sin\theta \\ I(\Omega\cos\theta + \dot{\phi}) \end{bmatrix} = \Omega\sin\theta \begin{bmatrix} 0 \\ 1 \\ \Omega\cos\theta \end{bmatrix} \times \begin{bmatrix} 0 \\ I_\perp \\ I(\Omega\cos\theta - \dot{\phi} \end{bmatrix}$$

$$= \Omega\sin\theta \begin{bmatrix} (I - I_\perp)\,\Omega\cos\theta + I\,\dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

Om $M_x = 0$ så är $(I - I_\perp)\,\Omega\cos\theta - I\,\dot{\phi} = 0$

$$\Longrightarrow \Omega = I\,\frac{\dot{\phi}}{(I_\perp - I)\cos\theta}$$

Allmäna fallet för symetrisk snurra!

(Fortsätter ifrån $\vec{M} = I \frac{\partial \vec{\omega}}{dt} + \vec{\Omega} \times (I \times \omega)$)

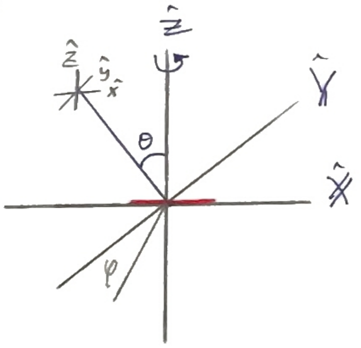
$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = I \frac{\partial \vec{\omega}}{dt} + \vec{\Omega} \times (I \times \vec{\omega}) = \begin{bmatrix} I_\perp \frac{d}{dt} \theta \\ I_\perp \frac{d}{dt}(\Omega \sin\theta) \\ I \frac{d}{dt}(\Omega \cos\theta + \dot{\phi}) \end{bmatrix}$$

$$\begin{bmatrix} \dot{\theta} \\ \Omega \sin\theta \\ \Omega \cos\theta \end{bmatrix} \times \begin{bmatrix} I_\perp \dot{\theta} \\ I_\perp \Omega \sin\theta \\ I(\cos\theta + \dot{\phi} \end{bmatrix}$$

$$\begin{cases} M_x = I_\perp(\ddot{\theta} - \Omega^2 \sin\theta \cos\theta + I \Omega \times (\Omega \cos\theta + \dot{\phi}) \sin\theta \\ M_y = --- \\ M_z = --- \end{cases}$$

finns i boken

Lös



Koordinattransformation till $\hat{x}, \hat{y}, \hat{z}$ från $\hat{X}$ $\hat{Y}$, $\hat{Z}$

$$\frac{d\vec{H}}{dt} = \frac{\partial}{\partial t} I \omega_{tot} + \Omega \times (I \vec{\omega}_{tot})$$

Ω är rotationen av koordinatsystem

Välj koordinatsystem så att $I \equiv$ konstant

$$\frac{d\vec{H}}{dt} = I \frac{\partial \vec{H}}{\partial t} + \Omega \times (I \vec{\omega}_{tot})$$

Symmetrisk snurra $\Rightarrow$ $I = \begin{bmatrix} I_\perp & & \\ & I_\perp & \\ & & I \end{bmatrix}$ roterar inte runt kroppens $\hat{z}$ axel

$$\Longrightarrow \vec{\Omega} = \dot{\psi} \hat{Z} + \dot{\theta} \hat{X} = \dot{\psi}(\cos(\theta) \hat{z} + \sin\theta \, \hat{y} + \dot{\theta} \hat{x})$$

$$\frac{d \vec{\omega}_{tot}}{dt} = \frac{\partial \vec{\omega}_{tot}}{dt} + \vec{\Omega} \times \vec{\omega}_{tot} = \dot{\vec{\omega}} + \vec{\Omega} \times (z \dot{\phi})$$

Koordinattransformationer

1) symmetrisk snurra : $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_x(\theta)\, R_z(\psi) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ stora

2) Asymmetrisk snurra : $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\phi)\, R_x(\theta)\, R_z(\psi) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$

där $R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

$R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Asymmetriska snurror

- $\vec{\Omega} = \vec{\omega}_{tot} = \vec{\omega}$

Euler ekvationen för en asymetrisk snurra

tänk limma fast koordinatsystemet på det roterande objektet

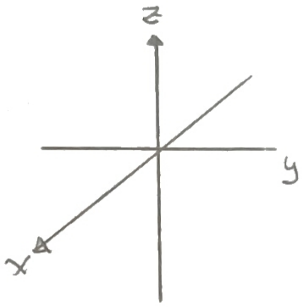
$$\frac{d\vec{\omega}}{dt} = \frac{\partial \vec{\omega}}{\partial t} + \underbrace{\vec{\omega} \times \vec{\omega}}_{=0} = \frac{\partial \vec{\omega}}{\partial t}$$

$$\vec{M} = \frac{d\vec{H}}{dt} = \frac{\partial}{\partial t} I\vec{\omega} + \vec{\omega} \times (I\vec{\omega}) = I\dot{\vec{\omega}} + \vec{\omega} \times (I\vec{\omega})$$

Teorem: Vi kan alltid välja en koordinat så att

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Runt en huvudaxel blir $\vec{H} \parallel \vec{\omega} \Rightarrow I_0 \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I\omega \\ 0 \\ 0 \end{bmatrix}$



$$I = \iint \begin{bmatrix} y^2+z^2 & -xy & -zx \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

$$I_{ij} = \iiint (\delta_{ij} r^2 - r_i - r_j) \rho d^2 r$$

$$= \iint \begin{bmatrix} y^2+z^2 & 0 & 0 \\ 0 & z^2 & \cancel{-yz} \\ 0 & \cancel{-yz} & y^2 \end{bmatrix} = \iint \begin{bmatrix} y^2+z^2 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & y^2 \end{bmatrix}$$

symmetri

$$I = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix} \quad I_1 = I_2 + I_3$$

$$\vec{M} = I\dot{\vec{\omega}} + \vec{\omega} \times \left(\begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix} \right) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \Rightarrow \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} I_1 \dot{\vec{\omega}}_1 \\ I_2 \dot{\vec{\omega}}_2 \\ I_3 \dot{\vec{\omega}}_3 \end{bmatrix} + \begin{bmatrix} \vec{\omega}_1 \\ \vec{\omega}_2 \\ \vec{\omega}_3 \end{bmatrix} \times \begin{bmatrix} I_1 \vec{\omega}_1 \\ I_2 \vec{\omega}_2 \\ I_3 \vec{\omega}_3 \end{bmatrix}$$

$$M_1 = I_1 \dot{\vec{\omega}}_1 + \vec{\omega}_2 \vec{\omega}_3 (I_3 - I_2)$$
$$M_2 = I_2 \dot{\vec{\omega}}_1 + \vec{\omega}_3 \vec{\omega}_1 (I_1 - I_3)$$
$$M_3 = I_3 \dot{\vec{\omega}} + \vec{\omega}_1 \vec{\omega}_2 (I_2 - I_1)$$

- Diffekationerna måste lösas matematisk
- De beskriver momenten för asymetrisk snurra

Specialfall: låt rotationen vara nästan runt en huvudaxel $\hat{x}$

$$\vec{\omega} = \begin{bmatrix} \omega_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \quad \text{antag } \vec{M} = 0$$

$$L \begin{cases} I_2\dot{\delta}_2 = \delta_3\,\omega_1\,(I_3-I_1) \\ I_3\dot{\delta}_3 = \omega_1\,\delta_2\,(I_1-I_2) \\ I_1\dot{\vec{\omega}}_1 = \delta_2\delta_3\,(I_2-I_3) \end{cases}$$

Utveckla till linjära ordning $\delta_2\delta_3$

$\Longrightarrow$ ω_1 = konstant (ty beror av $\delta_2\cdot\delta_3 = O(\delta^2) \rightarrow$ oändligt litet

$$I_3\ddot{\delta}_3 = \omega_1\,\delta_2\,(I_1-I_2)+O(\delta^2)$$

$$= \omega_1\frac{1}{I_2}\,\omega_1\,(I_3-I_1)(I_1-I_2) \Rightarrow \ddot{\delta}_3 = \frac{\omega_1^2}{I_2I_1}(I_3-I_1)(I_1-I_2)\delta_3$$

$\Longrightarrow$ diff-ekvationen är på formen $\ddot{x} = kx$

där $k = \dfrac{\omega_1}{I_2I_3}(I_3-I_1)(I_1-I_3)$

Vi vet $m\ddot{x} = kx$ $\Longrightarrow$ $F = kx$

Diff-ekvationen ($\ddot{x} = kx^2$) är

stabil om $(I_3-I_1)(I_1-I_2)<0$

ostabil om $(I_3-I_1)(I_1-I_2)>0$

$\Longrightarrow$ Om $I_2<I_1<I_3$ $\Rightarrow$ ostabil

Om I_1 är störst/minst $\Longrightarrow$ stabil

Intermediate axis theorem

Kap 8

Svängningar:

diff ekv för fjädrar

$$F = ma = \underbrace{m\ddot{x} = -kx}$$

$$\Rightarrow \ddot{x} = \frac{-k}{m}x = -\omega^2 x$$

där $\omega^2 = \frac{k}{x}$

Enheter:

$$[kg]\left[\frac{m}{s^2}\right] = k\,[m] \Rightarrow \text{enheten för } k = \left[\frac{kg}{s^2}\right]$$

Lösningar till $\ddot{x} = -\omega^2 x$

$$\left.\begin{aligned} x &= \cos(\omega t) \\ \dot{x} &= -\omega \sin(\omega t) \\ \ddot{x} &= -\omega^2 \cos(\omega t) \end{aligned}\right\}$$

Lösning till diff i $x(t) = \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{\text{alla existerande lösningar}}$

Randvilkor

$x(0) = x_0$

$\dot{x}(0) = v_0$

stoppa in i lösning

får ut

$x_0 = A$

$v_0 = B\omega$

$$\Longrightarrow \boxed{x(t) = x_0 \cos(\omega t) + B\omega \sin(\omega t)}$$

Annan Representation: $x(t) = A\sin(\omega t + \varphi)$

$= A(\sin \omega t \cos\varphi + \sin\varphi \cos \omega t)$

$= \underbrace{(A\cos\varphi)}_{A} \sin \omega t + \underbrace{(A\sin\varphi)}_{B} \cos(\omega t)$

Randvillkon:

$$\left.\begin{aligned} x_0 &= x(0) = A\sin(\varphi) \\ v_0 &= \dot{x}(0) = A\omega\cos(\varphi) \end{aligned}\right\} \quad \frac{x_0}{v_0} = \frac{1}{\omega}\tan(\varphi)$$

$$\Rightarrow \varphi = \arctan\left(\frac{\omega\, x_0}{v_0}\right)$$

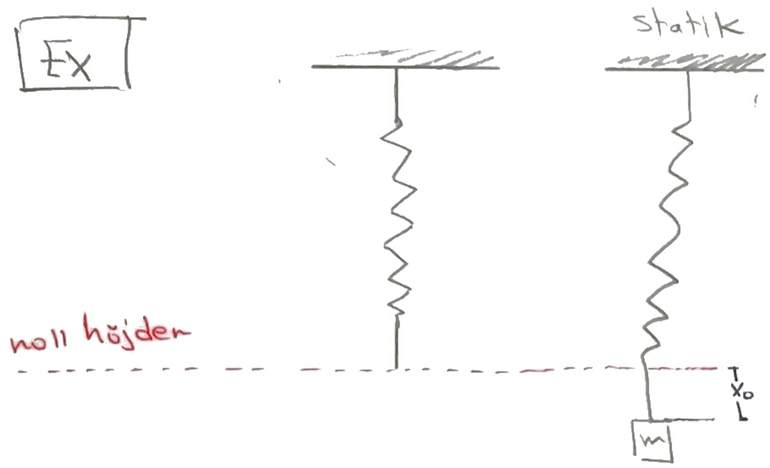
Ekv-system:

$$\begin{cases} A\sin(\varphi) = x_0 \\ A\cos(\varphi) = \dfrac{x_0}{\omega} \end{cases} \Rightarrow \quad \begin{aligned} A^2 &= x_0^2 + \left(\frac{v_0}{\omega}\right)^2 \\ A &= \pm\sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \end{aligned}$$

* funkar och blir på "rätt form"

* måste kompensera med en fasförskjutning π till φ

$\boxed{\text{Ex}}$

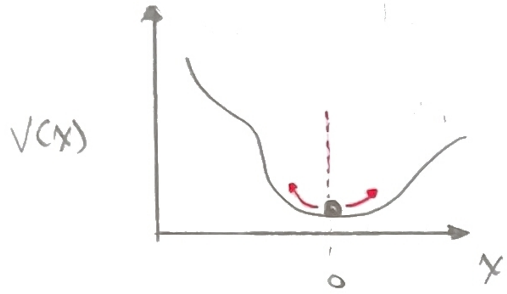


$$F = -kx + mg$$

$$m\ddot{x} = -k(x_0 + x) + mg$$

$$\Rightarrow m\ddot{x} = -kx$$

Harmoniska oscilatorer är väldigt allmäna !



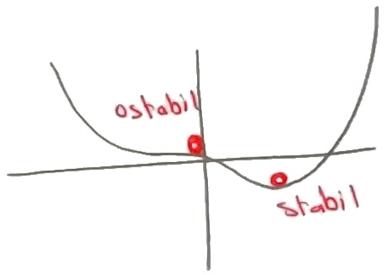
$V(x) = V(0) + x V'(0) + \frac{1}{2} x^2 V'(0) \ldots$

$= V(0) + 0 + \frac{1}{2} x^2 k$

där $k = V''(0)$

$$\boxed{\begin{aligned} F(x) &= -\frac{dV}{dx} = -kx \\ \Rightarrow\ m\ddot{x} &= -\ddot{V}(0)\, x \end{aligned}}$$

Ex



$V'' = 0$

$V''' \neq 0$ nytt jämviktsläge

där $V'' > 0$

$$\left.\begin{array}{ll} V'(0) = & \text{jämvikt} \\ V''(0) = & \text{specialsituationen} \\ V'''(0) = 0 & \text{symmetri} \end{array}\right\} \text{ger stabil}$$

$V''''(0) > 0$

Allmän lösning:

$$x(t) = A\cos(\omega t + \varphi)$$
$$= \mathrm{Re}\, e^{i(\omega t+\varphi)} A = \mathrm{Re}((e^{i(\omega t)} \cdot e^{i\varphi}) A)$$

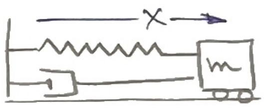
$$x(t) = \mathrm{Re}\, A e^{i\omega t}(\cos\varphi + i\sin\varphi)$$
$$= \mathrm{Re}\, e^{i\omega t}(\overbrace{A\cos\varphi + A i\sin\varphi}^{a})$$

$$\Longrightarrow \boxed{x(t) = \mathrm{Re}(a e^{i\omega t})}$$

$z(t) = a e^{i\omega t}$ uppfyller oxå diff-ekv ty:

$$\ddot{z}(t) = -\omega^2 e^{i\omega t} = -\omega^2 z(t)$$

Svängning och Dämpad rörelse



$$F_{visk} = c\dot{x}$$

stötdämpare

$$F = m \cdot a \Longrightarrow m\ddot{x} = -kx - c\dot{x} \Longrightarrow \boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0}$$

$$2\omega\xi = \frac{c}{m}, \quad \omega^2 = \frac{k}{m}, \quad \xi = \frac{c}{2\sqrt{mk}}$$

$$\Longrightarrow \ddot{x} + 2\omega\xi\dot{x} + \omega^2 x = 0 \longrightarrow \lambda^2 A e^{\lambda t} + 2\omega\xi\lambda A e^{\lambda t} + \omega^2 A e^{\lambda t} = 0$$

Ansätt:

$$\left.\begin{aligned} x(t) &= A e^{\lambda t} \\ \dot{x} &= \lambda A e^{\lambda t} \\ \ddot{x} &= \lambda^2 A e^{\lambda t} \end{aligned}\right\}$$

gemensam faktor $Ae^{\lambda t} \neq 0$

$$\lambda^2 + 2\omega\xi\lambda + \omega^2 = 0$$

$$\lambda = -\omega\xi \pm \sqrt{(\omega\lambda)^2 - \omega^2}$$

$$= -\omega(\xi \pm i\sqrt{1-\xi^2}$$

$$\lambda = -\omega\xi \pm i\omega\sqrt{1-\xi^2}$$

$$x(t) = A(e^{\omega\xi t})e^{(i\omega\sqrt{1-\xi^2}\, t)} + B(e^{-\omega\xi t})e^{i\omega\sqrt{1-\xi^2}\, t}$$

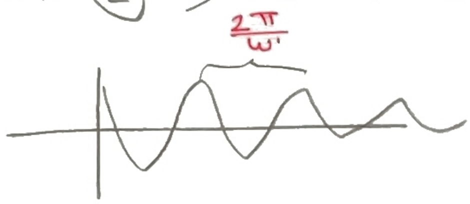
$$= e^{-\omega\xi t}(Ae^{i\omega(\sqrt{1-\xi^2})t} + Be^{-i\omega\sqrt{1-\xi^2}\, t})$$

$$= e^{-\gamma t}(Ae^{i\omega' t} + Be^{-i\omega' t}$$

$\gamma = \omega\xi$ $\qquad$ $\omega' =$ frekvens

$\omega' = \omega\sqrt{1-\xi^2}$

Fall (I) $\xi < 1 \Longrightarrow$ underdampad

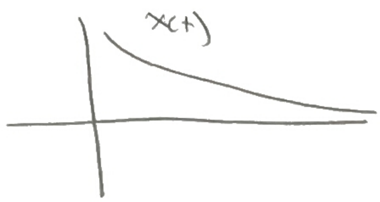


(II) över dampad: $\xi > 1$

$-\lambda_1 = -\omega\xi - \sqrt{1-\xi^2}$

$-\lambda_2 = -\omega\xi + \sqrt{\xi^2 - 1}$

$x(t) = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t}$



III kritisk dämpad

$\xi = 1$

$$x(t) = e^{-\omega \xi t}\left(Ae^{-i\sqrt{1-\xi^2}\,\omega t} + Be^{i\sqrt{1-\xi^2}\,\omega t}\right)$$

$$= e^{-\omega t}(A+B)$$

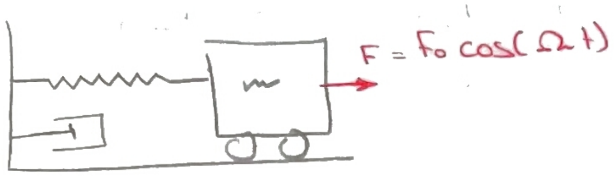
En lösning saknas

$$e^{-\omega \xi t}\left(a \cos\sqrt{(1-\xi^2)}\,\omega t + b \frac{\sin\left(\sqrt{1-\xi^2}\,\omega t\right)}{\sqrt{1-\xi^2}\quad \omega}\right)$$

Bakar in i konstanterna

$$\lim_{\xi = 1} = e^{-\omega t}(a + bt)$$

utnyttja $\sin \delta t \simeq \delta t$ för $\delta t \approx 0$



$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\Omega t) = \frac{F_0}{m} Re(e^{i\Omega t})$$

använd komplexa variabler

$x = Re\, a e^{i\Omega t}$

speciallösning ~~Partiku~~

Allmänalösning : $a e^{\lambda_1 t} + b e^{\lambda_2 t} + y_p$

Vi får
$$\left.\begin{aligned} \ddot{x} &= -a\Omega^2 e^{i\Omega t} \\ \dot{x} &= a i \Omega e^{i\Omega t} \\ x &= a e^{i\Omega t} \end{aligned}\right\}$$ stoppa i diff ekva

$$\left[-\Omega^2 + \frac{c}{m} i \Omega + \frac{k}{m} \right) a = \frac{F_0}{m} \right]$$

$$a = \frac{F_0/m}{(\Omega^2 + \frac{k}{m}) + i\frac{\Omega c}{m}}$$

$$x = a e^{i\Omega t} \Longrightarrow x = |a| e^{i(\Omega t + \phi)}$$

Hur hittar vi fasen ?

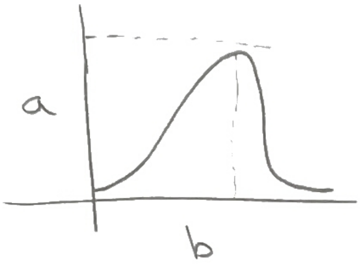
$$\tan\phi = \frac{Im}{Re} = \frac{i\frac{\Omega c}{m}}{-\Omega^2 + \frac{k}{m}} = \frac{2\xi\frac{\Omega}{\omega}}{1 - (\frac{\Omega}{\omega})^2}$$

Vi vet om $Z = \frac{1}{x+iy} = \frac{x - iy}{x^2+y^2} \Longrightarrow |Z| = \sqrt{Z^2}$

$$\Longrightarrow |a| = \frac{F_0/m}{\sqrt{(\omega^2 - \Omega^2)^2 + (\frac{\Omega c}{m})^2}}$$ där $\omega^2 = \frac{k}{m}$

$$\Longrightarrow x(t) = y_p = \frac{F_0/m}{\underbrace{(\omega^2-\Omega^2)^2 + i\left(\frac{\Omega c}{m}\right)}_{\text{bara speciella}}}$$

addera med resten



Energimetoden: (Kan användas då energin är bevarad !)

Ex: $E_{tot} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

Energi bevarad $\Longrightarrow \frac{dE}{dt} = 0 = \frac{d}{dt}\left(\frac{1}{2}kx^2\right) + \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$

$= kx\dot{x} + mv\dot{v} = 0$ ty $x = x(t)$, $v = v(t)$

$\Longrightarrow kx\dot{x} + m\dot{x}\ddot{x} = 0 \Longrightarrow \dot{x}(kx + m\ddot{x})$

2 lösn
1. $\dot{x} = 0$ (Partikeln står stilla)
2. $\boxed{m\ddot{x} + kx = 0}$

Ex: $E = \frac{1}{2}\lambda\dot{q}^2 + \frac{1}{2}kq^2$ (Det är bara att ersätta m med λ och k med k och x med q

Enheter: (Avstickare)

$m\ddot{x} = -kx$ Vad har k för enhet? $[k] = \frac{kg}{s^2}$

$$[\omega] = [m][k][a]$$

$$\left[\frac{1}{s}\right] = [kg]\left[\frac{kg}{s^2}\right]^{\frac{1}{2}}[m]^0$$

$$\Longrightarrow \omega = c\,\frac{k}{m}$$

↓
konstant

Vi vill hitta ω beroende på L:

$$[\omega] = [L][\rho][g]$$

$$\Longrightarrow \left[\frac{1}{s}\right] = [m]^{-\frac{1}{2}}\left[\frac{kg}{m^3}\right]^0\left[\frac{m}{s^2}\right]^{\frac{1}{2}}$$

$$\Longrightarrow \omega = c\left(\frac{g}{L}\right)^2$$

Är $\vec{F} = m\vec{a}$ ett cirkulär argument?

Newtons lagar:

I: Isolerad kropp förblir i rörelse

II: $\vec{F} = m\vec{a}$

III: Varje kraft har en motkraft (lika stor)

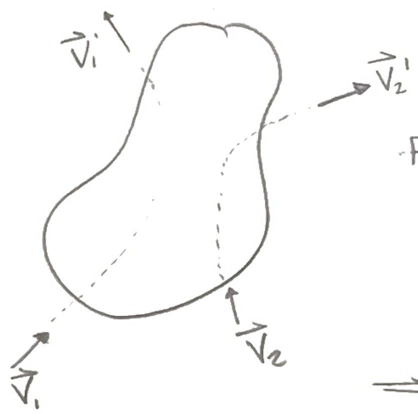
Alternativ:

Rymden → kartesiska koordinater, tid

Galileo lag I: En isolerad kropp i vila förblir i vila

Galileo lag II: $\vec{x} \longrightarrow \vec{x} + v$ lämna fysiken invariant

III: Det finns λ_i så att



Försök: $\lambda_1 \vec{V}_1 + \lambda_2 \vec{V} = \lambda_1 \vec{V}_1' + \lambda_2 \vec{V}'$

man kan visa $\lambda_i \propto m$

proportionell

$\Longrightarrow \vec{F} = m\vec{a} \longrightarrow$ Def av F

Lemma: $G\mathrm{II} \Rightarrow \vec{V}_i \longrightarrow \vec{V}_i + \vec{V}$

$G\mathrm{III}: \sum_i m_i \vec{V}_i = \sum m_i' \vec{V}_i'$

$$\sum_i m_i (\vec{V}_i + \vec{V}) = \sum_i m_i' (\vec{V}_i + \vec{V})$$

Energi:

Elastisk kollision: Energi bevarad

Lemma: Energi i ett koordinatsystem $\Rightarrow$ E är bevarad i alla koordinatsystem

Antag: $\sum_i m_i \vec{v}_i^2 = \sum_i m_i (\vec{v}_i')^2$

i ny koordinat

G II: $\sum_i m_i (\vec{v}_i + \vec{V})^2 \overset{?}{=} \sum_i m_i (\vec{v}_i' + \vec{V}^2)$

$\Longrightarrow \sum_i m_i (\vec{v}_i^2 + 2\vec{v}_i \vec{V} + \vec{V}^2) \overset{?}{=} \sum_i m_i (\vec{v}_i'^2 + 2\vec{v}_i' \cdot \vec{V} + \vec{V}^2)$

$$\sum_i m_i \vec{v}_i^2 + 2\vec{V} \sum_i m_i \vec{v}_i + \vec{V}^2 \sum_i m_i = \sum_i \underbrace{m_i \vec{v}_i'^2}_{\text{antag}} + \underbrace{2V \sum m_i v_i'}_{\text{G III rörelsemängd bevarad}} + \underbrace{\vec{V}^2 \sum_i m_i}_{\text{samma}}$$

Galileoprincip $\Longrightarrow$ Inget experiment kan avslöja någon absolut rörelse

(Einstein sa att detta omfattar ljus)!

Kopplade fjädrar: fler dimensionella fler frihetsgrader

Newton ekv:

$$\vec{F_1} = -kx_1 - k(x_1 - x_2) = m\ddot{x}_1$$

$$\vec{F_2} = -kx_2 - k(x_2 - x_1) = m\ddot{x}_2$$

$$\Longrightarrow m\ddot{x}_1 = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -2kx_2 + kx_1$$

Ω = frekvens   Ansats: $x_1 = a_1 e^{i\Omega t}$, $x_2 = a_2 e^{i\Omega t}$ } Vi antar att de är bundna med samma frekvens

$$\Longrightarrow \left.\begin{aligned} m\ddot{x}_1 &= -m\Omega^2 a = -2ka_1 + ka_2 \\ m\ddot{x}_2 &= -m\Omega^2 a = -2ka_2 + ka_1 \end{aligned}\right\}$$ gör om till matris

$$(2k - m\Omega^2)a_1 - ka_2 = 0$$
$$-k\,a_1 + (2k - m\Omega^2)a_2 = 0$$

$$\begin{bmatrix} 2k - m\Omega^2 & -k \\ -k & 2k - m\Omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

$M\bar{x} = \vec{y}$
$\bar{x} = M^{-1}\bar{y}$

icke-trivial lösning omm $\det|M| \neq 0$

För att icke-trivial lösning skall finnas så krävs

$$(2k - m\Omega^2)^2 - k = 0 \Longrightarrow (2k - \Omega^2 m)^2 = k^2$$

$$2k - m\Omega^2 = \pm k$$

$$\Longrightarrow \Omega^2 = \frac{2k \pm k}{m} \Longrightarrow \Omega^2 = \begin{cases} \frac{k}{m} \\ \frac{3k}{m} \end{cases}$$

 $\Omega_1 = \sqrt{\frac{k}{m}} \rightarrow$ om den rör sig "tillsammans" så sträcks aldrig mittenfjädern

 $\Omega_2 = \sqrt{\frac{3k}{m}} \rightarrow$ om 3st fjädrar påverkar en kropp

I

$k = m\Omega^2$ $$\begin{bmatrix} 2k-k & -k \\ -k & 2k-k \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Longrightarrow ka_1 - ka_2 = 0 \Longrightarrow a_1 = a_2$$

$$\Longrightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = a_+ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} e^{i\Omega t} + a_- \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} e^{-i\Omega t}$$

II $3k = m\Omega^2$ $$\begin{bmatrix} 2k-3k & -k \\ -k & 2k-3k \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

$$\Longrightarrow a_1 + a_2 = 0 \Longrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{a_+}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{\frac{3k}{m}}t} + \frac{a_-}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\sqrt{\frac{3k}{m}}t}$$

Kan man använda energimetoden för att lösa fjäderproblemet?

$$E = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_1 - x_1)^2 + \frac{1}{2} k x_2^2$$

$$0 = \frac{dE}{dt} = m\dot{x}_1 \ddot{x}_1 + m\dot{x}_2 \ddot{x}_2 + kx_1 \dot{x} + kx_2\dot{x}_2 + k(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) = 0$$

Nej: Energimetoden funkar bara för 1 parameter

Alternativ

Lagrangian: Om vi har en fjäder (som exempel)

Vi har då kinetisk energi $T = \frac{1}{2} m \dot{x}^2$

Potentiell energi $V = \frac{1}{2} k x^2$

$$\mathcal{L}(x, \dot{x}) = T - V = \underbrace{\underset{E_k}{\frac{1}{2} m \dot{x}^2} - \underset{E_p}{\frac{1}{2} k x^2}}_{\text{Kallas lagrangian}}$$

$$\text{Lagrange-Ekv:} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

Tillämpa lagrange: $\frac{d}{dt}(m\dot{x}) + kx = 0$

$$\Longrightarrow m\ddot{x} + kx = 0 \Longrightarrow \underbrace{m\ddot{x} = -kx}_{\text{Newton}}$$

Tillbaka till problemet med 2 frihetsgrader

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2 - \left(\frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k(x_1 - x_2)^2\right)$$

Lagrange ekvation: $\frac{d}{dt}\left(\frac{\mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x}$

$$\frac{d}{dt}(m\dot{x}_1) + kx_1 + k(x_1 - x_2) = 0$$

$$\Longrightarrow m\ddot{x}_1 = -2kx_1 + kx_2$$

$$\frac{d}{dt}(m\dot{x}_2) + kx_2 + k(x_2 - x_1) = 0$$

$$\Longrightarrow m\ddot{x}_2 = -2kx_2 + kx_1$$

Varför måste man ha – i både lagrangian och lagrange-ekv?

$$\mathcal{L} = K - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(r), \quad r = \sqrt{x^2 + y^2}$$

$$x = r\cos(\theta) \qquad \dot{x} = \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)$$

$$y = r\sin(\theta) \qquad \dot{y} = \dot{r}\sin(\theta) + r\dot{\theta}\cos\theta$$

$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + \ddot{\theta}r^2) - V(r)$, frihetsgrad r, θ, map r

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r} \Longrightarrow \frac{d}{dt}m\dot{r} = m\dot{\theta}^2 r - \frac{\partial V}{\partial r}$$

$$\Longrightarrow \boxed{m\ddot{r} = m\theta^2 r - \frac{\partial V}{\partial r}}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \Longrightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

bevarad

15/5

$\boxed{\text{Ex}}$ Polära koordinat. lagrangian

$$T = \frac{1}{2} m(\dot r^2 + \dot\theta r^2)$$

$$V = V(r)$$

$$\mathcal{L} : T - V = \frac{1}{2} m(\dot r^2 + \dot\theta^2 r^2) - V(r)$$

$$\begin{cases} A \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot r} = \frac{\partial \mathcal{L}}{\partial r} \\ B \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{\partial \mathcal{L}}{\partial \theta} \end{cases}$$

A) $\frac{d}{dt} m\dot r = m\dot\theta^2 r - \frac{d}{dr}V(r)$

$$m\ddot r = m\dot\theta^2 r - \frac{d}{dr}V(r)$$

$\boxed{m\ddot r - m\dot\theta^2 r = F}$ Newton i Polära koord

B) $\frac{d}{dt}(mr^2\dot\theta) = 0$

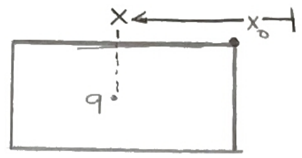
$\Rightarrow mr^2\dot\theta = P_\theta = $ bevarad

↳ rörelsemängdsmoment

$$\frac{1}{2} r\, r \frac{d\theta}{dt} = \frac{dA}{dt}$$

$\frac{1}{2} r^2 \dot\theta = \frac{dA}{dt}$ Konstant

Ex: 2 $\quad q = x - x_0(t)$



$$x = q + x_0(t)$$

Accelererat koordinatsystem

$$T = \frac{1}{2} m \dot{x}^2$$

$$V = V(q)$$

$$\mathcal{L} = \frac{1}{2} m (\dot{q} + \dot{x}_0(t))^2 - V(q)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

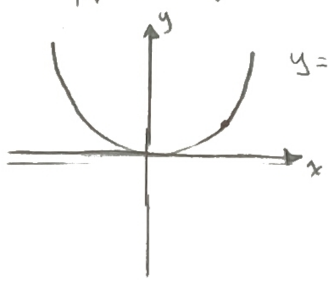
$$m \frac{d}{dt} (\dot{q} + \dot{x}_0) = -\frac{\partial V}{\partial q}$$

$$m \ddot{q} + m \ddot{x}_0 = -\frac{\partial V}{\partial q}$$

$$m \ddot{q} = \underbrace{-m \ddot{x}_0}_{\text{fiktiv-kraft}} - \frac{\partial V}{\partial q}$$

Ex3: förenkling av ex 23 i kompendiet

kroppen följer en bana



$y = \frac{1}{2} k x^2$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

1 frihetsgrad pga tvång
$y = f(x) \equiv y(x)$
ett generaliserat koordinat

Välj x eller y som generaliserad koord

ersätt $y(x)$, $\dot{y}(x, \dot{x})$

$$\boxed{\begin{aligned}\mathcal{L} &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \\ &= \frac{1}{2}m\dot{x}^2(1+k^2x^2) - \frac{1}{2}mgkx^2\end{aligned}}$$

$y = \frac{1}{2}kx^2$

$\dot{y} = kx\dot{x}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\, m\dot{x}(1+k^2x^2) = m\dot{x}^2k^2x - mgkx$$

$$m\ddot{x}(1+k^2x^2) + \cancel{2}m\dot{x}k^2\dot{x}x = m\dot{x}^2\cancel{k^2}x - mgkx$$

$$\boxed{m\ddot{x} = \frac{-mgxk - m\dot{x}^2xk^2}{1+k^2x^2}}$$

för små svängningar

Linearisera R.E

$$m\ddot{x} = -gmkx + \mathcal{O}(x^2, \dot{x}^2, xx')$$

$$\ddot{x} = -gkx \Rightarrow \omega = \sqrt{gk}$$

* Analysera $\mathcal{L}$ för att komma fram till samma svar

$\mathcal{L}$ för harmonisk oscillator är andragrads polynom

x, x':

$$\mathcal{L} \simeq \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\underbrace{mgk}_{\text{effektiv fjäderkonstant}}x^2$$

$$\omega^2 = \frac{mgk}{m} = gk$$

Ex: 4 Roterande koordinatsystem

$$\vec{v} = \frac{d\vec{r}}{dt} = (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})$$

$\mathcal{L}(\vec{r}, \dot{\vec{r}})$ $\quad \mathcal{L} = \frac{1}{2} m (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})^2$

La Grange Ekv.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}}\right) = \frac{\partial \mathcal{L}}{\partial \vec{r}}$$

$$\frac{d}{dt}\left(m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r})\right) = m(\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) \frac{\partial}{\partial \vec{r}} (\vec{\Omega} \times \vec{r})$$

$$m\ddot{r} + m\vec{\Omega} \times \dot{\vec{r}} = m \frac{\partial}{\partial \vec{r}} (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})(\vec{\Omega} \times \vec{r})$$

$$= -m \frac{\partial}{\partial \vec{r}} \left(\vec{r} (\vec{\Omega} \times (\dot{\vec{r}} + \vec{\Omega} \times \vec{r}))\right) = m(\vec{\Omega} \times (\dot{\vec{r}} + \vec{\Omega} \times \vec{r})$$

$$\boxed{m\ddot{r} + 2m\vec{\Omega} \times \dot{\vec{r}} + m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = 0}$$

coriolisk. Centrikpitat kraft

Ex 4 Sfäriska koordinat

$x = r\sin\theta\,\cos\varphi$

$y = r\sin\theta\,\sin\varphi$

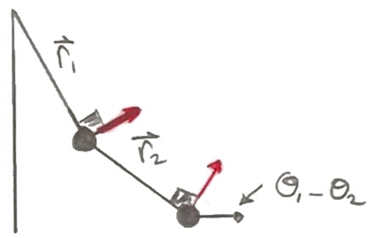
$z = r\cos\theta$

$$\vec{r}\,r = \dot{r}\hat{z} + r\dot{\theta}\hat{x} + r\sin\theta\,\dot{\varphi}\,\hat{y}$$

$$\dot{\vec{r}}^2 = (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2$$

Ex 5 Dubbel pendel



$$\mathcal{L} = \frac{1}{2}m(\dot{\vec{r}}_1^{\,2} + (\dot{\vec{r}}_1 + \dot{\vec{r}}_2)^2) + mg(\vec{r}_1\hat{z} + (\vec{r}_1 + \vec{r}_2)\hat{z}$$

$$= \frac{1}{2}m(2\dot{\vec{r}}^{\,2} + \dot{r}_2^2 + 2\dot{\vec{r}}_1\cdot\dot{\vec{r}}_2) + mg(2r_1\cdot z + \vec{r}_2\hat{z})$$

$$\mathcal{L} = \frac{1}{2}m l^2(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + mgl(2\cos\theta_1 + \cos\theta_2)$$

$$\mathcal{L} = \frac{1}{2}(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + g/l\,(2\cos\theta_1 + \cos\theta_2)$$

$\theta_1: \frac{d}{dt}\left(2\dot{\theta}_1 + \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) = -\dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - 2g/L \sin\theta_1$

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2(\theta_1 - \theta_2) \sin(\theta_1 - \theta_2)$$

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = -\frac{2}{L} g \sin\theta_1$$

$$\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{2}\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + \frac{g}{L}\sin\theta_1 = 0$$

θ_2: görs på samma sätt

Hamilton ekuationer

Energi princip: $\mathcal{L} = T - V \qquad E = T + V = 2T - (T - V)$

$$= 2T - \mathcal{L}$$

$$2T = m\dot{x}^2 = \dot{x} \cdot \frac{\partial \mathcal{L}}{\partial \dot{x}} \quad (= \dot{x} p)$$

$$\frac{dE}{dt} = \frac{d}{dt}\left(\dot{x} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L}\right) = \ddot{x} \frac{\partial \mathcal{L}}{\partial \dot{x}} + \dot{x} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt} \mathcal{L} =$$

$$= \ddot{x} \frac{\partial \mathcal{L}}{\partial \dot{x}} + \dot{x} \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{L}}{\partial x} \dot{x} - \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial \dot{x}} \ddot{x} = 0$$

$\boxed{\text{Kom ihåg:}}$ $dA = \frac{\partial A}{dx} dx + \frac{\partial A}{dy} dy = A_x dx + A_y dy = F dx + G dy$

$\mathcal{L}(q, \dot{q}) \qquad p = \frac{d\mathcal{L}}{d\dot{q}} = p(q, \dot{q}) \quad (\text{ex}: T = \frac{1}{2} m\dot{x}^2 \quad p = m\dot{x})$

$H(p, q) = p\dot{q}(q, p) - \mathcal{L}(q, \dot{q}(q, p)) \qquad (p = m\dot{x} \Rightarrow \dot{x} = p/m$

Variera H map p & q

$$dH = \dot{q}\, dp + p\, d\dot{q} - \frac{\partial \mathcal{L}}{\partial q} dq - \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q} = \dot{q}\, dp - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) dq =$$

$(p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad \frac{\partial \mathcal{L}}{\partial q} dq = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) dq)$

$= \dot{q}\, dp - \dot{p}\, dq \Longrightarrow \frac{\partial H}{\partial p} = \dot{q} \qquad \frac{\partial H}{\partial q} = -\dot{p}$ Hamilton ekuation

<u>Specialfall</u>: $T = \frac{1}{2} m\dot{q}^2$ $V = V(q)$

I : $\mathcal{L}(q, \dot{q}) = \frac{1}{2} m\dot{q}^2 - V(q)$

II : $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$

III : $\dot{q} = \dot{q}(p, q) = p/m$

IV : $H(p, q) = p\dot{q} - \mathcal{L}(q, \dot{q}(p, q)) = \frac{p^2}{m} - \frac{1}{2} m\dot{q}^2 + V(q)$

$$= \frac{p^2}{m} - \frac{1}{2} m \left(\frac{p}{m}\right)^2 + V(q)$$

$$H(p, q) = \frac{1}{2} p^2/m + V(q)$$

<u>Generaliserat</u>: $p \to \vec{p}$, $q \to \vec{q}$ $\Longrightarrow$ $H(\vec{p}, \vec{q}) = \frac{1}{2} \vec{p}\, M^{-1} \vec{p} + V(\vec{q})$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

Fullständigt dynamisk beskrivning

Newton: $\ddot{q} = F(q)$, andra ordning diff-ekv

Hamilton: 2 frihetsgrader, första ordning diff-ek i tiden t

Poisson braket

Elegant beskrivning av dynamisk variabler

ex. $\vec{x}, \vec{p} \Longrightarrow \vec{J} = \vec{x} \times \vec{p}$

Allmäna funktioner: $A(\vec{p}, \vec{q})$, $B(\vec{p}, \vec{q})$

$$[A\ B] = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$$

Specialfall: $[q_i, p_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right)$

$$= \sum_k \delta_{ki} \delta_{kj} = \sum_k \delta_{ik} \delta_{kj} = \delta_{ij}$$

$$\Longrightarrow [q_i, p_j] = \delta_{ij}$$

$$[q, H] = \frac{\partial q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} = \dot{q}$$

$$[p, H] = \frac{\partial p}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} = \dot{p}$$

$[p, H] = \dot{p}$ $\quad [q\ H] = \dot{q}$

$$[A(p, q, t), H] = \frac{\partial A}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial H}{\partial q} = \frac{\partial A}{\partial q} \dot{q} + \frac{\partial A}{\partial p} \dot{p} =$$

$$\frac{\partial A}{\partial q} \frac{dq}{dt} - \frac{\partial A}{\partial p} \frac{\partial p}{\partial t} = \frac{d}{dt} A(p, q, t) - \frac{\partial A}{\partial t}$$

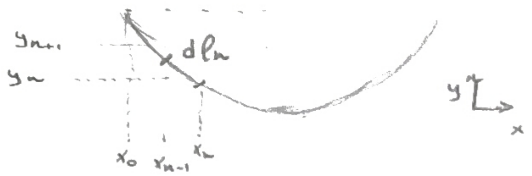
$$\frac{dA}{dt} = [A\ H] + \frac{\partial A}{\partial t}$$

(Exemplet: $J_i = \epsilon_{ijk} x_j p_k$ $[J_i\ J_j] = \epsilon_{ijk} = \epsilon_{ijk} J_k$)

(Kvantmekanik: $[q_i, p_j] = i\hbar \delta_{ij}$ $\frac{d}{dt} A = \frac{q}{i\hbar} [A H] + \frac{\partial A}{\partial t}$)

matriser kommuterar $[A B] = AB - BA$

Hängande kedja (variations-räkning)



- Kedjan rör sig tills den är i en viss form
- Minimering av lägesenergi behövs

$$S = \sum_n \rho g \,(d\ell_n) y_n = \sum_n \rho g \sqrt{\Delta^2 + (y_n - y_{n-1})^2}\; y_n = \ldots$$

$$\sum_n \rho g \sqrt{1 + ((y_n - y_{n-1})(\Delta))^2}\; y_n \Delta = \rho g \sum_n y_n \sqrt{1+\dot{y}_n^2}\,\Delta = \sum_n W(y_n, \dot{y}_n)\Delta$$

(definition: $\dot{y}_n = (y_n - y_{n-1})/\Delta \qquad W = y_n\sqrt{1+\dot{y}_n^2}\cdot \rho g$

Minimering av S: $\frac{\partial S}{\partial y_k} = 0$

$$\frac{\partial S}{\partial y_k} = \frac{\partial W}{\partial y_k} + \frac{\partial W}{\partial \dot{y}_k}\frac{\partial \dot{y}_k}{\partial y_k} + \frac{\partial W}{\partial \dot{y}_{k+1}}\frac{\partial \dot{y}_{k+1}}{\partial y_k} = \frac{\partial W}{\partial y_k} + \frac{\partial W}{\partial y_k}\cdot\frac{1}{\Delta} + \frac{\partial W}{\partial \dot{y}_{k+1}}\left(-\frac{1}{\Delta}\right)$$

$$= \frac{\partial W}{\partial y_k} + \frac{1}{\Delta}\left(\frac{\partial W}{\partial \dot{y}_k} - \frac{\partial W}{\partial \dot{y}_{k+1}}\right) = \frac{\partial W}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial W}{\partial \dot{y}_k}\right)$$

$$-\frac{1}{\Delta}\left(\frac{\partial W}{\partial \dot{y}_n}\Big|_{n=k+1} - \frac{\partial W}{\partial \dot{y}_n}\Big|_{n=k}\right) \approx \frac{d}{dx}\left(\frac{\partial W}{\partial \dot{y}_k}\right)$$

Ta kontinuum-gränsen:

$$\frac{\partial S}{\partial y(x)} = \frac{\partial W}{\partial y} - \frac{\partial}{dx}\left(\frac{\partial W(y\dot{y})}{\partial \dot{y}}\right) = 0 \Rightarrow S = \int_0^k \rho g \sqrt{1+\dot{y}^2}\; y$$

$(y_k \rightarrow (y(\cancel{x}_k) \rightarrow y(x))$ (ändrar notation $x \rightarrow t \quad w \rightarrow -\ell \quad y \rightarrow q$

Nu är S en funktional: S: funktion $\rightarrow$ tal

$$S = \int_0^T \mathcal{L}(q, \dot{q})\,dt \Rightarrow \frac{\partial S}{\partial q(t)} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} \text{ lagrange-ekv}$$