HOME EXAM

–GRAVITATION AND COSMOLOGY (FFM071)–
SPRING SEMESTER 2014

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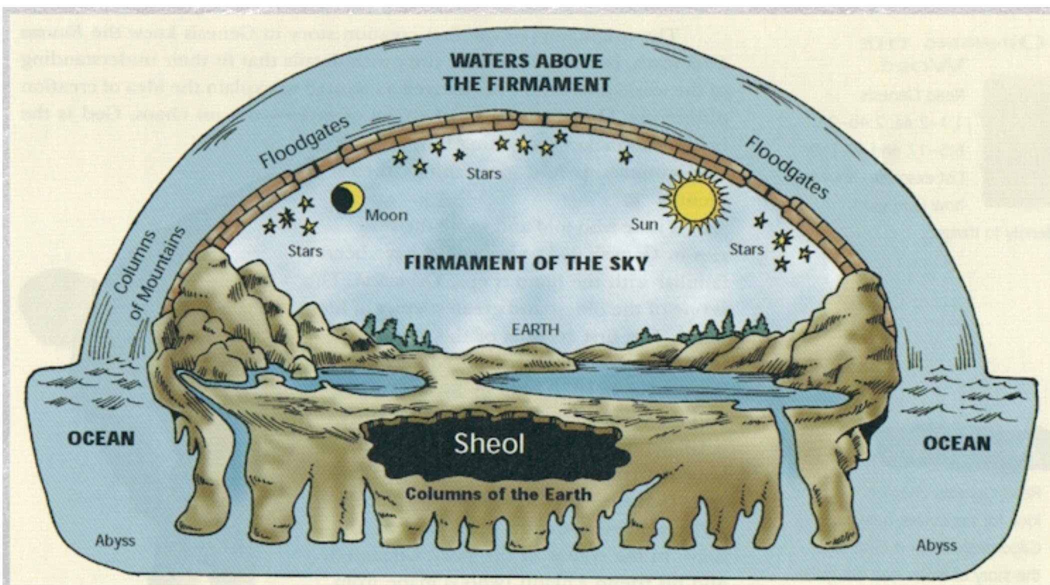
ASSISTANT TEACHER: CHRISTIAN VON SCHULTZ



Instructions: Please use the conventions of Weinberg for all problems[1], and write out explicitly the formulas you are using. Unless otherwise specified, write all derivations explicitly.

There are 9 problems, with a total maximum score of 75p.

Deadline is Friday 21/3 at 10 am.
If your solutions are TEXed you may email them directly to Christian, and otherwise deliver them in Christian's mailbox on the sixth floor of the Origo building.

[1] For covariant derivatives you may use Weinberg's notation, e.g. $V_{\mu;\nu}$, or the more standard one $D_\nu V_\mu$.

Problem 1. Indicate whether the following statements are true or false. (only answer **true** or **false**; no calculations or motivations needed!) For every wrong answer you get -0.5p (but the minimum score is still 0p).

• The energy density of radiation scales like the inverse volume of the spatial universe.

• The particle horizon gives an estimate of the size of our universe.

• The FRW-metric describes a maximally symmetric 4-dimensional spacetime.

• If the universe was contracting, a distant supernova Type IA would appear to be blue-shifted.

• Observations indicate that our universe is currently matter dominated.

• If we would conclude from observations that the total density parameter Ω is equal to 1 then our universe is described by the 4-dimensional Minkowski metric.

• The singularity in the Reissner-Nordstrm metric is timelike.

• Most of the energy in the universe is made up of dark matter.

• If you cross the horizon of a charged black hole you are doomed to hit the singularity.

• The FRW metric with $k = 0$ describes a flat 4-dimensional spacetime.

• The general theory of relativity is an accurate description of Nature at all energy scales.

• The metric tensor $g_{\mu\nu}$ is invariant under general coordinate transformations.

• The vacuum energy violates the condition the the pressure is positive semi-definite.

• A gravitational plane wave has as many independent components as a light wave.

• If you interchange the second and third indices on the Riemann tensor you get back the same tensor with an overall minus sign.

(5p)

Problem 2. A *Killing vector* is a vector Y_μ that satisfies the equation

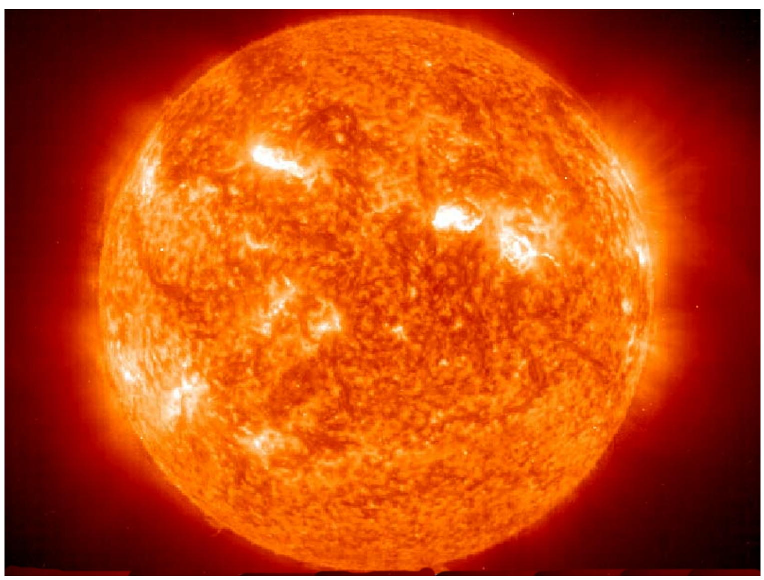
$$D_{(\mu}Y_{\nu)} = 0, \tag{1}$$

where D_μ is the covariant derivative. Suppose we are in an arbitrary space-time with metric $g_{\mu\nu}$ admitting N Killing vectors $Y_\mu^{(I)}$, with $I = 1, \ldots, N$. Let $x^\mu(\tau)$ be a geodesic of a freely falling particle in the geometry described by $g_{\mu\nu}$ along a path parametrized by τ. The particle 4-velocity along the path is $\mathrm{d}x^\mu(\tau)/\mathrm{d}\tau$. Show that the collection of N scalars defined by

$$V_{(I)} := Y_\mu^{(I)} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{2}$$

are constants of motion for the freely falling particle.

(3p)



Problem 3. Find all Killing vectors of the Schwarzschild metric in $D = 4$. For full score you need to use a systematic procedure that shows you have *all* the Killing vectors. You may get some points for educated guesses.

(7p)

Problem 4. A beacon radiating at a fixed frequency ν_0 is released at time $t = 0$ towards a black hole of mass M by an observer situated very far away from the black hole. The observer stays at constant distance from the black hole while the probe is falling. Show that the frequency of the beacon (when it is close to the event horizon) as measured by the observer can be written as $\nu \sim \mathrm{e}^{-t/K}$ for some constant K and relate the constant K to the mass of the black hole.

(12p)

Problem 5. Consider the action for the coupled Einstein-Maxwell system

$$S[g_{\mu\nu}, A_\mu] = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{g}\Big(R + 2\Lambda\Big) - \frac{1}{4}\int \mathrm{d}^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}, \tag{3}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ and $F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}$. Derive the complete Einstein equations for this system (including the stress tensor for Maxwell theory on the right hand side) by varying the action S with respect to the metric tensor $g_{\mu\nu}$. Provide details of all the calculations, including the variations of $g, g^{\mu\nu}$, and $R_{\mu\nu}$.

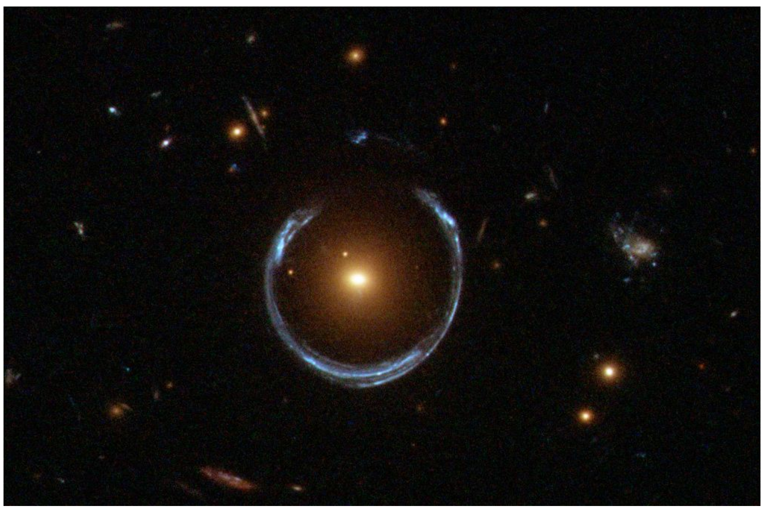
(8p)

Problem 6. Consider a spherically symmetric and static solution of Einsteins equations in three space-time dimensions, describing a point particle with mass M. Give the explicit solution. Is this space-time flat? Will there be gravitational lensing?

(10p)

Problem 7. A galaxy acts a gravitational lens for a very distant quasar. The galaxy is 100 Mpcs away, and the distance to the quasar can be assumed to be much greater. The image is a ring with radius 1.0 arcsec. What is the mass of the galaxy? Express your result in solar masses.

(12p)



Problem 8.

(a) Derive the Reissner-Nordström solution describing the metric and the electric field outside of an electrically charged black hole. You may assume that this is a static and spherically symmetric solution to the coupled Einstein-Maxwell equations:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= -8\pi G T_{\mu\nu}, \\ g^{\mu\nu} D_\mu F_{\nu\sigma} &= 0, \\ D_{[\mu} F_{\nu\sigma]} &= 0, \end{aligned} \tag{4}$$

where $T_{\mu\nu}$ is the stress tensor for Maxwell theory.

(b) Discuss the horizon structure of the Reissner-Nordström solution.

(12p)

Problem 9. According to the Equivalence Principle the horizon of a black hole is not a special place in the universe; if you were to fall through the horizon you would not experience anything special happening. This point of view is currently being challenged in the so called *firewall proposal*, which is the topic of a very recent intense scientific debate. At the heart of the controversy lies the so called *black hole information paradox*, originally pointed out by Hawking in the 1970's. Try to find information about the information paradox and the firewall proposal and then:

(a) *summarize in three bullet points the main arguments behind the black hole information paradox;*

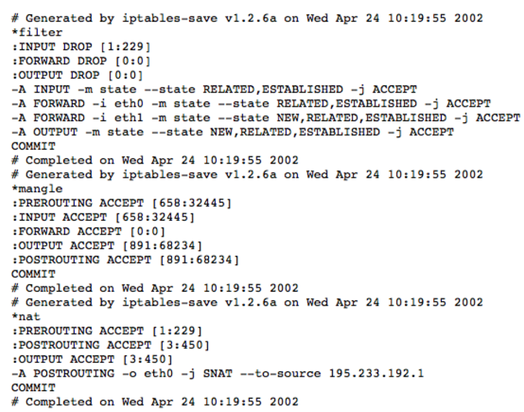
(b) *summarize in three bullet points the key ingredients in the firewall proposal.*

You may find the following survey article by Matt Strassler useful as a starting point:

`http://profmattstrassler.com/articles-and-posts/relativity-space-astronomy-and-cosmology/`

`black-holes/black-hole-information-paradox-an-introduction/`

(6p)



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-A FORWARD -i eth1 -m state --state NEW,RELATED,ESTABLISHED -j ACCEPT
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Good Luck!