HOME EXAMINATION

–GRAVITATION AND COSMOLOGY (FFM071)–
SPRING SEMESTER 2012

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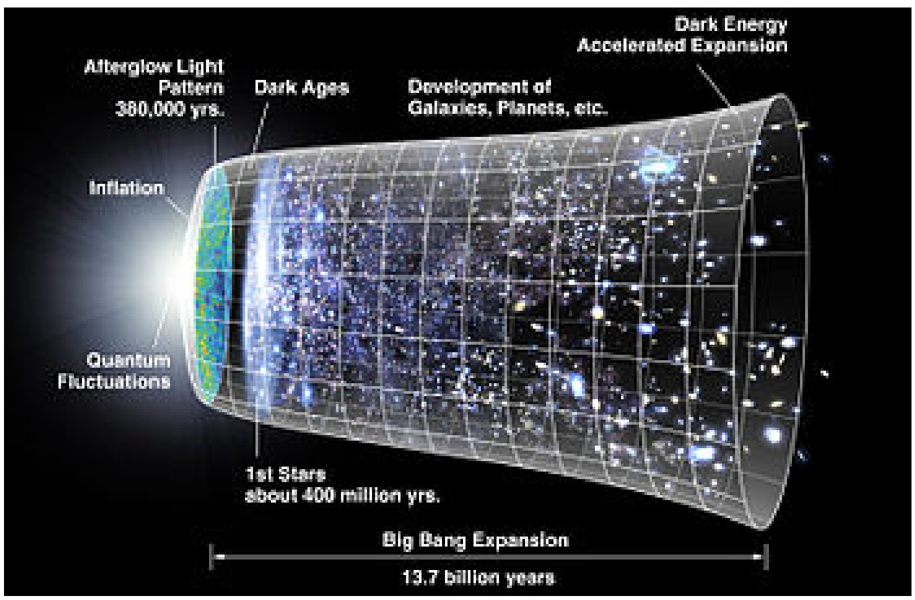

Instructions. Please use the conventions of Weinberg for all problems, and write out explicitly the formulas you are using, e.g. for the metric, affine connection, Ricci, Riemann tensors etc. Also, unless otherwise specified, write all derivations explicitly. E.g. for the curvature tensors, provide details of your calculations for all components. In other words, expressions like "analogously we find" are not approved.

There are 11 problems, with a maximum score of 80p.

Deadline is Monday 16/4 at 10 am.

Problem 1. Indicate whether the following statements are true or false. (only answer **true** or **false**; no calculations or motivations needed!) For every correct answer you get 0.5p, while for every wrong answer you get -0.5p (but the minimum score is still 0p).

• If $T_{\mu\nu}$ is a covariant tensor, then $T^{\mu\nu} = g^{\mu\sigma}g^{\nu\rho}T_{\rho\sigma}$ is the inverse of $T_{\mu\nu}$.

• $D_\lambda D_\rho g_{\mu\nu} = D_\rho D_\lambda g_{\mu\nu}$

• The principle of equivalence states that physics is invariant under Lorentz transformations.

• Neutrinos are a plausible candidate for the cold dark matter of the universe.

• The particle horizon gives an estimate of the size of our universe.

• The Schwarzschild solution describes a maximally symmetric 4-dimensional spacetime.

• If the universe was contracting, a distant supernova Type IA would appear to be blue-shifted.

• Observations indicate that our universe is currently matter dominated.

• An observer at infinity would measure that it takes an infinite amount of proper time for a probe particle to cross the horizon of a Schwarzschild black hole.

• When our sun runs out of hydrogen to burn, it will start to grow and eventually collapse into a black hole.

• The affine connection $\Gamma^\lambda_{\mu\nu}$ is a mixed $(1,2)$-tensor.

• Most of the energy in the universe is made up of dark matter.

• If you cross the horizon of a charged black hole you are doomed to hit the singularity.

• The FRW metric with $k = 0$ describes a flat 4-dimensional spacetime.

(7p)

Problem 2. Let $g_{\mu\nu}$ be the metric tensor in an arbitrary non-inertial frame x^μ. Show that $D_\lambda g_{\mu\nu} = 0$, where D_λ is the covariant derivative constructed from the affine connection.

(6p)

Problem 3. Consider a maximally symmetric metric $g_{\mu\nu}$ in 3 dimensions. Show that the Riemann tensor, Ricci tensor and Ricci scalar take the simple forms:

$$R_{\lambda\rho\sigma\nu} = K(g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}), \tag{1}$$

$$R_{\sigma\rho} = -2Kg_{\sigma\rho}, \tag{2}$$

$$R = -6K, \tag{3}$$

where K is a constant.

(7p)

Problem 4. Consider the metric $g_{\mu\nu}$ defined by:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - 4(\cosh\frac{x}{2})\left[(\cosh\frac{x}{2})(\mathrm{d}t + \mathrm{d}x) - (\sinh\frac{x}{2})\mathrm{d}y\right]\mathrm{d}x. \tag{4}$$

(a) Write out the components $g_{\mu\nu}$ in matrix form.

(b) Does the metric describe a maximally symmetric space?

(7p)

Problem 5. The Weyl tensor $C_{\mu\nu\rho\sigma}$ is defined in terms of the Riemann tensor as follows:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(D-1)(D-2)}g_{\mu[\rho}g_{\sigma]\nu}R, \tag{5}$$

where D is the dimension of spacetime, and, as usual, $[\mu\nu]$ denote antisymmetrization.

(a) Show that the Weyl tensor is traceless, i.e. that $C_{\rho\mu}{}^{\rho}{}_{\nu} = 0$.

(b) How many algebraically independent components does the Weyl tensor have?

(c) Show that $D = 4$ is the lowest dimension with a *non-trivial* Weyl tensor. Discuss the implications of this for gravitational waves in $D = 3$ versus $D = 4$ dimensions.

(d) Rescale the metric by $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\phi(x)}g_{\mu\nu}$, and find the relation between the corresponding Weyl tensors $C_{\mu\nu\rho\sigma}$ and $\tilde{C}_{\mu\nu\rho\sigma}$.

(e) The angle θ between two spacelike vectors X^μ and Y^μ is defined by the formula

$$g_{\mu\nu}\, X^\mu Y^\nu = \sqrt{X^2}\sqrt{Y^2}\,\cos\theta. \tag{6}$$

Show that the map $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ is indeed conformal, i.e. that it preserves the angle θ.

(9p)

Problem 6.

(a) Derive the Reissner-Nordström solution describing the metric and the electric field outside of an electrically charged black hole. You may assume that this is a static and spherically symmetric solution to the coupled Einstein-Maxwell equations:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= -8\pi G T_{\mu\nu}, \\ g^{\mu\nu}D_\mu F_{\nu\sigma} &= 0, \\ D_{[\mu}F_{\nu]\sigma} &= 0, \end{aligned} \tag{7}$$

where $T_{\mu\nu}$ is the stress tensor for Maxwell theory.

(b) Discuss the horizon structure of the Reissner-Nordström solution, and compare with that of the Schwarzschild solution. Discuss and compare also the fate of an observer falling behind the horizons in both solutions.

(9p)

Problem 7. (In the following questions, "brief" means no more than 10-15 lines of text.)

(a) Briefly describe the standard model of cosmology.

(b) Give a brief account of the main problems of the standard model of cosmology.

(c) Describe briefly in words what cosmological inflation is. Which of the problems discussed in **(b)** are solved by cosmological inflation? Sketch the main arguments for how inflation solves these problems.

(6p)

Problem 8. Consider the 4-dimensional spacetime described by the *Kasner metric*:

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2. \tag{8}$$

(a) Show that Einstein's vacuum equations enforce the following constraints:

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \tag{9}$$

(b) Is there any cosmological singularity in the metric (8)?

(c) Using the results of **(a)**, comment on the dynamics of the spatial slices as $t \to 0$. Make a comparison with the FRW metric.

(7p)

Problem 9.

(a) Using the continuity equation, derive how the energy density ρ evolves with the scale factor a, for matter with the general equation of state $p = w\rho$.

(b) Restrict the result of **(a)** to the 3 special cases when the energy density is completely due to dust, radiation or vacuum energy. Compare the behavior in the 3 cases, and explain why they differ.

(b) Derive the form of the time evolution of the scale factor $a(t)$ for matter with the equation of state $p = w\rho$ and restricted to $\rho > 0, p \geq 0$ using the FRW equations and the continuity equation. You may assume the general form $a(t) \sim t^\beta$.

(7p)

Problem 10. A rotating black hole is described by the Kerr metric, which can be written as follows in so called Boyer-Lindquist coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) + (r^2+a^2)\sin^2\theta\mathrm{d}\phi^2 + \frac{2Mr}{\Sigma}\left(a\sin^2\theta\mathrm{d}\phi - \mathrm{d}t\right)^2, \tag{10}$$

where M is the mass and a is a parameter such that $J = Ma$ is the angular momentum, and we have defined the quantities

$$\Delta \equiv r^2 - 2Mr + a^2, \qquad \Sigma \equiv r^2 + a^2\cos^2\theta. \tag{11}$$

In the questions below, please provide motivations and the calculational details that underlie your answer.

(a) Is the Kerr metric static?

(b) Does it have spherically symmetric spacelike slices?

(c) Is the metric asymptotically flat in the same sense as the Schwarzschild metric?

(d) What is the geometry of a massless Kerr black hole?

(e) Show that there is a limit in which Kerr $\to$ Schwarzschild.

(f) Comment on the singularity structure of the Kerr metric. You may find it useful to study the limit in **(d)**. Does the Kerr metric have a horizon?

(g) Find all the Killing vectors of the Kerr metric. Intelligent guesses give partial credit. For full credit, use a method that guarantees you *all* Killing vectors in a systematic manner.

(8p)

Problem 11. The cosmological constant problem refers to the discrepancy of about 120 orders of magnitude between the observed value of the cosmological constant Λ and the value predicted from quantum field theory estimates of the vacuum energy. The main question is then: *Why is the vacuum energy today so small?*

Read sections 1-3 of the survey article by Bousso available at this link:

`http://arxiv.org/pdf/0708.4231v2.pdf`

and address the following questions (you may of course use other sources as well if you like).

(a) Give the main arguments for why it is clear that the value of Λ today is very constrained, and must be extremely small.

(b) Could one imagine solving the cosmological constant problem by some kind of modification of gravity at large or small distances?

(c) One speculative proposal for a resolution of the cosmological constant problem has been to invoke some unknown mechanism that correlates the vacuum energy with the energy density of dust and radiation so that they are always be of comparable size. In other words, this would be a mechanism that ensures that vacuum energy is transferred into matter and radiation energy, keeping the total vacuum energy lower than expected. What is Bousso's argument against this idea? Do you agree with his argument?

(7p)

Good Luck!