Home examination, Gravitation & Cosmology, 2010

To be handed in January 14, 2011

The maximum score for home assignments (10+10 points) and home examination (80 points) is 100 points. 50 points is minimum requirement (see the course web page for Chalmers and GU grading).

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1. Show that for any tensors A and B,

$$\int d^D x \sqrt{|g|} D_{\mu_1} A_{\mu_2 \ldots \mu_n} B^{\mu_1 \ldots \mu_n} = -\int d^D x \sqrt{|g|} A_{\mu_2 \ldots \mu_n} D_{\mu_1} B^{\mu_1 \ldots \mu_n}$$

if boundary terms vanish.
(6 points)

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2. The Schwarzschild metric, which is static and spherically symmetric, describes the geometry outside a non-rotating spherically symmetric mass distribution. If the mass (planet, star, etc.) also rotates, this will change the metric.
How many isometries would be expected from such a solution?
Find information about the *Kerr metric*, which describes this type of space-time (you don't need to check that Einstein's equations are fulfilled).
Was it correct to neglect effects from the rotation of the earth in *e.g.* the first problem of the second home assignment?
(6 points)

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3. The Weyl tensor is defined as

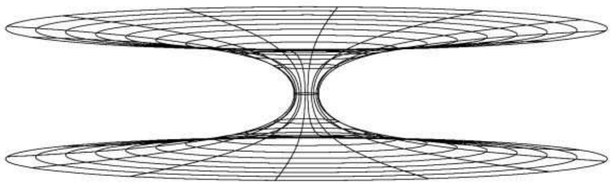
$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(D-1)(D-2)} g_{\mu[\rho}g_{\sigma]\nu}R \ .$$

Check that $C_{\mu\rho\nu}{}^{\rho} = 0$. How many algebraically independent components does such a tensor have? Check that $D = 4$ is the lowest dimensionality where the Weyl tensor is not automatically zero, and discuss the implications for the existence of gravitational waves. If two metrics g and $\tilde{g}$ are conformally equivalent, *i.e.*, if $\tilde{g}_{\mu\nu}(x) = e^{2\phi(x)} g_{\mu\nu}(x)$, how are the corresponding Weyl tensors related? (Brute force calculation does not necessarily provide the most efficient solution of the last problem.)
(12 points)

4. Consider the space-time geometry

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)d\Omega^2 \ ,$$

where $d\Omega^2$ is the maximally symmetric metric on S^2. Similar geometries have been proposed to describe "wormholes", tunnels between different regions of space-time. The "radial coordinate" r can take negative as well as positive values. Calculate the affine connection and investigate whether or not there are time-like geodesics traversing the wormhole (*i.e.*, going from large positive r to large negative r or vice versa). Calculate the Riemann tensor. Find all isometries of the metric. Find the energy-momentum tensor needed in order to make the geometry a solution to Einstein's equations. Are there any problems (in principle, not technical) with engineering such a solution?
(12 points)



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5. When one observes distant luminous objects, their emitted light becomes redshifted due to the expansion of the universe. We assume that both the emitting object and the observer are at rest in the standard coordinates. The redshift is commonly defined by the parameter

$$z = \frac{\lambda - \lambda_0}{\lambda_0} \ ,$$

where λ_0 is the emitted wavelength and λ the observed one. Show that it can be expressed in terms of the scale factor of the universe at the times of emission and observation as

$$z = \frac{a(t)}{a(t_0)} - 1 \ .$$

Is it correct to interpret the redshift as a Doppler shift corresponding to the relative velocity of emitter and observer due to the expansion of the universe? What is the value of z for the cosmic background radiation? Also, find information of the value for the most distant observed galaxies.
(6 points)

6. The mathematician and philosopher Kurt Gödel invented a strange solution to Einsteins's equations, which is said to have been presented to Einstein on the occasion of his 70'th birthday. It requires an energy-momentum tensor coming from a combination of dust and a cosmological constant. In a suitably chosen coordinate system, the Gödel metric reads

$$ds^2 = \frac{1}{2\omega^2}\left[-(dt + e^x dz)^2 + dx^2 + dy^2 + \tfrac{1}{2}e^{2x}dz^2\right]$$

Find all the Killing vectors of this metric.
(The Gödel metric has some remarkable properties, including the presence of closed time-like curves. Read more, for example on wikipedia, if you are interested.)
(16 points)

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7. A beacon radiating at a fixed frequency ν_0 is released at time $t = 0$ towards a black hole of mass M by an observer situated very far away from the black hole. The observer stays at a constant distance from the black hole while the probe is falling. Show that the frequency of the beacon (when it is close to the event horizon) as measured by the observer can be written as $\nu \sim e^{-\frac{t}{K}}$ for some constant K and relate the constant K to the mass of the black hole.
(10 points)

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8. Describe, either using the geodesic equation, or by symmetry methods, or any way you prefer, time-like and light-like geodesics in an anti-de Sitter space-time.
(12 points)