2ND HOMEWORK ASSIGNMENT

–GRAVITATION AND COSMOLOGY (FFM071)–
SPRING SEMESTER 2014

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Instructions: Please use the conventions of Weinberg for all problems[1], and write out explicitly the formulas you are using. Unless otherwise specified, write all derivations explicitly.

There are 5 problems, with a maximum score of 10p.

Deadline is Wednesday 26/2 at 10 am.
If your solutions are TeXed you may email them directly to Christian, and otherwise deliver them in Christian's mailbox on the sixth floor of the Origo building.

[1] For covariant derivatives you may use Weinberg's notation, e.g. $V_{\mu;\nu}$, or the more standard one $D_\nu V_\mu$.

Problem 1. Indicate whether the following statements are true or false. (only answer **true** or **false**; no calculations or motivations needed!) For every wrong answer you get -0.5p (but the minimum score is still 0p).

• If $X_{\mu\nu}$ is a $(0,2)$-tensor then $X^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} X_{\mu\nu}$ is the inverse of $X_{\mu\nu}$.

• $D_\lambda D_\rho g_{\mu\nu} = D_\rho D_\lambda g_{\mu\nu}$

• An observer at infinity would measure that it takes a probe particle an infinite amount of proper time to cross the horizon of a Schwarzschild black hole.

• When our sun runs out of hydrogen to burn, it will start to grow and eventually collapse into a black hole.

• Let $T^{\mu\nu}$ be a $(2,0)$-tensor. Each individual component of $T^{\mu\nu}$ is then a scalar.

• The Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ in an curved coordinate frame x^μ is a tensor with respect to arbitrary coordinate transformations.

• The Principle of General Covariance implies that the equations of motion of a physical theory in a curved background must be invariant under general coordinate transformations.

• In vacuum, Einstein's equations reduce to $g^{\mu\nu} R_{\mu\nu} = 0$.

• Let $F_{\mu\nu}$ be the field strength in Maxwell theory in a spacetime with metric $g_{\mu\nu}$. Then $g^{\mu\nu} F_{\mu\nu} = 0$.

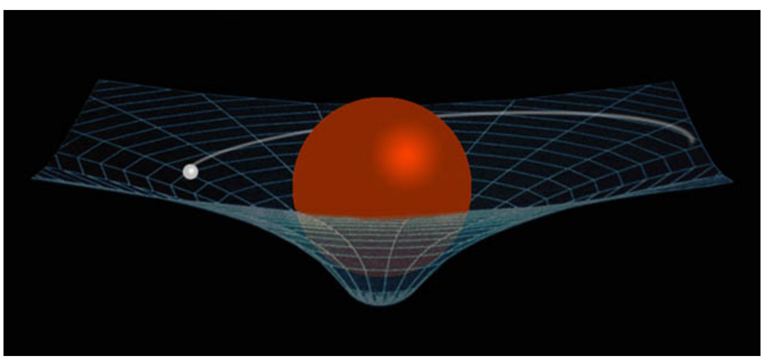
• If you fall through the horizon of a Reissner-Nordström black hole you are doomed to hit the singularity.

(2p)

Problem 2. Do any of the following equations fail to make sense for generic[2] coordinate systems? If yes, indicate which ones. ($\Gamma^{\lambda}_{\mu\nu}, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, g_{\mu\nu}, g, \eta_{\mu\nu}, D_\mu, F_{\mu\nu}$ are the objects defined in class. Primes indicate arbitrary coordinate transformations.)

$$
\begin{array}{lrcl}
(1) & D_\mu V_\nu + D_\nu V_\mu & = & \partial_\mu V_\nu + \partial_\nu V_\mu \\
(2) & D_\mu \phi & = & \partial_\mu \phi \\
(3) & \Gamma^{\mu}_{\mu\lambda} & = & \dfrac{1}{\sqrt{g}}\partial_\lambda \sqrt{g} \\
(4) & g^{\mu\nu}\left(D_\mu T^{\lambda}{}_{\lambda\nu} - D_\rho T^{\rho}{}_{\mu\nu}\right) & = & D_\sigma\left(T^{\lambda}{}_{\lambda}{}^{\sigma} - T^{\sigma\nu}{}_{\nu}\right) \\
(5) & \dfrac{\partial x^\rho}{\partial x'^\mu}\dfrac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} & = & g_{\mu\nu} \\
(6) & \partial_\mu V^\mu & = & \partial^\mu V_\mu \\
(7) & \Gamma^{\lambda}_{[\mu\nu]} & = & 0 \\
(8) & D^\rho(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) & = & 0 \\
(9) & R_{[\mu\nu\rho\sigma]} & = & 0 \\
(10) & \mathrm{d}^4 x' \sqrt{g'(x')} & = & \mathrm{d}^4 x \sqrt{g(x)}
\end{array}
$$

(1p)



[2] "generic" implies that the equations should be sensible for an arbitrary coordinate frame of the type indicated by the index structure. For instance, the components of a 4-vector are generically non-vanishing.

Problem 3. Let $R^{\lambda}{}_{\mu\nu\kappa}$ as defined in eq. 6.1.5 of Weinberg.

(a) Define its covariant version as $R_{\lambda\mu\nu\kappa} := g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\kappa}$ and derive the following properties:

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \tag{11}$$

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} \tag{12}$$

$$R_{\lambda\mu\nu\kappa} = -R_{\lambda\mu\kappa\nu} \tag{13}$$

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0 \tag{14}$$

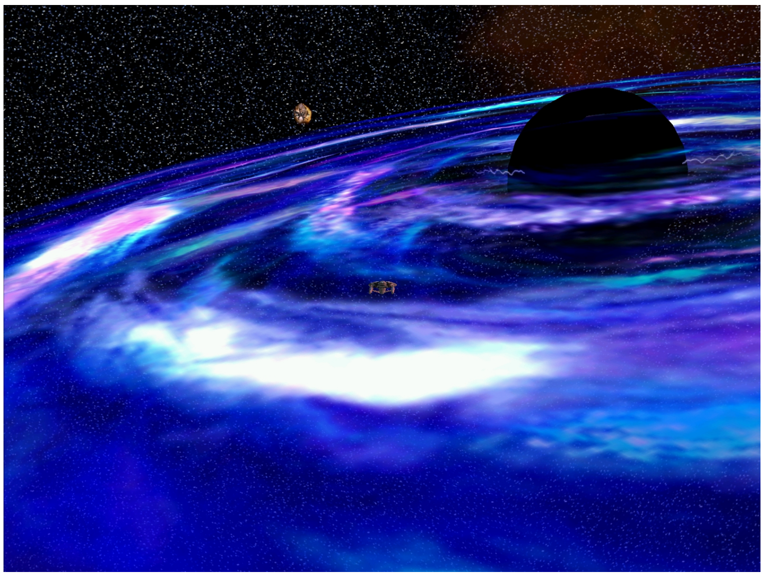
(b) Use your results in **(a)** to show that

$$\varepsilon^{\lambda\mu\nu\kappa} R_{\lambda\mu\nu\rho} = 0. \tag{15}$$

(2p)

Problem 4. Imagine that you are standing just outside the horizon of a black hole. Is it possible to see oneself from behind for some radius? Put differently, are there any closed photon orbits?

(2p)



Problem 5. Consider the action of a matter system in some curved background with metric $g_{\mu\nu}$:

$$S_M = \int \mathrm{d}^4 x \sqrt{g} \mathcal{L}_M. \tag{16}$$

(a) The stress-tensor for the matter system with this action is defined as

$$T^{\mu\nu} := -\frac{1}{\sqrt{g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \tag{17}$$

Show that invariance of the action $\delta S = 0$ under a diffeomorphism $\delta g_{\mu\nu}$ of the metric (i.e. induced by an infinitesimal coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu(x)$) implies the covariant conservation law for the stress tensor:

$$D_\mu T^{\mu\nu} = 0. \tag{18}$$

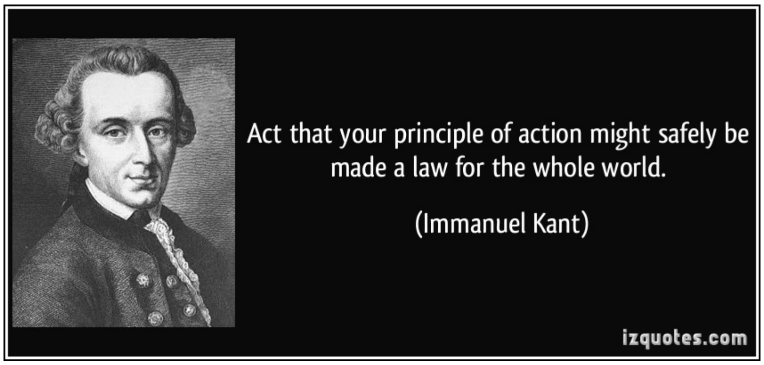
This is Noether's law.

(b) Consider now the special case when the matter component is that of Maxwell theory:

$$S_M[A_\mu] = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}, \tag{19}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ and $F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}$. Derive the stress tensor $T_{\mu\nu}$ of Maxwell theory from this action and verify that it is covariantly conserved.

(3p)



Good Luck!