1ST HOMEWORK ASSIGNMENT

–GRAVITATION AND COSMOLOGY (FFM071)– SPRING SEMESTER 2014

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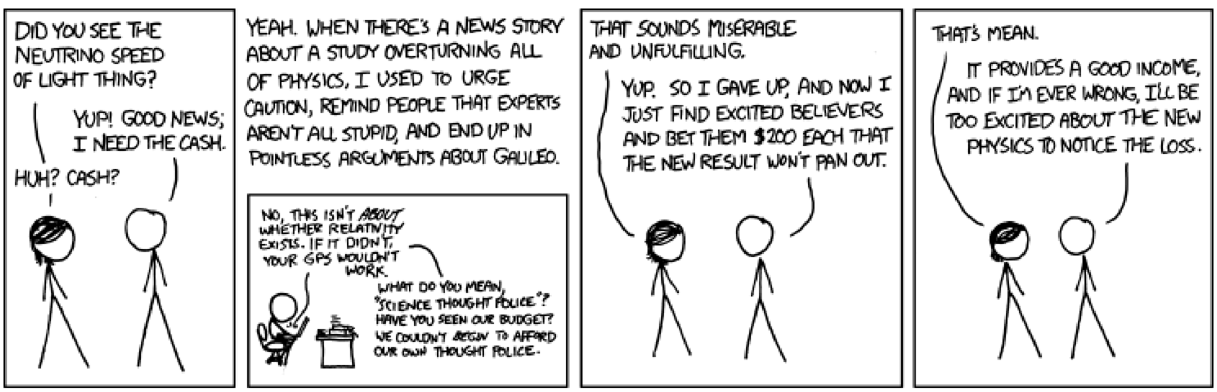


Instructions: Please use the conventions of Weinberg for all problems[1], and write out explicitly the formulas you are using. Unless otherwise specified, write all derivations explicitly.

There are 5 problems, each of which gives a maximum score of 2p.

Deadline is Wednesday 5/2 at 10 am.
If your solutions are TEXed you may email them directly to Christian, and otherwise deliver them in Christian's mailbox on the sixth floor of the Origo building.

[1]For covariant derivatives you may use Weinberg's notation, e.g. $V_{\mu;\nu}$, or the more standard one $D_\nu V_\mu$.

Problem 1. Indicate whether the following statements are true or false. (only answer **true** or **false**; no calculations or motivations needed!) For every wrong answer you get -0.5p (but the minimum score is still 0p).

- The Minkowski metric $\eta_{\alpha\beta}$ is invariant under Lorentz transformations.
- The speed of light c is invariant under Galilean transformations.
- The first postulate of special relativity states that the outcome of a physical experiment will depend on the coordinate system you use.
- Your past worldline lies outside of the lightcone.
- An observer at rest measures the lifetime of a muon in motion to be longer than that of a muon at rest.
- An arbitrary spacetime metric $g_{\mu\nu}$ is invariant under general coordinate transformations.
- The affine connection $\Gamma^{\lambda}_{\mu\nu}$ is a mixed $(1,2)$-tensor with respect to general coordinate transformations.
- The phenomenon of time dilation states that the coordinate time intervals of a clock at rest are longer than for a clock in motion.
- The components of the Kronecker delta symbol $\delta^{\mu}{}_{\nu}$ are the same in all coordinate systems.
- When we take the Newtonian limit we may assume that $\mathrm{d}x/\mathrm{d}\tau \ll \mathrm{d}t/\mathrm{d}\tau$.

(2p)

Problem 2. The following is a collection of small problems which concerns simple manipulations with tensors and indices. You don't need to provide derivations here, simply give the answer.

Do any of the following equations fail to make sense for generic[2] coordinate systems? If yes, indicate which ones.

$$\begin{array}{llcl}
(1) & \eta_{\alpha\beta}\Phi^{\beta} & = & \Phi_{\alpha} \\
(2) & \Lambda^{\alpha_1}{}_{\beta_1}\Lambda^{\alpha_2}{}_{\beta_2}\eta_{\alpha_1\alpha_2}\Lambda^{\beta_1}{}_{\gamma_1}\Lambda^{\beta_2}{}_{\gamma_2}\mathrm{d}x^{\gamma_1}\mathrm{d}x^{\gamma_2} & = & \eta_{\delta_1\delta_2}\mathrm{d}x^{\delta_1}\mathrm{d}x^{\delta_2} \\
(3) & V^{\alpha}{}_{\alpha} & = & \mathrm{Tr}(V) \\
(4) & V_{\mu}V^{\mu} & = & V^{\alpha}V_{\alpha} \\
(5) & \mathrm{d}\tau^2 & = & 0.000345 \\
(6) & dx'^{\mu} & = & \Lambda_{\mu}{}^{\nu}dx^{\nu} \\
(7) & \partial_{\alpha}F_{\alpha\beta} & = & -J_{\beta} \\
(8) & D_{\nu}V_{\mu} & = & \partial_{\nu}V_{\mu} - \Gamma^{\sigma}_{\mu\nu}V_{\rho} \\
(9) & g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} & = & \eta_{\alpha\beta}\mathrm{d}\xi^{\alpha}\mathrm{d}\xi^{\beta} \\
(10) & T^{\mu\nu} & = & pg^{\mu\nu} + p + \rho U^{\mu}U^{\nu}
\end{array}$$

(2p)

[2] "generic" implies that the equations should be sensible for an arbitrary coordinate frame of the type indicated by the index structure. For instance, the components of a 4-vector are generically non-vanishing.

Problem 3.

(a) Suppose first that we are in a flat coordinate frame ξ^α. Show that the Kronecker delta $\delta^\alpha{}_\beta$ and the Levi-Civita tensor $\varepsilon^{\alpha\beta\gamma\delta}$ are invariant under Lorentz transformations.

(b) In contrast to **(a)** we now consider an *arbitrary* (curved) coordinate frame x^μ. Show that in such an arbitrary coordinate frame the object $\varepsilon^{\mu\nu\rho\sigma}$ defined by

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ even permutation of } 0,1,2,3 \\ -1 & \mu\nu\rho\sigma \text{ odd permutation of } 0,1,2,3 \\ 0 & \text{two or more indices equal} \end{cases} \tag{11}$$

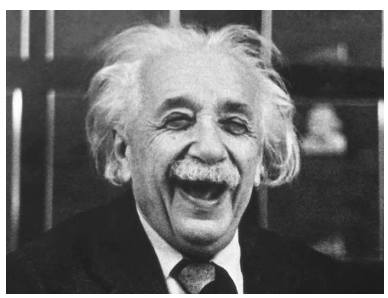
is no longer a tensor, but rather a *tensor density* of weight -1.

(c) Show that the covariant version $\varepsilon_{\mu\nu\rho\sigma} = g_{\mu\kappa} g_{\nu\delta} g_{\rho\gamma} g_{\sigma\lambda} \varepsilon^{\kappa\delta\gamma\lambda}$ satisfies

$$\varepsilon_{\mu\nu\rho\sigma} = -g \varepsilon^{\mu\nu\rho\sigma}, \tag{12}$$

where $g \equiv -\det g_{\mu\nu}$.

(2p)



Problem 4. Let $F_{\alpha\beta}$ be the electromagnetic field strength.

(a) Show that the tensorial Maxwell equation

$$\varepsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0, \tag{13}$$

is equivalent to two of the four Maxwell equations when written in their standard 3-vector form.

(b) Show that eq. (13) can be solved by introducing a vector potential A_α and writing $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

(c) Show that eq. (13) can be equivalently expressed as

(14) $$\partial_\alpha F_{\beta_\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.$$

(d) Explain the concept of *gauge symmetry* in Maxwell theory. Use the gauge symmetry to show that in the absence of sources Maxwell's equation $\partial_\alpha F^{\alpha\beta} = -J^\beta$ reduces to a wave equation:

(15) $$\partial^\alpha \partial_\alpha A_\beta = 0.$$

(2p)

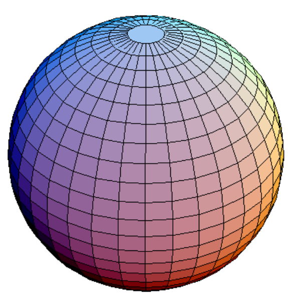
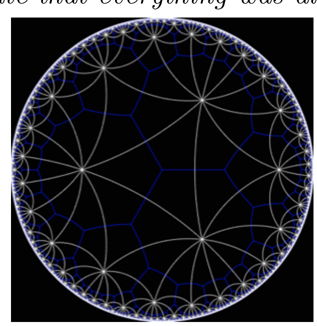
Problem 5. Consider the metrics for two-dimensional hyperbolic space (Poincaré disc model) and for a 2-dimensional unit sphere in polar coordinates:

(16) $$\mathrm{d}s^2 = \frac{\mathrm{d}r^2 + r^2 \mathrm{d}\phi^2}{\left(1 - r^2/a^2\right)^2},$$

(17) $$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2.$$

Calculate the Riemann tensor, Ricci tensor and Ricci scalar for the two metrics above. Compare the results.

Note: You should present all steps of your calculations! Computer programs may be used to check your calculations, but you must demonstrate that everything was also done by hand.

Good Luck!