2ND HOMEWORK ASSIGNMENT

–GRAVITATION AND COSMOLOGY (FFM071)–
SPRING SEMESTER 2013

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Instructions: Please use the conventions of Weinberg for all problems[1], and write out explicitly the formulas you are using. Unless otherwise specified, write all derivations explicitly.

There are 5 problems, each of which gives a maximum score of 2p.

Deadline is Monday 4/3 at 08 am.

[1] For covariant derivatives you may use Weinberg's notation, e.g. $V_{\mu;\nu}$, or the more standard one $D_\nu V_\mu$.

Problem 1. Indicate whether the following statements are true or false. (only answer **true** or **false**; no calculations or motivations needed!) For every wrong answer you get -0.5p (but the minimum score is still 0p).

• If $X_{\mu\nu}$ is a $(0,2)$-tensor then $X^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} X_{\mu\nu}$ is the inverse of $X_{\mu\nu}$.

• $D_\lambda D_\rho g_{\mu\nu} = D_\rho D_\lambda g_{\mu\nu}$

• An observer at infinity would measure that it takes an infinite amount of proper time for a probe particle to cross the horizon of a Schwarzschild black hole.

• When our sun runs out of hydrogen to burn, it will start to grow and eventually collapse into a black hole.

• The components of a tensor do not change under general coordinate transformations.

• The Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ in a non-inertial frame x^μ is a tensor with respect to arbitrary coordinate transformations.

• The Levi-Civita symbol $\epsilon^{\alpha\beta\gamma\delta}$ in a flat inertial frame ξ^α is a tensor density with respect to Lorentz transformations.

• In vacuum, Einstein's equations reduce to $g^{\mu\nu} R_{\mu\nu} = 0$.

• Let $F_{\mu\nu}$ be the field strength in Maxwell theory. Then $g^{\mu\nu} F_{\mu\nu} = 0$.

• Gravity in 3 dimensions admits no gravitational waves.

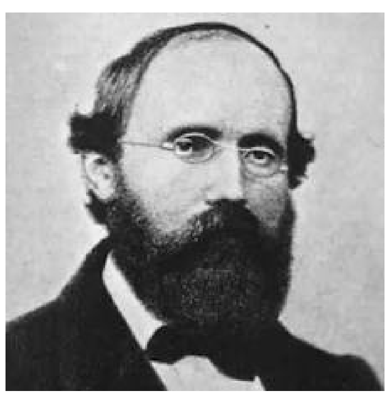
(2p)

Problem 2. The following is a collection of small problems which concerns simple manipulations with tensors and indices. You don't need to provide derivations here, simply give the answer.

Do any of the following equations fail to make sense for generic[2] coordinate systems? If yes, indicate which ones. ($\Gamma^\lambda_{\mu\nu}, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, g_{\mu\nu}, g, \eta_{\mu\nu}, D_\mu, F_{\mu\nu}$ are the objects defined in class.)

$$
\begin{array}{lrcl}
(1) & D_\mu V_\nu - D_\nu V_\mu &=& \partial_\mu V_\nu - \partial_\nu V_\mu \\
(2) & D_\mu \phi &=& \partial_\mu \phi \\
(3) & \Gamma^\mu_{\mu\lambda} &=& \frac{1}{\sqrt{g}}\partial_\lambda \sqrt{g} \\
(4) & g^{\mu\nu}\left(D_\mu T^\lambda{}_{\lambda\nu} - D_\rho T^\rho{}_{\mu\nu}\right) &=& D_\sigma\left(T^\lambda{}_\lambda{}^\sigma - T^{\sigma\nu}{}_\nu\right) \\
(5) & \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} &=& \eta_{\mu\nu} \\
(6) & F_{\mu\nu} g^{\mu\nu} &=& 0 \\
(7) & \Gamma^\lambda_{\mu\nu} &=& \Gamma^\lambda_{[\mu\nu]} \\
(8) & D^\rho(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) &=& 0 \\
(9) & R_{[\mu\nu\rho\sigma]} &=& 0 \\
(10) & \mathrm{d}^4 x' \sqrt{g'(x')} &=& \mathrm{d}^4 x \sqrt{g(x)}
\end{array}
$$

(2p)



[2] "generic" implies that the equations should be sensible for an arbitrary coordinate frame of the type indicated by the index structure. For instance, the components of a 4-vector are generically non-vanishing.

Problem 3. A *Killing vector* is a vector Y_μ that satisfies the equation

$$D_{(\mu}Y_{\nu)} = 0, \tag{11}$$

where D_μ is the covariant derivative. Let $x^\mu(\tau)$ be a geodesic in the geometry described by $g_{\mu\nu}$ for a path parametrized by τ. We shall denote by $U^\mu \equiv \mathrm{d}x^\mu(\tau)/\mathrm{d}\tau$ the tangent to the path. Since $x^\mu(\tau)$ is a geodesic, it obeys the *geodesic equation*

$$\frac{D}{D\tau}\frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau} \equiv \frac{D}{D\tau}U^\mu(\tau) = 0, \tag{12}$$

where $D/D\tau$ is the covariant derivative along the path. Using the information above, show that the quantity $U^\mu Y_\mu$ is covariantly conserved along the path.

(2p)

Problem 4. Consider the action for the coupled Einstein-Maxwell system

$$S[g_{\mu\nu}, A_\mu] = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{g}\Big(R + 2\Lambda\Big) - \frac{1}{4}\int \mathrm{d}^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}, \tag{13}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ and $F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}$. Derive the complete Einstein equations for this system (including the stress tensor for Maxwell theory on the right hand side) by varying the action S with respect to the metric tensor $g_{\mu\nu}$. Provide details of all the calculations, including the variations of $g, g^{\mu\nu}$, and $R_{\mu\nu}$.

(2p)

Problem 5. Consider the Schwarzschild solution

$$\mathrm{d}s^2 = -\left(1 - \frac{2MG}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2MG}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2. \tag{14}$$

(a) Calculate the components of the affine connection $\Gamma^\lambda_{\mu\nu}$ for the Schwarzschild metric.

(b) Use your result in **(a)** to calculate the quadratic curvature scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Interpret the result.

(2p)
