2ND HOMEWORK ASSIGNMENT

–GRAVITATION AND COSMOLOGY (FFM071)– SPRING SEMESTER 2012

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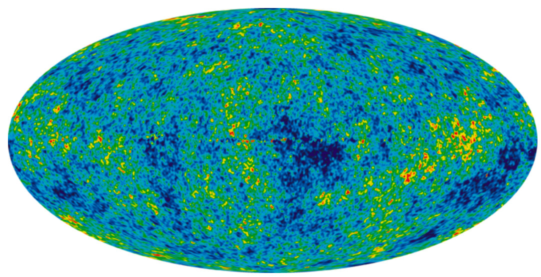


Instructions. Please use the conventions of Weinberg for all problems, and write out explicitly the formulas you are using, e.g. for the metric, affine connection, Ricci, Riemann tensors etc. Also, unless otherwise specified, write all derivations explicitly. E.g. for the curvature tensors, provide details of your calculations for all components. In other words, expressions like "analogously we find" are not approved.

Deadline is Monday 5/3 at 10 am.

Problem 1. The following is a collection of small problems which concerns manipulations with tensors and indices.

(a) Let A, B be matrices with components $A_{\mu\nu}, B_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$. Indices are lowered and raised with the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. Write out the following trace in component form:

$$\text{Tr}\left(ABA^{-1}\right), \tag{1}$$

and simplify the result as much as possible.

(b) Let $T_{\mu\nu\rho}$ be a tensor. Write out the explicit expressions for the completely symmetric part $T_{(\mu\nu\rho)}$ and the completely antisymmetric part $T_{[\mu\nu\rho]}$ in terms of $T_{\mu\nu\rho}$. Make sure you get the numerical normalization factors right.

(c) The Riemann tensor $R_{\mu\nu\rho\sigma}$ is an example of a tensor of mixed symmetry. Using your result of **1(b)** above, write out the explicit form of the antisymmetrization $R_{\mu[\nu\rho\sigma]}$. Then use the known symmetry properties of the Riemann tensor to simplify the expression. What is the result?

Problem 2. Let $\xi_\mu(x)$ be a Killing vector associated with a metric $g_{\mu\nu}$, and let $x^\mu(\lambda)$ be a geodesic in the geometry described by $g_{\mu\nu}$ for some path $\gamma(\lambda)$ parametrized by λ. We shall denote by $U^\mu \equiv \mathrm{d}x^\mu(\lambda)/\mathrm{d}\lambda$ the tangent to the path. Since $x^\mu(\lambda)$ is a geodesic, it obeys the *geodesic equation*

$$\frac{D}{D\lambda}\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} \equiv \frac{D}{D\lambda}U^\mu(\lambda) = 0, \tag{2}$$

where $D/D\lambda$ is the covariant derivative along the path $\gamma(\lambda)$.

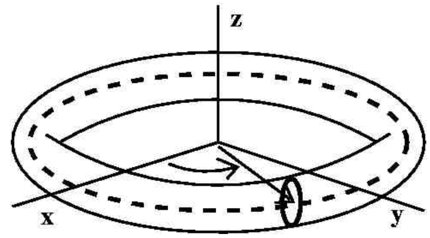
Using the information above, show that the quantity $U^\mu\xi_\mu$ is covariantly conserved along the path $\gamma(\lambda)$.

Hint: Use that the covariant derivative along a path is given by $D/D\lambda = (dx^\mu/d\lambda)D_\mu$, where D_μ is the standard covariant derivative.

Problem 3. Consider a torus T^2 with angular coordinates ϕ_1, ϕ_2. You can think of the torus as a cylinder whose endpoints have been glued together. Let ϕ_1 be the coordinate along the circle which surrounds the hole, and ϕ_2 the coordinate along the circle of the cylinder. We embed the torus into 3-dimensional Euclidean space $\mathbb{R}^3$, and in Cartesian coordinates the embedding is described explicitly by:

$$\begin{aligned} x &= (r_1 + r_2\cos\phi_2)\,\cos\phi_1, \\ y &= (r_1 + r_2\cos\phi_2)\sin\phi_1, \\ z &= r_2\sin\phi_2, \end{aligned} \tag{3}$$

where r_1 is the radius of the circle surrounding the hole of the torus, while r_2 is the radius of the cylinder. We restrict to $r_1 > r_2$.



(a) Find the metric of the torus T^2 using the embedding into $\mathbb{R}^3$ defined by (3).

(b) Compute the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ for the metric on T^2 obtained in **(a)**. Comment on the result for different points on the torus.

Problem 4. Consider the Schwarzschild solution

$$\mathrm{d}s^2 = -\left(1 - \frac{2MG}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2MG}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{4}$$

(a) Discuss the physical interpretation of the various special limits of the radius r, namely $r = \infty, r > 2MG, r = 2MG, r < 2MG$ and $r = 0$.

(b) You are an observer at infinity $r = \infty$ looking at a light ray on a radial null geodesic towards the Schwarzschild horizon. What would you see as the light ray approaches the horizon?

(c) Discuss briefly the meaning of the diagram in Figure 1.

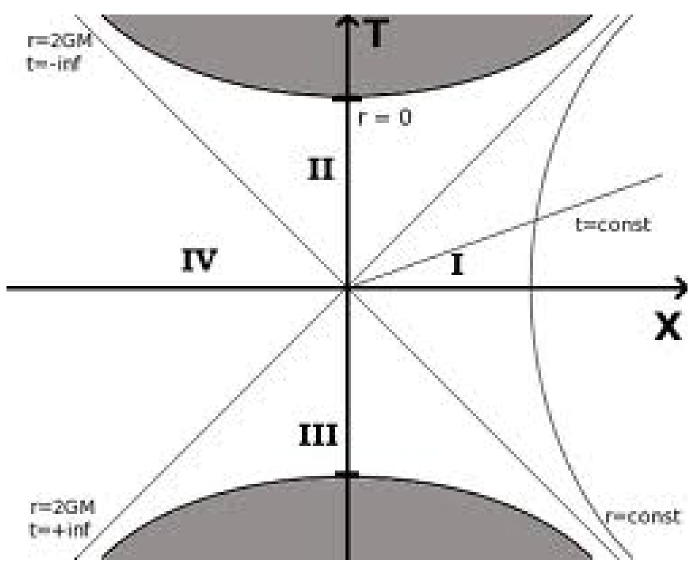


Figure 1. The Kruskal diagram of the Schwarzschild solution.