1ST HOMEWORK ASSIGNMENT

–GRAVITATION AND COSMOLOGY (FFM071)– SPRING SEMESTER 2012

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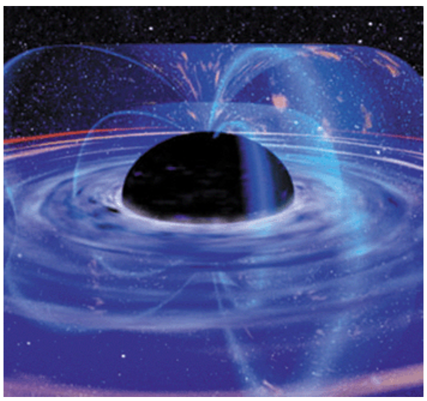
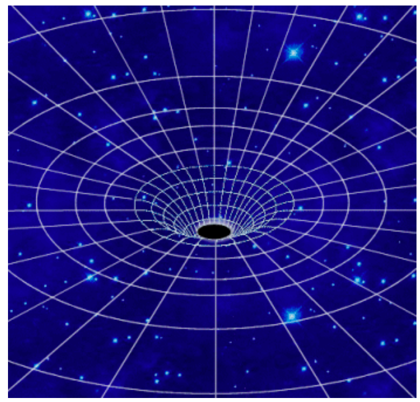




Problem 1. The following is a collection of small problems which concerns simple manipulations with tensors and indices. You don't need to provide derivations here, simply give the answer.

(a) Do any of the following equations make sense for generic[1] coordinate systems? If yes, indicate which ones.

$$
\begin{aligned}
&(1) \qquad & \eta_{\alpha\beta}V^{\beta} &= V^{\alpha} \\
&(2) \qquad & \Lambda^{\alpha_1}{}_{\beta_1}\Lambda^{\alpha_2}{}_{\beta_2}\eta_{\alpha_1\alpha_2}\Lambda^{\alpha_3}{}_{\beta_3}\Lambda^{\alpha_4}{}_{\beta_4}\eta_{\alpha_3\alpha_4}V^{\beta_1} &= \eta_{\beta_3\beta_4}V_{\beta_2} \\
&(3) \qquad & V_{\alpha}V^{\alpha} &= V^{\mu}V_{\mu} \\
&(4) \qquad & V_iV^i &= V_{\alpha}V^{\alpha} \\
&(5) \qquad & \mathrm{d}\tau^2 &= 5763 \\
&(6) \qquad & T^{\mu\nu} &= 37
\end{aligned}
$$

[1] "generic" implies that the equations should be sensible for an arbitrary coordinate frame of the type indicated by the index structure. For instance, the components of a 4-vector are generically non-vanishing.

(b) Let $T^{\alpha\beta}{}_{\gamma\delta}$ be a tensor, and define the combination $\widetilde{T}^{\alpha\beta}{}_{\gamma\delta} \equiv \frac{1}{2}(T^{\alpha\beta}{}_{\gamma\delta} - T^{\alpha\beta}{}_{\delta\gamma})$. Evaluate the expression:

$$\widetilde{T}^{\alpha\beta}{}_{\gamma\delta}\eta^{\gamma\delta} \tag{7}$$

and identify the resulting object. Is it a tensor?

Problem 2. Show that Maxwell's equations in four-dimensional Minkowski space can be written as

$$\frac{\partial}{\partial x^\alpha}F^{\alpha\beta} = -J^\beta, \tag{8}$$

$$\varepsilon^{\alpha\beta\gamma\delta}\frac{\partial}{\partial x^\beta}F_{\gamma\delta} = 0, \tag{9}$$

where $F^{\alpha\beta}$ is the electromagnetic field strength tensor and $\varepsilon^{\alpha\beta\gamma\delta}$ is the 4-dimensional totally antisymmetric Levi-Civita tensor. Use conventions where the Minkowski metric is given by

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Problem 3. Consider an arbitrary (non-inertial) coordinate frame x^μ, $\mu = 0, 1, 2, 3$.

(a) Consider the integral

$$\int d^4x\sqrt{g}\Phi(x). \tag{11}$$

Which constraints must you put on the object $\Phi(x)$ for the integral to be invariant under general coordinate transformations?

(b) If we want to construct an invariant integral like in (11), not for a scalar $\Phi(x)$, but for a tensor $V_{\mu\nu\rho\sigma}(x)$ which is antisymmetric in all its indices, which object must replace $\sqrt{g}$?

(c) Consider now 2-dimensional spacetime with arbitrary coordinate frame $x^\mu, \mu = 0, 1$. Is the following integral

$$\int d^2x g_{\mu\nu}\varepsilon^{\mu\nu}\Upsilon(x) \tag{12}$$

well-defined? Does the value of the integral depend on the properties of the object $\Upsilon(x)$?

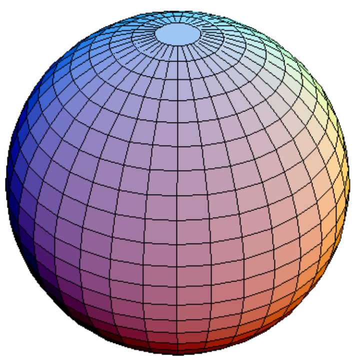
Problem 4. Consider the metrics for two-dimensional hyperbolic space (Poincaré disk model) and for a 2-dimensional unit sphere in polar coordinates:

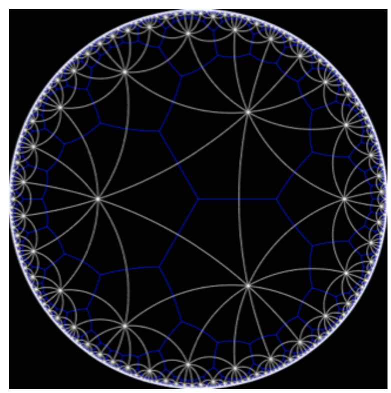
$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2 + r^2\mathrm{d}\phi^2}{(1 - r^2/a^2)^2}, \tag{13}$$

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2. \tag{14}$$

(a) What is the interpretation of the parameter a in the hyperbolic metric? Why is there no such parameter in the metric for a unit sphere?

(b) Calculate the Riemann tensor, Ricci tensor and Ricci scalar for the two metrics above. Compare the results.





Problem 5. Consider the action of a scalar field $\phi(x)$ in 4-dimensional spacetime

$$S[\phi] = \int d^4x \sqrt{g} g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x). \tag{15}$$

For this problem you will need to use the *functional derivative*, i.e. derivative with respect to functions rather then just coordinates. The functional derivative $\frac{\delta}{\delta J(x)}$ with respect to some function $J(x)$ is defined by either of the following two equivalent statements:

$$\frac{\delta}{\delta J(x)} J(y) \equiv \delta^{(4)}(x - y), \qquad \frac{\delta}{\delta J(x)} \int d^4y J(y) f(y) = f(x), \tag{16}$$

where x and y denote spacetime points and $\delta^{(4)}(x)$ is Dirac's delta function in 4 dimensions. This definition of the functional derivative is a natural generalization of the familiar rule for derivatives of 3-vectors:

$$\frac{\partial}{\partial x^i}x^j = \delta^{ij}, \qquad \frac{\partial}{\partial x^i}\sum_j x^j k^j = k^i. \tag{17}$$

(a) Find the equation of motion for $\phi(x)$ by varying the action (15) with respect to $\phi(x)$, i.e. calculate the functional derivative $\frac{\delta S}{\delta \phi}$ and set it to zero. Show that the resulting equation of motion can be written in terms of the covariant derivative D_μ, i.e. as $D^\mu D_\mu \phi(x) = 0$.

(b) Calculate the Laplacian $D^\mu D_\mu$ on the 2-sphere S^2 and compare it to the standard expression for the 2-dimensional Laplacian in spherical coordinates, as found e.g. in the Mathematics Handbook (Beta). Do the results agree? If, not, why?

(c) Compute the stress energy tensor $T^{\mu\nu}$ of a scalar field ϕ by varying the action (15) with respect to the metric $g_{\mu\nu}$, as explained during Lecture 6:

$$\delta S[\phi] = \int d^4x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu}. \tag{18}$$

(d) Show that the $T^{\mu\nu}$ so obtained is covariantly constant. What is the physical interpretation of this?