Home assignments 1, Gravitation & Cosmology, 2008

To be handed in Friday, November 21

1. Show that the invariant integration measure is $d^4x\sqrt{|\det g|}$, *i.e.*, that

$$\int d^4x\sqrt{|\det g|}\,\Phi(x)$$

is independent of the choice of coordinates when Φ is a scalar. Give the integration measure for 2- and 3-dimensional euclidean space in polar and spherical coordinates.

Show also that

$$\tfrac{1}{24}\int d^4x\epsilon^{\mu\nu\kappa\lambda}\Psi_{\mu\nu\kappa\lambda}(x)$$

is invariant, where Ψ is a tensor which is antisymmetric in all four indices.

2. Construct the Laplacian (in the case of euclidean signature) / the wave operator (in the case of lorentzian signature), acting on a scalar, from covariant derivatives, and give an explicit form in terms of derivatives and the affine connection.

As an alternative, one may consider the action for a massless scalar field,

$$S=-\tfrac{1}{2}\int d^4x\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

(g denotes the determinant of $g_{\mu\nu}$), whose variation will give the wave equation. Show that this gives the same equation as the construction with covariant derivatives.

Calculate the Laplacian on S^2 and compare it to the known explicit expression for the square of the angular momentum in three dimensions (used *e.g.* when separating variables in the solution of the wave function of the hydrogen atom).

3. Show that the tensor $T^{\mu\nu}$, defined from the action for a scalar field in the previous exercise as $T^{\mu\nu}(x) = \frac{1}{\sqrt{|\det g|}} \frac{\delta S}{\delta g_{\mu\nu}(x)}$, identically satisfies $D_\nu T^{\mu\nu} = 0$. Give a physical interpretation of $T^{\mu\nu}$ and of the differential equation.

4. As an example of a space with constant negative curvature, take the simplest example, the two-dimensional "Poincaré disk". Its metric can be written as

$$ds^2 = \frac{dr^2 + r^2 d\phi^2}{(1 - \frac{r^2}{a^2})^2} \ ,$$

where $0 \leq r < a$ and $0 \leq \phi < 2\pi$ (ϕ is an angle). What is the distance from a point with coordinates (r, ϕ) to the "boundary" $r = a$? Calculate the curvature tensor, the Ricci tensor and the curvature scalar. Compare to the corresponding results for a two-dimensional sphere.